Proof of the Pósa-Seymour Conjecture

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Abstract

In 1974 Paul Seymour conjectured that any graph \( G \) of order \( n \) and minimum degree at least \((k-1)n\) contains the \((k-1)^{th}\) power of a Hamiltonian cycle. In [20] this conjecture was proved with the help of the Regularity Lemma – Blow-up Lemma method for \( n \geq n_0 \) where \( n_0 \) is very large. Here we present another proof that avoids the use of the Regularity Lemma and thus resulting in a much smaller \( n_0 \).

Equally important is that we prove a stability result about this conjectures, namely that if our graph does not contain an almost independent set of size \( n/\kappa \), then Seymour conjecture is true even if the minimum degree of our graph \( G \) is at least \((k-1-\epsilon)n\).

The main ingredient is a new kind of connecting lemma.

1. Introduction

1.1. Notations and Definitions

\( V(G) \) and \( E(G) \) denote the vertex-set and the edge-set of the graph \( G \). \((A, B, E)\) denotes a bipartite graph \( G = (V, E) \), where \( V = A \cup B \), \( A \) and \( B \) are disjoint and \( E \subset A \times B \). For a graph \( G \) and a subset \( U \) of its vertices, \( G|_U \) is the restriction of \( G \) to \( U \). \( N(v) \) is the set of neighbors of \( v \) in \( V \), and \( N_S(v) \) is the set of neighbors of \( v \) in \( S \). \(|N_S(v)|\) is the degree of \( v \) into \( S \), denoted by \( \deg_S(v) \). \( \delta(G) \) stands for the minimum and \( \Delta(G) \) for the maximum degree of a vertex in \( G \). \( K_r(t) \) is the balanced complete \( r \)-partite graph with color

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classes of size $t$. We write $N(p_1, p_2, \ldots, p_\ell) = \bigcap_{i=1}^\ell N(p_i)$ for the set of common neighbors of $p_1, p_2, \ldots, p_\ell$, and, more generally, $N(X) = \bigcap_{x \in X} N(x)$. When $A$ and $B$ are subsets of $V(G)$, we denote by $e(A, B)$ the number of edges of $G$ with one endpoint in $A$ and the other in $B$. For non-empty $A$ and $B$,

$$d(A, B) = \frac{e(A, B)}{|A||B|}$$

is the density of the graph between $A$ and $B$. In particular, we write $d(A) = d(A, A)$. A graph $G$ on $n$ vertices is $\gamma$-dense if it has at least $\gamma \left(\frac{n}{2}\right)$ edges. A bipartite graph $G(A, B)$ is $\gamma$-dense if it contains at least $\gamma |A||B|$ edges. Throughout the paper log denotes the base 2 logarithm.

A graph $G$ is $\alpha$-extremal, if there exists an $A \subseteq V(G)$ for which

(a) $(\frac{1}{k} - \alpha)n \leq |A| \leq (\frac{1}{k} + \alpha)n$

(b) $d(A) < \alpha$

(We say that $G$ is $\alpha$-non extremal if no set $A \subseteq V(G)$ satisfies (a) and (b))

$K_{k+1}(t)$ is a complete $k+1$-partite graph where each color class has size $t$

We call a graph a small multipartite graph if it is either $K_{k+1}(t)$, or $K_k(t)$.

A path $P_m$ means a path of $m$ vertices. Let $C$ be a cycle. Then the $k^{th}$ power of $C$, denoted by $C^k$, is defined as follows: $V(C^k) = V(C)$ and $uv$ is an edge in $C^k$ if the distance between $u$ and $v$ in $C$ is at most $k$. The $k^{th}$ power of a path $P$ is defined in an analogous manner. For notational convenience we call the $k^{th}$ power of a path a $k$-path.

1.2. History

A classical result of Dirac [4] asserts that if $\delta(G) \geq n/2$, then $G$ contains a Hamiltonian cycle. A natural question analog to Dirac’s theorem was asked by Pósa (see Erdős [5]) in 1962:

**Conjecture 1** (Pósa). Let $G$ be a graph on $n$ vertices. If $\delta(G) \geq \frac{2}{3}n$, then $G$ contains the square of a Hamiltonian cycle.

This conjecture was generalized by Seymour in 1974 [25]:

**Conjecture 2** (Seymour). Let $G$ be a graph on $n$ vertices. If $\delta(G) \geq \left(\frac{k-1}{k}\right)n$, then $G$ contains the $(k - 1)^{th}$ power of a Hamiltonian cycle.
Substantial amount of work has been done on these problems. Jacobson (unpublished) first established that the square of a Hamiltonian cycle can be found in any graph $G$ given that $\delta(G) \geq 5n/6$. Later Faudree, Gould, Jacobson and Schelp [12] improved the result, showing that the square of a Hamiltonian cycle can be found if $\delta(G) \geq (3/4 + \varepsilon)n$. The same authors further relaxed the degree condition to $\delta(G) \geq 3n/4$. Fan and Häggkvist lowered the bound first in [6] to $\delta(G) \geq 5n/7$ and then in [7] to $\delta(G) \geq (17n+9)/24$. Faudree, Gould and Jacobson [11] further lowered the minimum degree condition to $\delta(G) \geq 7n/10$. Then Fan and Kierstead [8] achieved the almost optimal bound: they proved that if $\delta(G) \geq \left( \frac{2}{3} + \varepsilon \right)n$, then $G$ contains the square of a Hamiltonian cycle. They also proved in [9] that already $\delta(G) \geq (2n - 1)/3$ is sufficient for the existence of the square of a Hamiltonian path. Finally, they proved in [10] that if $\delta(G) \geq 2n/3$ and $G$ contains the square of a cycle with length greater than $2n/3$, then $G$ contains square of a Hamiltonian cycle.

For Conjecture 2 in the above mentioned paper of Faudree, Gould, Jacobson, and Schelp, they proved that for any $\varepsilon > 0$ and positive integer $k$, if the graph $G$, on $n$ vertices, satisfies $\delta(G) \geq \left( \frac{2k-1}{2k} + \varepsilon \right)n$, then $G$ contains the $k^{th}$ power of a Hamiltonian cycle.

Using the Regularity Lemma – Blow-up Lemma method first Komlós, Sárközy and Szemerédi [19] proved Conjecture 2 in asymptotic form, then in [17] and [20] they proved both conjectures for $n \geq n_0$. The proofs used the Regularity Lemma [26], the Blow-up Lemma [18, 21] and the Hajnal-Szemerédi Theorem [14]. Since the proofs used the Regularity Lemma, the resulting $n_0$ is very large (it involves a tower function). The use of the Regularity Lemma was removed by Levitt, Sárközy and Szemerédi in a new proof of Pósa’s conjecture in [23].

The purpose of this paper is to present a new proof of the Pósa-Seymour conjecture that avoids the use of the Regularity Lemma, thus resulting in a simpler proof and a much smaller $n_0$; and to prove a stability result, namely that if our graph does not contain an almost independent set of size $\frac{n}{n_0}$, then Seymour conjecture is true even if the minimum degree of our graph $G$ is at least $\left( \frac{2k-1}{k} - \varepsilon \right)n$.

We would like to mention the main ingredient in our proof, a new kind of connecting lemma, which we believe will have a lot of applications.

While proving the Pósa-Seymour Conjecture, we do not want to determine the optimal constants.
1.3. Main Results

Theorem 1. There exists an integer $n_0(\alpha)$, and $\epsilon(\alpha)$ such that any $\alpha$-non extremal graph $G$, with $|V(G)| = n > n_0(\alpha)$, and $\delta(G) \geq \left(\frac{k-1}{k} - \epsilon(\delta)\right)n$, contains a $(k-1)^{th}$ power of a hamiltonian cycle.

Theorem 2. There exists an integer $n_0$ such that any graph $G$, with $|V(G)| = n > n_0$ and $\delta(G) \geq \left(\frac{k-1}{k}\right)n$, contains a $(k-1)^{th}$ power of a hamiltonian cycle.

2. A brief sketch of the proof of theorem 1

In section 4, using the tools developed in section 3, we first cover a constant fraction of the vertices in $G$ by $K_{k+1}(t)$’s, and then we cover as much vertices as we can with $K_k(t)$’s, where $t = c \log n$, for a constant $0 < c < 1$. We refer to the sets $K_{k+1}(t)$’s and $K_k(t)$’s by $C$ and $K$ respectively.

We would inevitably be left with a set $I$ that cannot be covered in such a manner. However we show that the number of such vertices is small.

Denote the complete graphs $K_{k+1}(t)$ in our collection $C$ by $C_1, C_2, \ldots$, and by $K_1, K_2, \ldots$, the graphs $K_k(t)$ in our collection $K$.

For a graph $C_j \in C$, denote its color classes by $V_j^1, V_j^2, \ldots, V_j^{k+1}$. In any color class $V_j^i$, consider an ordering of its vertices $v_{0,i}, v_{1,i}, v_{2,i}, \ldots, v_{l-1,i}$. Finally let us call the $l^{th}$ column of $C_j$ to the sequence $v_{l,1}, v_{l,2}, \ldots, v_{l,k+1}$.

For a graph $K_j \in K$, denote color classes by $W_j^i$, and vertices by $w_{l,i}$.

To build a $(k-1)$-path in each of the small multipartite graphs $C_j, K_j$, sequentially connect the vertices within a column, and then connect its last vertex, to the first vertex of the following column (see figure 1).

![Diagram](image-url)
We will prove in section 3.2 that given two cliques \(\{a_1, a_2, \ldots, a_{k-1}\}\) and \(\{b_{k-1}, b_{k-2}, \ldots, b_1\}\), we can connect \(a_{k-1}\) to \(b_{k-1}\) with a \((k-1)\)-path of length at most \(9(k+1)!\), even if we cannot use \(o(n)\) vertices of the graph, given in advance. We use this lemma to connect with \((k-1)\)-paths (of length at most \(9(k+1)!\)) the last vertex of the last column of a small multipartite graph with the first vertex of the first column of the succeeding one. We impose further that such paths do not use vertices from the first or last columns of any of the small multipartite graphs (which can be done, since \(t = \Omega(\log n)\), and therefore the number of vertices in those columns is at most \(O\left(\frac{n}{\log n}\right) = o(n)\)).

