Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics

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Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics

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Dedicated to the 150th anniversary
of the birthday of the genius
Norwegian mathematician
Marius Sophus Lie (1842–1899).

We must be grateful to God that he
created the world in such a way
that everything simple is true and
everything complicated is untrue.

Gregory Skovoroda (an 18th-century
Ukrainian philosopher)

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Chapter 5. Some Special Questions

Solving concrete problems an investigator enriches himself, and in this way he discovers new methods and extends his mental outlook. He who searches new methods without any specific problem in mind just wastes time for nothing.

— D. Hilbert

Preface

As far back as D. Bernoulli, Euler, Laplace, d'Alembert, Fourier, Lame, Riemann, Liouville, and many other scientists, the symmetry properties of differential equations (DEs) were used, although implicitly, for finding their exact solutions. But the mathematical foundations of the theory of the symmetry of DEs were developed only at the end of the last century by the eminent Norwegian mathematician Sophus Lie. He was the first to successfully apply this theory—the theory of continuous (Lie) groups—to specific equations.

In recent decades Lie's ideas and methods have become widespread. It is impossible to overestimate the importance of Lie's contribution to modern science and mathematics. At present there are many articles and several monographs devoted to the application and development of Lie's methods. To give a complete bibliography on the subject is impossible. Suffice it to say that more than half of the articles of such popular science journal as "Journal of Mathematical Physics" deal with group (algebraic) investigations which we shall call symmetry analysis. The applications of Lie's theory include such diverse fields as algebraic topology, differential geometry, invariant theory, bifurcation theory, special functions, numerical analysis, control theory, classical mechanics, quantum mechanics, relativity, continuum mechanics, and so on.

This book is devoted to the development and diverse applications of Lie's theory. The main object of investigation is multidimensional systems of nonlinear partial differential equations (PDEs) of hyperbolic and parabolic types which admit the Poincare group, the Schrödinger group, and some other important groups. A key question considered is the construction of exact solutions of nonlinear systems of PDEs by means of symmetry reduction. For this purpose we use the theory of Lie groups and Lie algebras, the theory of representations, subalgebraic structure of Lie algebras, special ansatze, and so on. The table of contents indicates the equations studied. The especially important role played by spinor (spin $s = 1/2$) field equations is emphasized because their solutions can be used for constructing solutions of other field equations insofar as fields with any spin may be constructed from spin $s = 1/2$ fields. A brief account of the main ideas of the book is presented in the Introduction.

The book is largely based on the authors' works [55–109, 176–189, 13–16, 7*–14*, 23*, 24*] carried out in the Institute of Mathematics, Academy of Sciences of the Ukraine. References to other sources is not intended to imply completeness. As a rule, only those works used directly are cited.

The authors wish to express their gratitude to Academician Yu.A. Mitropolsky, and to Academician of Academy of Sciences of the Ukraine O.S. Parasyuk, for basic support and stimulation over the course of many years; to our coworkers in the Department of Applied Studies, I.A. Egorchenko, R.Z. Zhdanov, A.G. Nikitin, I.V. Revenko, V.I. Lagno, and I.M. Tsifra for assistance with the manuscript.

Ukraine, Kiev, October 1988 The Authors

Preface to the English Edition

For the English edition of our book *Symmetry Analysis and Exact Solutions of Nonlinear Equations of Mathematical Physics*, published in Russian in Kiev (Ukraine) in 1989 we have specially prepared some new paragraphs which naturally supplement and extend the basis text. The new paragraphs deal with solutions of the Navier-Stokes equations, the Fokker-Planck equations, conditional and approximate symmetry, nonlocal Galilei invariance of Maxwell equations, dual Poincare invariance and connection between Dirac and Maxwell equations, solutions of the Schrödinger equation invariant under the Lorentz algebra and some other topics. This, we hope, makes the text more interesting and useful.

Nowadays there arise new effective and constructive methods of mathemnatical description and analysis of nonlinear processes. The collection of these methods can be considered as new scientific direction—*nonlinear mathematical physics (NMP)*. The main distinguishing feature of NMP is nonfulfillment of classical superposition principle. This means that the majority of methods of linear mathematical physics are useless in NMP. Of course, sources of NMP can be discerned in works of classics from Bernoulli, Euler, and especially from Sophus Lie and Poincare, yet only at the end of the 20th century NMP has become a separate, wide and diversified scientific branch with its specific problems and methods.

We are glad that on professor Michiel Hazewinkel's initiative our book is published by Kluwer. We would like to take this opportunity to thank him for his interest in our work. This book should be considered as a natural continuation and further development and application of the ideas and methods of our first book *Symmetries of Maxwell's Equations* (W.I.Fushchich and A.G.Nikitin, D.Reidel, 1987) to nonlinear equations of mathematical and theoretical physics. In this series of books on symmetry and its applications in mathematical physics, we have three other books written in Russian, which have been or are yet to be published: *Symmetry of Quantum Mechanics Equations* (W.I.Fushchich and A.G.Nikitin, Moscow: Nauka, 1990), *Subgroup Analysis of the Galilei and Poincare Groups and Reduction of Nonlinear Equations* (W.I.Fushchich, L.F.Barannik and A.F.Barannik, Kiev: Naukova Dumka, 1991), *Nonlinear Spinor Equations: Symmetry and Exact Solutions* (W.I.Fushchich and R.Z.Zhdanzov, Kiev: Naukova Dumka, 1992).

Introduction

Since the time of Newton the search for exact solutions of differential equations describing genuine physical phenomena has been the most important issue in the mathematical description of nature. During the past 300 and more years a great number of effective and elegant methods for solving DEs have been developed: the method of special substitutions, the method of separation of variables, the Poisson method, the method of Fourier series expansion, the saddle point method, the method of the inverse scattering transform, and so on. If one looks at these methods from the point of view of group theory, then one sees that they are essentially based on symmetry, and effectively solve those problems which actually possess explicit or implicit symmetry.

Sophus Lie advanced many fundamental ideas and worked out basic methods for studying group properties of DEs. He also obtained many concrete results of great importance in applied mathmatics. In particular he was the first to establish the maximal group of point (local) transformations admitted by the one-dimensional heat equation, and discovered the so-called projective representation of the Galilei group. These results were rediscovered only recently. The well-known Noether theorem on conserved laws is based on Lie's theory of continuous groups. Nowadays, in connection with the modern development of mathematical and theoretical physics, numerous results of Lie are being recognized and rediscovered, and we are witnessing the triumph of Lie theory throughout all of the mathematically based sciences.

A crucial point in the recognition of Lie theory is the fact, established for the first time by Poincare in 1905, that the Lorentz transformations, which leave Maxwell's equations invariant, form a Lie group. In 1909 Bateman [25] and Cunningham [43] established that Maxwell's equations are invariant with respect to the conformal group which includes the Lorentz group as a subgroup. Bateman utilized the symmetry of the linear wave equation to construct its solutions. Later on such solutions were called functionally invariant (V.I. Smirnov and S.L. Sobolev, 1932). Some important ideas on searching for exact solutions of PDEs were suggested by H. Birkhoff [29]. A great number of exact solutions of two-dimensional nonlinear PDEs are given in books of Forsyth [54] and Ames [7]. In Kiev, Lie's methods were developed by V.P. Ermakov (1890–1900), G.V. Pfeifer (1920–1935), and M.K. Kurensky (1930).

In the post-war era of the USSR the first papers on the symmetry of PDEs were published by L.V. Ovsyannikov (1958) (see [161]) and by V.G. Kostenko (1959) [135]. The modern treatment of Lie's theory is given in monographs [27, 123, 124, 159, 161, 162].

It should be noted that Poincare was the first to suggest using the group theory approach for the construction and analysis of a physical theory. Today, symmetry principles are guiding principles in theoretical and mathematical physics. They are often used as selection rules, allowing one to choose from a set of mathematically permissible models (equations) those which possess the desirable properties. Sometimes such symmetry selection leads to a unique equation. For example, among the set of linear systems of PDEs for the two vector functions $\vec{E}(x)$ and $\vec{H}(x)$ there is only one system that is invariant under the Poincare group P(1,3) (P(1,3) is the ten-parameter group that includes the Lorentz transformations and spacetime translations, which at Wigner's suggestion was named in honor of Poincare), namely Maxwell's equations [82]. Analogous properties have been found for many of the basic linear and nonlinear PDEs of theoretical and mathematical physics [66]. It should be noted that some nonlinear PDEs have much wider symmetry than any linear ones. For example, equations such as the Monge-Ampere equation, and the Hamilton-Jacobi and relativistic Hamilton (eikonal) equations, admit such wide groups of transformations which do not admit (and cannot admit in principle) any linear PDE.

From a mathematical point of view it is important to know the maximal (in a certain sense) group of invariance of a given PDE. Of special value is information that provides knowledge of nonlinear invariance transformations that allow the construction of highly nontrivial formulae for generating solutions which yield new families of solutions starting from a known and often trivial solution. Evidently the first result that should be considered important in this context is the discovery of the conformal invariance of Maxwell's equations made by Bateman [25] and Cunningham [43] in 1909.

In [57] it was noted that the Lie approach is highly restricted. The fact is that it falls far short of providing the possibility of finding all symmetries which a system of DEs possesses in as much as Lie symmetry generators, which are always differential operators of first order, are far from being the only symmetry generators. Using the non-Lie approach recently developed [55, 57–61], new invariance algebras (IAs) of Maxwell's equations, the Dirac equation, and many other relativistic as well as nonrelativistic PDEs have been obtained. The basis elements of such IAs are differential operators of any order and even integrodifferential ones. A largely complete review of non-Lie symmetries obtained within the framework of the non-Lie approach is given in reference [82]. We often use the term "non-Lie symmetry," introduced in [61, 63,64], in order to emphasize that this symmetry cannot be obtained within the framework of classical Lie's method.

Investigation of symmetry and the construction of exact solutions of nonlinear PDEs is the major but not the only task of the present book. In the 5th chapter we consider some related questions.

For studying local (point) symmetry of PDEs we use Lie's algorithm [161,159]. But in many cases, especially for multicomponent systems of PDEs, the direct application of the standard Lie algorithm is conjugated with rather cumbersome calculations. To simplify these cases we shall do the following. Consider an arbitrary nonlinear system of PDEs written in the following symbolic form:

$$L\left(x, \psi(x)\right) = 0, \tag{1}$$

where ψ is a multicomponent function with components $\psi = \{\psi^1, \ldots, \psi^m\}$, and $x \in R^n$. In the Lie approach the infinitesimal operators (IFOs) of the invariance algebra (IA) of Equation (1) are sought in the form

$$X = \xi^\mu(x,\psi)\frac{\partial}{\partial x^\mu} + \eta^k(x,\psi)\frac{\partial}{\partial \psi^k}, \quad \begin{matrix} \mu = \overline{0, n-1} \\ k = \overline{1, m} \end{matrix} \tag{2}$$

where functions ξ^μ and η^k are determined from the invariance condition

$$\underset{s}{X}L \Big|_{L=0} = 0, \tag{3}$$

$\underset{s}{X}$ is the sth prolongation of the operator (2) which is defined according to the Lie formulae [159, 161]:

$$\underset{s}{X} = X + \tau^k_{\nu_1}\frac{\partial}{\partial \psi^k_{\nu_1}} + \ldots + \tau^k_{\nu_1\ldots\nu_s}\frac{\partial}{\partial \psi^k_{\nu_1\ldots\nu_s}}, \tag{4}$$

$$\tau^k_{\nu_1} = D_{\nu_1}\eta^k - \psi^k_\nu D_{\nu_1}\xi^\nu,$$

$$\tau^k_{\nu_1\nu_2} = D_{\nu_2}\tau^k_{\nu_1} - \psi^k_{\nu\nu_1} D_{\nu_2}\xi^\nu,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots ,$$

$$\tau^k_{\nu_1\ldots\nu_s} = D_{\nu_s}\tau^k_{\nu_1\ldots\nu_{s-1}} - \psi^k_{\nu\nu_1\ldots\nu_{s-1}} D_{\nu_s}\xi^\nu,$$

$\nu, \nu_1, \ldots, \nu_s = \overline{0, n-1}$; s is the order of the PDE in question; D_ν is the total derivative operator,

$$D_\nu = \frac{\partial}{\partial x^\nu} + \psi^k_\nu \frac{\partial}{\partial \psi^k} + \psi^k_{\nu\nu_1} \frac{\partial}{\partial \psi^k_{\nu_1}} + \ldots; \tag{5}$$

$$\psi^k_\nu = \frac{\partial \psi^k}{\partial x^\nu}, \qquad \psi^k_{\nu\nu_1} = \frac{\partial^2 \psi^k}{\partial x^\nu \partial x^{\nu_1}}, \quad \text{and so on.}$$

Invariance condition (3) results in a linear system of PDEs, the so-called defining equations, the general solution of which determines the maximal in the Lie sense IA.

If we deal with a linear system of PDEs

$$L(x, \partial)\psi(x) = 0, \tag{6}$$

where L is a linear operator, we can essentially simplify the above algorithm. Since the space of solutions of any linear equation is also linear, then the symmetry operators as well as the invariance transformations may also be only linear in ψ. So we can write the general form of the Lie IFOs in this case as

$$Q = \xi^\mu(x)\partial_\mu + \eta(x), \tag{7}$$

where $\partial_\mu \equiv \dfrac{\partial}{\partial x^\mu}$, and $\eta(x)$ is an $m \times m$ matrix. Note that operator (7) operates in the linear space $\{\psi(x)\}$ of solutions of Equation (6) while its counterpart of the form (2)

$$X = \xi^\mu(x)\partial_\mu - (\eta(x)\psi)^k \frac{\partial}{\partial \psi^k} \tag{8}$$

operates on the manifold $\{(x, \psi)\}$. There is a further point to be made here. According to the Lie algorithm, dependent and independent variables enter IFOs (2) equally and this circumstance deprives us of the possibility of effectively using the operator (matrix) calculus within the framework of the standard Lie algorithm.

It should also be noted that the most general form of IFOs generating linear transformations is

$$X = \xi^\mu(x)\partial_\mu - \big(\eta(x)\psi + \beta(x)\big)^k \frac{\partial}{\partial \psi^k} \tag{8'}$$

where $\beta(x)$ is an m-component function. In the case of linear PDEs (6) $\beta(x)$ is just an arbitrary solution of the equation. Evidently all the essential information on point symmetry of the system (6) is contained in operators of the form (7) or (8).

By means of operator (7) the invariance condition of the system (6) can be written as

$$LQ\psi(x)\Big|_{L\psi(x)=0} = 0. \tag{9}$$

It clearly means that operator Q transforms solutions of the system to other solutions, that is, its action on the set of solutions does not go out of this set. The condition (9) can be rewritten equivalently as

$$[L,Q]\psi \equiv (LQ - QL)\psi = 0, \tag{10}$$

or

$$[L,Q] = \lambda(x)L, \tag{11}$$

where $\lambda(x)$ is a certain $m \times m$ matrix. (Generalization of conditions (9)–(11); see Paragraphs 5.3, 5.7.) The general solution of the Equation (11) leads to the maximal (in sense of Lie) IA of the Equation (6). As one can see, using the algorithm based on Equation (11) is much simpler than using that of Lie based on Equations (2), (3).

Now consider the system of nonlinear PDEs

$$L(x,\partial)\psi(x) + F(x,\psi) = 0, \tag{12}$$

where $F(x,\psi)$ is a smooth m-component function depending on x and ψ. If relation (11) still holds for the IFOs (7) and the linear operator L of Equation (12), and, in addition, the following equality is fulfilled (see paragraph 5.5)

$$\xi^\mu(x)\frac{\partial F}{\partial x^\mu} - (\eta\psi)^k\frac{\partial F}{\partial \psi^k} + \big(\lambda(x) + \eta(x)\big)F = 0, \tag{13}$$

then our nonlinear system (12) is invariant under the IFO (7). The general solutions of Equations (11) and (13) determine the maximal IA of the system (12) in the class of operators (7). It will be noted that, generally speaking, the maximal, in the Lie sense, IA of the nonlinear PDEs (12) is apparently determined by IFOs of the form (8′) (for an example, see system (2.8.35)).

The greater part of the present book is devoted to the construction of exact solutions of nonlinear systems of PDEs. The method we are using is explained in Paragraphs 1.4 and 2.1. It is based on the representation of a solution in a special form called ansatz, which is defined as a rule by means of symmetry operators. The generalization of this method is given in Paragraph 5.7, where the concept of conditional invariance is introduced.

So, if an equation (linear or nonlinear) admits an IFO of the type (7), then its solutions can be sought in the form [63]

$$\psi(x) = A(x)\phi(x), \tag{14}$$

where the $m \times m$ matrix $A(x)$ and new variables $\omega = \omega(x)$ are determined from the equations [95]

$$\begin{aligned} &QA(x) \equiv (\xi^\mu(x)\partial_\mu + \eta(x))A(x) = 0 \\ &\xi^\mu(x)\partial_\mu\omega(x) = 0. \end{aligned} \tag{15}$$

The ansatz (14) reduces the initial system of PDEs to a system of PDEs with a smaller number of independent variables for the function $\phi(\omega)$. The idea of using the symmetry of equations in searching for their solutions goes back to Lie, and in the past 30 years was considerably advanced by H. Birkhoff [29], Ovsyannikov [161, 162], Ibragimov [123, 124] and others [7, 10, 11, 27, 159]. Numerous important results have been obtained by Winternitz with collaborators [114, 25*, 38*, 39*, 85*].

Symmetry transformations can be used to construct new solutions from known solutions. In particular, if the transformations

$$\begin{aligned} &x_\mu \to x'_\mu = f_\mu(x,\theta), \\ &\psi(x) \to \psi'(x') = R(x,\theta)\psi(x) \end{aligned} \tag{16}$$

($R(x,\theta)$ is a certain nonsingular matrix, θ are group parameters) leave a given PDE invariant, then the function

$$\psi_{II}(x) = R^{-1}(x,\theta)\psi_I(x') \tag{17}$$

will be a solution of the equation as soon as $\psi_I(x)$ is a solution of the equation. Expressions like (17) shall be called *formulae of generating solutions* [95]. It is appropriate to note that in connection with the operation of generating solutions the concept of G-ungenerative solutions naturally arises [65]. So, a family of solutions is called G-ungenerative if it is transformed into itself when it is subjected to the above-described group multiplication by means of all transformations of the group G. In this case the action of the operation of generating solutions is reduced to transformations of group parameters. (For more details see Paragraphs 2.3 and 4.1. Nonlocal transformations are considered in Paragraph 5.3.)

In discussing exact solutions of DEs one cannot fail to note two more methods which have been intensively developed recently, namely the method of separation of variables [11, 153] and the method of the inverse scattering transform [1, 154, 210]. The basic idea of monograph [153] is that the systems of coordinates which admit separation of variables for linear second-order PDEs can be described by means of sets of second-order differential operators $Q^{(2)}$. Solutions with separable variables are just eigenfunctions of operators $Q^{(2)}$ and the constant of separation k^2 is the corresponding eigenvalue, that is

$$Q^{(2)}\psi_k = k^2\psi_k. \qquad (18)$$

Comparing this relation with the conditions in (15), one easily sees that it is very similar to that of (15). Indeed, (18) can be rewritten as

$$\widetilde{Q}\psi_k = 0, \quad \widetilde{Q} = Q^{(2)} - k^2 I \qquad (19)$$

and therefore the solutions in (14) and solutions with separable variables ψ_k are invariant functions of the symmetry operators (7) and (19), respectively. Now it is clear why the method of separation of variables in its standard formulation is inapplicable to nonlinear PDEs. The point is that, firstly, nonlinear PDEs do not admit, as a rule, the unit operator I, and secondly, they may not be invariant under the second-order differential operators even though the latter belong to the enveloping algebra of the first-order symmetry operators.

It seems to us that the most natural way of describing separation of variables can be made in the framework of conditional invariance which is considered in Paragraph 5.7. For example, the classical variants of separation of variables, multiplicative and additive

$$\psi(x,t) = \phi(x)\chi(t), \quad \psi(x,t) = \phi(x) + \chi(t)$$

can be represented in the form of additional differential equations

$$\psi\psi_{xt} = \psi_x\psi_t, \quad \psi_{xt} = 0,$$

which may be added to the original equation. Similar ideas were discussed in [160].

The term and concept *conditional symmetry* (or *conditional invariance*) was suggested by one of us [67, 82, 7*, 59*] and represents by itself a natural generalization of classical Lie concept on invariance of differential equations. In general terms, the idea of this generalization consists in joining some additional equation(s)

$$L_1(x, \psi(x)) = 0 \qquad (20)$$

to the equation in question (1). Equation (20) is constructed so as to enlarge or just to change the symmetry of the basic Equation (1). For example, the maximal invariance algebra, in the Lie sense, of Maxwell equations without additional conditions $\operatorname{div}\vec{E} = \operatorname{div}\vec{H} = 0$ is the 10-dimensional Lie algebra which does not contain generators of Lorentz and conformal transformations [82, §42]. And only the full system of Maxwell equations, which is overdetermined, possesses the 17-dimensional invariance algebra which includes the Poincare and conformal algebras.

Another example of additional condition (20) is the equation [7*, 59*]

$$[a^{\mu\nu}(x,u,\underset{1}{u})J_{\mu\nu} + b^{\mu}(x,u,\underset{1}{u})P_{\mu} + c^{\mu}(x,u,\underset{1}{u})K_{\mu} + \\ + r(x,u,\underset{1}{u})D]u = F(x,u,\underset{1}{u}), \quad (21)$$

where $a^{\mu\nu}$, b^{μ}, c^{μ}, r, F are smooth functions of $x \in$ R(1,n), u, $\underset{1}{u} = \{\partial u/\partial x^{\mu}\}$; $J_{\mu\nu}$, P_{μ}, K_{μ}, D are basis elements of the conformal algebra (see §2.3). The particular case when elements of Equation (21) are arbitrary constants, solutions of Equation (1), satisfying (21), may coincide with those obtained by Lie's method. Symmetry of Equation (1) under the additional condition (21) we called conditional symmetry [82, 7*, 59*]. Numerous results on the study of conditional symmetry of nonlinear equations of mathematical physics are contained in [44*, 45*, 56*–60*, 82*, 83*, 90*–110*] where, in particular, non-Lie ansatze are constructed which reduce PDEs to ODEs.

The simplest example of an additional condition for nonlinear wave equation

$$\Box u = F(u) \quad (22)$$

is the equation

$$\frac{\partial u}{\partial x^{\mu}}\frac{\partial u}{\partial x_{\mu}} = \begin{cases} \pm 1, \\ 0. \end{cases} \quad (23)$$

Equation (23) is a particular case of Equation (21). Indeed, letting $a_{\mu\nu} = 0$, $c_{\mu} = r = 0$, $F = \{0, \pm 1\}$, $b_{\mu} = \partial u/\partial x^{\mu}$ in (21) we get (23). Equation (23) has an important property: it possesses a wider symmetry than Equation (22), which is why it can be used effectively to define new (non-Lie) ansatze for Equation (22). The general solution of system (22), (23) is obtained in [60*, 130*] (See Appendix 6).

Let us briefly note the method of the inverse scattering transform. In spite of numerous important results obtained within the framework of this method, it remains as yet applicable mainly to one-dimensional nonlinear PDEs of special form. The generalization to multidimensional cases faces difficulties which, unfortunately, have not yet been overcome.

In conclusion let us make one final remark. In many cases we give symmetry IFOs multiplied by i. This is done to make the operators Hermitian. But, since the independent variables considered are always real, all calculations should be done with real coordinates ξ^{μ} and η. This is especially important for nonlinear PDEs.

Chapter 1

Poincare-Invariant Nonlinear Scalar Equations

In the present chapter we describe the first- and second-order n-dimensional nonlinear PDEs which are invariant under the groups $\widetilde{\mathrm{P}}(1,n-1)$, $\widetilde{\mathrm{P}}(1,n)$. We investigate local and tangent symmetry of the relativistic Hamilton equation, of the nonlinear d'Alembert equation, of the Euler-Lagrange-Born-Infeld equation, the Monge-Ampere equation, and some other PDEs. For this purpose the Lie method has been used with the exception of Sec. 1.3, where the symmetry of the polywave equation is investigated by the operator method expounded in Sec. 5.5.

Sec. 1.4 is devoted to the description of the method of the construction of exact solutions for nonlinear PDEs, which is applied further to the above-mentioned equations and in the following chapters to systems of nonlinear PDEs.

1.1. *Nonlinear n-dimensional wave equations invariant under the groups* $\widetilde{\mathrm{P}}(1,n-1)$, $\widetilde{\mathrm{P}}(1,n)$

1. Let us solve the following problem: to describe all nonlinear equations of the form

$$L(x,u(x)) \equiv \Box u + F(x,u) = 0, \tag{1.1.1}$$

where $F(x,u)$ is some smooth function of x and u, invariant under the extended Poincare group which is generated by the operators

$$\begin{aligned} &P_0 = \partial_0 \equiv \frac{\partial}{\partial x_0}, \quad P_a = -\partial_a \equiv -\frac{\partial}{\partial x_a}, \quad a = \overline{1,n-1}; \\ &J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu; \quad \mu,\nu = \overline{0,n-1}; \end{aligned} \tag{1.1.2}$$

$$D = x^\nu P_\nu + \eta(u)\frac{\partial}{\partial u}, \tag{1.1.3}$$

($\eta(u)$ is an arbitrary differentiable function) which satisfy the commutation relations

$$\begin{aligned}
&[P_\mu, P_\nu] = 0, \quad [P_\sigma, J_{\mu\nu}] = g_{\sigma\mu}P_\nu - g_{\sigma\nu}P_\mu, \\
&[J_{\mu\nu}, J_{P\sigma}] = g_{\nu\rho}J_{\mu\sigma} + g_{\mu\sigma}J_{\nu\rho} - g_{\mu\rho}J_{\nu\sigma} + g_{\nu\sigma}J_{\mu\rho}, \\
&[P_\mu, D] = P_\mu, \quad [J_{\mu\nu}, D] = 0.
\end{aligned} \tag{1.1.4}$$

Theorem 1.1.1. [88] *Equation* (1.1.1) *is invariant under the group* $\widetilde{\mathrm{P}}(1, n-1)$ *iff*

$$F(x, u) = \lambda u^k, \quad k \neq 1 \tag{1.1.5}$$

or

$$F(x, u) = \lambda_1 e^{\lambda_2 u}, \quad \lambda_2 \neq 0 \tag{1.1.6}$$

($\lambda, \lambda_1, \lambda_2, k$ are arbitrary constants).

Proof. The necessary and sufficient condition of the invariance of Equation (1.1.1) under the group G according to S.Lie [161] is the condition (3). To write it explicitly it is necessary to construct, via formulae (4) and (5), the second prolongation of infinitesimal operators (IFOs) (2). For the scalar equation the twice-prolonged operator has the form

$$\begin{aligned}
\underset{2}{X} &= \xi^\mu(x,u)\partial_\mu + \eta(x,u)\partial_u + \tau_\mu \partial_{u_\mu} + \tau_{\mu\nu}\partial_{u_{\mu\nu}}, \\
\tau_\mu &= \eta_\mu + u_\mu \eta_u - u_\nu \xi^\nu_\mu - u_\mu u_\nu \xi^\nu_u, \\
\tau_{\mu\nu} &= \eta_{\mu\nu} + u_\mu \eta_{\nu u} + u_\nu \eta_{\mu u} + u_\mu u_\nu \eta_{uu} + u_{\mu\nu}\eta_u - \\
&- u_{\mu\sigma}\xi^\sigma_\nu - u_{\nu\sigma}\xi^\sigma_\mu - u_{\mu\sigma}u_\nu \xi^\sigma_u - u_{\nu\sigma}u_\mu \xi^\sigma_u - u_{\mu\nu}u_\sigma \xi^\sigma_u - \\
&- u_\sigma \xi^\sigma_{\mu\nu} - u_\mu u_\sigma \xi^\sigma_{\nu u} - u_\nu u_\sigma \xi^\sigma_{\mu u} - u_\mu u_\nu u_\sigma \xi^\sigma_{uu},
\end{aligned} \tag{1.1.7}$$

where $u_\mu \equiv \dfrac{\partial u}{\partial x_\mu}$, $u_{\mu\nu} \equiv \dfrac{\partial^2 u}{\partial x_\mu \partial x_\nu}$, $\eta_\nu \equiv \dfrac{\partial \eta}{\partial x_\nu}$, $\xi^\sigma_\nu \equiv \dfrac{\partial \xi^\sigma}{\partial x_\nu}$, and so on.

