Homework 1

Combinatorics I (Fall 2017)
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Due Date: October 3, 2017.

1. Let $S(n, k)$ denote Stirling numbers of the second kind.
   (a) Give a bijection to show that $S(n, k) = kS(n-1, k) + S(n-1, k-1)$.
   (b) Consider the following generating function: $F_k(X) = \sum_{n \geq k} S(n, k) X^n$. Show that $F_k(X) = \frac{X^k}{\prod_{j=1}^{k}(1-jX)}$.
   (c) Show that
   $$S(n, k) = \sum_{\substack{a_1, \ldots, a_k \geq 0 \\ \sum a_i = n-k}} \prod_{j=1}^{k} j^{a_j}.$$
   (d) Give a direct bijective proof of the above equation.

2. Let Σ be a finite set, with $|\Sigma| = q$. For a sequence s, let $|s|$ denote its length.

   Let $s_1, s_2, \ldots, s_k$ be (nonempty) sequences composed of elements from Σ.

   (a) We say that $s_1, \ldots, s_k$ is *prefix-free* if there are no $i \neq j$ such that s_i is a prefix of s_j.
   If $s_1, \ldots, s_k$ is prefix-free, show that $\sum_{i=1}^{k} q^{-|s_i|} \leq 1$.
   (b) We say that $s_1, \ldots, s_k$ is *uniquely decodable* if the following holds: for every m, n, and every $(i_1, \ldots, i_m) \in [k]^m$ and $(j_1, \ldots, j_n) \in [k]^n$, if
   $$s_{i_1} \cdot s_{i_2} \cdot \cdots \cdot s_{i_m} = s_{j_1} \cdot s_{j_2} \cdot \cdots \cdot s_{j_n},$$
   then we must have $m = n$ and $i_\ell = j_\ell$ for each ℓ. (For two sequences a, b, we use $a \cdot b$ to denote the concatenation of the sequences a and b).
   Notice that if $s_1, \ldots, s_k$ is prefix-free, then it is uniquely decodable.
   Show that if $s_1, \ldots, s_k$ is uniquely decodable, then $\sum_{i=1}^{k} q^{-|s_i|} \leq 1$.

3. This problem will lead to another proof of the formula for Catalan numbers.

   Let S denote the set of all strings in $\{+1, -1\}^n$ consisting of exactly $n+1$ +1's and n -1's. We call an element x of S *good* if all nonempty prefixes of x have strictly positive partial sum.

   Define the operation $T : S \to S$ which sends the string $x = x_1 x_2 \ldots x_{2n+1}$ to the string $T(x) = x_{2n+1} x_1 x_2 \ldots x_{2n}$. We say a string y is a rotation of a string x if there is some integer r with $T^r(x) = y$.

   Show that for every x in S, there is exactly one rotation of x which is good.

   Deduce a formula for the Catalan numbers from this.

4. Let a_n denote the number of labelled undirected simple graphs where each vertex has degree exactly 2. Let $F(X) = \sum_{n=0}^{\infty} \frac{a_n}{n!} X^n$.

   (a) Show that
   $$F(X) = \frac{1}{\sqrt{(1-x)}} \cdot e^{-x/2-x^2/4}.$$

   (b) Show that $\frac{F'(X)}{F(X)}$ is a rational function.

   (c) Use this to find a recurrence for a_n of the form: $a_n = P_1(n)a_{n-1}+P_2(n)a_{n-2}+P_3(n)a_{n-3}$, where P_1, P_2, P_3 are polynomials.

Hints.

1. top-heavy strings: Lay a string out on a circle. Suppose a string has "+1-1" occurring as a substring in it. What can you say about which rotations of the string are top-heavy?

2. 2-regular graphs: how do graphs with all degrees exactly 2 look? use the exponential formula to find $F(X)$.