After connecting the graphs in \(\mathcal{C}\) and \(\mathcal{K}\) we get a \((k-1)\)th power of a cycle covering the vertices in \(V(\mathcal{C}) \cup V(\mathcal{K})\) (see figure 2).

![Figure 2: Dashed lines represent the \((k-1)\)-paths constructed via the Connecting Lemma](image)

Unfortunately, since the \((k-1)\)-path connecting the small complete multipartite graphs might use a small number of vertices from some columns (at most \(9(k+1)!/(k+1)^{\frac{1}{2}}\frac{n}{\log n}\), for some constant \(0 \leq \eta \leq 1\)), we have to remove the vertices of those columns, and put them in \(I\). This will not increase significantly the size of \(I\).

Each time we remove a column from a small multipartite graph, we have to reconstruct the path inside the graph. We do this by connecting the column preceding the deleted one, directly to the one following it.

Let \(\mathcal{C}^*, \mathcal{K}^*\) denote the collection of multipartite graphs obtained from \(\mathcal{C}\) and \(\mathcal{K}\) respectively, after the columns removal. It will be important the fact that \(|\mathcal{C}^*| \geq |\mathcal{C}| - 9(k+1)!/(k+1)^{\frac{1}{2}}\frac{n}{\log n}\), which is still much bigger than \(I\).

To obtain a \((k-1)\)-hamiltonian cycle, we have to insert the vertices of \(I\) in the cycle we constructed, in such a way that it remains a \((k-1)\)-cycle.
Given a vertex $a \in I$, by the minimum degree condition, it sends at least
$$(k - 1 - \epsilon)n$$
edges to $C^* \cup K^*$. We will first try to insert the vertex $a$ in $C^*$.

If for some graph $C_j \subseteq C^*$ the degree of $a$ in $C_j$ is at least $(k - \frac{1}{k+1} + \delta)|C_j|$ we can insert it easily in the path inside $C_j$. Indeed, without loss of generality, there is a vertex $v_{x,y}$ in the path in $C_j$ such that $a$ is connected to all vertices in the path at distance at most $(k - 1)$ from $v_{x,y}$ (Otherwise, we just reorder the color classes of $C_j$ other than the first and last one, and reconstruct the path inside $|C_j|$ as in the initial procedure. Notice that this does not change the connecting paths, nor the vertices from the multipartite graphs they intersect, nor $I, C^*, K^*$).

We proceed by replacing vertex $v_{x,y}$ by $a$ in the path inside $C_j$. Notice that $v_{x,y}, v_{x+1,y-1}, v_{x+2,y-2}, \ldots, v_{x+k,y-k}$ (where indices are taken (mod $k+1$)) form a clique because they belong to different color classes in $C_j$. We can remove each one of them, and reconnect the one preceding it directly to the one succeeding it in the original path, still obtaining a $(k - 1)$-path.

Finally we insert the path formed by the removed vertices between any two columns of the graph other than the one from which $a$ was removed.

When inserting other vertices in $C_j$ we will be careful not to place them in any of the $k$ columns neighboring each of the affected ones $(x+1, \ldots, x+k)$ to guarantee that the resulting path remains a $(k - 1)$-path.

Figure 3: Inserting $a$ into the $(k - 1)$-path being constructed in the complete balanced $(k + 1)$-partite graph $C^*_j$, where $k = 5$, and $v_{x,y} = v_{1,4}$. 
If a vertex \( a \in \mathcal{I} \) cannot be inserted in any \((k+1)\)-partite graph, the minimum degree condition implies that the vertex has at least \((\frac{k-1}{k} + \delta) |K|\) neighbors for a big fraction of the graphs \( K \in \mathcal{K}^* \). In this case we can assume that \( a \) is connected to all vertices in three consecutive columns of \( K \), and just insert it between any two consecutive vertices of the middle column. The resulting path remains a \((k-1)\)-path. Registering the three columns used as not available to place new vertices in \( K \), we can insert in the path inside this graph a new element from \( I \).

Repeating this until all elements are used (which is feasible because \(|\mathcal{K}^*| \gg |\mathcal{I}|\)), we obtain a \((k-1)\)-hamiltonian cycle.

3. Main Tools

We shall assume that \( n \) is sufficiently large and use the following main parameters:

\[ 0 < \eta \ll \alpha \ll 1, \]

where \( a \ll b \) means that \( a \) is sufficiently small compared to \( b \). In order to present the results transparently we do not compute the actual dependencies, although it could be done.

3.1. Complete \( k \)-Partite Subgraphs

In [20] the Regularity Lemma [26] was used to prove the Pósa-Seymour conjecture, however, here we use more elementary methods using only the Bollobás-Erdős-Simonovits bound [22].

**Lemma 3** (Theorem 3.1 on page 328 in [1]). There is an absolute constant \( \beta_1 > 0 \) such that if \( 0 < \varepsilon < 1/s \) and we have an \( n \)-graph \( G \) with

\[ |E(G)| \geq \left( 1 - \frac{1}{s} + \varepsilon \right) \frac{n^2}{2} \]

then \( G \) contains a \( K_{s+1}(t_1) \), where

\[ t_1 = \left\lfloor \frac{\beta_1 \log n}{s \log 1/\varepsilon} \right\rfloor. \]

The following two observations will be useful later on.
Lemma 4. If $G(A, B)$ is an $\eta$-dense bipartite graph, then there must be at least $\eta |B|/2$ vertices in $B$ for which the degree in $A$ is at least $\eta |A|/2$.

Indeed, otherwise the total number of edges would be less than

$$\frac{\eta}{2} |A||B| + \frac{\eta}{2} |A||B| = \eta |A||B|,$$

a contradiction to the fact that $G(A, B)$ is $\eta$-dense.

Lemma 5. Let $G(A, B)$ be a bipartite graph such that $|A| = c_1 \log n$, $|B| = c_2 n^{c_3}$ where $0 < c_1, c_2, c_3 < 1$ are constants and $c_1 \ll c_3$. If for all $b \in B$ we have $\deg_A(b) \geq \eta |A|/2$, then we can find a complete bipartite subgraph $K(A', B')$ of $G$ such that $A' \subset A, B' \subset B, |A'| \geq \eta |A|/2$ and $|B'| \geq c_2 n^{(c_3 - c_1)}$.

To see this consider the neighborhoods in $A$ of the vertices in $B$. Since there can be at most $2|A| = n^{c_1}$ such neighborhoods, by averaging there must be a neighborhood that appears for at least $\frac{c_2 n^{c_3}}{n^{c_1}} = c_2 n^{(c_3 - c_1)}$ vertices of $B$. This means that we can find the desired complete bipartite graph.

Lemma 6. Let $G$ be a graph with $V(G)$ partitioned into $A_1, A_2, \ldots, A_k$ and $B$ such that the subsets $A_1, A_2, \ldots, A_k$ form a complete $k$-partite graph, and for $1 \leq i \leq k$, $|A_i| = c_1 \log n$, $|B| = c_2 n$ for constants $0 < c_1, c_2 < 1$. If for every $b \in B$, $\deg_A(b) \geq \eta |A_i|/2$, then we can find a complete $(k+1)$-partite graph $G(A_1', A_2', \ldots, A_k', B')$ such that $A_i' \subset A_i, B' \subset B, |A_i'| \geq \eta |A_i|/2$ and $|B'| \geq c_2 n^{(1-kc_1)}$.

Proof. First consider the bipartite graph $G_1(A_1, B)$. By Lemma 5 we have $A_1' \subset A_1, B_1 \subset B, |A_1'| \geq \eta |A_1|/2$ and $|B_1| \geq c_2 n^{(1-c_1)}$, such that $G_1(A_1', B_1)$ is a complete bipartite subgraph. Now consider the bipartite graph $G_2(A_2, B_1)$. Applying again Lemma 5 we find $A_2' \subset A_2, B_2 \subset B_1, |A_2'| \geq \eta |A_2|/2$ and $|B_2| \geq c_2 n^{(1-2c_1)}$, such that $G_2(A_2', B_2)$ is a complete bipartite subgraph. Note that this gives us a complete tripartite graph with color classes $A_1', A_2'$ and $B_2$. Proceeding similarly, we can find a complete $(k+1)$-partite graph $G_k(A_1', A_2', \ldots, A_k', B)$ such that for $1 \leq j \leq k$, $A_j' \subset A_j, B_i \subset B, |A_j'| \geq \eta |A_j|/2$ and $|B_k| \geq c_2 n^{(1-kc_1)}$. □

Lemma 7. There exist two constants $n_0$ and $\beta_2 > 0$ such that if $G$ is a non-extremal graph on $n \geq n_0$ vertices with $\delta(G) \geq (\frac{k-1}{k} - \sqrt{\eta})n$, then $G$ contains a $K_{k+1}(t)$, where $t = \lfloor \beta_2 \log n \rfloor$. Here $\beta_2$ and $n_0$ depend on $\alpha$ and $\eta$. 

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Proof. We apply Lemma 3 to $G$ to get $k$ disjoint sets $A_1, A_2, \ldots, A_k$, each of size $t_1 = \left\lceil \frac{\beta_1 \log n}{k \log \frac{1}{\alpha}} \right\rceil$ such that they form a complete balanced $k$-partite graph. Define $A := \bigcup_{i=1}^{k} A_i$ and let $B \subset V(G) \setminus A$ be the set of vertices that have more than $\eta |A_i|$ neighbors in each $A_i$. Our first observation is that we can assume that $|B| \leq \eta^2 n$. Indeed otherwise by Lemma 6 we get our desired $K_{k+1}(t)$.

Let $C = V(G) \setminus (A \cup B)$ and for $1 \leq i \leq k$ let $C_i = \{ c \in C : deg_{A_i}(c) < \eta |A_i| \}$. By definition of $B$ it follows that $C = \bigcup_{i=1}^{k} C_i$. From the minimum degree condition and the definition of $C_i$ we have that for every $i$:

\[
\left( \left( \frac{k-1}{k} - \sqrt{\eta} \right) n - |B| - |A| \right) |A_i| \leq e(A_i, C) \leq \eta |A_i||C_i| + |A_i||C| - |C_i|)
\]

which gives us that $|C_i| \leq (1 + 3\sqrt{\eta})n/k$.

We will show that from $|C_j| \leq (1 + 3\sqrt{\eta})\frac{n}{k}$, $(1 \leq j \leq k)$, it follows that for every $j$ $(1 \leq j \leq k)$

\[
|C_j| \geq (1 - 4k\sqrt{\eta})\frac{n}{k} 
\]

(1)

Assume (1) does not hold.