If we take as IFOs the linear combination of the operators (1.1.2), (1.1.3), i.e.,

$$X = (C_{\mu\nu}x_\nu + dx_\mu + a_\mu)\frac{\partial}{\partial x_\mu} + d \cdot \eta(u)\frac{\partial}{\partial u},$$

where $C_{\mu\nu} = -C_{\nu\mu}$, d, a_μ are arbitrary constants; then from the invariance condition (3) we shall get the defining equations for the functions $\eta(u)$ and $F(x, u)$

$$\eta_{uu} = 0, \quad \partial_\mu F(x,u) = 0, \quad (\eta_u - 2)F = \eta\frac{\partial F}{\partial u},$$

the general solution of which has the form $\eta(u) = au + b$, where a and b are arbitrary constants,

$$F(x,u) = \begin{cases} \lambda(au+b)^{(a-2)/a}, & a \neq 0, 2, \\ \lambda_1 \exp\{-\frac{2}{b}u\}, & b \neq 0. \end{cases} \tag{1.1.8}$$

Thus, in the class of Equations (1.1.1) there are only two essentially different equations that are invariant under the group $\widetilde{\mathrm{P}}(1, n-1)$ which correspond to the nonlinearities (1.1.5), (1.1.6)

$$\Box u + \lambda u^k = 0, \quad k \neq 1, \tag{1.1.9}$$

and

$$\Box u + \lambda_1 e^u = 0. \tag{1.1.10}$$

Remark 1.1.1. By using S.Lie's method it is easy to show that the group $\widetilde{\mathrm{P}}(1, n-1)$, $n > 2$ is the maximal invariance group of Equations (1.1.9), (1.1.10). The basis IFOs of the corresponding Lie algebra have the form (1.1.2), (1.1.3), $\eta(u)$ having the form $(\frac{-2}{k-1})u$ for Equation (1.1.9) and $\eta(u) = -2$ for Equation (1.1.10).

The multiparameter families of exact solutions of Equations (1.1.9), (1.1.10) are constructed in §§1.5, 1.6.

2. Studying group properties of the relativistic Hamilton equation

$$u_\mu u^\mu \equiv \frac{\partial u}{\partial x^\mu}\frac{\partial u}{\partial x_\mu} = 1, \qquad \mu = \overline{0,3}, \tag{1.1.11}$$

we discovered [92] that its invariance group is much wider than $\widetilde{\mathrm{P}}(1,3)$ and in particular contains the $\widetilde{\mathrm{P}}(1,4)$ group (for more details on the symmetry of (1.1.11) see §1.2) the action of which is defined in 5-dimensional Poincare-Minkowsky space R(1,4) with coordinates $x = (x_0, x_1, x_2, x_3, x_4 \equiv u)$. In this connection it is natural to consider the problem of describing n-dimensional first- and second-order PDEs

$$L_1(x, u, \underset{1}{u}) = 0, \tag{1.1.12}$$

$$L_2(x, u, \underset{1}{u}, \underset{2}{u}) = 0, \tag{1.1.13}$$

which are invariant under the group $\widetilde{\mathrm{P}}(1,n)$. The solution of this problem gives

Theorem 1.1.2. [86] *Equation* (1.1.12) *is invariant under the Lie algebra* $A\tilde{P}(1,n)$ *whose basis elements are given by the operators* (1.1.2), (1.1.14)

$$\begin{aligned} &P_n = -\partial_u, \quad J_{n\mu} = x_n P_\mu - x_\mu P_n; \quad \mu = \overline{0, n-1}, \\ &D = x^\alpha P_\alpha, \quad \alpha = \overline{0,n}, \quad (x_n \equiv u), \end{aligned} \tag{1.1.14}$$

iff it has the form (1.1.11).

Proof. Let us use Lie's algorithm. The invariance condition in this case takes the form

$$\underset{1}{X} L_1 \Big|_{L_1=0} = \left(\xi^\mu(x,u)\partial_\mu + \eta(x,u)\partial_u + \tau_\nu \partial_{u_\nu}\right) L_1 \Big|_{L_1=0} = 0. \tag{1.1.15}$$

where $\xi^\alpha = c^{\alpha\beta}x_\beta + d^\alpha$, $(\xi^n = \eta)$, $c^{\alpha\beta} = -c^{\beta\alpha}$, and τ_ν are constructed via formulas (1.1.7). From (1.1.15) we get the defining equations

$$\frac{\partial L_1}{\partial x_\mu} = \frac{\partial L_1}{\partial u} = 0, \quad u_\mu \frac{\partial L_1}{\partial u_\nu} - u_\nu \frac{\partial L_1}{\partial u_\mu} = 0,$$

$$\left(\frac{\partial L_1}{\partial u^\mu} - u_\mu u_\nu \frac{\partial L_1}{\partial u_\nu} \right) \Big|_{L_1=0} = 0,$$

whose general solution is the function $L_1 = u^\nu u_\nu - 1$. Thus in the class of PDEs defined by (1.1.12) there exists only one equation, namely the n-dimensional relativistic Hamilton equation, which is invariant under the group $P(1,n)$. The theorem is proved.

Theorem 1.1.3. *The two-dimensional Equation* (1.1.13) *is invariant under the algebra* $A\tilde{P}(1,2)$ *if it has the form*

$$\lambda_1[(1 - u_\nu u^\nu)\Box u + u^\mu u^\nu u_{\mu\nu}] + \lambda_2[(1 - u_\nu u^\nu)\det(u_{\mu\nu})]^{1/2} = 0, \tag{1.1.16}$$

where λ_1, λ_2 *are arbitrary constants*, $\mu, \nu = 0, 1$.

Proof. Necessity. From the invariance condition

$$\underset{2}{X} L_2 \Big|_{L_2=0} = 0,$$

where $\underset{2}{X}$ is defined in (1.1.7), $\xi^A = c^{AB}x_B + d^A$, $c^{AB} = -c^{BA}$, $A, B = \overline{0,n}$ we get the following system of equations:

$$\frac{\partial L_2}{\partial x_\mu} = \frac{\partial L_2}{\partial u} = 0,$$

$$\left[(u_\sigma u_\nu - g_{\sigma\nu})\frac{\partial L_2}{\partial u_\nu} + (u_\nu u_{\mu\sigma} + u_\mu u_{\nu\sigma})\frac{\partial L_2}{\partial x_{\mu\nu}}\right]\Bigg|_{L_2=0} = 0,$$

$$\left[u_\mu \frac{\partial L_2}{\partial u_\nu} - u_\nu \frac{\partial L_2}{\partial u_\mu} + (u_{\rho\mu} g^{\sigma\nu} + u_{\sigma\mu} g^{\rho\nu} - u_{\rho\nu} g^{\sigma\mu} - u_{\sigma\nu} g^{\rho\mu})\frac{\partial L_2}{\partial u_{\rho\sigma}}\right]\Bigg|_{L_2=0} = 0.$$

$$\left(u_{\rho\sigma}\frac{\partial L_2}{\partial u_{\rho\sigma}}\right)\Bigg|_{L_2=0} = 0.$$

We managed to find the general solution of this system when $n = 2$; it has the form (1.1.16).

Sufficiency. One can make sure by verification that Equation (1.1.16) is invariant under the algebra $A\widetilde{P}(1,2)$.

From (1.1.16) we can single out three different equations that are invariant under the group $\widetilde{P}(1,2)$: the Euler-Lagrange-Born-Infeld (ELBI) equation

$$(1 - u_\nu u^\nu)\Box u + u^\mu u^\nu u_{\mu\nu} = 0, \tag{1.1.17}$$

the Monge-Ampere (MA) equation

$$\det(u_{\mu\nu}) = 0. \tag{1.1.18}$$

and the eikonal equation (1.1.11).

When $n \geq 2$ the following statement holds true.

Theorem 1.1.4. [86] *The equation*

$$\lambda_1[(1 - u_\nu u^\nu)\Box u + u^\mu u^\nu u_{\mu\nu}] + \lambda_2[(1 - u_\nu u^\nu)\det(u_{\mu\nu})]^{\frac{3}{n+4}} = 0, \tag{1.1.19}$$

where $\mu, \nu = \overline{0, n-1}$ *is invariant under the group* $\widetilde{P}(1,n)$.

The proof is earried out by direct verification. When $n = 2$, Equation (1.1.19) coincides with (1.1.16). Using Lie's method we can show that $\widetilde{P}(1,n)$ is the maximal symmetry group of Equation (1.1.19) [86].

1.2. *The point and tangent symmetry of the relativistic Hamilton equation*

1. The relativistic analoque of the classical Hamilton equation is the equation

$$u_\nu u^\nu \equiv \frac{\partial u}{\partial x^\nu}\frac{\partial u}{\partial x_\nu} = 1, \tag{1.2.1}$$

In the preceeding paragraph we had shown that Equation (1.2.1) is invariant under the group $\widetilde{\mathrm{P}}(1,4) \supset \widetilde{\mathrm{P}}(1,3)$. It so happens that the point symmetry of Equation (1.2.1) is much wider than $\widetilde{\mathrm{P}}(1,4)$.

Theorem 1.2.1. [92] *The maximal local invariance group of Equation* (1.2.1) *is 21-parameter conformal group* C(1,4). *The basis elements of the corresponding Lie algebra* AC(1,4) *have the form*

$$\begin{aligned} &\partial_\alpha = \frac{\partial}{\partial x_\alpha}, \quad J_{\alpha\beta} = x_\alpha \partial_\beta - x_\beta \partial_\alpha, \quad D = x^\alpha \partial_\alpha, \\ &K_\alpha = 2x_\alpha D - s^2 \partial_\alpha, \quad s^2 \equiv x^\alpha x_\alpha = x_0^2 - x_1^2 - x_2^2 - x_3^2 - u^2, \\ &x^{(4)} = -x_4 \equiv u, \quad x^\alpha = g^{\alpha\beta} x_\beta, \quad g^{\alpha\beta} = g_{\alpha\beta} = (1,-1,-1,-1,-1)\,\delta_{\alpha\beta} \end{aligned} \tag{1.2.2}$$

(here and below the indices μ, ν range over 0,1,2,3 and the indices $\alpha, \beta, \rho, \sigma$ range over 0,1,2,3,4) and satisfy the commutation relations

$$\begin{aligned} &[\partial_\alpha, \partial_\beta] = 0, \quad [\partial_\alpha, J_{\beta\rho}] = g_{\alpha\beta}\partial_\rho - g_{\alpha\rho}\partial_\beta, \\ &[J_{\alpha\beta}, J_{\rho\sigma}] = g_{\alpha\sigma}J_{\beta\rho} + g_{\beta\rho}J_{\alpha\sigma} - g_{\alpha\rho}J_{\rho\sigma} - g_{\beta\sigma}J_{\alpha\rho}, \\ &[\partial_\alpha, D] = \partial_\alpha, \quad [J_{\alpha\beta}, D] = 0, \\ &[K_\alpha, K_\beta] = 0, \quad [K_\alpha, J_{\beta\rho}] = g_{\alpha\beta}K_\rho - g_{\alpha\rho}K_\beta, \\ &[\partial_\alpha, K_\beta] = 2(g_{\alpha\beta}D - J_{\alpha\beta}), \quad [D, K_\alpha] = K_\alpha. \end{aligned} \tag{1.2.3}$$

Proof. Let us use Lie's algorithm. From the invariance condition (3) which in the present case takes the form (see formulas (4), (1.1.7))

$$[\xi^\mu \partial_\mu + \eta \partial_u + \tau^\mu \partial_{u_\mu}](u^\nu u_\nu - 1) \Big|_{u^\nu u_\nu = 1} = 0,$$

we get the system of defining equations

$$\frac{\partial \eta}{\partial x_\nu} = \frac{\partial \xi^\nu}{\partial u}, \quad \frac{\partial \xi^0}{\partial x^0} = \frac{\partial \xi^1}{\partial x^1} = \frac{\partial \xi^2}{\partial x^2} = \frac{\partial \xi^3}{\partial x^3} = \frac{\partial \eta}{\partial u} \stackrel{\text{def}}{=} f(x,u),$$
$$\frac{\partial \xi^\mu}{\partial x_\nu} + \frac{\partial \xi^\nu}{\partial x_\mu} = 0, \quad \mu \neq \nu$$

or, in the more compact form,

$$\xi_{\alpha,\beta} + \xi_{\beta,\alpha} = g_{\alpha\beta} f, \quad \left(\xi_{\alpha,\beta} \equiv \frac{\partial \xi^\rho}{\partial x^\beta} g_{\alpha\rho} \right). \tag{1.2.4}$$

Equations (1.2.4) are known as *Killing* equations.

Having convoluted (1.2.4) with the metric tensor $g^{\alpha\beta}$ we get

$$\partial_\alpha \xi^\alpha = 2f$$

Acting with the operator ∂^β on (1.2.4) we find

$$\partial_\beta\partial^\beta\xi_\alpha + \partial_\alpha\partial^\beta\xi_\beta = \partial_\beta\partial^\beta\xi_\alpha + 2\partial_\alpha f, \quad (\xi_\alpha = g_{\alpha\rho}\xi^\rho),$$

i.e.,

$$\partial_\beta\partial^\beta\xi^\alpha = -\partial^\alpha f. \tag{1.2.5}$$

Taking the divergence of (1.2.5) we get

$$\partial_\beta\partial^\beta f = 0. \tag{1.2.6}$$

After applying the operator $\partial_\sigma\partial^\sigma$ to (1.2.4) and using the equalities (1.2.5), (1.2.6) we find that

$$\partial_\alpha\partial_\beta f = 0.$$

Thus f can be only the linear function of x

$$f(x) = 2d + 4c_\alpha x^\alpha$$

Finally, we find that the general solution of Equation (1.2.4) has the form

$$\xi^\alpha = 2x^\alpha c_\beta x^\beta - c^\alpha x^\beta x_\beta + b^{\alpha\beta}x_\beta + dx^\alpha + a^\alpha, \tag{1.2.7}$$

where c_β, $b_{\alpha\beta} = -b_{\beta\alpha}$, d, a_α are arbitrary constants.

From (1.2.7) we get the basis operators (1.2.3). By direct substitution we confirm that the operators (1.2.3) satisfy the commutational relations of the conformal algebra AC(1,4). The theorem is proved.

2. Let us study the invariance of Equation (1.2.1) under contact transformations, which are the natural generalization of the point transformations and the most closely related to them. We shall adduce some facts on contact transformations (for the more detailed information see the books [124, 161, 166, 168]).

Definition 1.2.1. The transformations of the form

$$\begin{aligned} &x_\nu \to x'_\nu = f_\nu(x, u, \underset{1}{u}, \theta), \quad \theta = \text{const}, \quad \underset{1}{u} = \left\{\frac{\partial u}{\partial x_\mu}\right\}, \\ &u \to u'(x') = g(x, u, \underset{1}{u}, \theta), \\ &u_\nu \to u'_\nu = h(x, u, \underset{1}{u}, \theta) \end{aligned} \tag{1.2.8}$$

are called *contact* if they leave invariant the expressions

$$du - u_\nu dx_\nu, \tag{1.2.9}$$

or, in other words, the following relations hold true

$$u'_\nu = \frac{\partial u'(x')}{\partial x'_\nu}$$

Relations (1.2.9) lead to quite strong restrictions on the coordinates of the generator of the contact transformations

$$X = \xi^\mu(x, u, \underset{1}{u})\partial_\mu + \eta(x, u, \underset{1}{u})\partial_u + \tau_\nu(x, u, \underset{1}{u})\partial_{u_\nu}, \qquad (1.2.10)$$

namely

$$\begin{aligned} &\xi^\mu = -\frac{\partial W}{\partial u_\mu}, \quad \eta = W - u_\mu \frac{\partial W}{\partial u_\mu}, \\ &\tau_\mu = D_\mu \eta - u_\nu D_\mu \xi^\nu = \frac{\partial W}{\partial x^\mu} + u_\mu \frac{\partial W}{\partial u}, \end{aligned} \qquad (1.2.11)$$

where $W = W(x, u, \underset{1}{u})$ is the characteristic function, and D_μ is the operator of full differentiation (5). Note that the more general statement is true: the group of the contact transformations coincides with the prolonged group of the point transformation if the dimension of the space of dependent variables $u(x)$ is more than one, and only for scalar PDEs do these transformations have nontrivial coordinates of the corresponding IFOs having the form (1.2.11). The proof of this fact, known since the time of Lie, may be found in [124].

To find the explicit form of the finite transformations generated by the operators (1.2.10), (1.2.11), it is necessary to solve the following system of Lie's equations

$$\begin{aligned} &\frac{\partial x'_\nu}{\partial \theta} = \xi^\nu(x', u', \underset{1}{u'}) = -\frac{\partial W'}{\partial u'_\nu}, \quad x'_\nu \Big|_{\theta=0} = x_\nu, \\ &\frac{\partial u'}{\partial \theta} = \eta(x', u', \underset{1}{u'}) = W' - u'_\mu \frac{\partial W'}{\partial u'_\mu}, \quad u' \Big|_{\theta=0} = u, \\ &\frac{\partial u'_\nu}{\partial \theta} = \tau_\nu(x', u', \underset{1}{u'}) = \frac{\partial W'}{\partial x'_\nu} + u'_\nu \frac{\partial W'}{\partial u'}, \quad u'_\nu \Big|_{\theta=0} = u_\nu. \end{aligned} \qquad (1.2.12)$$

The particular case of the contact transformations (1.2.7) are canonical or homogeneous contact transformations when $g(x, u, \underset{1}{u}) = u$, and f_ν, h_ν do not depend on u (this is equivalent to the case when the characteristic function W does not depend on u). For the homogeneous contact transformations, Lie's Equations (1.2.12) have the Hamiltonian structure

$$\frac{\partial x'_\nu}{\partial \theta} = -\frac{\partial W}{\partial u'_\nu}, \; \frac{\partial u'_\nu}{\partial \theta} = \frac{\partial W}{\partial x'_\nu}, \quad x'_\nu \Big|_{\theta=0} = x_\nu, \quad u'_\nu \Big|_{\theta=0} = u_\nu. \qquad (1.2.13)$$

Let us turn again to Equation (1.2.1).

Theorem 1.2.2. [187] *The maximal invariance algebra of Equation* (1.2.1) *in the class of operators* (1.2.10), (1.2.11) *is infinite-dimensional and is given by the characteristic function*

$$W = W(u_\nu, x_\mu - uu_\mu). \tag{1.2.14}$$

Proof. The necessary and sufficient condition of the invariance of Equation (1.2.1) under the contact transformations is the fulfillment of the conditions

$$\widetilde{X}(u_\nu u^\nu - 1)\Big|_{u_\nu u^\nu = 1} = 0,$$

with the operator $\widetilde{X}$ from (1.0.10), (1.2.11), or, at greater length

$$\left[u^\nu\left(\frac{\partial W}{\partial x^\nu} + u_\nu\frac{\partial W}{\partial u}\right)\right]\Bigg|_{u_\nu u^\nu = 1} = u_\nu\frac{\partial W}{\partial x^\nu} + \frac{\partial W}{\partial u} = 0, \quad u_\nu u^\nu = 1. \tag{1.2.15}$$

The general solution of Equations (1.2.15) is defined in (1.2.14). The theorem is proved.

3. Let us consider the relativistic Hamilton equation with the nonlinear right-hand part

$$W^\nu W_\nu = f(W). \tag{1.2.16}$$

If $f(W) \neq 0$ then, having carried out the transformation

$$W \to u = \int_0^W \frac{dz}{\sqrt{|f(z)|}}$$

we get as a result Equation (1.2.1), or the equation

$$u_\nu u^\nu = -1. \tag{1.2.17}$$

It is not difficult to understand that the maximal group of the point transformations of Equation (1.2.17) is the conformal group $C(2,3)$ whose generators have the form (1.2.3), but in this case $g_{\alpha\beta}$ has the signature $(++---)$.

If in (1.2.17) $f(W) = 0$ then we get the eikonal equation

$$u_\nu u^\nu = 0. \tag{1.2.18}$$

Theorem 1.2.3. *The maximal (in Lie's sense) invariance group of Equation* (1.2.18) *is the infinite-dimensional group generated by the operators* $C^{\infty}(1,3)\otimes G^{\infty}$,

$$X = (2x^{\mu}cx - c^{\mu}x^2 + dx^{\mu} + b^{\mu\nu}x_{\nu} + a^{\mu})\partial_{\mu} + \eta\partial_u, \qquad (1.2.19)$$

where $c_{\mu}, b_{\mu\nu} = -b_{\nu\mu}, d, a_{\mu}, \eta$ *are arbitrary differentiable functions of* u.

We omit the proof because of its similarity to the proof of Theorem 1.2.1.

Theorem 1.2.4. *The maximal invariance algebra of Equation* (1.2.18) *in the class of operators* (1.2.10), (1.2.11) *is infinite-dimensional and is defined by the characteristic function*

$$W = W(u, u_{\nu}, x_0u_a - x_au_0), \quad (a = 1,2,3)$$

The proof is the same as in the case of Theorem 1.2.2.

1.3. *The polywave equations invariant under the conformal group*

Consider the equation

$$\Box^l u + F(x,u) = 0, \qquad (1.3.1)$$

where $\Box^l = \Box^{l-1}\Box$, $l = 1,2,3,\ldots$; $\Box = \partial_{\mu}\partial^{\mu}$, $F(x,u)$ is a smooth function. We would like to describe all functions $F(x,u) \not\equiv 0$, which ensure the conformal symmetry of Equation (1.3.1). When $l > 1$ we call Equation (1.3.1), according to [63], the nonlinear polywave equation.

It is well known that the maximal local invariance group of the wave equation $\Box u = 0$ is the conformal group $C(1, n-1)$ generated by the operators (1.1.2) and (1.3.2)

$$\begin{aligned} D &= x^{\nu}P_{\nu} + k, \\ K_{\mu} &= 2x_{\mu}D - x^2P_{\mu}, \end{aligned} \qquad (1.3.2)$$

where k (called the conformal degree (see §2.3)) is equal to $n/2 - 1$.

Let us answer the question of whether the polywave equation

$$\Box^l u = 0 \qquad (1.3.3)$$

has conformal symmetry.

Theorem 1.3.1. [176] *The polywave Equation* (1.3.3) *is invariant under the group of conformal transformations* $C(1, n-1)$ *generated by the operators*

(1.1.2), (1.3.2), *the conformal degree k being equal to* $k = n/2 - l$ (*i.e.,* $D = D^{(l)} = x^\nu P_\nu + \frac{n}{2} - l$).

Proof. Let us write down the invariance condition of Equation (1.3.3) in the operator form [61,64] (see also §5.5):

$$[\Box^l, Q] = \lambda(x)\Box^l, \tag{1.3.4}$$

where Q is one of the operators (1.1.2), (1.3.2); $\lambda(x)$ is some function. When $l = 1$ the conditions (1.3.4) have the form

$$[\Box, P_\mu] = [\Box, J_{\mu\nu}] = 0,$$

$$[\Box, D] = 2\Box, \quad [\Box, K_\mu] = 4x_\mu\Box$$

Now we shall use the method of mathematical induction. Let us suppose that when $l = s - 1$ the following equalities take place

$$\begin{gathered} [\Box^{s-1}, P_\mu] = [\Box^{s-1}, J_{\mu\nu}] = 0 \\ [\Box^{s-1}, D^{(s-1)}] = 2(s-1)\Box^{s-1}, \\ [\Box^{s-1}, K_\mu^{(s-1)}] = 4(s-1)x_\mu\Box^{s-1} \end{gathered} \tag{1.3.5}$$

If we show that relations (1.3.5) hold true also when $l = s$ then we shall have proven the theorem. Thus, we have

$$[\Box^s, P_\mu] = [\Box^s, J_{\mu\nu}] = 0$$

$$[\Box^s, D^{(s)}] = [\Box\cdot\Box^{s-1}, D^{(s-1)} - 1] = \Box[\Box^{s-1}, D^{(s-1)}] + [\Box, D^{(s-1)}]\Box^{s-1} =$$

$$= 2(s-1)\Box^s + 2\Box^s = 2s\Box^s;$$

$$[\Box^s, K_\mu^{(s)}] = [\Box\cdot\Box^{s-1}, K_\mu^{(s-1)} - 2x_\mu] = \Box\left([\Box^{s-1}, K_\mu^{(s-1)}] -\right.$$

$$\left.-2[\Box^{s-1}, x_\mu]\right) + \left([\Box, K_\mu^{(s-1)}] - 2[\Box, x_\mu]\right)\Box^{s-1} = \Box\left(4(s-1)x_\mu\Box^{s-1} +\right.$$

$$\left.4(s-1)P_\mu\Box^{s-1}\right) + \left(4x_\mu\Box + 4(s-2)P_\mu + 4P_\mu\right)\Box^{s-1} =$$

$$= 4sx_\mu\Box^s.$$

Here we have made use of the equality

$$[\Box^s, x_\mu] = 2sP_\mu\Box^{s-1} \tag{1.3.6}$$

whose correctness may be easily verified.

Thus the theorem is proved (let us note that a more elegant proof may be derived using the formulas (A.3.5), (A.3.6)).

Consider the following question: what functions $F(x,u)$ from (1.3.1) preserve the conformal symmetry of Equation (1.3.3) established in Theorem 1.3.1? When $l = 1$ the answer is well-known [123]:

$$F(x,u) = cu^{(n+2)/(n-2)}, \quad n \neq 2$$

(where c is an arbitrary constant).

When $l \geq 1$ the following statement holds true [176].

Theorem 1.3.2. *Equation* (1.3.1) *is invariant under the conformal group* C$(1, n-1)$ *iff*

$$F(x,u) = cu^{(n+2l)/(n-2l)}, \quad n \neq 2l \tag{1.3.7}$$

Proof. Under the transformations generated by the operator (translations)

$$x_\mu \to x'_\mu = x_\mu + a_\mu, \quad u(x) \to u'(x') = u(x)$$

Equation (1.3.1) takes the form

$$\Box^l u = F(x+a, u).$$

Whence, due to the demand of invariance, we get

$$F(x+a,u) = F(x,u)$$

The solution of this functional relation is

$$F(x,u) = F(u)$$

Furthermore, we shall use Theorem 5.5.1, namely, the conditions (5.5.3) $1°, 2°$ which in our case have the form (1.3.4) and

$$[\eta(x) + \lambda(x)]\, F - \eta(x) u F \frac{\partial F}{\partial u} = 0. \tag{1.3.8}$$

As it follows from Theorem 1.3.1: $\eta(x) = \lambda(x) = 0$ for the operators $P_\mu, J_{\mu\nu}$; $\eta(x) = \dfrac{n}{2} - l$, $\lambda(x) = 2l$ for the operator $D^{(l)}$ (1.3.3), and at last $\eta(x) = cx(n-2l), \lambda(x) = 4lcx$ for the operator $K = c^\mu K_\mu$.

Substituting these expressions into (1.3.8) we get

$$(n+2l)F - (n-2l)u\frac{\partial F}{\partial u} = 0 \tag{1.3.9}$$

The general solution of this equation has the form (1.3.7). The theorem is proved.

The more general result may be obtained if we suppose that formula (A.3.6) is true for an arbitrary differentiable operator-valued function $F(\hat{x})$. Then the following statement holds true.

Remark 1.3.1. When $n = 2l$ Equation (1.3.1) is invariant under AC(1,$n-1$) with basis elements (1.1.2) and

$$D = x^{\nu} P_{\nu} + 2l\partial_u, \qquad K_{\mu} = 2x_{\mu} D - x^2 P_{\mu},$$

if it has the form

$$\Box^l u = \lambda e^u.$$

Theorem 1.3.3. *A pseudodifferential nonlinear wave equation*

$$\Box^r u = F(u) \tag{1.3.10}$$

(where r is an arbitrary number) is invariant under the conformal group $C(1, n-1)$ *iff*

$$F(u) = cu^{(n+2r)/(n-2r)}, \qquad n \neq 2r. \tag{1.3.11}$$

The basis IFOs of the corresponding Lie algebra have the form (1.1.2), (1.3.2), *the conformal degree k being equal to*

$$k = \frac{n}{2} - r \tag{1.3.12}$$

Proof. Let us make use of the invariance conditions in the form (5.5.3) 1°, 2° which coincide with (1.3.4) and (1.3.8) when $l = r$. We have:

$$\eta(x) = \begin{cases} 0 & \text{for } P_{\mu}, J_{\mu\nu}; \\ k & \text{for } D; \\ 2kcx & \text{for } K = K_{\mu} c^{\mu} \end{cases} \tag{1.3.13}$$

It is evident that $[\Box^r, P_{\mu}] = 0$. Furthermore, via formula (A.3.6) we find

$$[\Box^r, J_{\mu\nu}] = [\Box, J_{\mu\nu}] r\Box^{r-1} + \cdots = 0,$$

$$[\Box^r, D] = [\Box, D] r\Box^{r-1} + \frac{1}{2!}[\Box, [\Box, D]]\, r(r-1)\Box^{r-2} + \cdots = 2r\Box^r,$$

$$[\Box^r, K] = [\Box, K] r\Box^{r-1} + \frac{1}{2!}[\Box, [\Box, K]]\, r(r-1)\Box^{r-2} + \cdots =$$

$$= 4r\left(cx\Box^r + \left(r + k - \frac{n}{2}\right) c\partial\Box^{r-1}\right),$$

whence it can be seen that condition (1.3.4) is fulfilled if k has the form (1.3.12) with

$$\lambda(x) = \begin{cases} 0 & \text{for } P_\mu, J_{\mu\nu}; \\ 2r & \text{for } D; \\ 4rcx & \text{for } K = c^\mu K_\mu. \end{cases} \tag{1.3.14}$$

Having substituted (1.3.13) and (1.3.14) into (1.3.8), we shall obtain the equation for the function $F(u)$ which coincides with (1.3.9) when $l = r$. The theorem is proved.

