Homework 2

Combinatorics I (Fall 2017)
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Due Date: October 26, 2012.

1. Matchings and flows:

   (a) Show that every bipartite k-regular graph has a perfect matching.

   (b) Show that every doubly stochastic $n \times n$ matrix can be written as a convex combination of permutation matrices.

   (c) Prove Menger's theorem: in any directed graph, the maximum number of edge-disjoint s to t paths is equal to the minimum number of edges one has to delete in order to make t unreachable from s.

2. **Tight cases of the LYM inequality**

   (a) Let $\mathcal{A} \subseteq \binom{[n]}{k}$. Let

   $$\partial_{k,r}\mathcal{A} = \{B \in \binom{[n]}{r} \mid \exists A \in \mathcal{A} \text{ with } B \subseteq A\}.$$

   Show that $\frac{|\partial_{k,r}\mathcal{A}|}{\binom{n}{r}} \geq \frac{|\mathcal{A}|}{\binom{n}{k}}$, with equality iff $\mathcal{A} = \binom{[n]}{k}$.

   (b) Suppose $\mathcal{F} \subseteq 2^{[n]}$ is an antichain. Let $\mathcal{F}_k = \mathcal{F} \cap \binom{[n]}{k}$.
   What can you say about the relationship between $\partial_{k,r}\mathcal{F}_k$ and $\partial_{k',r}\mathcal{F}_{k'}$?

   (c) Use this to give another proof of the LYM inequality:

   $$\sum_{k=1}^{n} \frac{|\mathcal{F}_k|}{\binom{[n]}{k}} \leq 1,$$

   and show that equality holds iff $\mathcal{F}_k = \binom{[n]}{k}$ for some k.

3. In this problem, we will see a generalization of the Littlewood-Offord problem to vectors in $\mathbb{R}^d$.

   A chain $A_1 \subseteq A_2 \ldots \subseteq A_k$ in $2^{[n]}$ is called a symmetric chain if $|A_{i+1} = |A_i| + 1$ for each i, and $|A_k| = n - |A_1|$. A symmetric partition of $2^{[n]}$ is a partition consisting entirely of symmetric chains.

   (a) Show that every symmetric partition of $2^{[n]}$ has exactly $\binom{n}{\lfloor n/2 \rfloor}$ parts. (In fact, a symmetric partition has 1 part of cardinality $n+1$, and for each $1 \leq i \leq n/2$ it has exactly $\binom{n}{i} - \binom{n}{i-1}$ parts cardinality $n + 1 - 2i$).

   (b) Show how to construct a symmetric partition of $2^{[n]}$ given a symmetric partition of $2^{[n-1]}$. (Note that this now gives us another proof of Sperner's theorem.)

(c) Let $v_1, \ldots, v_n \in \mathbb{R}^d$ with $|v_i| \geq 1$ for each i (here $|\cdot|$ denotes the ℓ_2 norm). For $A \subseteq [n]$, let v_A denote $\sum_{i \in A} v_i$.

A family $\mathcal{A}$ of subsets of $2^{[n]}$ is called sparse if for all $A, B \in \mathcal{A}$, we have $|v_A - v_B| \geq 1$. A partition of $2^{[n]}$ is called pseudo-symmetric if it has exactly 1 part of cardinality $n+1$, and for each $1 \leq i \leq n/2$ it has exactly $\binom{n}{i} - \binom{n}{i-1}$ parts cardinality $n + 1 - 2i$.

Show (by induction on n) that $2^{[n]}$ has a pseudo-symmetric partition where each part is sparse.

(d) Deduce that

$$|\{A \subseteq [n] \mid |v_A| < 1/2| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

4. Let $A_1, \ldots, A_m \subseteq [n]$ and $B_1, \ldots, B_m \subseteq [n]$ be such that:
   - $A_i \cap B_i = \emptyset$ for each i.
   - $A_i \cap B_j \neq \emptyset$ for each $i \neq j$.

   Show that

   $$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1.$$

5. Use the Kruskal-Katona theorem to prove the Erdos-Ko-Rado theorem about 1-intersecting families.

6. Look up Stirling's formula and its proof (Not to be submitted).

   For the rest of this problem, submit only the answers and do not submit your calculations.

   Use Stirling's formula to find an asymptotic formula for $\binom{n}{\alpha n}$, where $\alpha \in [0, 1]$ is constant, and $n \to \infty$.

   Express your answer in terms of the "binary entropy function" $H : [0, 1] \to [0, 1]$ defined by

   $$H(\alpha) = \alpha \log_2 \frac{1}{\alpha} + (1 - \alpha) \log_2 \frac{1}{1 - \alpha},$$

   and $H(0) = H(1) = 0$. Draw of graph of H. Note the special case of $\alpha = 1/2$ and think about why that seems reasonable in terms of tossing n independent coins.

   How large should c be for $\sum_{i=0}^{c} \binom{n}{i}$ to be $\Omega(2^n)$? How large should c be for $\sum_{i=0}^{c} \binom{n}{i}$ to be at least $\Omega(2^n/n^k)$ for a given $k > 0$?

Hints

Littlewood-Offord: Without loss of generality, we may take v_n to be the vector $(\alpha, 0, 0, \ldots, 0)$ where $|\alpha| \geq 1$. For each part $\mathcal{A}$ of a given sparse pseudo-symmetric partition of $2^{[n-1]}$, produce up to two parts of a sparse pseudo-symmetric partition of $2^{[n]}$; these produced parts will depend on the first coordinates of the vectors $\{v_A \mid A \in \mathcal{A}\}$.

A_i's and B_i's: This generalizes one of the inequalities we saw in class. Try to adapt the proof of that inequality.

Kruskal-Katona: For an intersecting family $\mathcal{F} \subseteq \binom{[n]}{k}$, consider the family $\mathcal{G} = \{[n] \setminus A \mid A \in \mathcal{F}\}$. How do $\mathcal{F}$ and $\mathcal{G}$ relate?