Then $n - |B| - kt_1 = \sum_{i \neq j, i=1}^{k} |C_i| \leq (k-1)(1 + 3\sqrt{\eta})n/k + (1 - 4k\sqrt{\eta})\frac{n}{k}$

which is a contradiction because $|B| \leq \eta^2 n$, and $t_1 = O(\log n)$.

From the minimum degree condition and from

\[
|B| \leq \eta^2 n, \quad e(A_j, C_j) \leq \eta |A_j||C_j| \quad (1 \leq j \leq k) \quad (2)
\]

after a little calculation it follows that

\[
e(A_j, C_1) \geq (1 - 100k\sqrt{\eta})|A_j||C_1|, \quad (3)
\]

for $2 \leq j \leq k$.

Let us denote by $C_{1,i}$ the following set:

\[
C_{1,i} = \left\{ x \in C_1, \deg_{A_j}(x) \leq \frac{2}{3} |A_j| \right\}.
\]

Then, by double counting, from (3), we get

\[
|C_{1,i}| \leq 300k\sqrt{\eta}|C_1|
\]

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Define $C^*_i = \bigcup_{j=2}^{k} C^j_i$. Then $|C^*_i| \leq 300k^2 \sqrt{\eta}|C_1|$. We omit $C^*_i$ from $C_1$. Denote the remaining set by $C^*$. Obviously, $|C^*| \geq (1 - 300k^2 \sqrt{\eta})|C_1|$. We group the vertices in $C^*$ according to their neighborhoods in $A \setminus A_1$. The number of groups is at most $2^{kt_1}$. We again omit the groups containing at most $\sqrt{\eta}n/k2^{kt_1}$ elements. The union of the remaining groups has size greater than $(1 - 400\sqrt{\eta}k^2) \cdot \frac{n}{k}$. Let’s denote these groups by $D_1, D_2, \ldots, D_m$, and $D = \bigcup_{i=1}^{m} D_i$. We know that $|D| > (1 - 400\sqrt{\eta}k^2) \cdot \frac{n}{k} > (1 - \alpha) \frac{n}{k}$, and that $|D| \leq (1 + 3\sqrt{n}) \frac{n}{k}$.

Since $G$ is not $\alpha$-extremal, and $(1 - \alpha) \frac{n}{k} < |D| < (1 + \alpha) \frac{n}{k}$, we get that

$$e(G|D) \geq \alpha \left(\frac{n}{k}\right)^2.$$ 

There are two cases.

**Case 1.1.** There is a group, say $D_1$, such that $e(G|D_1) \geq \eta^2|D_1|^2$. Then, by Lemma 3 with $s = 1$ in $G|D_1$, we have a complete bipartite graph, spanned by two sets, say $Q$ and $R$, of size greater than $\beta_2 \log n$. Since $N_{A_1}(D_1) > \frac{2}{3}|A_1|$ (for $i \neq 1$) we can find $K_{k-1}(t) \subset N_{A \setminus A_1}(D_1)$ which together with $Q$ and $R$ gives us the required $K_{k+1}(t)$.

**Case 1.2.** There are two sets, say, $D_1$ and $D_2$, such that $e(D_1, D_2) \geq \eta^2|D_1||D_2|$. Then by Lemma 2, we have a set $Q \subset D_1$ and $R \subset D_2$ of size greater than $\beta_2 \log n$ and $Q$ and $R$ form a complete bipartite graph. Since $N_{A_1}(D_1) \cap N_{A_1}(D_2) > \frac{1}{3}|A_1|$ for $i \neq 1$ we can find $K_{k-1}(t) \subset N_{A \setminus A_1}(D_1) \cap N_{A \setminus A_1}(D_2)$ which together with $Q$ and $R$ gives us the required $K_{k+1}(t)$. □

We will also use the following simple fact on the size of a maximum set of vertex disjoint paths in $G$ (see [2]).

**Lemma 8.** In a graph $G$ on $n$ vertices, we have

$$\nu_1(G) \geq \max \left\{ \delta(G), \delta(G) - 1, \frac{n}{4\Delta(G)}, \frac{n}{6\Delta(G)} \right\} \quad \text{and} \quad \nu_2(G) \geq (\delta(G) - 1) \frac{n}{6\Delta(G)}$$

where $\nu_i(G)$ denotes the size of maximum set of vertex disjoint paths of length $i$ in $G$.

**3.2. The Connecting Lemma**

**Definition 1** (Eligible vertices). We call a vertex $v \in V(G)$, eligible, if for each $\ell \in [1, 9(k+1)!]$, the number of paths of length $\ell + 1$ (edges) between any $v_1, v_2 \in N(v)$, completely in $N(v)$, is at least $\eta^{16} \cdot n^\ell$. 

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(The paths contain $\ell + 2$ vertices, $v_1, v_2$ included. The endpoint are fixed, so we may have $c_k n^\ell$ connecting paths and we require a “positive” percentage of them. Here $c_k = k^{-k}$.)

It is easy to see that in $G$ there are at least $\frac{n}{k^2}$ eligible vertices.

**Definition 2.** A path of length $\ell$ is **good** if it is in $N(v)$ for at least $\eta^k n^\ell$ vertices $v$. Otherwise it is bad. We call a vertex $v$ **good** if $N(v)$ contains at most $100k^2 \eta^k n^\ell$ bad paths $P_\ell$, for each $\ell \in [1, 9k!]$.

An easy calculation shows that the number of good vertices is

$$n \left( 1 - \frac{1}{100k^2} \right)$$

and the number of vertices which are both eligible and good is at least

$$\frac{n}{4k} \left( 1 - \frac{1}{100k^2} \right)$$

After these definitions we formulate the Connecting Lemma.

**Lemma 9** (Connecting Lemma).

Given a clique $\{a_1, a_2, \ldots, a_{k-1}\}$ and $\{b_{k-1}, b_{k-2}, \ldots, b_1\}$, there is an $\ell \leq 9(k+1)!$ such that we can connect these two cliques with at least $\eta^k n^\ell (k-1)$-paths, even if we forbid using $o(n)$ vertices, given in advance.

**Lemma 10.** If $G$ is not an $\alpha$-extremal graph, then there are $\sqrt{\eta} n$ vertices $v \in V(G)$ such that $v$ is good and eligible and (for each $v$) there is a $T_v \subseteq N(v)$ for which $|T_v| = \sqrt{\eta} m$, $T_v = \{t_{1,v}, t_{2,v}, \ldots, t_{\sqrt{\eta} n,v}\}$ and $|N(t_{i,v}) \cap \overline{N(v)}| \leq \frac{n}{k} - \sqrt{\eta} m$ for $t_{i,v} \in T_v$ ($1 \leq i \leq \sqrt{\eta} n$).

**Proof.** Notice that the lemma trivially holds if there is a vertex $v \in V(G)$ for which $|\overline{N(v)}| \leq \frac{n}{k} - \sqrt{\eta} m$.

Assume that there is a $v_0 \in V(G)$ so that for at least $|N(v_0)| - \sqrt{\eta} m$ vertices $w_i \in N(v_0)$,

$$|N(w_i) \cap \overline{N(v_0)}| > \frac{n}{k} - \sqrt{\eta} m,$$

Denote the set of these vertices by $W = \{w_1, w_2, \ldots\}$.  

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Obviously the vertices in \( W \) are eligible, because \((W, N(v_0))\) is an \( \alpha \)-extremal graph. Omit from \( N(v_0) \) the vertices \( z \in N(v_0) \) for which \(|N(z) \cap N(v_0)| \leq \frac{n}{k} - \sqrt{\eta n} \). We omitted at most \( \sqrt{\eta n} \) vertices. Denote the set of the omitted vertices by \( T_1 \). Now we choose a vertex \( v_1 \in N(v) \setminus T_1 \). If there are at least \( \sqrt{\eta n} \) vertices in \( N(v_1) \setminus T_1 \), such that their neighbors intersect \( N(v_1) \) in at most \( \frac{n}{k} - \sqrt{\eta n} \) points, then we are done. (We can assume that \( v_1 \) is good, too. The number of vertices that are not good in \( N(v_1) \) is at most \( \sqrt{\eta n} \).) If not, then we omit the vertices \( z \in N(v_1) \setminus T_1 \) for which \(|N(z) \cap N(v_1)| \leq \frac{n}{k} - \sqrt{\eta n} \). We again omitted at most \( \sqrt{\eta n} \) vertices. Denote this set by \( T_2 \). Notice that \(|N(v_1) \cap N(v) \setminus T_1| \geq \frac{n}{k} - 2\sqrt{\eta n} \) and \(|N(v) \cap N(v_1)| \leq 2\sqrt{\eta n} \). Now choose a vertex \( v_2 \in N(v_1) \setminus (T_1 \cup T_2) \). We argue as previously and continue this argument until we find a \( v_i \) such that there are at least \( \sqrt{\eta n} \) vertices \( z \in N(v_i) \) for which \(|N(z) \cap N(v_i)| \leq \frac{n}{k} - \sqrt{\eta n} \). Then we stop and \( v_i \) is our vertex which we mentioned in the Lemma. If we can not find such a \( v_i \) (\( i < k \)), renaming \( N(v_{i-1}) \) to \( A_i \), then our graph is the union of \( A_1, A_2, \ldots, A_k \) where \( X \) is a small vertex set (\(|X| \leq 10k\sqrt{\eta n}\)); and \(|A_i| \geq \frac{n}{k} - \sqrt{\eta n} \) and the bipartite graphs \((A_i, A_j), (i \neq j)\) are almost complete (namely every vertex has degree at least \( \frac{n}{k} - 5k\sqrt{\eta n} \)). Adding vertices from \( X \), we balance the sets \( A_i \) so that all of them will have size \( \frac{n}{k} \). Denoting the new classes by \( A_1^*, A_2^*, \ldots, A_k^* \), then \((A_i^*, A_j^*), (i \neq j)\), is still an almost complete bipartite graph.