Corollary. *The equation*

$$\Box^r u + cu^q = 0 \tag{1.3.15}$$

is invariant under the conformal algebra $C(1, n-1)$ (1.3.2) *iff the numbers* r *and* q *are connected with the following equality:*

$$r + k - \frac{n}{2} = 0$$

$$2r + k = kq,$$

whence, supposing q *to be given, we find*

$$r = \frac{n}{2}\frac{q-1}{q+1}, \quad k = \frac{n}{q+1}$$

In particular, when $n = 4$ *and* $q = 2$ *we get the nonlinear conformally invariant equation*

$$\Box^{2/3} u = cu^2 \tag{1.3.16}$$

Note. The operator $\Box^r$, if r is a fractional number, is to be considered a pseudodifferential operator defined with the help of the integral Fourier transformation

$$\Box^r u = -\int\int (p_\nu p^\nu)^r e^{i(x-y)p} u(y) dy đp, \tag{1.3.17}$$

where $dy = dy_0 dy_1 dy_2 dy_3$, $đp = \frac{1}{(2\pi)^4} dp_0 dp_1 dp_2 dp_3$.

To conclude this paragraph let us prove the following statement:

Theorem 1.3.4. *If the equation*

$$\Box u = F(x, u, \underset{1}{u}) \tag{1.3.18}$$

is invariant under the conformal group C$(1, n-1)$ *then it is equivalent to the equation*

$$\Box W + \lambda W^{\frac{n+2}{n-2}} = 0 \tag{1.3.19}$$

Proof. The invariance of Equation (1.3.18) under the algebra AC$(1, n-1)$ means that there exists the set of operators $P_\mu, J_{\mu\nu}, D, K_\mu$ satisfying the commutational relations of the algebra AC$(1, n-1)$ (see §2.3). The formulas (1.1.2) and (1.3.2) define the linear representation of the conformal algebra acting in the space of scalar functions $u(x)$. Evidently the generalization of this representation is possible at the expense of altering the operator D. The most general form of the operator D which, when combined with P_μ, $J_{\mu\nu}$ (1.1.2) and $K_\mu = 2x_\mu D - x^2 P_\mu$, satisfies the commutational relations of the algebra AC$(1, n-1)$, has the form (1.1.3).

Note that if $\eta(u) \neq 0$ then it is possible to transform the function u

$$u \to w = \exp\left\{\frac{2-n}{2}\int \frac{du}{\eta(u)}\right\}, \quad \eta(u) \neq 0 \tag{1.3.20}$$

so that the scale transformations generator D (1.1.3) will change into D (1.3.2), $k = \frac{n}{2} - 1$. That is why we can further use the representation of the algebra AC$(1, n-1)$ in the form (1.1.2), (1.3.2) without loss of generality.

Evidently the function F from (1.3.18), due to relations $[\Box, P_\mu] = [\Box, J_{\mu\nu}] = 0$, has to be constructed of the absolute invariants of the Poincare group, and there are only two such invariants: u and $u_\nu u^\nu$.

By the action of the scale transformations

$$x_\mu \to x'_\mu = e^\theta x_\mu, \quad u(x) \to u'(x') = e^{-\theta(n/2-1)} u(x), \tag{1.3.21}$$

generated by the operator D (1.3.2) we have

$$\Box u - F(u, u_\nu u^\nu) = 0 \to e^{-\theta(\frac{n}{2}+1)}\Box u - F\left(e^{-\theta(\frac{n}{2}-1)}u, e^{-n\theta}u_\nu u^\nu\right) = 0.$$

From the demand of invariance we get the functional relation for F

$$F(u, u_\nu u^\nu) = e^{\theta(\frac{n}{2}+1)} F\left(e^{-\theta(\frac{n}{2}-1)}u, e^{-n\theta}u_\nu u^\nu\right).$$

Differentiating F with respect to θ and then putting $\theta = 0$ we get the differential equation

$$(n+2)F - (n-2)u\frac{\partial F}{\partial u} - 2nv\frac{\partial F}{\partial v} = 0, \quad (v \equiv u_\mu u^\mu),$$

whose general solution has the form

$$F = u^{\frac{n+2}{n-2}} f\left(u(u_\mu u^\mu)^{\frac{2-n}{2n}}\right), \qquad (1.3.22)$$

where f is an arbitrary differentiable function of the mentioned argument.

Using the explicit form of the conformal transformations (see §2.3) we can show that Equation (1.3.18) with the function f (1.3.22) remains invariant with respect to these transformations only when $f = \text{const}$.

In the case where the coordinate $\eta(u)$ of the operator D (1.1.3) is equal to zero, i.e., $D = x_\nu p^\nu$, it generates the transformations

$$x_\mu \to x'\mu = e^\vartheta x_\mu, \quad u(x) \to u'(x') = u(x). \qquad (1.3.23)$$

Having repeated the same reasoning used while obtaining the expressions (1.3.22), we adduce the final result.

The equation

$$\Box u + u_\nu u^\nu f(u) = 0 \qquad (1.3.24)$$

is invariant under the algebra $\widetilde{\mathrm{AP}}(1, n-1) = \{\mathrm{AP}(1, n-1)(1.1.2), \mathrm{D}(1.3.23)\}$.

We can verify that Equation (1.3.24) is not invariant under conformal transformations generated by the operators $K_\mu = 2x_\mu xP - x^2 P_\mu$ (see §2.3) with any $f(u)$. By the way, let us note that with the transformation $u = g(w)$, where $\ddot{g} + \dot{g}^2 f(g) = 0$, Equation (1.3.24) is reduced to the form (1.3.19) with $\lambda = 0$.

1.4. *Ansätze and reduction of PDEs. The extended Poincare group* $\widetilde{\mathrm{P}}(1,2)$ *and its invariants*

The method of constructing exact solutions of PDEs is expounded. This method is widely used throughout the book.

1. Let us consider as an example the nonlinear wave equation

$$\Box u + \lambda u^k = 0, \qquad (1.4.1)$$

where $\Box = \partial_\mu \partial^\mu$, $\mu = \overline{0, n-1}$, $u = u(x)$. Solutions of Equation (1.4.1), according to [63], are sought in the form

$$u(x) = f(x)\varphi(\omega), \qquad (1.4.2)$$

where $\varphi(w)$ are unknown functions to be found. The explicit form of the new variables $\omega = \omega(x)$ and function $f(x)$ is defined from the conditions described below. Expressions such as (1.4.2) shall be called *ansatze.* *

* The word "ansatz" in German means "substitution," although to stress that expressions (1.4.2) are not purely substitutions *ad hoc* but are also a method of calculating functions $f(x)$, $\omega(x)$ in explicit form they shall be referred to by the word "ansatz."

Substituting (1.4.2) into (1.4.1) gives

$$(\Box f)\varphi + f\left(\frac{\partial\omega_i}{\partial x_\mu}\frac{\partial\omega_k}{\partial x^\mu}\varphi_{\omega_i\omega_k} + \Box\omega_i\varphi_{\omega_i}\right) + 2\frac{\partial f}{\partial x_\mu}\frac{\partial\omega_i}{\partial x^\mu}\varphi_{\omega_i} + \lambda f^k\varphi^k = 0. \quad (1.4.3)$$

If we take

$$\frac{\partial\omega_i}{\partial x_\mu}\frac{\partial\omega_k}{\partial x^\mu} = f^{k-1}a_{ik}(\omega),$$

$$f\Box\,\omega_i + 2\frac{\partial f}{\partial x_\mu}\frac{\partial\omega_i}{\partial x^\mu} = f^k b_i(\omega), \quad (1.4.4)$$

$$\Box f = f^k c(\omega)$$

where a_{ik}, b_i, c_i are some functions depending on ω, only then instead of (1.4.3) we shall have

$$a_{ik}\varphi_{\omega_i\omega_k} + b_i\varphi_{\omega_i} + c\varphi + \lambda\varphi^k = 0. \quad (1.4.5)$$

Conditions (1.4.4), called the *splitting conditions* [63], are the system of equations for defining the explicit form of the new variables $\omega = \omega(x)$ and the functions $f(x), a_{ik}, b_i, c$. The new number of variables $\omega = \{\omega_i\}$ may be $1, 2, \ldots, n-1$. In these cases Equations (1.4.5) will be ODEs or PDEs with the number of independent variables being equal to $1, 2, 3, \ldots, n-1$. The solutions of the reduced Equation (1.4.5), especially in the case of ODEs, can often be found by direct integration.

The simplest case, when the number of independent variables ω in ansatz (1.4.2) is one and $f = 1$, the splitting conditions (1.4.4) take the form

$$\Box\omega = b(\omega)$$
$$\omega_\mu\omega^\mu \equiv \frac{\partial\omega}{\partial x^\mu}\frac{\partial\omega}{\partial x_\mu} = a(\omega) \quad (1.4.4')$$

By means of a change of variables similar to that of (1.2.16)–(1.2.18), the system (1.4.4′) can be transformed to the canonical form

$$\Box\tilde{\omega} = F(\tilde{\omega})$$
$$\tilde{\omega}_\mu\tilde{\omega}^\mu \equiv \lambda, \qquad \lambda = 0, 1, -1. \quad (1.4.4'')$$

So, to describe all ansatze $u(x) = \varphi(\omega)$ which reduce Equation (1.4.1) to ODEs one has to find all solutions of system (1.4.4″). This latter problem is fully solved in [60*,130*] (see Appendix 6).

To determine the explicit form of the ansatz (1.4.2) one can effectively use the symmetry of Equation (1.4.1). As had been shown in §1.1, the generators of the invariance algebra of Equation (1.4.1) have the form

$$X = \xi^\mu(x)\partial_\mu + \eta(x), \quad (1.4.6)$$

where $\xi^\mu(x), \eta(x)$ are some scalar functions. Using the operator (1.4.6) we can determine the explicit expressions for the variables $\omega(x)$ and functions $f(x)$ as solutions of the following equations:

$$\xi^\mu(x)\partial_\mu \omega(x) = 0, \tag{1.4.7}$$

$$X f(x) \equiv (\xi^\mu(x)\partial_\mu + \eta(x))\, f(x), \tag{1.4.8}$$

or of the equivalent system of Lagrange-Euler equations

$$\frac{dx_0}{\xi^0} = \frac{dx_1}{\xi^1} = \cdots = \frac{dx_{n-1}}{\xi^{n-1}} = \frac{du}{-\eta u} \stackrel{\text{def}}{=} d\tau \tag{1.4.9}$$

Substituting the ansatz into (1.4.1) we get for $\varphi(\omega)$ an equation not containing $f(x)$. The number of variables of $w = w(x)$ which are the first integrals of (1.4.9) is one less than the number of variables of x.

Evidently the formulated algorithm may be applied to the PDEs for the functions $\varphi(\omega_1, \ldots, \omega_{n-1})$ repeatedly. Surely it is necessary for these equations to possess nontrivial symmetries (for more details see §2.1).

Let us point out that the ansatz (1.4.2) can also work effectively in the case when an equation does not possess a proper Lie symmetry [67]. For example, the Equation [108]

$$\Box u + \left(\frac{\lambda_0}{x_0}\frac{\partial u}{\partial x_0}\right)^2 - \left(\frac{\lambda_1}{x_1}\frac{\partial u}{\partial x_1}\right)^2 - \left(\frac{\lambda_2}{x_2}\frac{\partial u}{\partial x_2}\right)^2 - \left(\frac{\lambda_3}{x_3}\frac{\partial u}{\partial x_3}\right)^2 = 0$$

where $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ are arbitrary paramenters, $x_\mu \neq 0$ is not Lorentz-invariant, but the Lorentz-invariant ansatz

$$u(x) = \varphi(\omega), \quad \omega = x_\mu x^\mu$$

reduces it to the ODE

$$\omega\frac{d^2\varphi}{d\omega^2} + 2\frac{d\varphi}{d\omega} - \lambda^2\left(\frac{d\varphi}{d\omega}\right)^2 = 0$$

(for more details about the reduction and solution of equations with broken symmetry see §5.7).

Let us also note that in some cases (see, for example, Sec. 1.6) the following ansatze proved to be effective

$$u(x) = f(x)\varphi(\omega) + g(x) \tag{1.4.10}$$

Transformations leaving an equation invariant may be used to generate new solutions from known solutions. Thus, if a PDE is invariant with respect to transformations

$$x \to x' = f(x,\theta), \quad u(x) = u'(x') = r(x,\theta)u(x) \tag{1.4.11}$$

(θ is a parameter, $f(x,\theta)$, $r(x,\theta)$ are some smooth functions) and if $u_I(x)$ is a solution of this equation then the function

$$u_{II}(x) = r^{-1}(x,\theta)u_I(x') \tag{1.4.12}$$

will be a solution too.

In the case when the symmetry transformations are nonlinear with respect to u, i.e.,

$$x' = f(x,u,\theta), \; u'(x') = g(x,u,\theta) \tag{1.4.13}$$

then to obtain the generating formulas it is necessary to solve the functional equation

$$g\left(x, u_{II}(x), \theta\right) = u_I\left(f(x, u_{II}(x), \theta)\right) \tag{1.4.14}$$

with respect to $u_{II}(x)$.

In connection with group generating of solutions the notion of G-ungenerative solutions [65] (see also §2.3) arises.

Definition. A set of solutions M is called G-ungenerative if the group G generating does not take out of this set with any transformations from the group G.

2. Let us determine the invariant variables $\omega = \omega(x)$ for the operator that is the linear combination of the generators of the group $\widetilde{P}(1,2)$ (see for example (1.1.2),(1.1.3)).

$$\xi^\mu(x)\partial_\mu = (æx^\mu + c^{\mu\nu}x_\nu + d^\mu)\partial_\mu, \tag{1.4.15}$$

where $\mu, \nu = 0,1,2$; æ, $c_{\mu\nu} = -c_{\nu\mu}$, d^μ are arbitrary constants.

Let us rewrite (1.4.9) equivalently as a system of ODEs

$$\frac{dx_\mu}{d\tau} = \xi^\mu(x) = æx_\mu + c_{\mu\nu}x^\nu + d_\mu, \tag{1.4.16}$$

or in matrix form

$$\dot{X} = AX + D, \tag{1.4.17}$$

where

$$X = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}, \quad A = \begin{pmatrix} æ & c_{01} & c_{02} \\ c_{01} & æ & c_{12} \\ c_{02} & -c_{12} & æ \end{pmatrix}, \quad D = \begin{pmatrix} d_0 \\ d_1 \\ d_2 \end{pmatrix}.$$

The number of independent solutions of system (1.4.17) is determined by the number of different roots of the characteristic equation

$$\det(A - \lambda I) = 0 \tag{1.4.18}$$

(where I is the unit matrix.)

In the present case, Equation (1.4.18) has the form

$$(æ - \lambda)\left[(æ - \lambda)^2 + c_{\mu\nu}c^{\mu\nu}\right] = 0. \tag{1.4.19}$$

Whence, having obtained λ, we construct the corresponding solutions of system (1.4.9) and then, having excluded τ from it via (1.4.9), we find the sought-after invariants.

Without going into the details of these rather cumbersome calculations we shall present the final result. Depending on the value of the parameters $æ, c_{\mu\nu}, d_\mu$, we get the (nonequivalent) invariants of the group $\widetilde{\mathrm{P}}(1,2)$ [88].

Table 1.4.1 The invariant variables of the group $\widetilde{\mathrm{P}}(1,2)$.

N	Invariant variables ω	Conditions on parameters
1.	$(\alpha y)(\beta y)^a, \quad y^2(\beta y)^{-2}$	$\alpha^2 = \alpha\beta = 0, \quad \beta^2 = b < 0.$
2.	$(\beta y)(\alpha y)^{-1} + \ln(\alpha y), y^2(\alpha y)^{-2}$	$\alpha^2 = \alpha\beta = 0, \beta^2 = b < 0.$
3.	$\ln(\alpha y) + \arctan\frac{\delta y}{\beta y}, \quad y^2(\alpha y)^{-2}$	$\alpha^2 = -\beta^2 = -\delta^2 = 1,$ $\alpha\beta = \alpha\delta = \beta\delta = 0.$
4.	$\alpha y + \ln(\beta y), \quad (\delta y)(\beta y)^{-2}$	$\alpha^2 = \delta^2 = \alpha\beta = \beta\delta = 0, \quad \beta^2 = 1,$ $\alpha\delta = b \neq 0.$
5.	$(\beta y)(\alpha y)^{-1}, \quad (\delta y)(\alpha y)^{-1}$	$\alpha^2 = -\beta^2 = -\delta^2 = 1,$ $\alpha\beta = \alpha\delta = \beta\delta = 0.$
6.	$y^2 + (\beta y)^2, \quad \beta y + \ln(\alpha y)$	$\alpha^2 = \alpha\beta = 0, \quad \beta^2 = 1.$
7.	$(\beta y)^2 - y^2, \quad \beta y + \arctan\frac{\delta y}{\alpha y}$	$\alpha^2 = -\beta^2 = -\delta^2 = 1,$ $\alpha\beta = \alpha\delta = \beta\delta = 0.$
8.	$\frac{1}{2}(\alpha y)^2 + a(\beta y), \quad \frac{1}{3}(\alpha y)^3 +$ $+a(\alpha y)(\beta y) + a^2(\delta y)$	$\alpha^2 = \alpha\beta = \beta\delta = 0,$ $\alpha\delta = -\beta^2 = \delta^2 = b \neq 0.$
9.	$\alpha x, \quad x^2$	$\alpha^2 = 1.$
10.	$\alpha x, \quad \beta x.$	

Here $a, b, a_\mu, \alpha_\mu, \delta_\mu$ are arbitrary constants, satisfying the aforementioned conditions: $y_\mu = x_\mu + a_\mu$.

The invariants 1–10 from Table 1.4.1 are easily generalized for a greater number of variables. For this purpose it is necessary to sum on μ, ν from 0 to an arbitrary $n > 2$. This is done so that we do not exhaust all possible unequivalent invariants.

The alternate method of constructing unequivalent invariants of the group $\tilde{\mathrm{P}}(1,2)$, as well as any group G, is the following. Instead of solving Equation (1.4.7) or the equivalent system of ODEs (1.4.9) with the functions ξ^μ which are the linear combination of the generator's coordinates of the Lie algebra, all at least one-dimensional, non-isomorphic subalgebras of the AG should be found. Then the invariants of these subalgebras' generators are calculated. Certainly the very problem of describing non-isomorphic subalgebras of a given Lie algebra is difficult enough but for the fact that there is an effective algorithm [163] with which the complete description of non-isomorphic subalgebras of the most important Lie algebras is given: P(1,3) [163], P(1,4) [51], $\tilde{\mathrm{P}}(1,n)$ [14–16], P(2,n) [13], G$(1,n)$ [12,68], E(n) [69] (for a comprehensive review of this subject see [139*]).

The explicit form of the non-isomorphic, one-dimensional subalgebras of the algebra AP(1,3) and the corresponding invariants obtained in [163,114] are presented in Table 2.1.1.

In Appendix 2 the one-dimensional non-isomorphic subalgebras of $\mathrm{A}\tilde{\mathrm{P}}(1,3)$ are constructed. These results are used in §2.1 and §2.2 when constructing ansatzes for spinor or vector fields.

1.5. *Reduction and exact solutions of the nonlinear d'Alembert equation*

In the present paragraph we construct the families of exact solutions of the nonlinear wave equation [88]

$$\Box u + \lambda u^k = 0, \tag{1.5.1}$$

where $u = u(x_0, x_1, x_2)$, $\Box = \partial_0^2 - \partial_1^2 - \partial_2^2$, λ, $k \neq 1$ are arbitrary constants.

As shown in §1.1, Equation (1.5.1) is invariant under the algebra $\mathrm{A}\tilde{\mathrm{P}}(1,2)$. Using this result we take for the operator X (1.4.6) the generators' linear combination of the AI $\mathrm{A}\tilde{\mathrm{P}}(1,2)$

$$X = (æ x_\mu + c_{\mu\nu} x^\mu + d_\mu)\partial_\mu + \frac{2æ}{k-1} \tag{1.5.2}$$

and construct by the method elaborated in the preceding paragraph the ansatze (1.4.2). The invariant variables ω_1, ω_2 are presented in Table 1.4.1. The function $f(x)$ may be determined both from (1.4.8) and (1.4.9) using the operator (1.5.2), and from the splitting conditions, (1.4.4). Omitting easy but rather lengthy calculations we present the explicit form of $\widetilde{\mathrm{P}}(1,2)$-unequivalent ansatze for the scalar field $u(x)$ with symmetry operators (1.5.2) [88]

$$\begin{aligned}&1^\circ\text{–}5^\circ \quad u(x) = (\beta y)^{\frac{2}{1-k}}\phi(\omega_1,\omega_2),\\ &6^\circ\text{–}10^\circ \quad u(x) = \phi(\omega_1,\omega_2),\end{aligned} \tag{1.5.3}$$

Formulas (1.5.3) are to be considered together with Table 1.4.1 where the corresponding values of invariant variables are represented, and the relationships between arbitrary constants α_ν, β_ν are indicated.

Let us substitute the ansatze 1°–10° (1.5.3) into Equation (1.5.1). After some cumbersome calculations we get

$$\begin{aligned}1^\circ.\ & a^2\omega_1^2\varphi_{11} - 4\omega_1(a\omega_2 - a - 1)\varphi_{12} + 4\omega_2(\omega_2 - 1)\varphi_{22} +\\ & + a[a - 1 + 4(1-k)]\omega_1\varphi_1 + 2(k-1)^{-1}[(3k+1)\omega_2 - 2(k+1)]\varphi_2 +\\ & \qquad + 2(k+1)(k-1)^{-2}\varphi + \lambda\varphi^k = 0;\end{aligned}$$

$$2^\circ.\ \varphi_{11} + 4\varphi_{12} - 4\omega_2\varphi_{22} + 2(3+k)(1-k)^{-1}\varphi_2 + \lambda\varphi^k = 0;$$

$$\begin{aligned}3^\circ.\ & [1 - (\omega_2 - 1)^{-1}]\varphi_{11} - 4(\omega_2 - 1)(\varphi_{12} - \omega_2\varphi_{22}) + (3+k)(1-k)^{-1}\varphi_1 -\\ & - 4(1-k)[(3-k)\omega_2 - (1+k)]\varphi_2 + 2(1+k)(1-k)^{-2}\varphi + \lambda\varphi^k = 0;\end{aligned}$$

$$\begin{aligned}4^\circ.\ & \omega_1^2\varphi_{11} - \omega_1\omega_2(b\omega_2 - 2)\varphi_{12} + \omega_2^2\varphi_{22} + 4(1+k)^{-1}(\omega_1\varphi_1 + \omega_2\varphi_2) +\\ & \qquad + 2(k+1)(1-k)^{-2}\varphi + \lambda\varphi^k = 0;\end{aligned}$$

$$\begin{aligned}5^\circ.\ & (\omega_1^2 + 1)\varphi_{11} + 2\omega_1\omega_1\varphi_{22} + (\omega_2^2 + 1)\varphi_{22} + 2(k+1)(k-1)^{-1}\times\\ & \times(\omega_1\varphi_1 + \omega_2\varphi_2) + 2(k+1)(k-1)^{-2}\varphi + \lambda^k = 0;\end{aligned}$$

$$6^\circ.\ 4\omega_1\varphi_{11} + 4a\varphi_{12} - \varphi_{22} + 4\varphi_1 + \lambda\varphi^k = 0; \tag{1.5.4}$$

$$7^\circ.\ -4\omega_1\varphi_{11} + (1 + a^2\omega_1^{-1})\varphi_{22} + 4\varphi_1 + \lambda\varphi^k = 0;$$

$$8^\circ.\ -\varphi_{11} + 2(\omega_1 + a^2)\varphi_{22} + (\lambda/ba^2)\varphi^k = 0;$$

$$9^\circ.\ \varphi_{11} + 4\omega_1\varphi_{12} + 4\omega_2\varphi_{22} + \varphi_2 + \lambda\varphi^k = 0;$$

$$10^\circ.\ \alpha^2\varphi_{11} + 2\alpha\beta\varphi_{12} + \beta^2\varphi_{22} + \lambda\varphi^k = 0,$$

Here $\varphi_k = \dfrac{\partial\varphi}{\partial\omega_k}$, $\varphi_{kl} = \dfrac{\partial^2\varphi}{\partial\omega_k\partial\omega_l}$, N 1°–10° correspond the numbers of ansatze in (1.5.3) and of invariant variables in Table 1.4.1.

Let us pass to the more detailed analysis of the reduced Equations 1°–10° (1.5.4).

If in 1° (1.5.4) we put $\varphi_2 = 0$ then the substitution

$$\varphi(\omega_1) = \omega_1\frac{k+1}{k-1}V(\omega_1) \tag{1.5.5}$$

reduces 1° (1.5.4) to the Emden-Fauler equation

$$\omega_1^2\ddot{V} + 2\omega_1\dot{V} + \lambda\omega_1^{k+1}V^k = 0, \tag{1.5.6}$$

where dots denotes differentiation with respect to ω_1. The partial solution of (1.5.6) is sought in the form

$$V(\omega_1) = c\omega_1^s, \quad \text{with } c, s \text{ constant} \tag{1.5.7}$$

Substituting this expression into (1.5.6) we get

$$s = -\frac{k+1}{k-1}, \qquad c^{k-1} = -2\frac{k+1}{\lambda(k-1)^2}$$

As a result we find the following solution for Equation (1.5.1):

$$u(x) = (\beta x)^{\frac{2}{1-k}}, \quad \beta^2 = -\lambda\frac{(1-k)^2}{2(1+k)} \tag{1.5.8}$$

Putting in 2° (1.5.4) $\varphi_2 = 0$ we get the ODE

$$\varphi_{11} + \lambda\varphi^k = 0, \tag{1.5.9}$$

whose general solution is expressed through the integral

$$\int_0^\varphi \frac{d\tau}{\sqrt{c_1 + \tau^{k+1}}} = \pm\frac{2}{1-k}(\omega_1 + c_2), \tag{1.5.10}$$

where c_1, c_2 are constants of integration. Specifically, when $c_1 = 0$ we have

$$\varphi(\omega_1) = (\omega_1 + c_2)^{\frac{2}{1-k}} \tag{1.5.11}$$

(When $k = 3$, the solutions of Equation (1.5.9) are expressed with elliptic functions—see Appendix 1).

Returning to 2° (1.5.3) we find the following solution for Equation (1.5.1):

$$u(x) = [\beta x + \alpha x(c_2 + \ln \alpha x)]^{\frac{2}{1-k}}, \qquad \alpha^2 = \alpha\beta = 0, \quad \beta^2 = -\frac{\lambda}{2}\frac{(1-k)^2}{1+k}. \tag{1.5.12}$$

Putting in 2° (1.5.4) $\varphi_1 = 0$ we obtain

$$-4\omega_2\varphi_{22} + 2(3+k)(1-k)^{-1}\varphi_2 + \lambda\varphi^k = 0, \tag{1.5.13}$$

whose partial solution is the function

$$\varphi(\omega_2) = \left[\frac{\lambda}{2}\frac{(k-1)^2}{k-3}\omega_2\right]^{\frac{1}{1-k}}, \quad k \neq 3.$$

This leads to the solution of Equation (1.5.1)

$$u(x) = \left[\frac{\lambda(k-1)^2}{2(k-3)}x_\nu x^\nu\right]^{\frac{1}{1-k}}, \quad k \neq 3. \tag{1.5.14}$$

By a substitution analagous to (1.5.5), Equation 4° (1.5.4), when $\varphi_1 = 0$ or $\varphi_2 = 0$, is reduced to the Emden-Fauler Equation (1.5.6).

