Homework 3

Combinatorics I (Fall 2017)
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Due Date: November 16, 2012.

1. Let $f(c)$ equal the c-color Ramsey number $R(3, 3, \ldots, 3)$ (namely, the smallest integer n such that any c-coloring of the edges of K_n contains a monochromatic triangle).

   Show that $(f(c_1 + c_2) - 1) \geq (f(c_1) - 1) \cdot (f(c_2) - 1)$. Thus show that $f(c) \geq 5^{\lfloor c/2 \rfloor}$.

   What upper bound for $f(c)$ does the proof of the Ramsey theorem from class give? Today it is unknown what the correct behavior for $f(c)$ is.

2. Let $\delta \in (0, 1/2)$ be a constant. We want to find a subset C of $\{0, 1\}^n$ of size as large as possible such that no two elements have Hamming distance $\leq \delta n$. This is a discrete "sphere packing" problem. Express all your answers in terms of the binary entropy function H.

   (a) Pick C to be a random set of size K. How large can you take K such that C has the desired property with probability at least 1/2?

   (b) Use the method of alterations to find a larger set C with the desired property. How large can you make C this way?

   (c) By a volume packing argument, show that no such C can have $|C| \geq 2^{(1-H(\delta/2)+o(1))n}$.

   (d) Now we consider a dual covering problem. We want to find a subset C of $\{0, 1\}^n$ of size as small as possible such that every element of $\{0, 1\}^n$ is within Hamming distance ϵn of some element of C.

   (e) How small a C can you find with this property?

   (f) Show that no such C can have $|C| \leq 2^{(1-H(\epsilon)-o(1))n}$.

3. (Simultaneously big cuts): Let $G_1, G_2, \ldots, G_k$ be graphs with the same vertex set V, edge sets $E_1, E_2, \ldots, E_k \subseteq \binom{V}{2}$. Show that for each $\epsilon > 0$, there exists a t depending only on ϵ, k, such that if $|E_i| > t$ for all i, then there exists a partition $V = S \cup T$ such that for each i, at least $(1/2 - \epsilon)$ fraction of the edges in E_i go between S and T.

4. Let $\alpha, \beta \in (0, 1)$ be constants.

   (a) Show that if $\beta > \alpha^2$, then there exists a collection of exponentially many subsets of $[n]$ with size αn such that any two subsets of the collection have interesection size at most βn.

   (b) Show that if $\beta < \alpha^2$, then any such collection has size at most $n^{O(1)}$.