In particular, every vertex in \( A_1^* \setminus X \), has at least \( (\frac{k-1}{k} - 10k\sqrt{\eta})n \) neighbors in the union of \( A_2^*, A_3^*, \ldots, A_k^* \). Since we assumed that the graph \( G \) is not \( \alpha \)-extremal, and \( X \) is small, there is a vertex \( v \in A_1^* \) which has at least \( \frac{\sqrt{\eta n}}{2} \) neighbors in \( A_1^* \). Hence \(|N(v)| \geq (\frac{k-1}{k} + \frac{\sqrt{\eta}}{10})n \), and the lemma follows from the first observation in this proof. \( \square \)

Iterating the above procedure, we get our set \( M \) of size \( \sqrt{\eta n} \), such that for every \( v \in M \), \( v \) is good and eligible and there is a \( T_v \subseteq N(v) \) for which \(|T_v| = \sqrt{\eta n}, T_v = \{t_1, \ldots, t_{\sqrt{\eta n}}, v\} \) and \(|N(t_{i,v}) \cap N(v)| \leq \frac{n}{k} - \sqrt{\eta n} \).

### 3.2.1. Extending Lemma

We fix a vertex \( v_1 \in M \). For simplicity, we denote \( N(v_1) \) by \( F \).

**Lemma 11** (Extending Lemma). Given a clique \( A = \{a_1, \ldots, a_{k-1}\} \) one can extend it with vertices \( w_1 w_2 \ldots w_s x_1 x_2 \ldots x_{k-1} \) \((0 \leq s \leq k - 1)\) so that \( a_1 a_2 \ldots a_{k-1} w_1 w_2 \ldots w_s x_1 x_2 \ldots x_{k-1} \) is a \((k - 1)\)-path and \( x_1, x_2, \ldots, x_{k-1} \in F \). Moreover, the number of choices of \( w_1, \ldots, w_s, x_1, x_2, \ldots, x_{k-1} \) is at least \((\frac{m}{10})^{k-1+s} \).\(12\)
Proof. Let $W_0 = \{a_1, a_2, \ldots, a_{k-1}\}$. The size of $N(W_0)$ is at least $\frac{n}{k}$. We may assume that it is $\frac{n}{k}$, after possibly dropping some vertices from $N(W_0)$. If $|N(W_0) \cap F| \geq \frac{n}{10} n$, then we choose $w_1$ from $N(W_0) \cap F$ and we get a set $W_1 = \{a_2, \ldots, a_{k-1}, w_1\}$. Note that we have at least $\frac{9n}{10}$ choices for $w_1$. If $|N(W_1) \cap F| \geq \frac{n}{10} n$ then we choose $w_2$ from $N(W_1) \cap F$. We get a set $W_2 = \{a_3, a_4, \ldots, w_1, w_2\}$. Define $W_i = \{a_{i+1}, \ldots, a_{k-1}, w_1, \ldots, w_i\}$. We proceed the same way as long as

$$|N(W_i) \cap F| > \eta \frac{n}{10} n$$

(4)

holds. If (4) holds for all $i \leq k-1$, then we have at least $\left(\frac{n}{10}\right)^{k-1} n^{k-1}$ $(k-1)$-paths $a_1, a_2, \ldots, a_{k-1}, w_1, w_2, \ldots, w_{k-1}$. We rename $w_1, w_2, \ldots, w_{k-1}$ to $x_1, x_2, \ldots, x_{k-1}$. So, in this case we are done with the proof. If $i < k-1$ is the smallest integer for which (4) does not hold, then we are going to work with $W_i$ (meaning that instead of $\{a_1, a_2, \ldots, a_{k-1}\}$ we start with $W_i$). For the sake of simplicity, we rename $W_i$ to $W$. We define $W_{low}$ as follows.

$$W_{low} = \{w \in W : \deg_{N(W)}(w) \leq |N(W)| - \eta n\}.$$ 

Since $|N(W) \cap F| \leq \frac{n}{10} n$, we have that $|N(W)| \leq \frac{n}{k} + \frac{n}{10} n$, and so $\deg_{N(W)}(w) \leq \frac{n}{k} + \frac{n}{10} n - \eta n$, for any $w \in W_{low}$.

That implies that for any $w \in W_{low}$,

$$|N_F(w)| \geq \frac{k-2}{k} n + \frac{8}{10} \eta n$$

Therefore, if $S \subseteq F$, $|S| = \frac{n}{k}$, then $|N_S(w)| \geq \frac{8}{10} \eta n$. There are two cases.

**Case 2.1.** $|W_{low}| \geq \frac{n}{10} n$. Choose a vertex $l \in W_{low}$. Since $|N_F(l)| \geq k \frac{2}{k} n + \frac{8}{10} \eta n$, we get that $|N_F(l, z_1, \ldots, z_{k-2})| \geq \frac{8}{10} \eta n$, for any $k-2$ vertices $z_1, \ldots, z_{k-2}$.

From this remark, it’s easy to see that we can construct $(k-1)$-paths $a_1, \ldots, a_{k-1}, w_1, \ldots, v_i, l, x_1, x_2, \ldots, x_{k-2}$, such that each $x_j$ can be chosen in at least $\frac{8}{10} \eta n$ many ways. So we are done with Case 1.

**Case 2.2.** $|W_{low}| \leq \frac{n}{10} n$. Choose a clique $C_1 \subseteq N(\{a_1, a_2, \ldots, a_{k-1}\})$ with $C_1 = \{c_1, \ldots, c_{2k}\}$.

It is obvious that there are two distinct vertices, $c_i$ and $c_j$ for which $|N_F(c_i) \cap N_F(c_j)| > \frac{k-3}{k} n + \frac{n}{10} n^2$.

We can assume that $i = 1$ and $j = 2$. We will consider only the first $k-1$ elements and denote the set $C_2 = \{c_1, c_2, \ldots, c_{k-1}\}$. Then using an argument similar to the previous one, we can find $x_1, x_2, \ldots, x_{k-2}$, such that $a_1, a_2, \ldots, a_{k-1}, c_{k-1}, c_{k-2}, \ldots, c_2, c_1 x_1, x_2, \ldots, x_{k-2}$ is a $(k-1)$-path and for
each $x_i$, as before, we have at least $\frac{n}{10}$ choices (notice that $c_1, c_2, \ldots, c_k$, can also be chosen in at least $\frac{n}{10}$ many ways).

If $|N_F(x_1, x_2, \ldots, x_{k-3}, x_{k-2}, c_1)| \geq \eta n$, then we can choose $x_{k-1}$ from $N_F(x_1, x_2, \ldots, x_{k-3}, x_{k-2}, c_1)$ in at least $\frac{n}{10}$ different ways, such that $a_1, a_2, \ldots, a_{k-1}, c_{k-1}, c_{k-2} \ldots c_2, c_1, x_1, \ldots, x_{k-2}, x_{k-1}$ is a $(k-1)$-path.

If $|N_F(x_1, x_2, \ldots, x_{k-2}, c_2)| \geq \eta n$, then we can choose $x_{k-1}$ in at least $\frac{n}{10}$ different ways, so that $a_1, a_2, \ldots, a_{k-1}, c_{k-1}c_{k-2}, \ldots c_2, c_1, x_1, \ldots, x_{k-2}, x_{k-1}$ is a $(k-1)$-path.

If both $|N_F(x_1, \ldots, x_{k-2}, c_2)| < \frac{n}{10}$, and $|N_F(x_1, \ldots, x_{k-2}, c_1)| < \frac{n}{10}$, then since $|N_F(x_1, x_2, \ldots, x_{k-3}, x_{k-2})| \geq \frac{n}{k}$, we have that

$$N_F(c_1, c_2) \geq \frac{k-2}{k} n - \frac{2\eta n}{10},$$

therefore $|N_F(c_1, c_2, x_1, x_2, \ldots, x_{k-3})| > \frac{n}{k} - \frac{2\eta n}{10}$. Since our graph $G$ is $\alpha$-non-extremal, in $|V_F(c_1, c_2, x_1, x_2, \ldots, x_{k-3})|$ we have at least $\alpha \left( \frac{n}{k} \right)^2$ edges. For $x_{k-2}$ we consider the vertices which have degrees into $N_F(c_1, c_2, x_1, x_2, \ldots, x_{k-3})$ larger than $\frac{n}{k}$. We have at least $\frac{n}{k}$ such large degree vertices. We choose for $x_{k-2}$ the large degree vertices and for $x_{k-1}$ the endpoint of the edges incident to $x_{k-2}$. It is then obvious that $a_1, \ldots, a_{k-1}, c_{k-1}, \ldots, c_2, c_1, x_1, x_2, \ldots, x_{k-3}, x_{k-2}, x_{k-1}$ is a $(k-1)$-path, and for both $x_{k-2}$ and $x_{k-1}$ we have $\left( \frac{n}{10} \right)^2$ choices. So we have proved the extending Lemma.

We shall apply this to the $b_i$'s as well, to get a $(k-1)$-path $b_1, b_2, \ldots, b_{k-1}c_1, c_2, \ldots, c_yy_1y_2y_{k-1}$. □

![Figure 4: The $(k-1)$-path may be extended if $W \cap F$ is large.](image)

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3.2.2. Connecting \(a_1, a_2, \ldots, a_{k-1}\) and \(b_{k-1}, \ldots, b_2, b_1\) inside \(N(v)\), where \(v\) is a good vertex

First we choose \(z_1, z_2, \ldots, z_k \in M\). For every \(v_i\) we choose a \((k-1)\)-path

\[
\begin{align*}
u^{(i)}_{k-1}, \ldots, u^{(i)}_1, \ldots, u^{(i)}_{k+1}, \ldots, u^{(i)}_{2k-1}.
\end{align*}
\]

Because of the properties of \(z_i\), this can be done easily. These paths are vertex-disjoint. The Extending Lemma will be applied to \(a_1, \ldots, a_{k-1}\) and also to \(b_1, b_2, \ldots, b_{k-1}\): We consider the \((k-1)\)-paths

\[
a_1 a_2 \ldots a_{k-1} w_1 w_2 \ldots w_s x_1 x_2 \ldots x_{k-1} \ b_1 b_2 \ldots b_{k-1} w_1 w_2 w_3 \ldots w_t y_1 y_2 y \ldots y_{k-1}
\]

Now, using the induction hypothesis, we connect \(x_2 x_3 \ldots x_{k-1}\) to \(u^{(1)}_{k-1} u^{(1)}_{k-2} \ldots u^{(1)}_2\).