When $\varphi_1 = 0$ or $\varphi_2 = 0$, Equation 5° (1.5.4) takes the form

$$(\omega^2+1)\ddot{\varphi} + a\omega\dot{\varphi} + b\varphi + \lambda\varphi^k = 0 \tag{1.5.15}$$

where $\omega = \omega_1$ or ω_2; and the overdots denote differentiation with respect to ω. When $k = -3$ it is possible to integrate Equation (1.5.15) and the result has the form

$$\varphi = \pm\left[2c_1 + \frac{c_2^2\left(\omega+\sqrt{\omega^2+1}\right)^{\pm 2} + c_1^2 - \lambda}{2c_2\left(\omega+\sqrt{\omega^2+1}\right)^{\pm 1}}\right]^{1/2} \tag{1.5.16}$$

4° (1.5.3) and (1.5.16), when $\omega = \omega_1$ or $\omega = \omega_2$, results in the following solution for Equation (1.5.1):

$$u(x) = \begin{cases} \pm[(\alpha x)(c_1 + \mu \operatorname{ch} Q(x)]^{1/2}, & c_1^2 - 4\lambda = \mu^2 > 0, \\ \pm[(\alpha x)(c_1 + \mu \operatorname{sh} Q(x)]^{1/2}, & c_1^2 - 4\lambda = -\mu^2 > 0, \\ \pm[(\alpha x)Q(x)]^{1/2}, & c_1 - 4\lambda = 0, \end{cases} \tag{1.5.17}$$

where $Q(x) = c_2\left[(\beta x)(\alpha x)^{-1} + (\beta x)^2(\alpha x)^2 + 1^{1/2}\right]$, $\alpha^2 = -\beta^2 = 1, \quad \alpha\beta = 0$.

Putting in 6° (1.5.4) $\varphi_2 = 0$, we obtain the ODE

$$\omega_1\varphi_{11} + \varphi_1 + \frac{\lambda}{4}\varphi^k = 0, \tag{1.5.18}$$

whose partial solution is

$$\varphi(\omega_1) = \left[-\frac{\lambda}{4}(1-k)^2\omega\right]^{\frac{1}{1-k}} \tag{1.5.19}$$

Formulas (1.5.19) and 6° (1.5.3) determine the following solution for Equation (1.5.1):

$$u(x) = \left\{-\frac{\lambda}{4}(1-k)^2\left[(\beta y)^2 + y^2\right]\right\}^{\frac{1}{1-k}}. \tag{1.5.20}$$

Putting in 8° (1.5.4) $\varphi_2 = 0$, we get (1.5.9), and using (1.5.11) and 8° (1.5.3) we obtain one more family of solutions for Equation (1.5.1):

$$u(x) = \left[\beta y + \frac{1}{2a}(\alpha y)^2 + c_2\right]^{\frac{2}{1-k}}, \quad \alpha^2 = \alpha\beta = 0, \quad \beta^2 = -1 \tag{1.5.21}$$

Equation 10° (1.5.4) is the two-dimensional wave equation, if $\alpha^2 = -\beta^2 = 1$, $\alpha\beta = 0$.

In Equation 9° (1.5.4) let us make the substitution of variables

$$\tau_0 = \sqrt{\omega_1^2 - \omega_2}\,, \qquad \tau_1 = \omega_1, \tag{1.5.22}$$

which transform 9° into the nonlinear Darboux equation

$$\varphi_{\tau_0\tau_0} - \varphi_{\tau_1\tau_1} + \tau_0^{-1}\varphi_{\tau_0} - \lambda\varphi^k = 0 \tag{1.5.23}$$

To find solutions for Equation (1.5.23) we once again apply the algorithm of §1.4. But first let us investigate its symmetry.

Theorem 1.5.1. *The maximal local invariance group of the nonlinear Darboux Equation* (1.5.23) *is a 2-parameter Lie group generated by operators of the form*

$$X_1 = \frac{\partial}{\partial \tau_1}, \qquad X_2 = \tau_0\frac{\partial}{\partial \tau_0} + \tau_1\frac{\partial}{\partial \tau_1} - \frac{2}{1-k} \tag{1.5.24}$$

Theorem 1.5.2. *The maximal local invariance group of the n-dimensional nonlinear Darboux equation*

$$\Box u + \lambda_1 x_0^{-1} u_0 + \lambda_2 u^k = 0 \tag{1.5.25}$$

when $k = \frac{n+2+\lambda_1}{n-2+\lambda_1}$, $\lambda_1 \neq 2-n$ *is the conformal group* $\mathrm{C}(n-1)$ *whose generators have the form*

$$\begin{aligned} &P_a = -\partial_a, \quad J_{ab} = x_a P_b - x_b P_a; \quad a, b = \overline{1, n-1}, \\ &D = x_0\partial_0 + x_a\partial_a - \tfrac{1}{2}(n-2+\lambda_1), \\ &K_a = 2x_a D - (x_0^2 - x_b x^b)P_a, \end{aligned} \tag{1.5.26}$$

when $k \neq \frac{n+2+\lambda_1}{n-2+\lambda_1}$ *it is the extended Euclidean group* $\widetilde{\mathrm{E}}(n-1)$ *generated by the operators*

$$\begin{aligned} &P_a = -\partial_a, \qquad J_{ab} = x_a P_b - x_b P_a, \\ &D = x_0\partial_0 + x_a\partial_a - \frac{2}{1-k}. \end{aligned} \tag{1.5.27}$$

The proof of Theorems 1.5.1, 1.5.2 may be obtained by Lie's method.

We seek the solutions of Equation (1.5.23) in the form

$$\varphi(\tau_0, \tau_1) = \tau^{\frac{2}{1-k}}\Phi(V), \quad V = \frac{\tau_1 + a}{\tau_0}, \quad a = \text{const.} \tag{1.5.28}$$

Substituting (1.5.28) into (1.5.23) results in the ODE

$$(V^2-1)\Phi_{VV}+\frac{k+3}{k-1}V\Phi_V+\frac{4}{(k-1)^2}\Phi+\lambda\Phi^k=0. \tag{1.5.29}$$

Knowing the solutions of this equation, we construct by the formulae (1.5.28), (1.5.22), 9° (1.5.3), solutions for equation (1.5.1). Note that when $k=5$, (1.5.28) coincides with the nonlinear Legendre equation

$$\frac{d}{dV}\left[(1-V^2)\frac{d\Phi}{dV}\right]-\frac{1}{4}\Phi-\lambda\Phi^5=0. \tag{1.5.30}$$

To conclude this paragraph we obtain solutions for Equation (1.5.1) containing arbitrary functions. We shall seek these solutions in the form

$$u(x)=[f(x)+g(x)]^\alpha \tag{1.5.31}$$

where $\alpha=(1-k)^{-1}$ or $2(1-k)^{-1}$, $V=\{V^\nu\}$, and $W=\{W^\nu\}$ are some functions of x.

Let us substitute (1.5.31) into the n-dimensional Equation (1.5.1). As a result we obtain

$$\begin{aligned}(\alpha-1)F_\mu F^\mu+(f+g)\big(f_{\nu\lambda}V_\mu^{\lambda\mu}+g_{\nu\lambda}W_\mu^\nu W^{\lambda\mu}+f_\nu\Box V^\nu+\\+g_\nu\Box W^\nu\big)+\frac{\lambda}{\alpha}(f+g)^{\alpha(k-1)+2}=0,\end{aligned} \tag{1.5.32}$$

where

$$F_\mu=f_\nu V_\mu^\nu+g_\nu W_\mu^\nu,\quad f_\nu=\frac{\partial g}{\partial W^\nu},\quad f_{\mu\nu}=\frac{\partial^2 f}{\partial V^\mu\partial V^\nu},\quad g_{\mu\nu}=\frac{\partial^2 g}{\partial V^\mu\partial V^\nu}.$$

Let us adduce some partial solutions of Equation (1.5.32):

a). $n\geq 3$, $\alpha=(1-k)^{-1}$, $f(V)=(V^0+c)^2$, $g(W)=W^0W^1$,

$$V^0=\alpha x,\quad W^0=\beta x,\quad W^1=\delta x,\quad c=\text{const}, \tag{1.5.33}$$
$$\alpha\beta=\alpha\delta=\beta^2=\delta^2=0,\quad 2\alpha^2=\beta\delta=\lambda\frac{k-1}{k-2},\quad k\neq 3.$$

b). $n\geq 3$, $\alpha=2(1-k)^{-1}$, $f(V)=V^1\varphi(V^0)$, $g(W)=W^1\psi(W^0)$,

$$V^0=W^0=\alpha x,\quad V^1=\beta x,\quad W^1=\delta x, \tag{1.5.34}$$
$$\alpha^2=\alpha\beta=\alpha\delta=\beta\delta=0,\quad \beta^2=-\delta^2=-1,$$

where φ and ψ are arbitrary differentiable functions satisfying the condition $\varphi^2+\psi^2=\lambda(k-1)^2/2(k+1)$.

c). $n\geq 3$, $\alpha=2(1-k)^{-1}$, $f(V)=f(V^0)$ is an arbitrary smooth function;

$g(W) = W^0$, $V^0 = \alpha x$, $W^0 = \beta x$,

$$\alpha^2 = \alpha\beta = 0, \quad \beta^2 = -\frac{\lambda}{2}\frac{(1-k)^2}{1+k} \tag{1.5.35}$$

Returning to (1.5.31) we obtain from (1.5.33) through (1.5.35) corresponding solutions for the n-dimensional Equation (1.5.1) ($n \geq 3$)

$$\begin{aligned} &u(x) = \left[(\alpha x + c)^2 + (\beta x)(\delta x)\right]^{\frac{1}{1-k}}, \quad c = \text{const}, \ k \neq 3, \\ &\alpha\beta = \alpha\delta = \beta^2 = \delta^2 = 0, \quad 2\alpha^2 = \beta\delta = \lambda\frac{k-1}{k-3}; \end{aligned} \tag{1.5.36}$$

$$\begin{aligned} &u(x) = \left[\beta x \varphi(\alpha x) + \delta x \psi(\alpha x)\right]^{\frac{2}{1-k}}, \\ &\alpha^2 = \alpha\beta = \alpha\delta = \beta\delta = 0, \quad \beta^2 = -\delta^2 = -1; \end{aligned} \tag{1.5.37}$$

$$\begin{aligned} &u(x) = [f(\alpha x) + \beta x]^{\frac{2}{2-k}}, \\ &\alpha^2 = \alpha\beta = 0, \quad \beta^2 = -\frac{\lambda}{2}\frac{(k-1)^2}{k+1}. \end{aligned} \tag{1.5.38}$$

The solutions obtained, containing the arbitrary functions, may be useful while solving various problems of mathematical physics, such as the Cauchy problem or some boundary value problems.

In conclusion let us note that in the case of three spacial variables nonlinear wave equation

$$\Box u + \lambda u^3 = 0 \tag{1.5.39}$$

is invariant under the conformal group C(1,3) $\supset \widetilde{\mathrm{P}}(1,3)$. The detailed analysis of conformal symmetry will be given in the next chapter (§2.3), but here it is appropriate to consider this question in connection with Equation (1.5.39).

The conformal transformations leaving Equation (1.5.39) invariant have the form

$$x_\mu \to x'_\mu = \frac{x_\mu - c_\mu x^2}{\sigma(x,c)}, \quad u(x) \to u'(x') = \sigma(x,c)u(x), \tag{1.5.40}$$

where $\sigma(x,c) = 1 - 2cx + c^2x^2$, c_μ are arbitrary constants. According to (1.4.11), (1.4.12) one can easily derive from (1.5.40) the following formula of generating solutions

$$u_{II}(x) = \frac{u_I(x')}{\sigma(x,c)}. \tag{1.5.41}$$

The conformally invariant ansatz for Equation (1.5.39) has the form

$$u(x) = (x^2)^{-1}\varphi(\omega), \quad \omega = \frac{\beta x}{x^2}, \quad (\beta x \equiv \beta_\nu x^\nu) \tag{1.5.42}$$

where β_ν are arbitrary constants. The substitution of (1.5.42) into (1.5.39) gives the following ODE

$$\beta^2 \frac{d^2\varphi}{d\omega^2} + \lambda\varphi^3 = 0 \qquad (\beta^2 \equiv \beta^\nu \beta_\nu). \tag{1.5.43}$$

Solutions of this equation are expressed in terms of elliptic functions (see Appendix 1) when $\beta^2 = -1$, $\lambda > 0$, there is a simple partial solution of Equation (1.5.43)

$$\varphi = \sqrt{\frac{2}{\lambda}} \frac{1}{\omega}. \tag{1.5.44}$$

So, the expression (1.5.44) together with (1.5.42) result in the following solution of Equation (1.5.39)

$$u(x) = \sqrt{\frac{2}{\lambda}} \frac{1}{\beta x}, \quad \beta^2 = -1. \tag{1.5.45}$$

After application to (1.5.45) the formula of generating solutions by means of translational transformations

$$u_{II}(x) = u_I(x'), \quad x'_\mu = x_\mu + a_\mu, \qquad (a_\mu = \text{const}),$$

we get the C(1,3)-ungenerative family of solutions of Equation (1.5.39)

$$u(x) = \sqrt{\frac{2}{\lambda}} \frac{1}{\beta x + æ}, \quad \beta^2 = -1, \tag{1.5.46}$$

where æ is an arbitrary constant.

1.6. *Reduction and solutions of the Liouville equation*

The Liouville equation

$$\Box u + \lambda e^u = 0 \tag{1.6.1}$$

arises in the problems of differential geometry, the theory of nonlinear waves, and quantum field theory [18].

In the two-dimensional case the general solution of Equation (1.6.1) is

$$u(x_0, x_1) = \ln \left\{ -\frac{8}{\lambda} \frac{\dot{f}(x_0 + x_1)\dot{g}(x_0 - x_1)}{[f(x_0 + x_1) + g(x_0 - x_1)]^2} \right\}, \tag{1.6.2}$$

where f, g are arbitrary differentiable functions and $\dot{f}, \dot{g}$ are derivatives with respect to the corresponding argument ($\dot{f}\dot{g} < 0$), was constructed in 1853 by

Liouville. The singular solutions of Equation (1.6.1) were obtained in [127, 128].

Below we shall consider the three-dimensional Equation (1.6.1). As shown in §1.1 its maximal (in Lie's sense) invariance group when $n > 2$ is the extended Poincare group $\widetilde{\mathrm{P}}(1, n-1)$ and the general form of the symmetry IFO is as follows:

$$X = (æx^{\mu} + c^{\mu\nu}x_{\nu} + d^{\mu})\frac{\partial}{\partial x_{\mu}} - 2æ\frac{\partial}{\partial u}. \tag{1.6.3}$$

Because of the addend $-2æ\partial_u$ the operator (1.6.3) doesn't have the structure of (1.4.6). But if we make the transformation (see (1.3.20))

$$u = -2\ln W, \tag{1.6.4}$$

then (1.6.3) coincides with the operator (1.5.2) with $k = -1$ and it is possible to use ansatz (1.5.3). Substituting ansatze 1°–10° (1.5.3) with $k = -1$ into (1.6.4) instead of W, we obtain the ansatz for Equation (1.6.1) [88]:

$$\begin{aligned} &1^\circ - 5^\circ. \quad u(x) = \varphi(w_1, w_2) - 2\ln(\alpha y),\\ &6^\circ - 10^\circ. \quad u(x) = \varphi(w_1, w_2). \end{aligned} \tag{1.6.5}$$

Formulas (1.6.5) are to be considered together with Table 1.4.1 where the corresponding values of the invariant variables w_1, w_2 are adduced and the conditions for α_ν, β_ν *et al.* are indicated.

Substituting ansatz $1^\circ - 10^\circ$ (1.6.5) into Equation (1.6.1) we obtain, correspondingly

$$\begin{aligned}
&1^\circ. \quad a^2w_1^2\varphi_{11} + 4w_1(w_2 + a + 1)\varphi_{12} + 4w_2(w_2 - 1)\varphi_{22} + \\
&\qquad\qquad + a(a-1)w_1\varphi_1 + +2(3w_2 - 1)\varphi_2 + 2 + \lambda e^{\varphi} = 0;\\
&2^\circ. \quad \varphi_{11} + 4\varphi_{12} - 4w_2\varphi_{22} - 2\varphi_2 + \lambda e^{\varphi} = 0;\\
&3^\circ. \quad \left[1 - (w_2 - 1)^{-1}\right]\varphi_{11} - 4(w_2 - 1)\varphi_{12} + 4w_2(w_2 - 1)\varphi_{22} - \varphi_1 + \\
&\qquad\qquad + 2(3w_2 - 1)\varphi_2 + 2 + \lambda e^{\varphi} = 0;\\
&4^\circ. \quad \varphi_{11} + 2(2w_2 + b)\varphi_{12} + 4w_2^2\varphi_{22} - \varphi_1 + bw_2\varphi_2 + 2 + \lambda e^{\varphi} = 0;\\
&5^\circ. \quad (w_1^2 + 1)\varphi_{11} + 2w_1w_2\varphi_{12} + (w_2^2 + 1)\varphi_{22} + 2w_1\varphi_1 + 2w_2\varphi_2 + 2 + \lambda e^{\varphi} = 0;\\
&6^\circ. \quad 4w_1\varphi_{11} + 4a\varphi_{12} - \varphi_{22} + 4\varphi_1 + \lambda e^{\varphi} = 0;\\
&7^\circ. \quad -4w_1\varphi_{11} + (1 + a^2w_1^{-1})\varphi_{22} + 4\varphi_1 + \lambda e^{\varphi} = 0;\\
&8^\circ. \quad -\varphi_{11} + 2(w_1^2 + a^2)\varphi_{22} + \frac{\lambda}{a^2b}e^{\varphi} = 0;\\
&9^\circ. \quad \varphi_{11} + 4w_1\varphi_{12} + 4\omega_2\varphi_{22} + \lambda e^{\varphi} = 0;
\end{aligned} \tag{1.6.6}$$

10°. $\alpha\varphi_{11} + 2\alpha\beta\varphi_{12} + \beta\varphi_{22} + \lambda e^{\varphi} = 0.$

Here $\varphi_k = \dfrac{\partial\varphi}{\partial w_k}$, and $\varphi_{kl} = \dfrac{\partial^2\varphi}{\partial w_k \partial w_l}$; N $1° - 10°$ correspond to the numeration in Tables 1.6.1 and 1.4.1.

First of all we note that if $\alpha^2 = \alpha\beta = 0$, $\beta^2 = -\lambda/2$, then Equation 10°(1.6.6) takes the form

$$\varphi_{22} - 2e^{\varphi} = 0$$

and its general solution is

$$\varphi = \begin{cases} F(\omega_1) - 2\ln \,\mathrm{ch}\,[\omega_2 + G(\omega_1)], \\ F(\omega_1) - 2\ln \,\cos[\omega_2 + G(\omega_1)], \\ -2\ln\,[-\omega_2 + G(\omega_1)], \end{cases}$$

where F, G are arbitrary differentiable functions. Further, if $\alpha^2 = -\beta^2 = 1$, $\alpha\beta = 0$ then 10°(1.6.6) coincides with the two-dimensional Liouville equation.

It turns out that in the case $n = 2$ the symmetry of Equation (1.6.1) is essentially more rich than the extended Poincare group $\widetilde{\mathrm{P}}(1,1)$ [88].

Theorem 1.6.1. *The maximal (in Lie's sense) invariance group of Equation (1.6.1) with $n = 2$ is infinite-dimensional and generated by the operators*

$$X = (f + g)\partial_0 + (f - g)\partial_1 - 2(\dot{f} + \dot{g})\partial_u, \tag{1.6.7}$$

where $f = f(x_0 + x_1)$ and $g = g(x_0 - x_1)$ are arbitrary differentiable functions; and $\dot{f}, \dot{g}$ are derivatives with respect to the corresponding arguments.

Proof. From the invariance condition

$$\underset{2}{X}(u_{00} - u_{11} + \lambda e^u)\Big|_{u_{00}-u_{11}+\lambda e^u=0} = 0,$$

where $\underset{2}{X}$ is defined in (1.1.7), we have

$$\Big[\eta_{00} - \eta_{11} + \lambda\eta e^u + 2(u_0\eta_{0u} - u_1\eta_{1u}) + (u_{00} - u_{11})(\eta_u - u_\sigma\xi^\sigma_u) -$$
$$-2(u_{0\sigma}\xi^\sigma_0 - u_{1\sigma}\xi^\sigma_1) - 2(u_{0\sigma}u_0 - u_{1\sigma}u_1)\xi^\sigma_u - u_\sigma(\xi^\sigma_{00} - \xi^\sigma_{11}) -$$
$$-(u_0^2 - u_1^2)u_\sigma\xi^\sigma_{uu}\Big]\Big|_{u_{00}-u_{11}+\lambda e^u=0} = 0,$$

whence we obtain

$$\xi^\sigma_u = 0, \quad \xi^1_0 = \xi^0_1, \quad \xi^0_0 = \xi^1_1,$$

$$\eta_{u0} = \eta_{u1}, \quad \eta_{00} - \eta_{11} + \lambda\eta e^u - \lambda\eta_u + 2\lambda e^u\xi^0_0 = 0.$$

The general solution of this system has the form

$$\xi^0 = f(x_0 + x_1) + g(x_0 - x_1),$$
$$\xi^1 = f(x_0 + x_1) - g(x_0 - x_1),$$
$$\eta = -2\xi^0_0$$

and determines the operator (1.6.7). The theorem is proved.

Solution (1.6.2) can be easily generalized to the n-dimensional Equation (1.6.1) (certainly for $n > 2$ it is not general):

$$u(x) = \ln\left\{-\frac{8}{\lambda}\frac{\dot f(\alpha x)\dot g(\beta x)}{[f(\alpha x) + g(\beta x)]^2}\right\}, \qquad \alpha^2 = \beta^2 = 0, \qquad \alpha\beta = 2, \quad (1.6.8)$$

where α_ν and β_ν ($\nu = \overline{0, n-1}$) are arbitrary variables satisfying the mentioned conditions, and $f, g, \dot f, \dot g$ are arbitrary functions and their derivatives.

Putting in 1° (1.6.6) $\varphi_2 = 0$, we obtain the ODE

$$a^2 w_1^2 \varphi_{11} + a(a-1)w_1\varphi_1 + \lambda e^u = 0 \qquad (1.6.9)$$

whose general solution has the form

$$u(x) = \begin{cases} -2\ln\left[\frac{\sqrt{-\lambda}}{2c_1} w_1^{-1/a}\,\mathrm{sh}\,(c_1 w_1^{1/a} + c_2)\right], & \lambda < 0, \\ -2\ln\left[\frac{\sqrt{\lambda}}{2c_1} w_1^{-1/a}\,\mathrm{ch}\,(c_1 w_1^{1/a} + c_2)\right], & \lambda > 0, \\ -2\ln\left[\frac{\sqrt{-\lambda}}{2c_1} w_1^{-1/a}\cos(c_1 w_1^{1/a} + c_2)\right], & \lambda < 0, \\ -2\ln\left[\frac{\sqrt{-\lambda}}{2c_1} w_1^{-1/a}(c_1 w_1^{1/a} + c_2)\right], & \lambda < 0, \end{cases} \qquad (1.6.10)$$

The ansatz 1° (1.6.5) and the formulas (1.6.10) result in the following solutions for Equation (1.6.1):

$$u(x) = \begin{cases} -2\ln\left[\rho\mathcal{P}(x)\,\mathrm{sh}\,(c_1 Q(x) + c_2)\right], \\ -2\ln\left[\delta\mathcal{P}(x)\,\mathrm{ch}\,(c_1 Q(x) + c_2)\right], \\ -2\ln\left[\rho\mathcal{P}(x)\cos(c_1 Q(x) + c_2)\right], \\ -2\ln\left[\rho\mathcal{P}(x)(Q(x) + c_2)\right], \end{cases} \qquad (1.6.11)$$

where

$$-\rho^2 = \delta^2 = \frac{\lambda}{4c_1^2}, \quad \mathcal{P}(x) = (\alpha y)^{-1/a},$$
$$Q(x) = (\beta y)(\alpha y)^{1/a}, \quad \alpha^2 = \alpha\beta = 0, \quad \beta^2 \neq 0. \qquad (1.6.12)$$

We list some more solutions for Equation (1.6.1) having the structure of (1.6.11) but with other functions $P(x)$ and $Q(x)$:

$$\begin{aligned}
&\mathcal{P}(x) = \alpha y, \quad Q(x) = \sqrt{y^2}/\alpha y;\\
&\mathcal{P}(x) = (\beta y)^2 + y^2, \quad Q(x) = \ln P(x);\\
&\mathcal{P}(x) = 1, \quad Q(x) = \beta y + a \ln(\alpha y);\\
&\mathcal{P}(x) = 1, \quad Q(x) = \beta y; \qquad (1.6.13)\\
&\mathcal{P}(x) = F^{-1}(\alpha y), \quad Q(x) = (\beta y)F(\alpha y);\\
&\mathcal{P}(x) = F^{-1}(\alpha y), \quad Q(x) = \beta y F(\alpha y) - \ln F(\alpha y);\\
&\mathcal{P}(x) = 1, \quad Q(x) = \beta y + F(\alpha y).
\end{aligned}$$

In the formulas (1.6.13) α_ν, β_ν are arbitrary parameters satisfying the conditions $\alpha^2 = \alpha\beta = 0$, $\beta^2 \neq 0$; F is an arbitrary smooth function.

From the partial solution of Equation 9° (1.6.6) and the ansatz 9° (1.6.5) we obtain one more solution for Equation (1.6.1):

$$u(x) = -\ln\left(\frac{\lambda}{2}x_\nu x^\nu\right). \qquad (1.6.14)$$

The solutions of Equation (1.6.1) can evidently be generalized for an arbitrary number of variables.

It should be stressed that all solutions of the Liouville Equation (1.6.1) obtained here have a singularity at the point $\lambda = 0$. This means that with the regular methods of perturbation theory it is impossible to obtain the approximate solutions near the exact solutions (1.6.2), (1.6.11)–(1.6.14).

1.7. *Reduction and solutions of d'Alembert's equation with the nonlinearities* sin u, sh u

The wave equation

$$\Box u + F(u) = 0, \quad F(u) = \{\operatorname{sh} u, \sin u\} \qquad (1.7.1)$$

describes a wide class of physical phenomena: dislocation propagation through a crystal lattice, Bloch wall movement in magnetic crystals, propagation of a "skewed wave" along a lipid membrane, unitary theory of elementary particles, propagation of magnetic fluxes in Josephson line, *et al.* (See [1, 26, 210] and references therein.)

For the two-dimensional ($n = 2$) Equation (1.7.1) there are well-known soliton, soliton-antisoliton, and two-soliton solutions [170, §2.5].

As shown in §1.1 the maximal (in Lie's sense) invariance algebra of Equation (1.7.1) is the algebra AP(1,$n-1$) whose basis generators have the form (1.1.2). Thus we seek solutions for Equation (1.7.1) in the form

$$u(x) = \varphi(w) \qquad (1.7.2)$$

In the present paragraph we construct the families of solutions of the 4-dimensional ($n = 4$) Equation (1.7.1).

