Homework 4

Combinatorics I (Fall 2017)
Rutgers University
Swastik Kopparty

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1. Let S be a set of $\omega(q)$ lines in the projective plane over $\mathbb{F}_q$. We want to show that the union of all lines in S has size $(1 - o(1)) \cdot q^2$.
   (a) Using inclusion-exclusion (more precisely, the Bonferroni inequality), show that that the union of all lines in S has size $\geq \frac{q^2}{2}$.
   (b) Let A denote the incidence matrix of the projective plane over $\mathbb{F}_q$ (ie. the 0/1 matrix with rows indexed by points and columns indexed by lines, and a 1 in entry (p, ℓ) denotes that line ℓ passes through point p). Compute $A^T A$.
   (c) Let 1_S be the indicator vector of S. Compute the norms $\|A1_S\|_1$ and $\|A1_S\|_2$, and use this to show that the union of all lines in S has size $(1 - o(1)) \cdot q^2$.
2. Let Σ be a finite set of cardinality q. Formulate and prove a version of the Sauer-Shelah lemma for subsets of Σ^n (The original Sauer-Shelah lemma deals with the case $q = 2$).
3. (a) Let S be a set of n points in $\mathbb{R}^m$. Let $\mathcal{F}$ be the family of subsets of S which are of the form $S \cap \{x \in \mathbb{R}^d \mid Q(x) = 0\}$ for some degree $\leq d$ polynomial $Q(X_1, \ldots, X_m) \in \mathbb{R}[X_1, \ldots, X_m]$.
   Show that the VC dimension of $\mathcal{F}$ is at most $\binom{m+d}{d}$. Thus deduce an upper bound on $|\mathcal{F}|$. Bonus: Show that this bound is tight.
   (b) Let $Q_1, \ldots, Q_k$ be polynomials in $\mathbb{R}[X_1, \ldots, X_m]$ of degree at most d. Let $\mathcal{G}$ be the family of all subsets of $[k]$ which are of the form $\{i \in [k] \mid Q_i(x) = 0\}$ for some $x \in \mathbb{R}^m$.
   For each $G \in \mathcal{G}$, let $Q_G(X_1, \ldots, X_m)$ be the polynomial $\prod_{i \notin G} Q_i(X_1, \ldots, X_m)$. Show that the Q_G, as G varies in $\mathcal{G}$, are linearly independent. Thus deduce an upper bound on $|\mathcal{G}|$.
4. Let L be a set of s nonnegative integers. Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be such that $|A \cap B| \in L$ for distinct $A, B \in \mathcal{F}$. We will prove the uniform Ray-Chaudhuri Wilson inequality:
$$|\mathcal{F}| \leq \binom{n}{s}.$$
   (In class we showed the weaker inequality $|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1} + \ldots + \binom{n}{0}$).
   For $A \in \mathcal{F}$, let $f_A(X_1, \ldots, X_n) \in \mathbb{R}[X_1, \ldots, X_n]$ be the polynomial
$$f_A(X_1, \ldots, X_n) = \prod_{\ell \in L} \left(\sum_{i \in A} X_i - \ell \right).$$
   For $I \subseteq [n]$, let $v_I \in \{0,1\}^n$ be the indicator vector of I. For $J \subseteq [n]$, let $x_J : \{0,1\}^n \to \mathbb{R}$ be the function $x_J(v) = \prod_{j \in J} v_j$.

(a) Suppose $f : \{0,1\}^n \to \mathbb{R}$ is such that $f(v_I) \neq 0$ for each $I \subseteq [n]$ with $|I| \leq t$. Show that the functions $(f \cdot x_J)_{|J| \leq t}$ are linearly independent.

(b) Show that the $|\mathcal{F}| + \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{s-1}$ functions:

- f_A with $A \in \mathcal{F}$,
- $x_J \cdot (\sum_{i=1}^n x_i - k)$ with $|J| \leq s-1$,

are all linearly independent.

(c) Deduce that $|\mathcal{F}| \leq \binom{n}{s}$.