The number of connecting paths is at least \(\eta^{(k-1)(k-1)}\) \(n^{\ell_1}\), where \(\ell_1\) is the length of the connecting paths. (We can assume that all the paths have the same length \(\ell_1 \leq 10(k-1)(k-1)!\).) Now we connect \(u^{(1)}_{k+2}, \ldots, u^{(1)}_{2k-1}\) to \(u^{(2)}_{k-1}, u^{(2)}_{k-2}, \ldots, u^{(2)}_2\).

We continue connecting the \(i^{th}\) ending segment to the \((i+1)^{th}\) initial segment. Finally we connect \(u^{(k)}_{k+1} \ldots u^{(k)}_{2k-1}\) to \(y_{k-1} y_{k-2} \ldots y_2\). They are \((k-2)\)-paths and they remain \((k-2)\)-paths even after removing any number of \(z\)’s.

Let the number of vertices on the path from \(x_{k-1}\) to \(y_{k-1}\) be \((k-1)t + r\) where
We omit \( r \) vertices \( z_1, \ldots, z_r \) from our path, to get the divisibility with \( k - 1 \) and get \( \gamma = (k - 1)t \) vertices. If the corresponding lengths are \( \ell_1, \ldots, \ell_{k+1} \), then we have altogether at least \( \eta^{(k+1)(k-1)} n^\gamma(2k(k-1))^2 \eta^2 n^\gamma + 2(k-1) \)

If we consider all the paths for all possible \( x_1, x_2, \ldots, x_{k-1} \) and \( y_{k-1}, y_{k-2}, \ldots, y_1 \), then, obviously, most of them will be good paths, because the number of paths we constructed is at least

\[
Q := \eta^{(k+1)(k-1)(k-1)} (\frac{n}{\eta})^{2(k+1)(k-1)} n^\gamma + 2(k-1) \geq 100k^2 \eta^{kk} n^\gamma + 2(k-1).
\]

Fix a good path \( P \) joining \( x_1 \) to \( y_1 \). For this \( P \) we have a set \( T_P \) of size \( \eta^{kk} n \). We shall change this \( P \) into a \((k-1)\)-path by inserting \( t \) vertices from \( T_P \). We insert them in the following way. We move along \( P \), starting with \( y_{k-1} \) inserting the next vertex from \( T_P \) and then moving along \( P \) for \( k-1 \) vertices and then again insert a vertex from \( T_P \), again move on along the paths \( P, \ldots \) and continue this until we have inserted a vertex of \( T_P \) next to \( x_{k-1} \). This way our path will be a \((k-1)\)-path, and we created from a given \( P \) at least \( |T_P|^t (k-1)\)-paths. If we consider all the possible paths and do the same thing, we get at least \( Q \eta^{kk} n^t (k-1)\)-paths, which is more than the required number of connecting paths in the Lemma.

4. Covering Lemma

**Lemma 12** (Covering). We can partition \( V(G) \) into \( C, K, I \) such that \( C \) is the union of complete \((k+1)\)-partite graphs with color classes of size \( \eta^{(1/\eta)} \log n \), and \(|C| = \sqrt{\eta m} \); \( K \) is the union of complete \( k \)-partite graphs with color classes of size \( \eta^{(1/\eta)} \log n \); and \(|I| < \eta n \).

**Proof.** Using Lemma 7 we get \( C \). Notice that if a graph \( G^* \) on \( m \) vertices has density \( > \frac{k-2}{2} + \frac{1}{k} \), then by Lemma 6 we get that \( G^* \) contains a complete \( K_k(\eta \log m) \). So, using Lemma 6 we can construct a set \( K^{(1)} \) of complete \( k \)-partite graphs of size \( \eta \log n \), whose union has at least \( \eta n \) vertices. We are left with a set of vertices we denote by \( I^{(1)} \).

We will successively remove vertices from \( I^{(1)} \), and use them to construct more complete \( k \)-partite graphs (even if with slightly smaller color classes), until the set of remaining vertices has less than \( \eta n \) elements.

We proceed by rounds, as explained below. At the \( i^{th} \) round we get a set \( K^{(i)} \) of complete \( k \)-partite graphs with color classes of size \( t^{(i)} := \eta^i \log n \),
and whose union has \( i\eta \) vertices. Let \( \mathcal{I}^{(i)} = V(G) \setminus \mathcal{C} \cup \mathcal{K}^{(i)} \). If \(|\mathcal{I}^{(i)}| \geq \eta n\) we carry out the \((i+1)\)th round; otherwise we stop and \( \mathcal{I}^{(i)} \) is the set \( \mathcal{I} \) as in the lemma. Notice that such procedure terminates after at most \( \frac{1}{\eta} \) rounds.

When the set of remaining vertices \( \mathcal{I}^{(i)} \) has much more elements than \( \mathcal{C} \), say \(|\mathcal{I}^{(i)}| \geq \eta^\frac{1}{2} n\), we can easily complete a round in the following way.

**Case** \( d(\mathcal{I}^{(i)}) \geq \frac{k-2}{k-1} + \frac{1}{k} \): then, since \(|\mathcal{I}^{(i)}| \geq \eta^\frac{1}{2} n\), by lemma 3 we can get complete \( k \)-partite graphs of size \( \eta^{i+1} \log n \).

**Case** \( d(\mathcal{I}^{(i)}) \leq \frac{k-2}{k-1} + \frac{1}{k} \): then, because of the minimum degree condition, we get that \( \deg_{\mathcal{K}^{(i)}}(x) \geq \frac{k-1}{k} n + \frac{\eta^\frac{1}{2}}{2^k} n \), for at least \( \frac{\eta^\frac{1}{2}}{2^k} \) of the vertices \( x \in \mathcal{I}^{(i)} \).

Therefore, by lemma 3 we can find a complete \( k \)-partite graph \( K \in \mathcal{K}^{(i)} \) with color classes \( V_1, V_2, \ldots, V_k \), and a set \( B \subseteq \mathcal{I}^{(i)} \) with \(|B| = k\eta^{i+1} \log n \), such that \( N_{V_1 \cup V_2 \cup \cdots \cup V_k}(v) \) is the same for every \( v \in B \), and it has size at least \( k\eta^{i+1} \log n \). But then some subsets \( A_j \subseteq N(B) \cap V_j \), \( 1 \leq j \leq k \), and \( B \) will give us a complete \((k+1)\)-partite graph with color classes of size \( k\eta^{i+1} \log n \).

We break this complete \((k+1)\)-partite graph into complete \( k \)-partite graphs of size \( \eta^{i+1} \log n \). And we also break the unbalanced complete \( k \)-partite graph obtained from \( K \) by removing the sets \( A_1, A_2, \ldots, A_k \), into \( k \)-partite graphs all with color classes of size \( \eta^{i+1} \log n \).

We can repeat the above procedure while \(|\mathcal{I}^{(i)}| \geq \eta^\frac{1}{2} n \) (Since \(|\mathcal{I}^{(i)}| \gg |\mathcal{C}| \) in this situation, and so almost all edges coming out of \( \mathcal{I}^{(i)} \) go into \( \mathcal{K}^{(i)} \)).

Assume now that \( \eta n \leq |\mathcal{I}^{(i)}| \leq \eta^\frac{1}{2} n \).

Let us show that for most of the complete \( k \)-partite graphs in \( \mathcal{K}^{(i)} \), a big fraction of the vertices in \( \mathcal{I}^{(i)} \) has a large neighborhood in it.

We can assume that for every \( K \in \mathcal{K}^{(i)} \), \( \{x \in \mathcal{I}^{(i)} : N_K(x) \geq \left( \frac{k-1}{k} + \eta \right) |K| \} \) has size at most \( n^\frac{1}{2} \) (Otherwise we find \( k\eta^{i+1} \log n \) elements which we can remove from \( \mathcal{I}^{(i)} \) and use to construct new complete \( k \)-partite graphs, just as in the previous case).

It follows by an elementary averaging that for at least \( (1 - \eta^\frac{1}{2}) \) fraction of the graphs \( K \) in \( \mathcal{K}^{(i)} \), \( |E(K, \mathcal{I}^{(i)})| \geq \left( \frac{k-1}{k} - 4\eta^\frac{1}{2} \right) |K||\mathcal{I}^{(i)}| \). Let \( \mathcal{K}^* \) be the set of such \( k \)-partite graphs in \( \mathcal{K}^{(i)} \) (for which a large fraction of the vertices in \( \mathcal{I}^{(i)} \) have large neighborhood in it).

Given \( K \in \mathcal{K}^* \), by definition of \( \mathcal{K}^* \) and averaging it follows that at least \( (1 - \eta^\frac{1}{2}) |\mathcal{I}^{(i)}| \) vertices in \( \mathcal{I}^{(i)} \) have more than \( \left( \frac{k-1}{k} - 5\eta^\frac{1}{2} \right) |K| \) neighbors in \( K \). Call \( \mathcal{I}^* \) the set formed by those vertices.

By the definition of \( \mathcal{I}^* \) (and the assumption that few elements of \( \mathcal{I}^{(i)} \)}
have degree larger than \((\frac{k-1}{k} + \eta) |K_1|\) in \(K\) we get that every vertex in \(\mathcal{I}^*\) has almost full degree to \(k - 1\) of the color classes of \(K\), and very few neighbors in the remaining one. Partition \(\mathcal{I}^*\) into sets \(I_1, I_2, \ldots, I_k\), where 

\[ I_j = \{ x \in \mathcal{I}^* : deg_{V_j}(x) \leq 6 \eta \frac{1}{16} \}. \]

If \(|I_1| \geq n^{\frac{1}{2}}\), by lemma 6 we can find a set \(I'_1 \subseteq I_1\) of \(\eta t\) elements, and \(A_2 \subseteq V^2, \ldots, A_k \subseteq V^k\), which form a complete \(k\)-partite graph. Therefore if we remove a set \(X_1\) of \(\eta t\) elements from the first color class \(V^1\), and replace it by \(I'_1\), we can break the graph into smaller \(k\)-partite graphs, with color classes of size \(\eta t\). The reader might feel discouraged to notice that we got as many new vertices excluded from \(\mathcal{K}^*\) as the ones we were able to remove from \(\mathcal{I}^*\). However, if we repeat this procedure for \(I_2, I_3, \ldots, I_k\), we obtain excluded sets \(X_1, X_2, \ldots, X_k\) from the different color classes in \(K\). And these form a \(K_k(\eta t^{(i)})\), which we can add to \(\mathcal{K}^{(i)}\).