Let us use the invariance variables listed in Table 2.1.1. This gives 13 unequivalent ansatze (1.7.2). Substituting into Equation (1.7.1) we obtain 13 reduced PDEs, correspondingly:

$$
\begin{aligned}
&1^\circ. \quad -\varphi_{11} - \varphi_{22} - \varphi_{33} + F(\varphi) = 0;\\
&2^\circ. \quad \varphi_{11} - \varphi_{22} - \varphi_{33} + F(\varphi) = 0;\\
&3^\circ. \quad -\varphi_{22} - \varphi_{33} + F(\varphi) = 0;\\
&4^\circ. \quad -\varphi_{11} - \frac{1}{w_1}\varphi_1 + \varphi_{22} - \varphi_{33} + F(\varphi) = 0;\\
&5^\circ. \quad \varphi_{11} + \frac{1}{w_1}\varphi_1 - \varphi_{22} - \varphi_{33} + F(\varphi) = 0;\\
&6^\circ. \quad \varphi_{22} + \frac{2}{w_2}\varphi_2 - \varphi_{33} + 2\frac{w_1}{w_2}\varphi_{12} + F(\varphi) = 0;\\
&7^\circ. \quad \varphi_{11} + \frac{1}{w_1}\varphi_1 - \frac{1}{w_3}\varphi_{22} - \varphi_{33} + \frac{1}{w_3}\varphi_3 + \frac{2}{w_1}\varphi_{12} + F(\varphi) = 0;\\
&8^\circ. \quad -\frac{1}{w_3^2}\varphi_{22} - \varphi_{33} - \frac{1}{w_3}\varphi_3 + 4\epsilon\varphi_{12} + F(\varphi) = 0;\\
&9^\circ. \quad -\varphi_{11} + \frac{1}{w_1}\varphi_1 + (1 - \alpha^2 w_1^{-2})\varphi_{12} - \varphi_{33} + F(\varphi) = 0;\\
&10^\circ. \quad -\varphi_{11} + \frac{1}{w_1}\varphi_1 + (1 + \alpha^2 w_1^{-2})\varphi_{12} + \varphi_{33} + F(\varphi) = 0;\\
&11^\circ. \quad \varphi_{11} + \frac{1}{w_1}\varphi_1 - \varphi_{22} - \varphi_{33} + \frac{2\alpha}{w_1}\varphi_{12} + F(\varphi) = 0;\\
&12^\circ. \quad 4w_2\varphi_{11} - \varphi_{22} - \varphi_{33} + F(\varphi) = 0;\\
&13^\circ. \quad \varphi_{22} + \frac{2}{w_2}\varphi_2 - (1 + w_1^2)\varphi_{33} + 2\frac{w_3}{w_2}\varphi_{23} + 2\frac{w_1}{w_2}\varphi_{12} + F(\varphi) = 0;
\end{aligned}
\tag{1.7.3}
$$

Equations $1^\circ - 13^\circ$ (1.7.3) correspond to the ansatz (1.7.2) with the invariants 1-13 from Table 2.1.1. The direct reduction of Equations $1^\circ - 12^\circ$ (1.7.3) results in ODEs of the form

$$
\ddot{\varphi} + \frac{k}{w}\dot{\varphi} = \epsilon F(u), \quad k = 0, 1, 2; \quad \epsilon = \pm 1 \tag{1.7.4}
$$

Putting in 13° (1.7.3) $\varphi_2 = 0$ we get the ODE

$$
-(1 + w_1^2)\,\varphi_{33} + F(\varphi) = 0,
$$

which, by the change of variables

$$
w_3 \to w = \frac{w_3}{\sqrt{1 + w_1^2}} + f(w_1),
$$

where $f(w_1)$ is an arbitrary differentiable function, results again in Equation (1.7.4) with $k = 0$ and $\epsilon = -1$:

$$-\ddot{\varphi} + F(\varphi) = 0, \quad w = \frac{x_2 + \epsilon(x_0 + x_1)x_3}{\sqrt{1 + (x_0 + x_1)^2}} + f(x_0 + x_1). \tag{1.7.5}$$

When $k = 0$, Equation (1.7.4) can be solved by quadrature:

$$w + c_1 = \int \frac{d\varphi}{\sqrt{2\epsilon \int F(\tau)d\tau + c}} \tag{1.7.6}$$

where c_1, c are arbitrary constants.

When $k = 1$ Equation (1.7.4) may be reduced to one of Painleve's Equations [114, 139]. If $k > 1$ then (1.7.4) cannot be solved by quadrature.

From (1.7.6), for $F(\varphi) = \sin\varphi$, we obtain [63, 114, 88]

$$\begin{aligned}
&\varphi = 4\arctan(\alpha e^{\epsilon_0 w}) - \tfrac{1}{2}(1-\epsilon)\pi, \quad \epsilon_0 = \pm 1, \quad \epsilon = \pm 1, \quad \alpha = \text{const};\\
&\varphi = 2\arccos[\,\mathrm{dn}\,(w + \alpha, m)] + \tfrac{1}{2}(1+\epsilon)\pi, \quad 0 < m < 1;\\
&\varphi = 2\arccos[\,\mathrm{cn}\,(w + \alpha, m)] + \tfrac{1}{2}(1+\epsilon)\pi, \quad 0 < m < 1,
\end{aligned} \tag{1.7.7}$$

where α is an arbitrary constant, and $\mathrm{dn}\,(x,y)$, $\mathrm{cn}\,(x,y)$ are Jacobi elliptic functions (see Appendix 1).

Analogously, for $F(\varphi) = \mathrm{sh}\,\varphi$ from (1.7.6) we have [83] (1.7.8)

$$\begin{aligned}
&\varphi = 2\mathrm{arcth}\,[\,\mathrm{sn}\,(z,x)], \quad z = \tfrac{1}{2}\sqrt{c+2}\,w, \quad k^2 = \frac{c-2}{c+2}, \quad c > 2;\\
&\varphi = 2\mathrm{arcth}\,(\,\mathrm{sn}\,z'), \quad z' = \sqrt{2}\,w, \quad c = 2;\\
&\varphi = \arccos\left[\frac{\frac{1}{2} - \mathrm{sn}^2(z,k)}{\mathrm{cn}^2(w,k)}\right], \quad z = \tfrac{1}{2}\sqrt{c+2}\,w, \quad k = \frac{4}{c+2}, \quad c > 2;\\
&\varphi = \arccos\left[\frac{2 - c\,\mathrm{sn}\,(w,k)}{2\,\mathrm{cn}^2(w,k)}\right], \quad k^2 = \frac{c+2}{4}, \quad 0 < c < 2;\\
&\varphi = 4\mathrm{arcth}\,(e^w), \quad c = 2;\\
&\varphi = \mathrm{arcth}\,[\,\mathrm{cn}\,(w,k)]^{-1}, \quad c = 0, \quad k^2 = \tfrac{1}{2};\\
&\varphi = 2\mathrm{arcch}\,\left[\frac{c}{2}\,\mathrm{cn}^2(z,k) + \mathrm{sn}^2(z,x)\right], \quad z = \tfrac{1}{2}\sqrt{c+2}\,w, \; k^2 = \frac{c-2}{c+2}, \; c > 2.
\end{aligned} \tag{1.7.8}$$

In formulas (1.7.7) and (1.7.8) it follows from the previous analysis that w may take the values

$$x_0, x_1, x_2 + f(x_0 + x_1), \quad \frac{x_2 + \epsilon(x_0 + x_1)x_3}{\sqrt{1 + (x_0 + x_1)^2}} + f(x_0 + x_1), \tag{1.7.9}$$

where $f(x_0 + x_1)$ is an arbitrary differentiable function.

Applying to (1.7.7) and (1.7.8) the operation of group generation it is easy to construct P(1,3)-ungenerative families of solutions for Equation (1.7.1). These solutions have the form (1.7.7), (1.7.8) where the variable ω takes one of the following values:

$$dy,\ ay,\ by + f(ay + dy),\quad \frac{by + \epsilon(ay + dy)cy}{\sqrt{1 + (ay + dy)^2}} + f(ay + dy),$$

where a_ν, b_ν, d_ν, d_ν are arbitrary constants satisfying the conditions (2.1.27), $y_\nu = x_\nu + \delta_\nu$, δ_ν are arbitrary constants.

1.8. *Solutions of eikonal equations*

In the present paragraph we shall obtain families of exact solutions of equations whose symmetry was studied in §1.2.

1. Let us consider Equation (1.2.18)

$$u_\nu u^\nu \equiv \frac{\partial u}{\partial x^\nu} \frac{\partial u}{\partial x_\nu} = 0. \tag{1.8.1}$$

Equation (1.8.1) admits the infinite-dimensional algebra generated by the operators (see Theorem 1.2.3)

$$X = \eta(u)\, \partial_u$$

with an arbitrary differentiable function $\eta(u)$, and therefore it possesses a remarkable property: an arbitrary differentiable function of a solution of Equation (1.8.1) is also a solution, which can be easily confirmed.

Let us seek the solution of Equation (1.8.1) in the form

$$u(x) = \varphi(w) + g(x), \tag{1.8.2}$$

where the variables w are listed in Table 1.4.1 and the corresponding expressions for the function $g(x)$ have the form

$$g(x) = \begin{cases} \ln(\alpha y), & \text{for } N1 - 5, \\ 0, & \text{for } N6 - 10. \end{cases}$$

Substituting ansatz (1.8.2) into Equation (1.8.1) we obtain the following reduced PDE for the function $\varphi(w)$:

1°. $a^2 w_1^2 \varphi_1^2 - 4w_1(aw_2 - a - 1)\varphi_1\varphi_2 + 4w_2(w_2 - 1)\varphi_2^2 +$
$+ 2aw_1\varphi_1 - 4(w_2 - 1)\varphi_2 + 1 = 0;$

2°. $\varphi_1^2 + 4\varphi_1\varphi_2 - 4w_2\varphi_2^2 + 4\varphi_2 = 0;$

3°. $[1-(w_2^2-1)^{-1}]\varphi_1^2 - 4(w_2-1)(\varphi_1 - w_2\varphi_2)\varphi_2 + 2\varphi_1 - 4(w_2-1)\varphi_2 + 1 = 0;$

4°. $\varphi_1^2 - 2(2w_2 - b)\varphi_1\varphi_2 + 4w_2\varphi_2^2 + 2\varphi_1 - 4w_2\varphi_2 + 1 = 0;$

5°. $(w_1^2+1)\varphi_1^2 + 2w_1w_2\varphi_1\varphi_2 + (w_2^2+1)\varphi_2^2 - 2(w_1\varphi_1 + w_2\varphi_2) + 1 = 0;$

6°. $4w_1\varphi_1^2 + 4a\varphi_1\varphi_2 - \varphi_2^2 - 2\varphi_2 = 1;$ (1.8.3)

7°. $-4w_1\varphi_1^2 + (1 - a^2w_1^{-1})\varphi_2^2 - 2\varphi_2 + 1 = 0;$

8°. $-\varphi_1^2 + (2w_1 + a^2)\varphi_2^2 + a^2\varphi_2 = 0;$

9°. $\varphi_1^2 + 4w_1\varphi_1\varphi_2 + 4w_2\varphi_2^2 + 4\varphi_2 = 0;$

10°. $\alpha\varphi_1^2 + 2\alpha\beta\varphi_1\varphi_2 + \beta^2\varphi_2^2 = 0.$

Here $\varphi_k = \dfrac{\partial\varphi}{\partial w_k}$. Equations 1° – 10° correspond to the reduced Equations (1.8.1) appearing as a result of substitution by ansatz 1°–10° from Table 1.4.1, (1.8.2).

Integrating (1.8.3) we obtain with (1.8.2) the following solutions of Equation (1.8.1) [88]:

$$u(x) = F(\alpha x), \quad \alpha^2 = 0; \tag{1.8.4}$$

$$u(x) = F\left(\frac{\alpha x}{x^2}\right), \quad \alpha^2 = 0; \tag{1.8.5}$$

$$u(x) = F\left(\beta x \pm \sqrt{(\beta x)^2 - x^2}\right), \quad \beta^2 = 1, \tag{1.8.6}$$

where F is an arbitrary function of the mentioned arguments (this arbitrary function appears as a consequence of the property of solutions of Equation (1.8.1) stressed above).

The arbitrary dependence on u of the coefficients æ, $c_{\mu\nu}$, d_ν of the infinitesimal operator (1.2.19) admitted by Equation (1.8.1) permits us to seek solutions of Equation (1.8.1) in the form

$$u = \varphi(v), \tag{1.8.7}$$

where $v = v(x,u) = \{v_1(x,u), v_2(x,u)\}$.

Substituting (1.8.7) into (1.8.1) we have

$$v_{1\nu}v_1^\nu\varphi_{v_1}^2 + 2v_{1\nu}v_2^\nu\varphi_{v_1}\varphi_{v_2} + v_{2\nu}v_2^\nu\varphi_{v_2}^2 = 0. \tag{1.8.8}$$

Owing to the fact that v_1 and v_2 are invariants with respect to some subgroup of the symmetry group of Equation (1.8.1) we can rewrite (1.8.8) in the form

$$A(v)\varphi_{v_1}^2 + 2B(v)\varphi_{v_1}\varphi_{v_2} + C(v)\varphi_{v_2}^2, \tag{1.8.9}$$

where the functions $A(v)$, $B(v)$, $C(v)$ are found from the conditions

$$\frac{v_{1\nu}v_1^\nu}{A(v)} = \frac{v_{1\nu}v_2^\nu}{B(v)} = \frac{v_{2\nu}v_2^\nu}{C(v)}$$

$$\varphi(v) = \Phi(J_1), \quad \varphi(v) = \Phi(J_2),$$

The set of solutions of Equation (1.8.9) is equivalent to the set of solutions of two equations

$$A(v)\varphi_{v_1} + \left[B(v) + \sqrt{B^2(v) - A(v)C(v)}\right]\varphi_{v_2} = 0,$$

$$A(v)\varphi_{v_1} + \left[B(v) - \sqrt{B^2(v) - A(v)C(v)}\right]\varphi_{v_2} = 0,$$

whose general solution may be represented in the form

$$\varphi(v) = \Phi(J_1), \quad \varphi(v) = \Phi(J_2),$$

where J_1 and J_2 are the first integrals of the equations

$$\frac{dv_1}{A(v)} = \frac{dv_2}{B(v) \pm \sqrt{B^2(v) - A(v)C(v)}},$$

It is possible to use for v_1 and v_2 the expressions for w_1 and w_2 from Table 1.4.1; however, in the present case the coefficients α_ν β_ν δ_ν should be considered as arbitrary functions of u. Let us list several solutions of Equation (1.8.1) obtained by this method:

$$\alpha_\nu(u)x^\nu = x^2, \quad \alpha_\nu(u)\alpha^\nu(u) = 0;$$

$$\alpha_\nu(u)x^\nu = x^2 - [\beta_\nu(u)x^\nu]^2, \quad \alpha^2 = \alpha\beta = 0, \quad \beta^2 = 1;$$

$$\phi(x,u) + \beta_\nu(u)x^\nu \ln\phi(x,u) = 0, \quad \beta^2 = -1,$$

where $\phi(x,u) = \beta_\nu(u)x^\nu \pm \sqrt{x^2 + (\beta_\nu(u)x^\nu)^2}$

$$\alpha_\nu(u)x^\nu - \ln\frac{\sqrt{\delta_\nu(u)x^\nu + (\beta_\nu(u)x^\nu)^2} \pm \beta_\nu(u)x^\nu}{(\beta_\nu(u)x^\nu)^2},$$

$$\alpha^2 = \delta = \alpha\beta = \beta\delta = 0, \quad \alpha\delta = 2, \quad \beta^2 = 1;$$

$$(\alpha_\nu(u)x^\nu)^3 + 3\alpha_\nu(u)x^\nu\beta_\sigma(u)x^\sigma + 3\delta_\nu(u)x^\nu \pm [(\alpha_\nu(u)x^\nu)^2 + 2\beta_\nu(u)x^\nu + 1]^{3/2} = 0,$$

$$\alpha^2 = \alpha\beta = \beta\delta = 0, \quad \tfrac{1}{2}\alpha\delta = -\beta^2 = \delta^2 = 1;$$

$$\alpha_\nu(u)x^\nu = x^2 + 1, \quad \alpha_\nu(u)\alpha^\nu(u) = 4.$$

2. The solutions of the relativistic Hamilton equation

$$u_\nu u^\nu \equiv \frac{\partial u}{\partial x^\nu}\frac{\partial u}{\partial x_\nu} = 1 \tag{1.8.10}$$

are sought in the form (1.5.3) when $k = -1$, i.e.,

$$u(x) = \begin{cases} (\alpha y)\varphi(w_1, w_2), & N\ 1^\circ - 5^\circ, \\ \varphi(w_1, w_2), & N\ 6^\circ - 10^\circ \end{cases} \tag{1.8.11}$$

The corresponding variables w_1 and w_2 are listed in Table 1.4.1.

As a result of substituting ansatz $1^\circ - 10^\circ$ from (1.8.11) into (1.8.10) we obtain the following reduced PDEs for the function φ:

$$\begin{aligned}
&1^\circ.\ \ a^2w_1^2\varphi_1^2 - 4w_1(aw_2 - a - 1)\varphi_1\varphi_2 - 4w_2(w_2 - 1)\varphi_2^2 + \\
&\qquad\qquad + 2aw_1\varphi\varphi_1 - 4(w_2 - 1)\varphi\varphi_2 + \varphi^2 = 1; \\
&2^\circ.\ \ \varphi_1^2 + 4\varphi_1\varphi_2 - 4w_2\varphi_2^2 + 4\varphi\varphi_2 = 1; \\
&3^\circ.\ \ [1 - (w_2 - 1)^{-1}]\varphi_1^2 + 4(1 - w_2)\varphi_1\varphi_2 + 4w_2(w_2 - 1)\varphi_2^2 + \\
&\qquad\qquad + 2\varphi\varphi_1 + 4(1 - w_2)\varphi\varphi_2 + \varphi^2 = 1; \\
&4^\circ.\ \ \varphi_1^2 + 2(b - 2w_2)\varphi_1\varphi_2 + 4w_2^2\varphi_2^2 + 2\varphi\varphi_1 - 4w_2\varphi\varphi_2 + \varphi^2 = 1; \\
&5^\circ.\ \ (w_1^2 + 1)\varphi_1^2 + 2w_1w_2\varphi_1\varphi_2 + (w_2^2 + 1)\varphi_2^2 - 2\varphi(w_1\varphi_1 + w_2\varphi_2) + \varphi^2 = 1; \\
&6^\circ.\ \ 4w_1\varphi_1^2 + 4a\varphi_1\varphi_2 - \varphi_2^2 - 2\varphi_2 = 2; \\
&7^\circ.\ \ -4w_1\varphi_1^2 + (1 + a^2w_1^{-1})\varphi_2^2 + 2\varphi_2 = 0; \\
&8^\circ.\ \ -\varphi_1^2 + (2w_1 + a^2)\varphi_2^2 + \varphi_2 = a^{-2}b^{-1}; \\
&9^\circ.\ \ \varphi_1^2 + 4w_1\varphi_1\varphi_2 + 4w_2\varphi_2^2 + 4\varphi_2 = 1; \\
&10^\circ.\ \ \alpha\varphi_1^2 + 2\alpha\beta\varphi_1\varphi_2 + \beta^2\varphi_2^2 = 1.
\end{aligned} \tag{1.8.12}$$

Having determined some partial solutions of Equations (1.8.12) we construct, with the corresponding formulas (1.8.11), the solutions of Equation (1.8.10). Below we list some of these solutions [88, 92]:

$$u(x) = \beta y \sin c + \sqrt{y^2 - (\beta)^2}\cos c, \quad \beta^2 = 1, \quad c = \text{const};$$

$$u(x) = (2c\alpha y)^{-1}(c^2y^2 + (\alpha y)^2), \quad \alpha^2 = 0;$$

$$u(x) = \sqrt{y^2+1} - \ln \frac{\sqrt{y^2+1}+1}{\alpha y}, \quad \alpha^2 = 0;$$

$$u(x) = -\beta y + \ln|(\beta y)^2 - y^2| - 2\arctan\frac{\delta y}{\alpha y}, \alpha^2 = \alpha\beta = \beta\delta = 0, \alpha\delta = 0;$$

$$u(x) = \left[(\alpha y)^2 + 2a(\beta y)^+ a^2\right]^{3/2} + (\alpha y)^3 + 3a(\alpha y)(\beta y) + 3a^2(\delta y) + \alpha y,$$

where $\alpha^2 = \alpha\beta = \beta\delta = 0, \alpha\delta = -\beta^2 = \delta = (6a^2)^{-1}$;

$$u(x) = F(\alpha y) + \beta y,$$

where $\alpha^2 = \alpha\beta = 0$, $\beta^2 = 1$, and F is an arbitrary differentiable function;

$$u(x) = \sqrt{(\alpha y)^2 + y^2}, \qquad \alpha^2 = -1;$$

$$u(x) = \sqrt{y_0^2 + (\vec{\alpha}\vec{y})^2}, \qquad \vec{\alpha}^2 = 1;$$

$$u(x) = \sqrt{y^2}$$

It is possible to apply the generating formulas (see §1.4) to the family of solutions (1.8.13) in such a way as to construct new families of solutions for Equation (1.8.10). Let us adduce two such solutions. For this purpose we write down the finite transformations generated by operators K_A (1.3.2) [92]:

$$x'_\mu = \frac{x_\mu - c_\mu s^2}{1 - 2cx + 2c_3u + c^A c_A s^2}, \qquad u'(x') = \frac{u(x) - c_3 s^2}{1 - 2cx + 2c_3u + c^A c_A s^2} \tag{1.8.14}$$

where $\mu, \nu = 0, 1, 2$; $s^2 = x^2 - u^2$, $A = 0, 1, 2, 3$; $x_3 \equiv u$. We take as initial solutions $u_I(x)$ the simplest of the solutions of the set (1.8.13)

$$u_I(x) = \beta x, \quad \beta^2 = 1$$

and

$$u_I(x) = \sqrt{x^2}$$

Solving the functional Equation (1.4.14) for the transformation (1.8.14) we have correspondingly [92]

$$u(x) = (2a)^{-1}\left(-1 \pm \sqrt{1 + 4a\beta x + 4a^2x^2}\right), \qquad a = c_3 - \beta c \neq 0, \quad \beta^2 = 1; \tag{1.8.15}$$

$$u(x) = (c^A c_A)^{-1} \left[c_3 \pm \left(c_3 + c^A c_A - 2c_A c^A (cx) + (c_A c^A)^2 x^2 \right)^{1/2} \right], \quad c_A c^A \neq 0, \quad A = 0,1,2,3. \tag{1.8.16}$$

A solution of Equation (1.8.10) may be also be sought in implicit form

$$v_1 = \varphi(v) \tag{1.8.17}$$

Let us consider several simple cases.

a).
$$v_1(x,u) = \beta_A x^A, \quad v(x,u) = \alpha_A x^A. \tag{1.8.18}$$

Substitution of (1.8.17) and (1.8.18) into (1.8.10) gives

$$\alpha_A \alpha^A \varphi_v^2 - 2\alpha_A \beta^A \varphi_v + \beta_A \beta^A = 0$$

and if $\alpha_A \alpha^A = \alpha_A \beta^A = \beta_A \beta^A = 0$ then

$$\beta_A x^A = \varphi(\alpha_A x^A). \tag{1.8.19}$$

Formula (1.8.19) defines the solution of Equation (1.8.10) with an arbitrary differentiable function φ.

b).
$$v_1(x,u) = x_A x^A, \quad v(x,u) = \alpha_A x^A.$$

In this case the equation for the function $\varphi(v)$ has the form

$$\alpha_A \alpha^A \varphi_v^2 - 4v\varphi_v + 4v = 0.$$

Integrating this equation and returning to (1.8.17) we obtain the solution of Equation (1.8.10):

$$x_A x^A - 2\alpha_A x^A + \alpha_A \alpha^A = 0.$$

c).
$$v_1(x,u) = \frac{x_A x^A}{\beta_A x^A}, \quad v(x,u) = \alpha_A x^A; \quad \alpha_A \alpha^A = \alpha_A \beta^A = 0, \quad \beta_A \beta^A = 1.$$

The equation for the function $\varphi(v)$ has the form

$$v\varphi_v + \varphi = \varphi^2.$$

Integrating this equation we obtain the solution of Equation (1.8.10)

$$\frac{(\beta_A x^A)^2}{x_A x^A} = \alpha_A x^A + 1.$$

Solutions of Equation (1.2.17) are obtained from solutions of (1.8.10) by changing in the latter the signature (1,-1,-1) to (1,-1,1).

The generalization of the solutions obtained here in the case of an arbitrary number of independent variables is quite evident.

1.9. *Symmetry and exact solutions of the Euler-Lagrange-Born-Infeld equation*

The equation

$$L(u) \equiv (1 - u_\nu u^\nu)\Box u + u^\mu u^\nu u_{\mu\nu} = 0, \qquad (1.9.1)$$

where $u = u(x)$, $x = (x_0, \ldots, x_{n-1})$, and

$$u_\mu \equiv \frac{\partial u}{\partial x_\mu}, \quad u_{\mu\nu} \equiv \frac{\partial^2 u}{\partial x_\mu \partial x_\nu}, \quad u^\mu = g^{\mu\nu} u_\nu, \; g^{\mu\nu} = g_{\mu\nu} = (1, -1, \ldots, -1)\, \delta_{\mu\nu}$$

which we considered in §1.1 while solving the problem of group classification of second-order PDEs invariant under the group P(1,n). Equation (1.9.1) in Euclidean space generalizes the n-dimensional case of the equation of minimal surfaces which was first obtained by Lagrange from the Euler-Lagrange variational principle.

In the present paragraph we shall determine the maximal (in Lie's sense) invariance group of Equation (1.9.1) and then obtain some of its solutions.

Theorem 1.9.1. [87] *The maximal local invariance group of Equation* (1.9.1) *is the extended Poincare group* $\widetilde{\mathrm{P}}(1, n)$

Proof. From the invariance condition

$$\underset{2}{X} L(u) \Big|_{L(u)=0} = 0$$

where $\underset{2}{X}$ is written in (1.1.7), we obtain the defining equations for the coordinates $\xi^\mu(x, u)$ and $\eta(x, u)$:

$$\partial_b \xi^a + \partial_a \xi^b = 0, \quad a \neq b; \quad \partial_a \xi^0 = \partial_0 \xi^a,$$

$$\partial_u \xi^0 = \partial_0 \eta, \quad \partial_u \xi^a + \partial_a \eta = 0,$$

$$\partial_0 \xi^0 = \partial_1 \xi^1 = \cdots = \partial_{n-1} \xi^{n-1} = \partial_u \eta$$

whose general solution has the form

$$\xi^A = c^{AB} x_B + d^A, \quad A = 0, 1, 2, \ldots, n; \quad x_n \equiv u,$$

$$c^{AB} = -c^{BA}, \quad d^A = \mathrm{const}$$

and establishes the basis of the algebra $\widetilde{\mathrm{AP}}(1,n)$ operators. Thus, the theorem is proved.

Let us seek the exact solutions of Equation (1.9.1). We consider first the case where $n=2$. By the way, let us note that the particular class of exact solutions of the two-dimensional Equation (1.9.1) had been obtained by Barbashov and Chernikov (see [17]); these solutions, as shown in [207], may be found by the godograph method.