Therefore, without loss of generality, we can assume that \(|I_k| \leq n^{\frac{1}{2}}\). This means that at least \((1 - 2 \eta \frac{1}{16}) |\mathcal{I}^{(i)}|\) of the vertices in \(\mathcal{I}^{(i)}\) have more than \((1 - 7 \eta \frac{1}{16}) |V^k|\) neighbors in \(V^k\).

Proceeding the same way with the remaining \(k\)-partite graphs in \(\mathcal{K}^*\), we can find for each one of them, a big subset of vertices in \(I^{(i)}\) which are connected to almost all elements of their last color class.

![Figure 6](image_url)

Figure 6: The bold circle represents elements in \(\mathcal{I}^{(i)}\), which neighbor almost all the vertices of the last class of the first graph.

Given a graph in \(\mathcal{K}^*\), we can repeat the above analysis, studying the neighborhood in its first \(k - 1\) color classes, of those vertices in \(I^{(i)}\) that are connected to almost all the last class.

Iterating the previous argument, we can assume that for each of the \(k\)-partite graphs in \(\mathcal{K}^*\), there is a big subset of (at least \((1 - 2 \eta \frac{1}{16})^{k-1} |\mathcal{I}^{(i)}|\)) vertices in \(I^{(i)}\) which are connected to almost all elements of the last \(k - 1\) color classes (precisely, with at least \((1 - 7 \eta \frac{1}{16}) t^{(i)}\) neighbors in each class).
Figure 7: The bold circle represents elements in $\mathcal{I}^{(i)}$, which neighbor almost all the vertices of the last $k - 1$ classes of the first graph.

At this point of our procedure, given a graph $K$ in $\mathcal{K}^*$, we can replace any $\eta t^{(i)}$ of the elements of its first color class, by a subset of elements from $\mathcal{I}^{(i)}$ which are connected to almost all vertices in the last $k - 1$ classes of $K$. We use lemma 6 as usual.

The reader might feel discouraged to notice that we exclude as many elements from $\mathcal{K}^{(i)}$ as the ones we insert from $\mathcal{I}^*$. The interest in such step is that we may choose any $\eta t^{(i)}$ elements we like to remove from the first color class of the graph $K$. Proceeding analogously with the remaining graphs in $\mathcal{K}^*$, we remove a random looking set of their first color classes and replace $\mathcal{I}^{(i)}$ by it. Since $G$ is $\alpha$-non extremal, we can chose $Y_1, Y_2, \ldots$ such that the set obtained from $\mathcal{I}^{(i)}$ after replacing most of its elements by $Y_1 \cup Y_2 \cup \ldots$ still has the same size than $\mathcal{I}^{(i)}$, but density at least $\alpha/2$.

Restarting the whole procedure with this new set of remaining vertices, and assuming that the algorithm does not the terminate, we reach the last step (as in figure 4) with a set of vertices with density at least $\alpha/4$. By abuse of notation denote it still by $\mathcal{I}^{(i)}$ (though many rounds might be completed).

At this point we can reduce the size of $\mathcal{I}^{(i)}$ in the following way. Given a graph $K$ in $\mathcal{K}^*$, with color classes $V^1, V^2, \ldots, V^k$, partition $\mathcal{I}^{(i)}$ into sets of vertices $D_1, D_2, \ldots, D_l$ that have the same neighborhood in $K$. We distinguish two cases.

**Case 1.1.** There is a group, say $D_1$ such that $e(G|_{D_1}) \geq \eta^2|D_1|^2$. Then, by Lemma 3 with $s = 1$ in $G|_{D_1}$, we have a complete bipartite graph, spanned by two sets $Q$ and $R$, of size greater than $\beta_2 \log n$. Since $N_{V_j}(D_1) > \frac{2}{3}|V_j|$
(for $j \neq 1$) we can find $K_{k-1}(t) \subset N_{V(K) \setminus V_i}(D_1)$ which together with $Q$ and $R$ gives us a $(k+1)$-partite graph with color classes of size $k \eta t^{(i)}$.

**Case 1.2.** There are two sets, say, $D_1$ and $D_2$, such that $e(D_1, D_2) \geq \eta^2 |D_1||D_2|$. Then by Lemma 2, we have a set $Q \subset D_1$ and $R \subset D_2$ of size greater than $\beta_2 \log n$ and $Q$ and $R$ form a complete bipartite graph. Since $N_{V_j}(D_1) \cap N_{V_j}(D_2) > \frac{1}{3} |V_j|$ for $j \neq 1$ we can find $K_{k-1}(t) \subset N_{V(K) \setminus V_i}(D_1) \cap N_{V(K) \setminus V_i}(D_2)$ which together with $Q$ and $R$ gives us a $K_{k+1}(k \eta t^{i})$.

In either case, we can consider sets $A_2 \subseteq V_2, \ldots, A_k \subseteq V_k$ such that $Q, R, A_2, \ldots, A_k$ form a complete $(k+1)$-partite graph. Breaking this complete $(k+1)$-partite graph, and also the graph obtained from $K$ by removing $A_1, \ldots, A_k$, into complete $k$-partite graphs of size $\eta t^{i+1} \log n$, we get $k$-partite graphs with color classes of size $\eta t^{(i)}$.

The key remark is that we inserted in $K^{(i)}$ new $2\eta tk$ elements from $I^{(i)}$ (namely the whole sets $Q, R$), and only remove $|A_1| = \eta tk$ of its vertices. Therefore the size of $I^{(i)}$ decreases by $\eta tk$.

We continue this procedure until the size of $I^{(i)}$ is smaller than $\eta n$, in which case we are done. This completes the proof of the Covering Lemma. \hfill $\square$

### 5. Constructing the $(k-1)$-cycle in the $\alpha$-non-extremal case

Let $t = \eta^{(1/\alpha)} \log n$. First we shall find a $(k-1)$-cycle covering $C \cup K$, and then insert all vertices from $I$, thus getting a $(k-1)^{th}$ power of a Hamiltonian cycle.

**Connecting the vertices in $C \cup K$**

We have covered the vertices of $C$ by vertex-disjoint copies of $K_{k+1}(t)$, and the vertices of $K$ by copies of $K_k(t)$. We call these blocks small multipartite graphs, some of them have $k+1$ classes, the others have $k$ classes. Denote the graphs in $C$ by $C_1, C_2, \ldots$, and by $K_1, K_2, \ldots$, the graphs in $K$.

For a graph $C_j \in C$, denote its color classes by $V_j^1, V_j^2, \ldots V_j^{k+1}$. In every color class $V_j^i$, consider an ordering of the vertices $v_{0,i}, v_{1,i}, v_{2,i}, \ldots, v_{t-1,i}$. Finally let us call the $l^{th}$ column of $C_j$ to the sequence $v_{l,1}, v_{l,2}, v_{l,3}, \ldots, v_{l,k+1}$.

For a graph $K_j \in K$, denote color classes by $W_j^1, W_j^2, \ldots W_j^{k+1}$, and vertices by $w_{l,i}$.

To build a $(k-1)$-path in each of the small multipartite graphs $C_j, K_j$, sequentially connect the vertices within a column, and then connect its last vertex, to the first vertex of the following column (see figure [1]).
Using lemma 9 we connect with \((k-1)\)-paths (of length at most \(9(k+1)!\)) the last vertex of the last column of a small multipartite graph with the first vertex of the first column of the succeeding one (see figure 2). We impose further that such paths do not use vertices from the first or last columns of any of the small multipartite graphs (which can be done since \(t = \Omega(\log n)\), and therefore the number of vertices in those columns is at most \(O\left(\frac{n}{\log n}\right) = o(n)\)).

Since the connecting paths might use a small number of vertices of the columns of the small multipartite graphs (at most \(9(k+1)!(k+1)\eta^{-\frac{1}{k}}\frac{n}{\log n}\)), we remove the vertices of those columns, and put them in \(I\). This will not increase significantly the size of \(I\).

Each time we remove a column from a small multipartite graph, we have to reconstruct the path inside the graph. We do this by connecting the column preceding the deleted one, directly to the one following it.

Let \(C^*, K^*\) denote the collection of multipartite graphs obtained from \(C\) and \(K\) respectively, after the removal of columns. Notice that \(|C^*| \geq |C| - 9(k+1)!(k+1)\eta^{-\frac{1}{k}}\frac{n}{\log n}\), which is still much bigger than \(I\). Let us now insert the vertices of \(I\) in the cycle in such a way that it remains a \((k-1)\)-path.

Adding the vertices of \(I\) to the cycle on \(C \cup K\)

**Definition 3** (Rich points). A vertex \(x \notin K_k(t)\) is rich for this \(K_k(t)\), if it is joined to each class of \(K_k(t)\) by at least \(k\eta t\) edges. A vertex \(y\) is rich for \(K_{k+1}(t)\) if it is joined to at least \(k\) of the classes by at least \(k\eta t\) edges.

We will first prove an easy consequence of the degree condition in \(G\):

**Lemma 13.** Every vertex \(a \in I\) is “rich” for at least \(\eta\) fraction of the cliques in \(C \cup K\).

**Proof.** For contradiction, assume that we are given a vertex \(a \in I\) that is not rich to at least an \(\eta\) fraction of the graphs in \(C \cup K\). Then

\[
\deg_G(a) < |I| + \eta n + (|V(C)| - \eta n) \left(\frac{k - 1}{k + 1} + 2\eta\right) + |K| \left(\frac{k - 1}{k} + \eta\right)
\]

\[
< \frac{k - 1}{k} n \quad (\text{since} \quad |C| \gg \eta n)
\]

A contradiction to the minimum degree condition. \(\square\)
We will have two cases.

**Case 1:** When $a$ is rich to a $(k+1)$-partite graph $C_j = (V_{j1}, \ldots, V_{jk+1}) \in \mathcal{C}^*$, assume that $a$ has at least $k\eta t$ neighbors in all the color classes of $C_j$ except $V_{jy}$. Without loss of generality, there is a vertex $v_{x,y}$ in the path in $C_j$, such that $a$ is connected to all vertices in the path at distance at most $(k - 1)$ from $v_{x,y}$ (Otherwise, we can reorder the vertices inside each color class of $C_j$ other than those in the first or last column, and reconstruct the path inside $|C_j|$ as in the first step. This does not change the connecting paths, nor the vertices of the small multipartite graphs they use, nor the sets $\mathcal{I}, \mathcal{C}^*, \mathcal{K}^*$).

Let us now replace the vertex $v_{x,y}$ by $a$ in the path inside $C_j$. Notice that $v_{x,y}, v_{x+1,y-1}, v_{x+2,y-2}, \ldots, v_{x+k,y-k}$ (where indices are taken (mod $k+1$)) form a clique, because they belong to different color classes in $C_j$. We can remove each one of them from the path, connecting the vertex preceding it directly to the one succeeding it in the original path. Finally, we insert the path formed by these $k + 1$ vertices between any two the columns of the graph, other than the one from which $a$ was removed.

Notice that the path obtained is still a $k - 1$ path, since each vertex was connected to the closest $k$-neighbors in the original one.

![Figure 8: Inserting $a$ into the $(k-1)$-path being constructed in the complete balanced $(k+1)$-partite graph $C_j$, where $k = 5$ and $b = 4.$](image)

We will however call the columns $x + 1, \ldots, x + k$ contaminated, as well as the $2k$ ones closer to each one of these after repeating the process and possibly reordering the vertices in $C_j$, because one no longer can use them.
to insert a new element, and still guarantee a \((k - 1)\)-path. The process of inserting \(a\) into our \((k - 1)\)-path is depicted in Figure 8, for \(k = 5\) and \(b = 4\).

When inserting a new vertex rich to the graph \(C_j\), at most \((k + 1)\) columns of the block got \textit{contaminated} by the previous vertex inserted. So we shall have enough space to insert all the vertices from \(I\) that are “rich” to graphs in \(C^*\).

**Case 2:** When \(a \in I\) is not “rich” for any \(K_{k+1}(t) \in C^*\) (since \(|C^*| \geq n^{1\over 2}\) an elementary computation shows that then \(\deg_{K^*}(a) \geq (k-1\over k + \frac{1}{4k})|K^*|\)) then \(a\) is rich for at least a \(\frac{n^{1\over 2}}{8k}\) fraction of the graphs \(K \in K^*\).

As a remark to the reader, this case is the very reason for considering \((k + 1)\)-partite graphs in the original covering, and for requiring their union to have size much bigger than \(|I|\).

Considering now a \(k\)-partite graph \(K = (W^1, \ldots, W^{k+1}) \in K^*\) for which \(a\) is rich, we can can assume without loss of generality that \(a\) is connected to all vertices in three consecutive columns of \(K\) (otherwise we just reorder the vertices inside each color class of \(K\) other than those in the first or last column, and reconstruct the path inside \(|K|\) as we did initially).

We insert the vertex \(a\) in the middle column of those three, between any two consecutive vertices. Again we refer to the three columns as being \textit{contaminated}, for the fact that we do not use them again when inserting in \(K\) a new vertex rich to the graph. It is clear that the paths thus obtained in the \(k\)-partite graphs after insertion of vertices in non contaminated columns, remain \((k - 1)\)-paths.

We can repeat this until all the vertices from \(I\) are used up (since \(|K^*| \gg |I|\)). This completes the case of the \(\alpha\)-non-extremal graph.

### 6. The Extremal Case

It is clear that \(\ell \leq k - 2\), otherwise our graph is an extremal graph, if we omit connecting two vertices of degree bigger than \(n^{1\over k} + 1\). In this case, the graph \(G\) satisfies the extremal condition. We take the maximum number of disjoint \(\alpha\)-extremal sets \(A_1, A_2, \ldots, A_\ell\). \(|A_i| = n^{1\over k}\) \((1 \leq i \leq \ell)\).

We let \(B = V(G) \setminus (A_1 \cup \cdots \cup A_\ell)\) for \(\ell \leq k\). Furthermore, we say that \(v \in A_i\) is \textit{bad} if we have

\[
\deg_{A_i}(v) \geq \alpha^{1/3}|A_i|.
\]
Note that by the fact that \(d(A_i) < \alpha\), there are at most \(\alpha^{2/3}|A_i|\) bad vertices in any \(A_i\). A vertex \(v \in A_i\) (or \(B\)) is exceptional for \(A_j\) (for \(j \neq i\)) if \(\text{deg}_{A_j}(v) < \alpha^{1/3}|A_j|\). For each \(v \in A_i\) (or \(B\)), there can be at most one \(j \neq i\), such that \(v\) is exceptional for \(A_j\). We denote the set of vertices in \(A_i\) (or \(B\)) that are exceptional for \(A_j\) by \(E_i(j)\) (or \(E_B(j)\)). By the minimum degree condition a vertex can be in \(E_i(j)\) for at most one \(j \neq i\). The following two remarks are easy to deduce.

**Remark 14.** If a vertex \(v\) is in \(E_i(j)\) for some \(j\) then it is bad, indeed \(\text{deg}_{A_i}(v) > (1 - \alpha^{1/3})|A_i|\).

**Remark 15.** Switching a bad vertex in \(A_i\) with a vertex in \(E_j(i)\) reduces the number of exceptional vertices. Hence we may assume that either there are no bad vertices in \(A_i\) or \(E_j(i)\) is empty for every \(j \neq i\).

### 6.1. Finding the Cycle in the extremal case

To convey the basic idea of the proof we deal separately with cases when \(\ell = k\) and when \(\ell < k\).

#### 6.1.1. \(G\) has \(k\) \(\alpha\)-extremal sets

In this case the vertex set \(V\) can be partitioned into \(A_1, A_2, \ldots, A_k\) such that \(|A_i| = \lfloor \frac{n}{k} \rfloor\) and \(d(A_i) < \alpha\) for \(1 \leq i \leq k\), that is, \(\ell = k\) (and hence \(B = \phi\)). We will further subdivide this case into two subcases.

**The Clean Case:** There are no bad or exceptional vertices in any \(A_i\), (hence \(E_i(j)\) is empty for all \(i, j\) by Remark 14). We will cover \(A_1 \cup \cdots \cup A_k\) with \(k\)-cliques such that every clique uses a vertex from each \(A_i\). For each \(v \in A_i\), we have \(\text{deg}_{A_i}(v) \geq (1 - \alpha^{1/3})|A_i|\) for all \(j \neq i\). Furthermore, since in this case there are no bad vertices it is relatively straightforward to find \(k\)-cliques by a simple greedy procedure that uses the König-Hall theorem as follows. We first find a perfect matching \(M_1\) between \(A_1\) and \(A_2\). Then we find a perfect matching between \(M_1\) and \(A_3\), such that \(e = \{x, y\} \in M_1\) is matched with a vertex \(z \in N(x, y) \cap A_3\). We can continue this process to find the desired \(k\)-cliques. Indeed, let \(M_{k-2}\) be the \((k - 1)\)-cliques made so far, from \(A_1, A_2, \ldots, A_{k-1}\). For any clique \((x_1, x_2, \ldots, x_{k-1})\), \(x_i \in A_i\) we have that \(|N(x_1, x_2, \ldots, x_{k-1}) \cap A_k| \geq (1 - \alpha^{1/4})|A_k|\), also, for \(y \in A_k\), \(y\) is connected to at least \((1 - \alpha^{1/10})n/k\) \((k - 1)\)-cliques of \(M_{k-2}\). Therefore, by König-Hall theorem there exists a perfect matching between the \((k - 1)\)-cliques and vertices in

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A_k, therefore we can extend these \((k-1)\)-cliques to \(k\)-cliques. Call this clique cover \(C_k = \{c_1, c_2, \ldots, c_{\lceil \frac{n}{k} \rceil} \} \).

**Figure 9:** Unfolding the cliques in the order defined by \(H^*\) gives us the required power of a Hamiltonian cycle.

Let \(c_1 = (x_1, x_2, \ldots, x_k)\) and \(c_2 = (y_1, y_2, \ldots, y_k)\) be any two such \(k\)-cliques in \(C_k\) (note that \(x_i, y_i \in A_i\)). We say that \(c_1\) precedes \(c_2\) if \(x_i\) is connected to \(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_k\) for \(1 \leq i \leq k\). \(c_1\) precedes \(c_2\) basically means that \(x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k\) is a \((k-1)\)-path. We say that \(\{c_1, c_2\}\) is a good pair, if \(c_1\) precedes \(c_2\) and \(c_2\) precedes \(c_1\). By the degree conditions above, any \(c_i \in C_k\) makes a good pair with at least \((1 - \alpha^{1/5})|C_k|\) other cliques in \(C_k\).

We define an auxiliary graph \(G^*\) in the following way: the vertex set of the graph \(G^*\) is \(C_k = \{c_1, c_2, \ldots, c_{\lceil \frac{n}{k} \rceil} \}\) and \(\{c_i, c_j\}\) is an edge in \(G^*\) if and only if \(\{c_i, c_j\}\) is a good pair. By the above observation \(\delta(G^*) > |C_k|/2\), hence there exists a Hamiltonian cycle \(H^*\) in \(G^*\). If we take the cliques in the order of \(H^*\) and unfold individual cliques in the natural order defined by \(A_1, A_2, \ldots, A_k\), it is easy to see that this gives us the \((k-1)^{th}\) power of a Hamiltonian cycle in \(G\).

**Handling the Exceptional Vertices:** In this case we have some \(E_j(i)\)'s that are non-empty. The main idea is to reduce this case to the clean case where there are no exceptional vertices. Handling of the bad vertices can be reduced to the handling of the exceptional vertices. So we shall discuss only the handling of the exceptional vertices.

Define \(X_i\) to be the set of all the vertices that are exceptional for \(A_i\), that is, \(X_i = \bigcup_{j=1}^k E_j(i)\).

**Case 1:**
Figure 10: Finding the *exceptional clique* when $|X_i| > 1$.

If $|X_i| > 1$, degree almost $x_i$, we would want to find paths of length 2 with endpoints in $A_i$ and centers at *exceptional* vertices in $E_j(i)$ for some $j$. For this purpose we note that $\delta(G|_{A_i \cup X_i}) \geq |X_i|$ by the minimum degree condition. Furthermore, since $d(A_i) < \alpha$ it follows that $\Delta(G|_{A_i \cup X_i}) \leq \alpha^{1/3}|A_i| + |X_i|$. Thus by Lemma 8 we can find more than $|X_i|$ vertex disjoint paths of length two. However, not all such paths may have their endpoints in $A_i$ or their centers in $X_i$. This can easily be handled by noting that any vertex in $X_i$ may be switched with any of the vertices in $A_i$ and the exchanged vertices become *exceptional* or not *bad* in their respective new sets. Therefore we may assume that there is a set, $P_i$, of $|X_i|$ disjoint paths of length 2, such that the two endpoints of each path are vertices in $A_i$ and the center is an *exceptional* vertex in some $E_j(i)$.