Following §1.4, we seek solutions for Equation (1.9.1) in the form

$$u(x) = f(x)\varphi(w) + g(x) \tag{1.9.2}$$

There are several cases [87].

a). $w = \alpha x,\ f(x) = 1,\ g(x) = \beta x; \quad \alpha_\nu, \beta_\nu$ – const; $\nu = 0, 1$

In this case, for the function $\varphi(w)$ we obtain the equation

$$\left[\alpha^2 + (\alpha_0\beta_1 - \alpha_1\beta_0)^2\right]\ddot{\varphi} = 0 \tag{1.9.3}$$

If $\alpha^2 + (\alpha_0\beta_1 - \alpha_1\beta_0)^2 \neq 0$, then the solution of (1.9.3) is a linear function, and consequently

$$u(x) = a_\nu x^\nu + c, \tag{1.9.4}$$

where a_ν and c are arbitrary constants. When $\alpha^2 + (\alpha_0\beta_1 - \alpha_1\beta_0)^2 = 0$, the solution of Equation (1.9.1) has the form

$$u(x) = \varphi(\alpha x) + \beta x \tag{1.9.5}$$

where φ is an arbitrary twice-differentiable function. The solution to (1.9.5) with $\beta_0 = \beta_1 = 0$ was also obtained in [17].

b). $w = x^2 \equiv x_0^2 - x_1^2, \quad f(x) = 1, \quad g(x) = a\ln(x_0 + x_1).$ (1.9.6)

In this case the ansatz (1.9.2) reduced (1.9.1) to the ODE

$$(w + a^2)\ddot{\varphi} - 2w\dot{\varphi}^3 - 3a\dot{\varphi}^2 + \dot{\varphi} = 0$$

whose general solution has the form

$$\varphi(w) = \begin{cases} \frac{1}{2}\ln\left[c\left(\frac{b\sqrt{w+a^2} - a\sqrt{w+b^2}}{bw\sqrt{w+a^2} + aw\sqrt{w+b^2}}\right)^a\left(\frac{\sqrt{w+a^2}+\sqrt{w+b^2}}{\sqrt{w+a^2}-\sqrt{w+b^2}}\right)^b\right], \\ \frac{a}{2}\ln\left[c\frac{b\sqrt{w+a^2}-a\sqrt{w+b^2}}{bw\sqrt{w+a^2}+aw\sqrt{w+b^2}}\right] + b\arctan\frac{\sqrt{w+a^2}}{\sqrt{b^2-w}}, \\ \sqrt{w+a^2} - a\ln\left[c(\sqrt{w+a^2}+a)\right]. \end{cases}$$

Whence, using (1.9.6) and (1.9.2), we obtain the solution of Equation (1.9.1):

$$u(x) = \begin{cases} \frac{1}{2}\ln\left\{c\left(\frac{x_0+x_1}{x_0-x_1}\right)^a \operatorname{th}^a\left[\frac{1}{4}\ln\left(\frac{a^2}{b^2}\frac{x^2+b^2}{x^2+a^2}\right)\right] \operatorname{cth}^b\left(\frac{1}{4}\ln\frac{x^2+b^2}{x^2+a^2}\right)\right\}, \\ \frac{1}{2}\ln\left\{c\left(\frac{x_0+x_1}{x_0-x_1}\right)^a \operatorname{th}^a\left[\frac{1}{4}\ln\left(\frac{a^2}{b^2}\frac{x^2+b^2}{x^2+a^2}\right)\right] + b\arctan\sqrt{\frac{a^2+x^2}{b^2-x^2}}\right\} \\ a\ln\left(c\frac{x_0+x_1}{\sqrt{x^2+a^2}+a}\right) + \sqrt{x^2+a^2}, \end{cases} \tag{1.9.7}$$

where a, b, and c are arbitrary constants.

c).
$$w = \frac{x_1}{x_0}, \quad f(x) = x_0, \quad g(x) = c = \text{const.} \tag{1.9.8}$$

Substitution of (1.9.8) and (1.9.2) into (1.9.1) gives the equation

$$(\varphi^2 + w^2 - 1)\ddot{\varphi} = 0$$

by integration we get the solutions of Equation (1.9.1) in the form (1.9.4) and

$$u(x) = \pm\sqrt{x_0^2 - x_1^2} + c \tag{1.9.9}$$

d).
$$w = x_0 + x_1, \quad f(x) = \sqrt{x_0 - x_1}, \quad g(x) = c = \text{const} \tag{1.9.10}$$

The corresponding equation for the function $\varphi(w)$ has the form

$$\varphi^2\ddot{\varphi} - 3\varphi\dot{\varphi}^2 + 2\dot{\varphi} = 0. \tag{1.9.11}$$

Changing variables in (1.9.11),

$$\dot{\varphi} = \phi(y), \quad y = \varphi \tag{1.9.12}$$

we obtain the linear equation

$$y^2\dot{\phi} - 3y\phi + 2 = 0$$

whose general solutions are given by the formula

$$\phi = \frac{c_1 y^4 + 1}{2y}.$$

Using this result and formulas (1.9.12), (1.9.10), and (1.9.2) we construct the solutions of Equation (1.9.1):

$$u(x) = \begin{cases} \pm\left[\frac{x_0-x_1}{c_1}\operatorname{th}\left(c_1(x_0+x_1)+c_2\right)\right]^{1/2} + c_3, \\ \pm\left[\frac{x_0-x_1}{c_1}\operatorname{cth}\left(c_1(x_0+x_1)+c_2\right)\right]^{1/2} + c_3, \\ \pm\left[\frac{x_0-x_1}{c_1}\tan(c_1(x_0^2+x_1^2)+c_2)\right]^{1/2} + c_3, \\ \pm\left[x_0 - x_1 + c_2(x_0-x_1)\right]^{1/2} + c_3, \end{cases} \tag{1.9.13}$$

where c_1, c_2, c_3 are arbitrary constants.

e). $w = x_0 - x_1 + a, \qquad \ln(x_0 + x_1), \qquad f(x) = \sqrt{x_0 + x_1},$

$$g(x) = c = \text{const} \tag{1.9.14}$$

Substituting (1.9.14) and (1.9.2) into (1.9.1) results in the ODE

$$(\varphi + 4a)\ddot{\varphi} - 4a\dot{\varphi}^3 - 3\varphi\dot{\varphi}^2 + 2\dot{\varphi} = 0$$

which, by transformation (1.9.12), may be transformed into the Riccati equation:

$$(y^2 + 4a)\dot{\phi} - 4a\phi^2 - 3y\phi + 2 = 0 \tag{1.9.15}$$

The general solution of Equation (1.9.15) is

$$\phi(y) = \frac{2a - y(c_1\sqrt{y^2 + 4a} - y)}{2a(c_1\sqrt{y^2 + 4a} - y)}.$$

Substituting this expression into (1.9.12) we obtain the following equation:

$$\frac{d\varphi}{dw} = \frac{2a - \varphi\left(c_1\sqrt{\varphi^2 + 4a} - \varphi\right)}{2a\left(c_1\sqrt{\varphi^2 + 4a} - \varphi\right)},$$

and integrating this we obtain, via (1.9.14) and (1.9.2), solutions for Equation (1.9.1):

$$u(x) = \pm\left[c_2 e^{c_3(x_0 - x_1)} + \frac{2}{c_3}(x_0 + x_1)\right]^{1/2} + c_4, \quad c_1 = 0;$$

$$c_2 = \left(u - \sqrt{u^2 + 4a(x_0 + x_1)}\right)^{-1} \times$$

$$\times \exp\left[\frac{u}{u - \sqrt{u^2 + 4a(x_0 + x_1)}} + \frac{x_0 + x_1}{2a}\right], \quad c_1 = 1;$$

$$c_2 = \frac{(v-1)^s (v+1)^{1/s} (v - k_+)^{b_-} (v - k_-)^{b_+}}{(x_0 + x_1)\exp(\frac{x_0 + x_1)}{a})}, \quad c_1^2 > 0; \tag{1.9.16}$$

$$c_2 = \frac{(v-1)^s (v+1)^{1/s} \left(v^2 - 2c_1 v + 1\right)^{(s^2+1)/2s}}{(x_0 + x_1)\exp\left\{\frac{2c_1}{\sqrt{1-c_1^2}}\arctan\frac{v - c_1}{\sqrt{1-c_1^2}} + \frac{x_0 - x_1}{a}\right\}}, \quad c_1^2 - 1 < 0,$$

where $v = u[u^2 + 4a(x_0 + x_1)]^{-1/2}, \quad s = (1 - c_1)(1 + c_1)^{-1},$

and $b_{\pm} = \dfrac{(c_1^2+1)(c_1^2-1) \pm (c_1^3-1)}{(c_1^2-1)^{3/2}}, \quad k_{\pm} = c_1 \pm \sqrt{c_1^2-1}.$

Let us consider Equation (1.9.1) when $n \geq 3$. As in the two-dimensional case we seek solutions in the form (1.9.2). Let

$$u(x) = (\beta x)\varphi(w), \quad w = \frac{(\alpha x)^2}{\beta x}, \quad \alpha^2 = \alpha\beta = 0, \quad \beta^2 \neq 0 \qquad (1.9.17)$$

where a, α_ν β_ν are arbitrary constants satisfying the indicated conditions. Substituting (1.9.17) into (1.9.1) results in the ODE

$$\ddot{\varphi} = 0$$

whence we obtain, if we return to (1.9.17), the solution of Equation (1.9.1)

$$u(x) = (\alpha x)^2 + \beta x, \quad \alpha^2 = \alpha\beta = 0, \quad \beta^2 \neq 0 \qquad (1.9.18)$$

One can directly verify that

$$u(x) = (\beta x)\varphi(\alpha x), \quad \alpha^2 = \alpha\beta = 0, \quad \beta^2 \neq 0 \qquad (1.9.19)$$

where φ is an arbitrary twice-differentiable function, is also a solution of Equation (1.9.1).

Let

$$u(x) = \varphi(w) + \beta x, \quad w = \alpha x,$$

where α_ν, β_ν are arbitrary constants. Substituting this ansatz into (1.9.1) results in the solution

$$u(x) = \varphi(\alpha x) + \beta x, \quad (\alpha x)^2 + \alpha^2(1-\beta^2) = 0 \qquad (1.9.20)$$

with an arbitrary twice-differentiable function φ.

Let us consider the ansatz

$$u(x) = \varphi(w), \quad w = x^2$$

For the function $\dot{\varphi}(w)$ from (1.9.1) follows the Bernoulli equation

$$2w\ddot{\varphi} + n\dot{\varphi} - 4(n-1)w\dot{\varphi}^3 = 0$$

integration of which we yields the solution of Equation (1.9.1)

$$u(x) = c_1 \int_0^{\sqrt{x^2}} \frac{d\tau}{\sqrt{1 + c_2\tau^{2n-2}}} \qquad (1.9.21)$$

where c_1, c_2 are constants. When $c_1 = 1$ or $c_2 = 0$ we find from (1.9.21)

$$u(x) = \sqrt{x^2}$$

Owing to the fact that Equation (1.9.1) is invariant under transformations from the group P$(1, n)$, where the variables $x_0, x_1, \ldots, x_{n-1}, u$ enter equally, solutions may be sought in implicit form. If, for example, we use the invariants

a). $$w_1 = \alpha_A x^A \equiv \alpha x - \alpha_n u, \quad w_2 = \beta_A x^A \equiv \beta x - \beta_n u, \tag{1.9.22}$$

or

b). $$w_1 = x_A x^A \equiv x^2 - u^2, \quad w_2 = \beta_A x^A$$

where α_A, β_A are arbitrary constants, and the substitution

$$w_1 = \phi(w_2) \tag{1.9.23}$$

then in case a) we obtain the solution

$$\alpha_A x^A = \phi(\beta_A x^A), \quad (\alpha_A \alpha^A)(\beta_A \beta^A) - (\alpha_A \beta^A)^2 = 0 \tag{1.9.24}$$

In case b) (1.9.22) the substitution (1.9.23) reduces Equation (1.9.1) to the ODE

$$2(w_2^2 - \beta^2 \phi)\ddot{\phi} + n(4\phi - 4w_2\dot{\varphi} + \beta_A \beta^A \dot{\phi}^2) = 0 \tag{1.9.25}$$

where n is the number of independent variables x.

The substitution

$$\beta_A \beta^A \phi(w_2) = \psi(w_2) + w_2^2, \quad \beta_A \beta^A \neq 0$$

in (1.9.25) yields the equation

$$2\psi\ddot{\psi} - n\dot{\psi}^2 - 4(n-1)\psi = 0$$

whose general solution is given by the formula

$$\int_0^{\sqrt{\psi}} \frac{d\tau}{\sqrt{c_1 \tau^{2n-2} - 1}} = w_2 + c_2 \text{ or } \psi = 0.$$

From here we obtain the solution of Equation (1.9.1) in the form of the functional relation

$$(\beta_A x^A)^2 - (\beta_A \beta^A) x_A x^A + \phi(\beta_A x^A) = 0, \quad \beta_A \beta^A \neq 0. \tag{1.9.26}$$

If $\beta_A\beta^A = 0$ then the substitutions (1.9.23) and (1.9.22) reduce Equation (1.9.1) to the linear Euler equation

$$w_2^2\ddot{\phi} - 2nw_2\dot{\phi} + 2n\phi = 0$$

whose solution is

$$\phi = c_1w_2 + c_2w_2^{2n}$$

The corresponding solution of Equation (1.9.1) is found from the algebraic relation

$$x_Ax^A = c_1\beta_Ax^A + c_2(\beta_Ax^A)^{2n}, \quad \beta_A\beta^A = 0.$$

1.10. *Symmetry and exact solutions of the Monge-Ampere equation*

In this section we study the symmetry and construct several classes of solutions of the multi-dimensional MA equation

$$\det(u_{\mu\nu}) = 0, \quad u_{\mu\nu} \equiv \frac{\partial^2 u}{\partial x_\mu \partial x_\nu}; \quad \mu,\nu = \overline{0, n-1} \tag{1.10.1}$$

and also make group classifications of equations like

$$\det(u_{\mu\nu}) = F(x, u, \underset{1}{u}), \quad \underset{1}{u} = \{u_\mu\}. \tag{1.10.2}$$

The MA equation was generalized in the n-dimensional case by Pogorelov and had been used by him to solve the multi-dimensional Minkowsky problem [165]. Nowadays Equation (1.10.1) is widely used in quantum field theory.

As shown in §1.1, Equation (1.10.1) is invariant under the group P(1,n). More complete information on its symmetry gives the following statement.

Theorem 1.10.1. [86]. *The maximal local invariance group of the MA equation* (1.10.1) *is the group* G = {IGL($n+1, R$),C($n+1$)}, *containing the group of general linear inhomogeneous transformations* IGL($n+1, R$) *of the space* $\mathrm{R}^{n+1} = \mathrm{R}^n(x) \times \mathrm{R}^1(u)$ *and the group of conformal transformations* C($n+1$). *The basis elements of the corresponding Lie algebra have the form*

$$\begin{aligned} &P_A = \frac{\partial}{\partial x_A}, \quad \tilde{J}_{AB} = x_AP_B; \quad A,B = \overline{0,n}; \\ &K_A = x_Ax_BP_B; \quad (x_n \equiv u). \end{aligned} \tag{1.10.3}$$

Proof. From the invariance condition (1.1.6)

$$\underset{2}{X}\det(u_{\mu\nu})\Big|_{\det(u_{\mu\nu})=0} = 0$$

where the operator $\underset{2}{X}$ is given in (1.1.7), we find the system of defining equations for the coordinates $\xi^\mu(x,u)$, $\eta(x,u)$ of the infinitesimal operator $X = \xi^\mu\partial_\mu + \eta\partial_u$:

$$\eta_{\mu\nu} = \xi^0_{uu} = 0, \quad 2\xi^\mu_{u\nu} = \delta_{\mu\nu}\eta_{uu}$$
$$\xi^0_{\mu\nu} = \delta_{\mu 0}\eta_{\nu u} + \delta_{\nu 0}\eta_{\mu u}$$

The general solution of this system is rather easy to derive and has the form

$$\xi^A = x^A c^B x_B + c^{AB} x_B + d^A, \quad \xi^n \equiv \eta.$$

The solution yields the formulae (1.10.3). The theorem is proved.

As the group of general linear transformations IGL contains as subgroups the Galilei group and the Lorentz group, it is possible to state that for the MA Equation (1.10.1) both the relativity principles of Galilei and Lorentz-Poincare-Einstein are true. The relativistic or the nonrelativistic symmetry of the MA equation can be singled out as follows.

Let us consider Equation (1.10.2).

Theorem 1.10.2. [86] *Equation* (1.10.2) *is invariant under algebra* AP(1, *n*)

$$P_A = \frac{\partial}{\partial x^A}, \quad J_{AB} = x_A P_B - x_B P_A \tag{1.10.4}$$

iff

$$F(x,u,\underset{1}{u}) = \lambda(1 - u_\nu u^\nu)^{\frac{n+2}{2}}, \quad \lambda = \text{const.} \tag{1.10.5}$$

Equation (1.10.2) *is invariant under the Galilei algebra* AG(2, *n* − 1)

$$\begin{aligned} &P_A = \frac{\partial}{\partial x^A}, \; J_{AB} = x_a P_b - x_b P_a; \; G_{1a} = x_0 P_a + m x_a P_n, \\ &G_{2a} = x_n P_a + m x_a P_0; \quad a, b = \overline{1, n-1} \end{aligned} \tag{1.10.6}$$

iff

$$F(x,u,\underset{1}{u}) = \lambda(u_0 + \frac{u_a u_a}{2m})^{\frac{n+2}{2}}; \quad \lambda, m = \text{const} \tag{1.10.7}$$

Proof. Necessity. As in the proof of the previous theorem we start from the invariance condition (1.10.8)

$$\underset{2}{X}\left[\det(u_{\mu\nu}) - F(x,u,\underset{1}{u})\right]\Big|_{\det(u_{\mu\nu}) = F} = 0 \tag{1.10.8}$$

where the operator $\underset{2}{X}$ is constructed via the formulae (1.1.7), ξ^μ and η are defined in (1.10.4) for the Poincare group or in (1.10.6) for the Galilei group. For the function $F(x,u,\underset{1}{u})$ from (1.10.8) we obtain the equation

$$\xi^\mu\left(\frac{\partial F}{\partial u_\mu}+\frac{\partial F}{\partial x_\mu}\right)+\eta\frac{\partial F}{\partial u}+$$
$$+\left[2\left(\frac{\partial \xi^\mu}{\partial x^\mu}+u_0\frac{\partial \xi^0}{\partial u}\right)-n\left(\eta_u-u_0\frac{\partial \xi^0}{\partial u}\right)\right]F=0 \quad (1.10.9)$$

In the case of the algebra (1.10.4) it follows that

$$F(x,u,\underset{1}{u})=F(w),\quad w=u_\nu u^\nu,$$

$$2(1-w)\frac{dF}{dw}+(n+2)F=0.$$

The general solution of these relations is given in (1.10.5). Analogously, for the set of operators (1.10.6) we obtain from (1.10.9)

$$F(x,u,\underset{1}{u})=F(W),\quad W=u_0+\frac{1}{2m}u_a u^a,$$

$$2W\frac{dF}{dW}+(n+2)F=0$$

whence follows the formula (1.10.7).

Sufficiency. It is easy to confirm directly that Equation (1.10.2) with the function F from (1.10.5) and (1.10.7) is invariant under the groups P(1,n) and G(2,$n-1$), respectively. The theorem is proved.

Let us note that the equation

$$\det(u_{\mu\nu})=\lambda(1-u_\nu u^\nu)^{\frac{n+2}{2}}$$

is no longer Galilei-invariant and the equation

$$\det(u_{\mu\nu})=\lambda\left(u_0+\frac{u_a u_a}{2m}\right)^{\frac{n+2}{2}} \quad (1.10.10)$$

is no longer Lorentz-invariant.

We seek the solutions of Equation (1.10.1) in the form [86]

$$u(x)=\varphi(w) \quad (1.10.11)$$

where $w=w(x)$ is some differentiable function. Ansatz (1.10.11) reduces (1.10.1) to the linear ODE

$$M(w)\ddot\varphi+N(w)\dot\varphi=0, \quad (1.10.12)$$

where $N(w)=\det(w_{\mu\nu})$,

$M(w) =$

$$= \begin{vmatrix} w_0^2 & w_{01} & \dots & w_{0\,n-1} \\ w_0 w_1 & w_{11} & \dots & w_{1\,n-1} \\ \dots & \dots & \dots & \dots \\ w_0 w_{n-1} & w_{n-1\,1} & \dots & w_{n-1\,n-1} \end{vmatrix} + \begin{vmatrix} w_{00} & w_0 w_1 & \dots & w_{0\,n-1} \\ w_{10} & w_1^2 & \dots & w_{1\,n-1} \\ \dots & \dots & \dots & \dots \\ w_{n-1\,0} & w_{n-1} w_1 & \dots & w_{n-1\,n-1} \end{vmatrix} +$$

$$+ \dots + \begin{vmatrix} w_{00} & w_{01} & \dots & w_0 w_{n-1} \\ w_{10} & w_{11} & \dots & w_1 w_{n-1} \\ \dots & \dots & \dots & \dots \\ w_{n-1\,0} & w_{n-1\,1} & \dots & w_{n-1}^2 \end{vmatrix} \equiv \frac{d}{d\tau} N(w) \Big|_{\frac{d}{d\tau} w_{\mu\nu} \overset{\text{def}}{=} w_\mu w_\nu}$$

The proof of this statement can be carried out by the method of mathematical induction. In the two-dimensional case, using direct calculation, we obtain

$$\det(\ddot{\varphi} w_\mu w_\nu + \dot{\varphi} w_{\mu\nu}) = \dot{\varphi}\ddot{\varphi}(w_0^2 w_{11} + w_1^2 w_{00} - 2w_0 w_1 w_{01}) + \dot{\varphi}^2 (w_{00} w_{11} - w_{01}^2) \equiv$$
$$\equiv \dot{\varphi}\left[\dot{\varphi} \det(w_{\mu\nu}) + \ddot{\varphi}(w_0^2 w_{11} + w_1^2 w_{00} - 2w_0 w_1 w_{01})\right]$$

The calculations for $n > 2$ are carried out analogously. Let

$$w = \alpha x, \quad \alpha_\mu = \text{const} \tag{1.10.13}$$

In this case it follows from (1.10.2) that the solution of Equation (1.10.1) is an arbitrary differentiable function φ, i.e.,

$$u(x) = \varphi(\alpha x) \tag{1.10.14}$$

Solution (1.10.14) can be generalized. It is easy to verify that the function

$$u(x) = \varphi(w_1, w_2, \dots, w_{n-1}) \tag{1.10.15}$$

where $w_k = \alpha_\nu^{(k)} x^\nu$, $\alpha_\nu^{(k)}$ are arbitrary constants and $k = \overline{1, n-1}$ satisfies (1.10.1).

Let us put $w = x^2$ in (1.10.11). Then

$$N(w) = 2^n, \quad M(w) = 2^{n+1} w$$

and Equation (1.10.12) takes the form

$$2w\ddot{\varphi} + \dot{\varphi} = 0 \tag{1.10.16}$$

The general solution of (1.10.16) is given by the expression

$$\varphi(\omega) = c_1 \sqrt{\omega} + c_2, \qquad (c_1, c_2) = \text{const}.$$

Thus we obtain the solution of Equation (1.10.1)

$$u(x) = c_1\sqrt{x_\nu x^\nu} + c_2$$

Let us list some more families of solutions for Equation (1.10.1) in explicit and implicit form:

$$\begin{aligned} &u(x) = (\alpha x)^2 - \alpha^2 x^2, \\ &u(x) = x^2/\alpha x, \\ &u^2 - x^2 = c(\alpha_n u - \alpha x)^2, \\ &\alpha x - \alpha_n u = \varphi(\beta x - \beta_n u). \end{aligned} \tag{1.10.17}$$

In the formulae (1.10.17) $c, \alpha_\nu, \beta_\nu, \alpha_n, \beta_n$ are arbitrary constants, with $\alpha x \equiv \alpha_\nu x^\nu$, $\nu = \overline{0, n-1}$, and φ is an arbitrary twice-differentiable function.

1.11.* *Symmetry of the scalar wave equation with interaction*

The standard approach of describing interaction of scalar particle (spin $s = 0$) with external electromagnetic field consists in the following: one has to substitute $\pi_\mu = \partial_\mu - eA_\mu$ (A_μ is the vector-potential of the electromagnetic field) for ∂_μ in the equation of motion of a free scalar particle. As a result one gets the equation

$$\begin{aligned} &\pi_\mu \pi^\mu u = m^2 u, \quad \text{that is} \\ &\partial_\mu \partial^\mu u - eu\partial_\mu A^\mu - 2eA_\mu \partial^\mu u + e^2 A_\mu A^\mu u = m^2 u. \end{aligned} \tag{1.11.1}$$

Let us generalize this equation as follows

$$\partial_\mu \partial^\mu u + \lambda_1 u \partial_\mu A^\mu + \lambda_2 A_\mu \partial^\mu u + \lambda_3 A_\mu A^\mu u = m^2 u, \tag{1.11.2}$$

where λ_1, λ_2, λ_3 are arbitrary constants.

Equations (1.11.1), (1.11.2) are invariant neither under the Poincare group P(1,3) nor under the Lorentz group O(1,3), the vector-potential A_μ being given as arbitrary functions of x. However, if we treat these equations as nonlinear ones, the vector-potential being considered as an arbitrary vector field equal in rights with the scalar field u, we get that the equations possess nontrivial symmetry. From this point of view, Equations (1.11.2) actually mean an infinite set of Equations (1.11.2) for different A_μ [59*]. The most physically interesting case, when $\lambda_2 = 2\lambda_1 = -2e$, $\lambda_3 = e^2$, is selected from the set of Equations (1.11.2) due to its symmetry properties.

* This section is written in collaboration with R.Z.Zhdanov.

Let us consider first the case $m = 0$. The symmetry operators are looked for in the form

$$Q = \xi^{\mu}(x,u,A)\frac{\partial}{\partial x^{\mu}} + \eta(x,u,A)\frac{\partial}{\partial u} + \eta^{\mu}(x,u,A)\frac{\partial}{\partial A^{\mu}}. \qquad (1.11.3)$$

Theorem 1.11.1 *Depending on* λ_1, λ_2, λ_3 *Equation* (1.11.2) *with* $m = 0$ *is invariant under the following Lie algebras:*

1) if

$$\begin{gathered} \lambda_1 \neq 0, \quad \lambda_3 \neq 0, \quad \lambda_2^2 - 4\lambda_3 = 0, \\ (\lambda_1\lambda_2 - 2\lambda_3)^2 + (\lambda_2 - 2\lambda_1)^2 \neq 0, \end{gathered} \qquad (1.11.4)$$

then IA is $\{AP(1,n), A_1^{\infty}\}$ with elements

$$\begin{aligned} &P_{\mu} = \partial_{\mu} \equiv \frac{\partial}{\partial x^{\mu}}, \quad J_{\mu\nu} = x_{\mu}P_{\nu} - x_{\nu}P_{\mu} + S_{\mu\nu}, \\ &D = x^{\nu}P_{\nu} - A^{\nu}\partial_{A^{\nu}} + ku\partial_u \\ &\quad (S_{\mu\nu} \equiv A_{\mu}\partial_{A^{\nu}} - A_{\nu}\partial_{A^{\mu}}) \end{aligned} \qquad (1.11.5)$$

and

$$Q = \lambda_2 F(x)u\partial_u - 2\frac{\partial F}{\partial x_{\mu}}\frac{\partial}{\partial A^{\mu}}, \qquad (1.11.6)$$

where $F(x)$ is an arbitrary solution of the wave equation $\Box F = 0$, k is an arbitrary constant;

2) if

$$\lambda_1 = -e, \quad \lambda_2 = -2e, \quad \lambda_3 = e^2, \quad e \neq 0 \qquad (1.11.7)$$

then IA is the conformal algebra AC(1,n) with basis elements (1.11.5), $k = (1-n)/2$ and

$$K_{\mu} = 2x_{\mu}D - x^2P_{\mu} + 2S_{\mu\nu}x^{\nu}, \qquad (1.11.8)$$

and A_1^{∞} generated by operators

$$Q = eF(x)u\partial_u + \frac{\partial F}{\partial x_{\mu}}\frac{\partial}{\partial A^{\mu}} \qquad (1.11.9)$$

where $F = F(x)$ is an arbitrary smooth function;

3) if

$$\lambda_1 \neq 0, \quad \lambda_3 \neq 0, \quad \lambda_2^2 - 4\lambda_3 \neq 0, \qquad (1.11.10)$$

then IA is AC(1,n) with basis elements (1.11.5), where

$$k = (1-n)\frac{\lambda_1\lambda_2 - 2\lambda_3}{\lambda_2^2 - 4\lambda_3}, \text{ and}$$

$$\widetilde{K}_\mu = K_\mu + \lambda\partial_{A^\mu}, \quad E = u\partial_u, \tag{1.11.11}$$

$$\left(\lambda = 2(1-n)(\lambda_2 - 2\lambda_1)/(\lambda_2^2 - 4\lambda_3)\right)$$

4) if

$$\lambda_1 \neq 0, \quad \lambda_2 = \lambda_3 = 0, \tag{1.11.12}$$

then IA is $\mathrm{A}\widetilde{\mathrm{P}}(1,n)$ (1.11.5) and A_3^∞ generated by operators

$$Q_1 = a(x)u\partial_u - \frac{2}{\lambda_1}\frac{\partial a}{\partial x_\mu}\ln u\frac{\partial}{\partial A^\mu}, \tag{1.11.13}$$

$$Q_2 = F^\mu(x)\frac{\partial}{\partial A^\mu}, \quad Q_3 = \partial_u + \frac{1}{u}A^\mu\frac{\partial}{\partial A^\mu},$$

where a, F^μ satisfy conditions

$$\Box a = 0, \quad \partial_\mu F^\mu(x) = 0;$$

5) if

$$\lambda_1 \neq 0, \quad \lambda_2 \neq 0, \quad \lambda_3 = 0, \tag{1.11.14}$$

then IA is AC(1,n) with basis elements (1.11.5), where $k = (1-n)\lambda_1/\lambda_2$, (1.11.11) with $\lambda = 2(1-n)\lambda_2^{-2}(\lambda_2 - 2\lambda_1)$ and A_1^∞ generated by operators

$$Q = u^{-\lambda_2/\lambda_1}F^\mu(x)\frac{\partial}{\partial A^\mu}, \qquad \partial_\mu F^\mu = 0; \tag{1.11.15}$$

6) if

$$\lambda_1 = \lambda_2 = 0, \qquad \lambda_3 \neq 0, \tag{1.11.16}$$

then IA is AC(1,n) with basis elements (1.11.5), (1.11.8), where $k = (1-n)/2$, and A_2^∞ generated by operators

$$E = u\partial_u, \qquad Q = b(x)\partial_u + B^\mu(x,u,A)\frac{\partial}{\partial A^\mu}, \tag{1.11.17}$$

where b, B^μ are arbitrary solutions of the equations

$$\Box b + \lambda_3 bA_\nu A^\nu + 2\lambda_3 uA_\nu B^\nu = 0; \tag{1.11.18}$$

7) if

$$\lambda_1 = 0, \quad \lambda_2 \neq 0, \quad \lambda_3 \neq 0, \quad \lambda_2^2 - 4\lambda_3 = 0, \tag{1.11.19}$$

then IA is $A\widetilde{P}(1,n)$ (1.11.5) and A_3^∞

$$Q_1 = a(x)u\partial_u - \frac{2}{\lambda_2}\frac{\partial a}{\partial x_\mu}\frac{\partial}{\partial A^\mu}, \tag{1.11.20}$$

$$Q_2 = B^\mu(x,u,A)\frac{\partial}{\partial A^\mu}, \quad \Box a = 0, \quad A_\mu B^\mu = 0.$$

Proof. Using the Lie algorithm one finds that coefficients of operator (1.11.3) must satisfy the following (defining) equations:

$$\xi^\mu_u = \xi^\mu_{A^\nu} = \eta_{A^\nu} = \eta_{uu} = 0; \tag{1.11.21}$$

$$\xi^\mu_\nu + \xi^\nu_\mu = \delta^\mu_\nu f(x) \tag{1.11.22}$$

($f(x)$ is an arbitrary differentiable function);

$$\lambda_1\left[(\eta^\mu_{A^\nu} + 2\xi^0_0\delta^\mu_\nu - \xi^\mu_\nu)u - \delta^\nu_\mu(\eta - u\eta_u)\right] = 0; \tag{1.11.23}$$

$$\lambda_2\left[(\eta^\mu + (2\xi^0_0\delta_{\mu\nu} - \xi^\mu_\nu)A^\nu\right] - \lambda u\eta^\mu_u + 2\eta_{u\mu} - \Box\xi^\mu = 0; \tag{1.11.24}$$

$$2\lambda_3 u\left[\eta^\mu + (2\xi^0_0\delta^\mu_\nu - \xi^\mu_\nu)A^\nu\right]A_\mu + \lambda_1 u\eta^\mu_u + \tag{1.11.25}$$
$$+\lambda_2 A_\mu\eta^\mu + \Box\eta + \lambda_3 A_\mu A^\mu(\eta - u\eta_u) = 0.$$

Equation (1.11.21) yields

$$\xi^\mu = \xi^\mu(x), \quad \eta = a(x)u + b(x). \tag{1.11.26}$$

The general solution of Killing equations (1.11.22) is given in (1.2.7).