We embed each of these paths in a distinct unit of three $k$-cliques as follows: let $(u_i, c_j, \bar{u}_i) \in P_i$ be one of the paths such that $u_i, \bar{u}_i \in A_i$, and $c_j \in A_j$. Select a clique in the natural order $S = (s_1, s_2, \ldots, s_k)$ such that $s_i = u_i$ and $s_j = c_j$, (so we use the $\{u_i, c_j\}$ edge). Now we select another
clique $T = (t_1, t_2, \ldots, t_k)$ such that $t_i = \bar{u}_i$ and $S$ precedes $T$. Then we select a clique $R = (r_1, r_2, \ldots, r_k)$ which precedes $S$.

It is easy to see that there are many cliques with the given restrictions such that only $s_i$ is the bad vertex among all the three cliques. The cliques, unfolded in the order $R, S, T$, make a $(k-1)$-path. We replace this set of three $k$-cliques by a single $k$-clique (which we call an exceptional clique) with one vertex each from $A_1, \ldots, A_k$. The new vertex of the exceptional clique in $A_m$ is connected to all the common neighbors of $r_m$ and $t_m$. Since $r_m$ and $t_m$ are not bad vertices for $1 \leq m \leq k$, therefore these new vertices have high degree in all the sets $A_i$ where $i \neq m$. We deal with all the exceptional vertices in this manner and get exceptional cliques for each of them. In the remaining graph, we use the procedure described in the previous section to find a cover consisting of $k$-cliques and add the exceptional cliques to the cover. Then, as previously, we find a Hamiltonian cycle of the cliques in the cover and unfold the vertices in the cliques in the order defined by the cycle to get $(k-1)$th power of a Hamiltonian cycle. In Figure 10 the relevant portion in the final $(k-1)$th power of a Hamiltonian cycle looks as follows: $(\ldots, v_5, v_6, r_1, r_2, r_3, r_4, r_5, r_6, s_1, s_2, s_3, s_4, s_5, s_6, t_1, t_2, t_3, t_4, t_5, t_6, v_7, v_8, \ldots)$

**Case 2:**

When $|X_i| = 1$, we may not be able to find the length 2 path as above, for example, when the exceptional vertex $c_j \in E_j(i)$ for some $j$ has exactly one neighbor $y \in A_i$ (it has to have at least one neighbor). Then all the vertices in $A_i$ (except $y$) may have exactly one neighbor inside $A_i$. Note that by the minimum degree condition this case can only happen if $|X_i| = 1$. Therefore we find a path $p_i = (u_i, c_j, u_j)$ of length 2, where $u_i \in A_i$ and $c_j, u_j \in A_j$ such that $c_j$ is an exceptional vertex for $A_i$. In addition, we select an edge $\{w_i, \bar{w}_i\}$ inside $A_i$ disjoint from all the paths of the length 2 which we may have already chosen.

Select a clique in the natural order $S = (s_1, s_2, \ldots, s_k)$ such that $s_i = u_i$ and $s_j = c_j$ so that we use the $\{u_i, c_j\}$ edge. Now select another clique $T = (t_1, t_2, \ldots, t_k)$ such that $t_i = w_i$ and $t_j = u_j$. However, we are going to consider $T$ in the following order:

$$ T' = (t_1, t_2, \ldots, t_i-1, t_j = u_j, t_{i+1}, \ldots, t_{j-1}, t_i = w_i, t_{j+1}, \ldots t_k) $$

(i.e. this order switches the positions of $t_i$ and $t_j$). Note that $T'$ utilizes the $\{c_j, u_j\}$ edge of $p_i$ and that $S$ precedes $T'$.

Next we find a clique $U$ such that $T'$ precedes $U$. Such a clique exists, because we can utilize the edge $\{w_i, \bar{w}_i\}$ and $w_i$ and $\bar{w}_i$ are not bad ver-
Figure 11: Finding the exceptional clique when $|X_i| = 1$.

tices. There are many cliques $U = (u'_1, \ldots, u'_k)$, and $u'_i = \bar{w}_i$, such that $T'$ precedes $U$. Then we find another clique $R$ which precedes $S$. We replace this set of four $k$-cliques by a single $k$-clique (the exceptional clique) with one vertex each from $A_1, \ldots, A_k$. As previously, the new vertex of the exceptional clique in $A_m$ is connected to all the common neighbors of $r_m$ and $u'_m$. Since $r_m$ and $u'_m$ are not bad vertices for $1 \leq m \leq k$, therefore these new vertices have high degree in all the sets $A_i$ where $i \neq m$. We deal with all the exceptional vertices in this manner and get exceptional cliques for each of them. We get $(k-1)^{th}$ power of a Hamiltonian cycle using the same method as was done in the previous cases. In Figure 11 the relevant portion in the final $(k-1)^{th}$ power of a Hamiltonian cycle looks as follows:

$$\left(\ldots, v_5, v_6, r_1, r_2, r_3, r_4, r_5, r_6, s_1, s_2, s_3, s_4, s_5, s_6, t_1, t_5, t_3, t_4, t_2, t_6, u'_1, u'_2, u'_3, u'_4, u'_5, u'_6, v_7, v_8, \ldots\right)$$

6.1.2. $G$ has less than $k$ extremal sets

We first assume that $A_1, A_2, \ldots, A_{\ell}$ are the extremal sets where $\ell < k$ and we let $A = \bigcup_{i=1}^{\ell} A_i$ and $B = V(G) \setminus A$. (We remark that $\ell \leq k-2$.) We say that $v \in B$ is bad if $\deg_A(v) \leq (\ell - 1 + \alpha^{1/3})|A|$. While the bad vertices in
For every $y$ remaining

$\in C$

of Hamiltonian path. By (R3), we can easily find $b$

cycle. Consider

As before, we construct cliques of size $\ell$. Let the set of these cliques be $\mathcal{C} = \{c_1, c_2, \ldots, c_{n/k}\}$, where $c_i = \{y_1, y_2, \ldots, y_\ell : y_i \in A_i, \ \text{for} \ 1 \leq i \leq \ell\}$.

Remarks.

(R1) $N_A(P_{t-1}(k-\ell)+1, \ldots, P_t(k-\ell)) \geq (1 - k\eta^{1/4})|A|$ where $|A| = \ell \frac{n}{k}$

(R2) For every $y \in A$, $N_B(y) \geq (1 - k\eta^{1/4})|B|$ where $|B| = (k - \ell) \frac{n}{k}$.

(R3) For every $c_s \in \mathcal{C}$, the number of good pairs is at least $(1 - k\eta^{1/4})|\mathcal{C}|$, where $|\mathcal{C}| = \frac{n}{k}$.

After these remarks, we start we start to build our $(k-1)^{th}$-Hamiltonian cycle. Consider $b_1, b_2, b_3, \ldots, b_{n/k}$, which forms in this order a $(k-1)^{th}$-Hamiltonian path. By (R3), we can easily find $\frac{1}{2} \frac{n}{k}$ $c_s$ so that all the points of $c_s$ are connected to all the points of $b_{2s-1} \cup b_{2s}$. The we have the following subpath:

$p(2s-1)(k-\ell)+1 \cdots p_{2s}(k-\ell), y_1^{(s)} \cdots y_\ell^{(s)} p_{2s}(k-\ell)+1 \cdots p(2s+1)(k-\ell)$.

Because of Remarks (R1-R3), using König-Hall Theorem, we can order the remaining $c_s$ so that this ordering say $c_{(n/2k)+1}, c_{(n/2k)+2}, \ldots, c_{(n/k)}$ have the
property that $c_{(n/2k)+s}$ is good to $c_s$ and $c_{s+1}$, and every point of $c_{(n/2k)+s}$ is connected to every point of $b_{2s}$ and $b_{2s+1}$.

The subpath looks like this.

\[ p(2s-1)(k-\ell)+1, \ldots, p_{2s}(k-\ell) y_1^{(s)} \cdots y_\ell^{(s)}, p_{2s}(k-\ell)+1, \ldots, p_{(2s+1)(k-\ell)} \cdots, \]
\[ y_1^{(n/2k+s)} \cdots y_\ell^{(n/2k+s)} p_{(2s+1)(k-\ell)+1} \cdots p_{(2s+2)(k-\ell)}. \]

We got our desired $(k-1)^{th}$-Hamiltonian cycle.

**Handling the Exceptional Vertices:** Let us denote by $B^+$ the exceptional vertices in $B$. Notice that $B^+$ is connected to at least $(k-\ell-5k\eta)n$ vertices of $B \setminus B^+$. Therefore we can insert them on the $(k-\ell-1)^{th}$-Hamiltonian cycle \( \square \) covering $B \setminus B^+$, which we have already constructed. We insert the points of $B^+$ into our cycle $b_s$, but the distance on the cycle is at least $20(k-\ell)$.

We also will be careful with inserting this vertices, meaning that if we have inserted $v \in B^+$, into $b_s$, then $v$ is connected to every point of $b_{s-10}, \ldots, b_s, b_{s+1}, \ldots, b_{s+10}$. Since the degree $d_B(v)$ of every $v \in B^+$ is extremely large, we can do this easily. Notice that our new cycle is still a $(k-\ell-1)^{th}$ Hamiltonian cycle.

Again, we break our Hamiltonian cycle into paths of length $(k-\ell)$. As in the clean case, we denote these paths by $b_1, b_2, \ldots, b_{n/k}$. Since $N_B(v)$ is extremely large for $v \in B^+$, we can insert these vertices so that they are always the initial points of some $b_i$’s.

Now we build above $v \in B^+$ an exceptional clique as we did for $\ell = k$, and we had exceptional vertices. Of course, the points of $b_i$ are also included in these exceptional cliques. So the length of this clique is $k$. From here the procedure is the same as when $B^+$ was empty. \( \square \)

**References**


\footnote{When we say that a Hamiltonian path in a smaller set, we mean that completely covers that set.}


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