Now it is easy to find the general solution of Equations (1.11.23)–(1.11.25) and thereby the general form of ξ^μ, η, η^μ. Depending on λ_1, λ_2, λ_3 there are seven different cases which are stated above in the theorem. So, the theorem is proved.

Theorem 1.11.2 *Equation* (1.11.2) *with* $m \neq 0$ *is invariant under* AP(1,n) *with basis elements*

$$P_\mu = \frac{\partial}{\partial x^\mu}, \quad \mu = \overline{0,n},$$
$$J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu + A_\mu\partial_{A^\nu} - A_\nu\partial_{A^\mu}. \tag{1.11.27}$$

So, the treatment of Equation (1.11.2) as nonlinear system gives us the possiblility to study its symmetry properties in detail.

Chapter 2

Systems of Poincare-invariant Nonlinear PDEs

In the present chapter we will consider systems of nonlinear PDEs that are invariant under the Poincare group P(1,3), extended Poincare group $\widetilde{\mathrm{P}}(1,3)$ and conformal group C(1,3). Ansatze for spinor fields will be constructed. The formula of generating solutions (GS) by conformal transformations is obtained for fields of arbitrary spin. Wide classes of exact solutions of nonlinear generalizations of the Dirac equation are found as well as solutions of the quantum electrodynamics nonlinear equations, coupled nonlinear equations for vector and scalar fields, the Yang-Mills equations, and some others.

2.1. *Reduction and exact solutions of the nonlinear massive Dirac Equation*

Let us consider a nonlinear Dirac equation for a massive spinor field

$$\left[i\gamma \cdot \partial - m - \lambda(\bar{\psi}\psi)^k\right] = 0, \tag{2.1.1}$$

where $\gamma \cdot \partial = \gamma_0\partial_0 + \gamma_1\partial_1 + \gamma_2\partial_2 + \gamma_3\partial_3$; $\partial_\nu = \dfrac{\partial}{\partial x_\nu}$, $\nu = \overline{0,3}$; $\psi = \psi(x)$ is a four-component spinor (column); $\bar{\psi} = \psi^+\gamma_0$, $x \in \mathrm{R}(1,3)$; and m, k, λ are arbitrary real constants; γ_ν are Dirac matrices

$$\gamma_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} \qquad \gamma_a = \begin{pmatrix} 0 & \sigma_a \\ -\sigma_a & 0 \end{pmatrix}, \quad (a = 1,2,3) \tag{2.1.2}$$

σ_0 and σ_a are 2×2 unit and Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.1.3}$$

Equation (2.1.1) under $\lambda = 0$ coincides with the linear Dirac equation discovered in 1928, which describes free particles and antiparticles with mass m

and spin $s = 1/2$. An interest in generalizing the Dirac equation was inspired by Louis de Broglie's ideas about the possibility of describing particles (fields) with spin $s = 0, 1, 3/2, \ldots$ by virtue of the field with spin $s = 1/2$. de Broglie developed for this purpose the method of "fusion" [35] according to which the wave function is represented as a product of "fusable" functions. From the point of view of unitary field theory, if we introduce only one field, then the field equations should be nonlinear, as the intersection generating exited states in the form of diverse particles can only be in this case a self-intersection. One of the first attempts at a nonlinear realization of de Broglie's idea was the work of Ivanenko and his collaborators dated to 1938–1953 (see, for example, [156], Introduction). In the early fifties Heisenberg [118] put forward a wide program on the development of unitary field theory based on the following nonlinear spinor equation

$$[i\gamma \cdot \partial + \lambda(\bar{\psi}\gamma_\mu\gamma_5\psi)\gamma^\mu\gamma_5]\,\psi = 0, \quad \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 \tag{2.1.4}$$

The simplest conformally invariant nonlinear spinor equation has been proposed by Gursey [115]

$$\left[i\gamma \cdot \partial + \lambda(\bar{\psi}\psi)^{1/3}\right]\psi = 0 \tag{2.1.5}$$

A wide class of conformally invariant nonlinear generalizations of the Dirac equation different from (2.1.4) and (2.1.5) was suggested in [101] and [102] (see §2.4). A simple method of constructing nonlinear spinor equations which essentially differ from (2.1.1) and (2.1.4) was pointed out in [65]. The simplest equation of this kind follows from (2.1.1) under $m = 0$ with the help of the replacement $\gamma_\mu \to \bar{\psi}\gamma_\mu\psi$:

$$(\bar{\psi}\gamma_\mu\psi)\partial^\mu\psi = 0.$$

This equation is invariant under the infinite-dimensional group of point transformations.

There are some works on exact solutions of nonlinear spinor equations: [5, 6, 19, 20, 132, 138, 149, 150]; all of them use the Heisenberg ansatz [119] only

$$\psi(x) = [f(\omega) + (\gamma \cdot x)g(\omega)]\,\chi, \tag{2.1.6}$$

where $w = x^\mu x_\mu$; χ is a constant spinor; and f and g are real, differentiable functions.

Below we construct new ansatze for spinor fields and then obtain multiparameter families of exact solutions of Equation (2.1.1) and, in the next sections, exact solutions of Equation (2.1.1) under $m = 0$, Equation (2.1.5), and a conformally invariant generalization of the Dirac-Heisenberg equation, (2.1.4).

To find exact solutions of the nonlinear Dirac equation (2.1.1) we use the ansatz suggested in [63]

$$\psi(x) = A(x)\varphi(\omega), \tag{2.1.7}$$

where $A(x)$ is 4×4 matrix; $\varphi(\omega)$ is an unknown four-component function (column) depending on new variables $\omega = \omega(x) = \{\omega_1, \omega_2, \omega_3\}$. After substituting ansatz (2.1.7) into (2.1.1) we obtain

$$(i\gamma \cdot \partial A)\varphi + i\gamma_\nu \frac{\partial \omega_a}{\partial x_\nu} A \frac{\partial \varphi}{\partial \omega_a} - \left[m + \lambda(\overline{\varphi}\overline{A}A\varphi)^k\right] A\varphi = 0 \qquad (2.1.8)$$

Furthermore we require Equation (2.1.8) to contain addends depending on the new variables ω only, whence follows

$$A^{-1}\gamma_\nu \frac{\partial A}{\partial x_\nu} = F(\omega), \qquad A^{-1}\gamma_\nu \frac{\partial \omega_a}{\partial x_\nu} = F_a(\omega), \qquad \overline{A}A = G(\omega) \qquad (2.1.9)$$

where F, F_a, and G are arbitrary 4×4 matrices. If one succeeds in finding a partial solution $A = A(x)$, $\omega_a = \omega_a(x)$ of system (2.1.9), then to determine the function we obtain from (2.1.8) the reduced equation

$$F(\omega)\varphi + F_a(\omega)\frac{\partial \varphi}{\partial \omega_a} + i\left[m + \lambda(\overline{\varphi}G(\omega)\varphi)^k\right]\varphi = 0. \qquad (2.1.10)$$

At first sight the system of PDEs (2.1.9) is not in the least simpler than the initial equation. Nevertheless algebraic-theoretic techniques give an effective algorithm for constructing vast classes of its exact solutions. Later, to determine the explicit form of ansatz (2.1.7), we shall proceed in the same way as in §1.4.

Let us use the fact that the maximal invariance algebra of Equation (2.1.1) is AP(1,3), with basis elements

$$\begin{aligned} &P_0 = i\partial_0, \quad P_a = -i\partial_a, \\ &J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu + \frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu). \end{aligned} \qquad (2.1.11)$$

If $A(x)$ and $\omega(x)$ satisfy the conditions

$$\left(a^\mu P_\mu + c^{\mu\nu} J_{\mu\nu}\right) A(x) = 0, \qquad (2.1.12)$$

$$\left[a^\mu P_\mu + c^{\mu\nu}(x_\mu P_\nu - x_\nu P_\mu)\right]\omega(x) = 0, \qquad (2.1.13)$$

where a_μ, $c_{\mu\nu} = -c_{\nu\mu}$ are arbitrary real constants, they also satisfy system (2.1.9). So the problem of describing ansatze (2.1.7) for spinor fields invariant under AP(1,3) (2.1.11) is reduced to the solution of systems (2.1.12) and (2.1.13). Using nonequivalent one-dimensional subalgebras of the Poincare algebra AP(1,3) and corresponding invariant variables constructed in [163] and [114], we write down the P(1,3)-nonequivalent ansatze (2.1.7) [100].

Table 2.1.1. P(1,3)-nonequivalent ansatze for a spinor field.

N	Algebra	Invar. var. $\omega = \{\omega_1, \omega_2, \omega_3\}$	Ansatze $\psi(x) =$
1.	P_0	x_1, x_2, x_3	$\varphi(\omega)$
2.	P_3	x_0, x_1, x_2	$\varphi(\omega)$
3.	$P_0 + P_1$	$x_0 + x_1, x_1, x_3$	$\varphi(\omega)$
4.	J_{12}	$(x_1^2 + x_2^2)^{1/2}, x_0, x_3$	$\exp\{-\frac{1}{2}\gamma_1\gamma_2 \cdot$ $\cdot \arctan \frac{x_1}{x_2}\}\varphi(\omega)$
5.	J_{03}	$(x_0^2 - x_3^2)^{1/2}, x_1, x_2$	$\exp\{+\frac{1}{2}\gamma_0\gamma_3 \cdot$ $\cdot \ln(x_0 + x_3)\}\varphi(\omega)$
6.	$J_{02} + J_{12}$	$x_0 + x_1,$ $(x_0^2 - x_1^2 - x_2^2)^{1/2}, x_3$	$\exp\{-\frac{x_2}{2(x_0+x_1)} \cdot$ $\cdot\gamma_2(\gamma_0 + \gamma_1)\}\varphi(\omega)$
7.	$\alpha J_{23} - J_{01}$	$(x_0^2 - x_1^2)^{1/2},$ $\alpha \ln(x_0 + x_1) +$ $+ \arctan \frac{x_2}{x_3}, (x_2^2 + x_3^2)^{1/2}$	$\exp\{\frac{1}{2}\gamma_0\gamma_1 \cdot$ $\cdot \ln(x_0 + x_1) -$ $-\frac{1}{2}\gamma_2\gamma_3 \cdot$ $\cdot \arctan \frac{x_2}{x_3}\}\varphi(\omega)$
8.	$J_{23} -$ $-\frac{\epsilon}{2}(P_0 + P_1)$	$x_0 + x_1, \epsilon(x_0 - x_1) +$ $+ \arctan \frac{x_2}{x_3}, (x_2^2 + x_3^2)^{1/2}$	$\exp\{-\frac{1}{2}\gamma_2\gamma_3 \cdot$ $\cdot \arctan \frac{x_2}{x_3}\varphi(\omega)$
9.	$J_{12} + \alpha P_0$	$(x_1^2 + x_2^2)^{1/2},$ $x_0 + \alpha \arctan \frac{x_1}{x_2}, x_3$	$\exp\{-\frac{1}{2}\gamma_1\gamma_2 \cdot$ $\cdot \arctan \frac{x_1}{x_2}\}\varphi(\omega)$
10.	$J_{12} + \alpha P_3$	$(x_1^2 + x_2^2)^{1/2},$ $x_3 + \alpha \arctan \frac{x_1}{x_2}, x_0$	$\exp\{-\frac{1}{2}\gamma_1\gamma_2 \cdot$ $\cdot \arctan \frac{x_1}{x_2}\}\varphi(\omega)$
11.	$J_{01} + \alpha P_2$	$(x_0^2 - x_1^2)^{1/2},$ $x_2 + \alpha \ln(x_0 + x_1), x_3$	$\exp\{\frac{1}{2}\gamma_0\gamma_1 \cdot$ $\cdot \ln(x_0 + x_1)\}\varphi(\omega)$
12.	$J_{02} + J_{12} +$ $+P_0 - P_1$	$x_0 - x_1 + (x_0 + x_1)x_2 +$ $+\frac{1}{6}(x_0 + x_1)^3,$ $x_2 + \frac{1}{4}(x_0 + x_1)^2, x_3$	$\exp\{\frac{1}{4}(x_0 + x_1) \cdot$ $\cdot\gamma_2(\gamma_0 + \gamma_1)\}\varphi(\omega)$
13.	$J_{02} + J_{12} -$ $-\epsilon P_3$	$x_0 + x_1, (x_0^2 - x_1^2 - x_2^2)^{1/2},$ $x_2 + \epsilon(x_0 + x_1)x_3$	$\exp\{-\frac{x_2}{2(x_0+x_1)} \cdot$ $\cdot\gamma_2(\gamma_0 + \gamma_1)\}\varphi$

Here $\alpha \neq 0$ is an arbitrary constant; $\epsilon = \pm 1$.

To illustrate how to construct the ansatze of Table 2.1.1 we consider, as an example, operator J_{12} (the fourth position of the table). In this case Equations (2.1.12) and (2.1.13) take the form

$$(x_1\partial_2 - x_2\partial_1 - \tfrac{1}{2}\gamma_1\gamma_2)A(x) = 0, \tag{2.1.14}$$

$$(x_1\partial_2 - x_2\partial_1)\omega(x) = 0. \tag{2.1.15}$$

We look for a solution of (2.1.14) in the form

$$A(x) = \exp\left\{\tfrac{1}{2}\gamma_1\gamma_2 f(x)\right\},$$

where $f(x)$ is a scalar function. Substitution of this expression into (2.1.14) gives

$$(x_1\partial_2 - x_2\partial_1 - \tfrac{1}{2}\gamma_1\gamma_2)\exp\left\{\tfrac{1}{2}\gamma_a\gamma_2 f(x)\right\} = \\ = \tfrac{1}{2}\gamma_1\gamma_2\left[(x_1\partial_2 - x_2\partial_1)f - 1\right]\exp\{\tfrac{1}{2}\gamma_1\gamma_2\} = 0,$$

that is

$$x_1\frac{\partial f}{\partial x_2} - x_2\frac{\partial f}{\partial x_1} = 1.$$

Integrating this equation one obtains

$$f(x) = -\arctan\frac{x_1}{x_2},$$

and therefore

$$A(x) = \exp\left\{-\tfrac{1}{2}\gamma_1\gamma_2\arctan\frac{x_1}{x_2}\right\}$$

Equation (2.1.15) is equivalent to the following Euler-Lagrange system

$$\frac{dx_0}{0} = \frac{dx_1}{-x_2} = \frac{dx_2}{x_1} = \frac{dx_3}{0},$$

which has first integrals of the form

$$x_0, \qquad x_1^2 + x_2^2, \qquad x_3$$

On putting

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \qquad \omega_2 = x_0, \qquad \omega_3 = x_3$$

we finally obtain ansatz N4. For the rest of the cases calculations can be done in much the same way.

Ansatze 1–13 of Table 2.1.1 are the complete set of P(1,3)-nonequivalent ansatze for spinor fields. This means that one of them cannot be changed into another by means of a group generating procedure.

Let us substitute ansatze 1–13 from Table 2.1.1 into Equation (2.1.1). After lengthy but elementary calculations we obtain the following reduced PDEs for the function $\varphi(\omega)$ [100]:

$$(1)\ \gamma_1\varphi_1 + \gamma_2\varphi_2 + \gamma_3\varphi_3 + iF\varphi = 0,$$

$$(2)\ \gamma_0\varphi_1 + \gamma_1\varphi_2 + \gamma_2\varphi_3 + iF\varphi = 0,$$

$$(3)\ (\gamma_0 + \gamma_1)\varphi_1 + \gamma_2\varphi_2 + \gamma_3\varphi_3 + iF\varphi = 0, \qquad (2.1.16)$$

$$(4)\ \gamma_2\left(\varphi_1 + \frac{1}{2\omega_1}\varphi\right) + \gamma_0\varphi_2 + \gamma_3\varphi_3 + iF\varphi = 0,$$

$$(5)\ \tfrac{1}{2}(\gamma_0 + \gamma_3)\varphi + \tfrac{1}{2}\left[(\gamma_0 + \gamma_3)\omega_1 + \frac{1}{\omega_1}(\gamma_0 - \gamma_3)\right]\varphi_1 + \gamma_1\varphi_2 + \gamma_2\varphi_3 + iF\varphi = 0;$$

$$(6)\ (\gamma_0+\gamma_1)\left(\varphi_1++\frac{1}{2\omega_1}\varphi\right)+\left[(\gamma_0+\gamma_1)\frac{\omega_1^2+\omega_2^2}{2\omega_1\omega_2}-\gamma_1\frac{\omega_1}{\omega_2}\right]\varphi_2+\gamma_3\varphi_3+iF\varphi=0,$$

$$(7)\ \tfrac{1}{2}(\gamma_0+\gamma_1)\varphi+\tfrac{1}{2}\left[(\gamma_0+\gamma_1)\omega_1+\frac{1}{\omega_1}(\gamma_0-\gamma_1)\right]\varphi_1+\left[\alpha(\gamma_0+\gamma_1)+\frac{1}{\omega_3}\gamma_2\right]\varphi_2+ +\gamma_3\left(\frac{1}{2\omega_3}\varphi+\varphi_3\right)+iF\varphi=0,$$

$$(8)\ (\gamma_0+\gamma_1)\varphi_1+\left[\epsilon(\gamma_0-\gamma_1)+\frac{1}{\omega_3}\gamma_2\right]\varphi_2+\gamma_3\left(\frac{1}{2\omega_3}\varphi+\varphi_3\right)+iF\varphi=0,$$

$$(9)\ \gamma_2\left(\frac{1}{2\omega_1}\varphi+\varphi_1\right)+\left(\frac{\alpha}{\omega_1}\gamma_1+\gamma_0\right)\varphi_2+\gamma_3\varphi_3+iF\varphi=0,$$

$$(10)\ \gamma_2\left(\frac{1}{2\omega_1}\varphi+\varphi_1\right)+\left(\frac{\alpha}{\omega_1}\gamma_1+\gamma_3\right)\varphi_2+\gamma_0\varphi_3+iF\varphi=0,$$

$$(11)\ \tfrac{1}{2}(\gamma_0+\gamma_1)\varphi+\tfrac{1}{2}\left[(\gamma_0+\gamma_1)\omega_1+\frac{1}{\omega_1}(\gamma_0-\gamma_1)\right]\varphi_1+ +\left[\gamma_2+\alpha(\gamma_0+\gamma_1)\right]\varphi_2+\gamma_3\varphi_3+iF\varphi=0,$$

$$(12)\ \left[(\gamma_0+\gamma_1)\omega_2+\gamma_0-\gamma_1\right]\varphi_1+\gamma_2\varphi_2+\gamma_3\varphi_3+iF\varphi$$

$$(13)\ (\gamma_0+\gamma_1)\left(\varphi_1-\frac{1}{2\omega_1}\varphi\right)+\left[(\gamma_0+\gamma_1)\frac{\omega_1^2+\omega_2^2}{2\omega_1\omega_2}-\gamma_1\frac{\omega_1}{\omega_2}\right]\varphi_2+ +\left[(\gamma_0+\gamma_1)\frac{\omega_3}{\omega_1}+\gamma_2+\epsilon_1\omega_1\gamma_3\right]\varphi_3+iF\varphi=0,$$

Here $\varphi_a\equiv\dfrac{\partial\varphi}{\partial\omega_a}$, $a=\overline{1,3}$; $F=\left[m+\lambda(\bar{\varphi}\varphi)^k\right]$; the equation of number n, $n=\overline{1,13}$ is obtained with the help of the nth ansatz of Table 2.1.1.

The next step in finding invariant solutions of Equation (2.1.1) should have been the investigation of symmetry properties of the reduced Equations 1–13 (2.1.16). But the problem of establishing the maximal group of invariance of these equations is very tedious and cumbersome in itself. Naturally a question arises as to whether there is a procedure for obtaining information about the symmetry of the reduced equations from the symmetry of the initial equation. An affirmative answer to this question is given by the following statement (as for its proof, see [161]).

Theorem 2.1.1. *Let G be an invariance group of some PDE and $H\subset G$ be a one parameter subgroup of G, (H being a normal divisor in G). Then the factor group G/H is an invariance group of the equation obtained through reduction of the initial equation by H-invariant solutions.*

It will be noted that this theorem does not guarantee the maximality of the group G/H, that is, the reduced equation can admit a wider symmetry group than G/H.

In our case we have G = P(1,3), and H is one of 13 subgroups with generators listed Table 2.1.1. Generally speaking, these subgroups are not normal divisors in P(1,3) and for this reason one has to construct normalizers of subgroups before using Theorem 2.1.1, that is, to find maximal subgroups of P(1,3) which normal divisors are $H_1 - H_{13}$. The next step is the construction of factor groups $\mathrm{Nor}\,H_1$ – $\mathrm{Nor}\,H_{13}$, where $\mathrm{Nor}\,H_i$ ($i = 1, \ldots, 13$) is a normalizer of a group H_i. According to Theorem 2.1.1 these factor-groups are symmetry groups of the reduced equations.

The complete realization of this program is rather cumbersome, making it the subject of a separate investigation. Therefore we will now simply make a direct reduction of Equations (2.1.16) to systems of ODEs or two-dimensional PDEs (if possible). This means we suppose that φ is dependent on one or two variables of the ω_1, ω_2, ω_3. As a result we obtain:

$$
\begin{aligned}
&(1)\ \gamma_1\varphi_1 + iF\varphi = 0,\\
&(2)\ \gamma_0\varphi_1 + iF\varphi = 0,\\
&(3)\ (\gamma_0+\gamma_1)\varphi_1 + \gamma_2\varphi_2 + iF\varphi = 0,\\
&(4)\ \gamma_2\left(\varphi_1 + \frac{1}{2\omega_1}\varphi\right) + iF\varphi = 0,\\
&(5)\ \tfrac{1}{2}(\gamma_0+\gamma_3)\varphi + \gamma_1\varphi_2 + iF\varphi = 0,\\
&(5')\ \tfrac{1}{2}(\gamma_0+\gamma_3)\varphi + \tfrac{1}{2}\left[(\gamma_0+\gamma_3)\omega_1 + \frac{1}{\omega_1}(\gamma_0-\gamma_3)\right]\varphi_1 + iF\varphi = 0,\\
&(6)\ (\gamma_0+\gamma_1)\left(\varphi_1 + \frac{1}{2\omega_1}\varphi\right) + \gamma_3\varphi_3 + iF\varphi = 0,\\
&(6')\ (\gamma_0+\gamma_1)\left(\varphi_1 + \frac{1}{2\omega_1}\varphi\right) + \left[(\gamma_0+\gamma_1)\frac{\omega_1^2+\omega_2^2}{2\omega_1\omega_2} - \gamma_1\frac{\omega_1}{\omega_2}\right]\varphi_2 + iF\varphi = 0,\\
&(7)\ \tfrac{1}{2}(\gamma_0+\gamma_1)\varphi + \gamma_3\left(\varphi_3 + \frac{1}{2\omega_3}\varphi\right) + iF\varphi = 0,\\
&(8)\ \left[\epsilon(\gamma_0-\gamma_1) + \frac{1}{\omega_3}\gamma_2\right]\varphi_2 + \left(\varphi_3 + \frac{1}{2\omega_3}\varphi\right) + iF\varphi = 0,\\
&(8')\ (\gamma_0+\gamma_1)\varphi_1 + \gamma_3\left(\varphi_3 + \frac{1}{2\omega_3}\varphi\right) + iF\varphi = 0,\\
&(9)\ \gamma_2\left(\varphi_1 + \frac{1}{2\omega_1}\varphi\right) + \left(\frac{\alpha}{\omega_1}\gamma_1 + \gamma_0\right)\varphi_2 + iF\varphi = 0,
\end{aligned}
\qquad (2.1.17)
$$

(10) $\gamma_2\left(\varphi_1+\frac{1}{2\omega_1}\varphi\right)+\left(\frac{\alpha}{\omega_1}\gamma_1+\gamma_3\right)\varphi_2+iF\varphi=0,$

(11) $\frac{1}{2}(\gamma_0+\gamma_1)\varphi+[\gamma_2+\alpha(\gamma_0+\gamma_1)]\,\varphi_2+iF\varphi=0,$

(12) $\gamma_2\varphi_2+iF\varphi=0,$

(13) $(\gamma_0+\gamma_1)\left(\varphi_1+\frac{1}{2\omega_1}\varphi\right)+\left[(\gamma_0+\gamma_1)\frac{\omega_3}{\omega_1}+\gamma_2+\epsilon\omega_1\gamma_3\right]\varphi_3+iF\varphi=0.$

Finding the general solutions of Equations (1), (2) from (2.1.17) presents no difficulties (see formulae (1) and (2) from (2.1.24). Consider (3) from (2.1.17) at greater length. We will rewrite it as follows:

$$\begin{aligned}&\frac{\partial\varphi}{\partial\omega_2}=i\gamma_2\frac{\partial f}{\partial\omega_2}\varphi+\gamma_2(\gamma_0+\gamma_1)\Phi,\\&\frac{\partial f}{\partial\omega_2}=F\equiv\lambda(\overline{\varphi}\varphi)^k+m,\quad f=f(\omega_1,\omega_2)\end{aligned}\tag{2.1.18}$$

$$\Phi=\Phi(\omega_1,\omega_2)=\frac{\partial\varphi}{\partial\omega_1}$$

The first equation from (2.1.18) is a linear inhomogeneous system of ODEs and its general solution has the form

$$\varphi=e^{i\gamma_2 f}\left[\chi(\omega_1)+\gamma_2(\gamma_0+\gamma_1)\int^{\omega_2}e^{i\gamma_2 f}\Phi d\omega_2\right],\tag{2.1.19}$$

where $\chi(\omega_1)$ is a four-component spinor depending on ω_1. Substituting (2.1.19) into (2.1.18), one can determine the explicit form of functions f and φ, but they are so unwieldy that we do not present them here. It will be noted that the maximal invariance group of Equation (3) from (2.1.17) is infinite-dimensional, and that is why one can find its general solution [212].

By the change of variables

$$\varphi(\omega_1)=\frac{1}{\sqrt{\omega_1}}\Phi(\omega_1)\tag{2.1.20}$$

Equation (4) from (2.1.17) is reduced to the form

$$\gamma_2\dot{\Phi}+i\left[m+\lambda\omega_1^{-k}(\overline{\phi}\phi)^k\right]\Phi=0,$$

or

$$\dot{\Phi}\equiv\frac{d\Phi}{d\omega_1}=i\gamma_2\left[m+\lambda\omega_1^{-k}(\overline{\phi}\phi)^k\right]\Phi=0,\tag{2.1.21}$$

whence follows

$$\dot{\overline{\phi}}\phi + \overline{\phi}\dot{\phi} \equiv \frac{d}{d\omega_1}(\overline{\phi}\phi) = 0,$$

that is $\overline{\phi}\phi = \text{const}$. This allows us to look for the general solution of Equation (2.1.21) in the form

$$\phi(\omega_1) = \exp\{i\gamma_2 g(\omega_1)\}\chi, \tag{2.1.22}$$

where χ is a constant spinor. Substituting (2.1.22) into (2.1.21) we obtain the ODE for the function $g = g(\omega_1)$

$$\dot{g} = \lambda\omega^{-k}(\overline{\chi}\chi)^k + m, \tag{2.1.23}$$

whose general solution is given in (2.1.25). The rest of the equations from (2.1.17) are solved in a similar way. Below we list their solutions.

$$
\begin{aligned}
&(1)\ \varphi(\omega) = \exp\{i æ\gamma_1\omega_1\}\chi,\\
&(2)\ \varphi(\omega) = \exp\{-i æ\gamma_0\omega_1\}\chi,\\
&(3)\ \varphi(\omega) = \exp\{i æ\gamma_2\}\exp\{i(\gamma_0+\gamma_1)f(\omega_1)\}\chi,\\
&(3')\ \varphi(\omega) = \exp\{im\gamma_2\omega_2 + f(\omega_1)\}(\gamma_0+\gamma_1)\chi,\\
&(4)\ \varphi(\omega) = \frac{1}{\sqrt{\omega_1}}\exp\{i\gamma_2 g(\omega_1)\}\chi,\\
&(5)\ \varphi(\omega) = \exp\left\{i\gamma_1\left[æ - \frac{i}{2}(\gamma_0+\gamma_3)\right]\omega_2\right\}\chi,\\
&(6)\ \varphi(\omega) = \exp\{i(\gamma_3\omega_3 m + f(\omega_1))\}(\gamma_0+\gamma_1)\chi,\\
&(7)\ \varphi(\omega) = \frac{1}{\sqrt{\omega_3}}(\gamma_0+\gamma_1)\exp\{-im\gamma_3\omega_3\}\chi,\\
&(8)\ \varphi(\omega) = \frac{1}{\sqrt{\omega_3}}\exp\{i\gamma_3 g(\omega_3)\}\exp\{i(\gamma_0+\gamma_1)f(\omega_1)\}\chi,\\
&(11)\ \varphi(\omega) = \exp\left\{i(\gamma_2+\alpha(\gamma_0+\gamma_1))\left(æ - \frac{i}{2}(\gamma_0+\gamma_1)\right)\omega_2\right\}\chi,\\
&(12)\ \varphi(\omega) = \exp\{i æ\gamma_2\omega_2\}\chi.
\end{aligned}
\tag{2.1.24}
$$

The solutions in (2.1.24) are enumerated in the same way as the ansatze in Table 2.1.1 and Equations (2.1.16) and (2.1.17); χ is a constant spinor; $æ = \lambda(\overline{\chi}\chi)^k + m$; and $f(\omega)$ is an artibrary differentiable function;

$$g(\omega) = \begin{cases} \frac{\lambda}{1-k}\omega^{1-k}(\overline{\chi}\chi)^k + m\omega, & k \neq 1;\\ \lambda(\overline{\chi}\chi)\ln\omega + m\omega, & k = 1. \end{cases} \tag{2.1.25}$$

One can apply the procedure of generating solutions to (2.1.24) (see Introduction and §1.4). Using the explicit form of the final transformations generated by operators (2.1.11) (see Appendix 3) we present the corresponding formulae of GS in Table 2.1.2.

Table 2.1.2 Final transformations of P(1,3) group and formulae of generating solutions.

N	Operator	Transformations		Formulae of GS
		$x \to x'$	$\psi(x) \to \psi'(x') =$	$\psi_{II} =$
1-4	P_μ	$x'_\mu = x_\mu + \delta_\mu$	$= \psi(x)$	$= \psi_I(x')$
5-7	$\alpha_a J_a \equiv$ $\equiv \frac{1}{2}\epsilon_{abc} J_{bc}$	$x'_0 = x_0,$ $\vec{x}' = \vec{x}\cos\alpha + {} + \frac{\vec{x}\times\vec{\alpha}}{\alpha}\sin\alpha + {} + \frac{\vec{\alpha}(\vec{\alpha}\cdot\vec{x})}{\alpha^2}(1-\cos\alpha)$	$= \exp\left\{\frac{1}{2}\vec{\alpha}\cdot\vec{S}\right\}\psi(x)$ $= \big(\cos\frac{\alpha}{2} + {} + i\vec{\alpha}\vec{S}\sin\frac{\alpha}{2}\big)\psi(x)$	$= (\cos\frac{\alpha}{2} - \vec{\alpha}\vec{S} \cdot {} \cdot \sin\frac{\alpha}{2})\psi_I(x')$
8-10	$\beta_a J_{0a}$	$x'_0 = x_0 \operatorname{ch}\beta + {} + \frac{\vec{\beta}\cdot\vec{x}}{\beta}\operatorname{sh}\beta$ $\vec{x}' = \vec{x} + \frac{\vec{\beta}(\vec{\beta}\cdot\vec{x})}{\beta^2} \cdot {} \cdot (\operatorname{ch}\beta - 1) + {} + x_0\frac{\vec{\beta}}{\beta}\operatorname{sh}\beta$	$= \exp\{-\frac{1}{2}\cdot\gamma_0\vec{\gamma}\cdot\vec{\beta}\} \cdot {} \cdot \psi(x)$ $= \big(\operatorname{ch}\beta - {} - \gamma_0\frac{\vec{\gamma}\cdot\vec{\beta}}{\beta}\operatorname{sh}\beta\big)\psi(x)$	$= \big(\operatorname{ch}\beta + {} + \gamma_0\frac{\vec{\gamma}\cdot\vec{\beta}}{\beta}\operatorname{sh}\beta\big) \cdot {} \cdot \psi_I(x')$

Here δ_μ; $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$; $\vec{\beta} = (\beta_1, \beta_2, \beta_3)$ are arbitrary constants;

$$\alpha = \left(\alpha_1^2 + \alpha_2^2 + \alpha_3^2\right)^{1/2}, \qquad \beta = (\beta_1^2 + \beta_2^2 + \beta_3^2)^{1/2}; \qquad \vec{S} = (S_1, S_2, S_3)$$

are 4×4 matrices, with $S_a = \frac{i}{2}\epsilon_{abc}\gamma_b\gamma_c$.

Inserting (2.1.24) into the corresponding ansatze of Table 2.1.1, and having applied the formulae of GS from Table 2.1.2, we obtain the solutions of Equation (2.1.1).

$$\psi(x) = \exp\{i æ(\gamma a)(ay)\}\chi; \tag{2.1.26}$$

$$\psi(x) = \exp\{-i æ(\gamma d)(dy)\}\chi;$$

$$\psi(x) = \exp\{i æ(\gamma b)(by)\}\exp\{i(\gamma a + \gamma d)f(ay + dy)\}\chi;$$

$$\psi(x) = \exp\{i\,(m(\gamma b)(by) + f(ay + dy))\}(\gamma a + \gamma d)\chi;$$

$$\psi(x) = \frac{1}{\sqrt{\omega}}\exp\left\{-\tfrac{1}{2}(\gamma a)(\gamma b)\arctan\frac{ay}{by}\right\}\exp\{i(\gamma a)g(\omega)\}\chi;$$

$$\omega = \left((ay)^2 + (by)^2\right)^{1/2};$$

$$\psi(x) = \exp\left\{-\tfrac{1}{2}(\gamma c)(\gamma d)\ln(cy+dy)\right\}\exp\left\{(\gamma a)\left(i æ + \tfrac{1}{2}(\gamma c + \gamma d)ay\right)\right\}\chi;$$

$$\psi(x) = \exp\left\{-\frac{by}{2(ay+dy)}(\gamma b)(\gamma a + \gamma d)\right\}\exp\left\{i\Big(m(\gamma c)(cy) + +f(ay+dy)\Big)\right\}(\gamma a + \gamma d)\chi;$$

$$\psi(x) = \frac{1}{\sqrt{\omega}}\exp\left\{-\tfrac{1}{2}(\gamma b)(\gamma c)\arctan\frac{by}{cy} - \tfrac{1}{2}(\gamma a + \gamma d)\times \times\ln(ay+dy)\right\}\exp\left\{i\left(m(\gamma c)\,\omega\right\}(\gamma a + \gamma d)\chi,$$

$$\omega = \left((by)^2 + (cy)^2\right)^{1/2};$$

$$\psi(x) = \frac{1}{\sqrt{\omega}}\exp\left\{-\tfrac{1}{2}(\gamma b)(\gamma c)\arctan\frac{by}{cy}\right\}\exp\left\{i(\gamma c)g(\omega)\right\}\times \times\exp\left\{i(\gamma a + \gamma d)f(ay+dy)\right\}\chi,$$

$$\omega = \left((by)^2 + (cy)^2\right)^{1/2};$$

$$\psi(x) = \exp\left\{\tfrac{1}{2}(\gamma d)(\gamma a)\ln(ay+dy)\right\}\exp\left\{i\big((\gamma b) + \alpha(\gamma a + \gamma d)\big)\times \times\left(æ - \frac{i}{2}(\gamma a + \gamma d)(by + \alpha\ln(ay+dy)\right)\right\}\chi$$

$$\psi(x) = \exp\left\{-\frac{1}{4}(\gamma a + \gamma d)(\gamma b)(ay+dy)\right\}\exp\left\{iæ(\gamma b)\left(by + \frac{1}{4}(ay+dy)^2\right)\right\}\chi$$

In (2.1.26) the following notations have been used: χ is a constant spinor, $æ = m + \lambda\left(\overline{\chi}\chi\right)^k$; $y_\mu = x_\mu + \delta_\mu$; $\delta_\mu, a_\mu, b_\mu, c_\mu, d_\mu$ are arbitrary real constants satisfying the conditions

$$\begin{aligned} a^2 = a_\nu a^\nu = b^2 = c^2 = -d^2 = -1; \\ ab = bc = dc = ac = da = db = 0 \end{aligned} \tag{2.1.27}$$

$f(w)$ is arbitrary differentiable function; and $g(w)$ is determined in (2.1.25). It will be noted that functions (2.1.27) are analytic in the constants λ and m. So, on setting $\lambda = 0$ one obtains from (2.1.26) solutions of the linear Dirac equation.

It is useful to know the covariant form of ansatze for the spinor fields represented in Table 2.1.1. Having applied to the ansatze from Table 2.1.1 the formulae of GS from Table 2.1.2, we obtain covariant P(1,3)-nonequivalent ansatze for spinor fields and list them in the following table.

Table 2.1.3. Covariant P(1,3)-nonequivalent ansatze for spinor fields.

N	Invariant variables ω	Ansatze $\psi(x) =$
1.	ax, bx, cx	$\varphi(\omega)$
2.	dx, ax, bx	$\varphi(\omega)$
3.	$ax + dx, bx, cx$	$\varphi(\omega)$
4.	$\left((ax)^2 + (bx)^2\right)^{1/2}, dx, cx$	$\exp\{-\frac{1}{2}(\gamma a)(\gamma b) \cdot \arctan\frac{ax}{bx}\}\varphi(\omega)$
5.	$\left((dx)^2 - (cx)^2\right)^{1/2}, ax, bx$	$\exp\{\frac{1}{2}(\gamma d)(\gamma c) \cdot \ln(dx + cx)\}\varphi(\omega)$
6.	$ax + dx,\ \left[(dx)^2 - (ax)^2 - (bx)^2\right]^{1/2}, cx$	$\exp\{-\frac{bx}{2(ax+dx)}(\gamma b) \cdot ((\gamma a) + (\gamma d))\}\varphi(\omega)$
7.	$\left[(dx)^2 - (ax)^2\right]^{1/2}, \alpha\ln(dx + ax) + \arctan(bx/cx),\ \left[(bx)^2 + (cx)^2\right]^{1/2}$	$\exp\{\frac{1}{2}(\gamma d)(\gamma a)\ln(ax + dx) - \frac{1}{2}(\gamma b)(\gamma c) \cdot \arctan(bx/cx)\}\varphi(\omega)$
8.	$ax + dx,\ \epsilon(dx - ax) + \arctan\frac{bx}{cx},\ \left[(bx)^2 + (cx)^2\right]^{1/2}$	$\exp\{-\frac{1}{2}(\gamma b)(\gamma c) \cdot \arctan\frac{bx}{cx}\}\varphi(\omega)$
9.	$\left[(ax)^2 + (bx)^2\right]^{1/2}, dx + \alpha\arctan\frac{ax}{bx}, cx$	$\exp\{-\frac{1}{2}(\gamma a)(\gamma b) \cdot \arctan\frac{ax}{bx}\}\varphi(\omega)$
10.	$\left[(ax)^2 + (bx)^2\right]^{1/2}, cx + \alpha\arctan\frac{ax}{bx}, dx$	$\exp\{-\frac{1}{2}(\gamma a)(\gamma b) \cdot \arctan\frac{ax}{bx}\}\varphi(\omega)$
11.	$\left[(dx)^2 - (ax)^2\right]^{1/2}, bx + \alpha\ln(ax + dx), cx$	$\exp\{\frac{1}{2}(\gamma d)(\gamma a) \cdot \ln(dx + ax)\}\varphi(\omega)$
12.	$dx - ax + (dx + ax)bx + \frac{1}{6}(dx + ax)^3,\ bx + \frac{1}{4}(dx + ax)^2, cx$	$\exp\{\frac{1}{4}(dx + ax)(\gamma b) \cdot (\gamma a + \gamma d)\}\varphi(\omega)$
13.	$dx + ax,\ \left[(dx)^2 - (ax)^2 - (bx)^2\right]^{1/2},\ bx + \epsilon(dx + ax)cx$	$\exp\{-\frac{bx}{2(ax+dx)}(\gamma b) \cdot (\gamma a + \gamma d)\}\varphi(\omega)$

In this table $\epsilon = \pm 1$; $\alpha, a_\nu, b_\nu, c_\nu, d_\nu$ are arbitrary constants satisfying conditions (2.1.27).

The ansatze given in Tables 2.1.1 and 2.1.3 do not exhaust all possible ansatze for reducing the nonlinear Dirac equation (2.1.1). For example, ansatz [100, 212, 31*]

$$\psi(x) = \left[f(\omega) + i(\gamma^\nu \frac{\partial\omega}{\partial x_\nu})g(\omega)\right]\chi, \qquad \bar{\chi}\chi = 1, \tag{2.1.28}$$

where f and g are real scalar differentiable functions, reduces (2.1.1) to the system of ODEs

$$\dot{f} = gF, \qquad F \equiv m + \lambda(f^2 + \epsilon g^2)^k,$$
$$\epsilon \dot{g} + \frac{N}{\omega} g = -fF. \tag{2.1.29}$$

In (2.1.29) dot means differentiation with respect to ω; $\epsilon = \pm 1$; $N = -2, -1, \dots, 3$; the variable $\omega = \omega(x)$ is determined from the system of PDEs of Collins type [41]

$$\frac{\partial \omega}{\partial x_\nu} \frac{\partial \omega}{\partial x^\nu} = \epsilon$$
$$\Box \omega = \frac{N}{\omega} \tag{2.1.30}$$

Solutions of this overdetermined system have the form [100, 30*, 31*, 60*, 122*, 130*]

$$w(x) = \begin{cases} \left[(ay)^2 + (by)^2 + (cy)^2\right]^{1/2}, & \epsilon = -1, \quad N = -2; \\ \left[(ay)^2 + (by)^2\right]^{1/2}, & \epsilon = -1, \quad N = -1; \\ ay + v(by + dy), & \epsilon = -1, \quad N = 0; \\ dy, & \epsilon = +1, \quad N = 0; \\ \left[(dy)^2 - (ay)^2\right]^{1/2}, & \epsilon = +1, \quad N = 1; \\ \left[(dy)^2 - (ay)^2 - (by)^2\right]^{1/2}, & \epsilon = +1, \quad N = 2; \\ (y_\nu y^\nu)^{1/2}, & \epsilon = +1, \quad N = 3; \end{cases} \tag{2.1.31}$$

where $y_\nu = x_\nu + \delta_\nu$, and $\delta_\nu, a_\nu, b_\nu, c_\nu, d_\nu$ are arbitrary constants satisfying (2.1.27); v is an arbitrary differentiable function. When ω is chosen as $\sqrt{x_\nu x^\nu}$ then ansatz (2.1.28) coincides with Heisenberg's [119] (2.1.6).

Let us obtain some solutions of system (2.1.29). Multiplying the first equation by f and the second by g, and then summing them we obtain

$$\frac{d}{d\omega}\left(f^2 + \epsilon g^2\right) = -\frac{2N}{\omega} g^2. \tag{2.1.32}$$

In particular, with $N = 0$, (2.1.32) yields

$$f^2 + \epsilon g^2 = \text{const}$$

and in this case system (2.1.29) can be easily integrated, and its general solution has the form

$$\epsilon = -1: \quad f(\omega) = \alpha \operatorname{sh} F\omega + \beta \operatorname{ch} F\omega, \quad F = m + \lambda(\alpha^2 - \beta^2)^k$$
$$g(\omega) = \alpha \operatorname{ch} F\omega + \beta \operatorname{sh} F\omega; \tag{2.1.33}$$

$$\epsilon = +1: \quad f(\omega) = \alpha \sin F\omega + \beta \cos F\omega, \quad F = m + \lambda(\alpha^2 + \beta^2)^k$$
$$g(\omega) = \alpha \cos F\omega - \beta \sin F\omega; \tag{2.1.34}$$

where α and β are arbitrary constants, and ω, as follows from (2.1.31), is given by the expressions $\omega = ay + v(by + dy)$ for (2.1.33) and $\omega = dy$ for (2.1.34). When $N \neq 0$ and $m \neq 0$ we did not succeed in integrating the system (2.1.29). If $m = 0, N \neq 0$ one can seek solutions of system (2.1.29) in the form

$$f(\omega) = \frac{\alpha}{\omega^s}, \quad g(\omega) = \frac{\beta}{\omega^s}; \quad (\alpha, \beta, s \text{ are constants}) \tag{2.1.35}$$

Substituting these expressions into (2.1.32) and (2.1.29) gives

$$s\left(\alpha^2 + \epsilon\beta^2\right) = N\beta^2,$$
$$2sk = 1, \qquad s\alpha + \lambda\left(\alpha^2 + \epsilon\beta^2\right)^k = 0,$$
$$\beta(N - \epsilon s) + \lambda\alpha\left(\alpha^2 + \epsilon\beta^2\right)^k = 0,$$

whence follows

$$s = \frac{1}{2k}, \quad 2kN - \epsilon > 0. \quad \alpha = -\beta\sqrt{2kN - \epsilon}$$
$$\beta = \frac{(2kN - \epsilon)^{1/4k}}{(2k\lambda)^{1/2k}\sqrt{2kN}} \tag{2.1.36}$$

It is seen from (2.1.36) that solutions of Equation (2.1.1), determined by formulae (2.1.28) and (2.1.35), unlike solutions (2.1.26) are not analytical in the coupling constant λ. Therefore it is of interest to consider the case when $\lambda = 0$, $m \neq 0$, that is, the standard Dirac equation. System (2.1.29) under $\lambda = 0$ yields

$$\ddot{f} + \frac{\epsilon N}{\omega}\dot{f} + \epsilon m^2 f = 0,$$
$$\epsilon\ddot{g} + \frac{N}{\omega}\dot{g} + \left(m^2 - \frac{N}{\omega^2}\right)g = 0 \tag{2.1.37}$$

Equations (2.1.37), when $\epsilon = 1$, are reduced by the change of variables

$$f(\omega) = c_1\omega^{(1-N)/2} Z_{\pm\frac{1-N}{2}}(m\omega),$$
$$g(\omega) = c_2\omega^{(1-N)/2} Z_{\pm\frac{1+N}{2}}(m\omega) \tag{2.1.38}$$

(c_1, c_2 are arbitary constants) to the Bessel equation for cylindrical function $Z_j(m\omega)$. Under $\epsilon = -1$ solutions of Equations (2.1.37) can be expressed in terms of modified cylindrical functions.

To conclude this section we write down P(1,3)-nonequivalent ansatze for vector fields calculated according to formula $A_\mu = \overline{\psi}\gamma_\mu\psi$ [100]. Invariant variables ω and other notations are the same as in Table 2.1.3.

Table 2.1.4. P(1,3)-nonequivalent ansatze for vector fields.

N	Ansatze $A_\mu(x) =$
1 − 3	$B_\mu(\omega)$
4	$\left[g_{\mu\sigma} + \left(1 - \frac{bx}{\omega_1}\right)(b_\mu b_\sigma + a_\mu a_\sigma) + \frac{ax}{\omega_1}(b_\mu a_\sigma - b_\sigma a_\mu)\right] B^\sigma(\omega)$
5	$\left[g_{\mu\sigma} + \frac{(dx+cx)^2-1}{2(dx+cx)}(d_\mu c_\sigma - c_\mu d_\sigma) + \frac{(dx+cx-1)^2}{2(dx+cx)}(d_\mu d_\sigma - c_\mu c_\sigma)\right] B^\sigma(\omega)$
6	$\left[g_{\mu\sigma} + \frac{bx}{ax+dx}((d_\mu + a_\mu)b_\sigma - b_\mu(d_\sigma + a_\sigma)) + \frac{1}{2}\left(\frac{bx}{ax+dx}\right)^2 \cdot (d_\mu + a_\mu)(d_\sigma + a_\sigma)\right] B^\sigma(\omega)$
7	$\left[g_{\mu\sigma} + \frac{(dx+ax-1)^2}{2(dx+ax)}(d_\mu d_\sigma - a_\mu a_\sigma) + \frac{(dx+ax)^2-1}{2(dx+ax)}(d_\mu a_\sigma - d_\sigma a_\mu) + \left(1 - \frac{cx}{\omega_3}\right)(b_\mu b_\sigma + c_\mu c_\sigma) + \frac{bx}{\omega_3}(c_\mu b_\sigma - b_\mu c_\sigma)\right] B^\sigma(\omega)$
8	$\left[g_{\mu\sigma} + \left(1 - \frac{cx}{\omega_3}\right)(b_\mu b_\sigma + c_\mu c_\sigma) + \frac{bx}{\omega_3}(c_\mu b_\sigma - b_\mu c_\sigma)\right] B^\sigma(\omega)$
9	$\left[g_{\mu\sigma} + \left(1 - \frac{bx}{\omega_1}\right)(b_\mu b_\sigma + a_\mu a_\sigma) + \frac{ax}{\omega_1}(b_\mu a_\sigma - a_\mu b_\sigma)\right] B^\sigma(\omega)$
10	$\left[g_{\mu\sigma} + \left(1 - \frac{bx}{\omega_1}\right)(b_\mu b_\sigma + a_\mu a_\sigma) + \frac{ax}{\omega_1}(b_\mu a_\sigma - a_\mu b_\sigma)\right] B^\sigma(\omega)$
11	$\left[g_{\mu\sigma} + \frac{(dx+ax-1)^2-1}{2(dx+ax)}(d_\mu a_\sigma - d_\sigma a_\mu) + \frac{(dx+ax-1)^2}{2(dx+ax)}(d_\mu d_\sigma - a_\mu a_\sigma)\right] B^\sigma(\omega)$
12	$\left[g_{\mu\sigma} + \frac{1}{2}(dx + ax)\big(b_\mu(a_\sigma + d_\sigma) - (a_\mu + d_\mu)b_\sigma + \frac{1}{8}(dx + ax)^2(a_\mu + d_\mu)(a_\sigma + d_\sigma)\right] B^\sigma(\omega)$
13	$\left[g_{\mu\sigma} + \frac{bx}{ax+dx}(b_\sigma(a_\mu + d_\mu) - (a_\sigma + d_\sigma)b_\mu) + \frac{1}{2}\left(\frac{bx}{dx+ax}\right)^2(a_\mu + d_\mu)(a_\sigma + d_\sigma)\right] B^\sigma(\omega)$

Below we consider, as an example, how to calculate ansatz N5 from Table 2.1.4. Using ansatz N5 from Table 2.1.3 and the identity $(\gamma d\gamma c)^2 = 1$ it is easy to find

$$\psi(x) = \exp\left\{\tfrac{1}{2}\gamma d\gamma c \ln(dx + cx)\right\}\varphi(\omega) = \left[\sum_{n=0}^{\infty} \frac{1}{(2n)!}\left(\frac{\ln(dx + cx)}{2}\right)^{2n} + \right.$$

$$+\gamma d\gamma c\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(\frac{\ln(dx+cx)}{2}\right)^{2n+1}\Bigg]\varphi(\omega)=(\,\mathrm{ch}\,\tau+\gamma d\gamma c\,\mathrm{sh}\,\tau)\varphi(\omega),$$

where $\tau\equiv\frac{1}{2}\ln(dx+cx)$.

Next, we substitute $\psi(x)$ into the formula $A_\mu=\overline{\psi}\gamma_\mu\psi$. This gives

$$\begin{aligned}A_\mu(x)&=\overline{\varphi}(\omega)(\,\mathrm{ch}\,\tau-\gamma d\gamma c\,\mathrm{sh}\,\tau)\gamma_\mu(\,\mathrm{ch}\,\tau+\gamma d\gamma c\,\mathrm{sh}\,\tau)\varphi(\omega)=\\&=\overline{\varphi}(\omega)\gamma_\mu\varphi(\omega)\,\mathrm{ch}^2\tau+\,\mathrm{sh}\,\tau\,\mathrm{ch}\,\tau\overline{\varphi}(\omega)[\gamma_\mu,\gamma d\gamma c]\varphi(\omega)+\\&\quad+\overline{\varphi}(\omega)\gamma c\gamma d\gamma_\mu\gamma d\gamma c\varphi(\omega)\,\mathrm{sh}^2\tau\end{aligned}$$

Further, we use the identities

$$\begin{aligned}[\gamma_\mu,\gamma d\gamma c]&=2(d_\mu\gamma c-c_\mu\gamma d);\\\gamma c\gamma d\gamma_\mu\gamma d\gamma c&=\gamma c\gamma d\,(\gamma d\gamma c\gamma_\mu+[\gamma_\mu,\gamma d\gamma c])=\\&=-\gamma_\mu+2(d_\mu\gamma d-c_\mu\gamma c);\end{aligned}$$

$$\mathrm{ch}^2\tau-\,\mathrm{sh}^2\tau=1;$$

$$2\,\mathrm{sh}\,\tau\,\mathrm{ch}\,\tau=\,\mathrm{sh}\,2\tau=\tfrac{1}{2}\left(dx+cx-\frac{1}{dx+cx}\right)=\frac{(dx+cx)^2-1}{2(dx+cx)};$$

$$\mathrm{sh}^2\tau=\,\mathrm{ch}\,2\tau-1=\tfrac{1}{2}\left(dx+cx+\frac{1}{dx+cx}\right)-1=\frac{(dx+cx-1)^2}{2(dx+cx)};$$

Denoting $B_\sigma(\omega)=\overline{\varphi}(\omega)\gamma_\sigma\varphi(\omega)$ we finally obtain ansatz N5 from Table 2.1.4.

In much the same way one can construct ansatze for tensor fields $F_{\mu\nu}$ via the formula $F_{\mu\nu}=\frac{i}{2}\overline{\psi}(\gamma_m u\gamma_\nu-\gamma_\nu\gamma_\mu)\psi$.

2.2 *Reduction and exact solutions of the nonlinear massless Dirac equation*

Provided $m=0$, the symmetry of Equation (2.1.1) becomes wider. The equation

$$\left[i\gamma\partial+\lambda(\overline{\psi}\psi)^{1/2k}\right]\psi=0 \tag{2.2.1}$$

with an arbitrary $k\neq0$ is invariant under the extended Poincare group $\widetilde{\mathrm{P}}(1,3)$, the generators of the corresponding Lie algebra having the form of (2.1.11) and

$$D=x^\mu P_\mu+ik. \tag{2.2.2}$$

Table 2.2.1. $\widetilde{\mathrm{P}}(1,3)$-nonequivalent ansatze for spinor fields

N	Algebra	Invar. var. $\omega = \{\omega_1, \omega_2, \omega_3\}$	Ansatze $\psi(x) =$
1.	P_0	x_1, x_2, x_3	$\varphi(\omega)$
2.	P_1	x_0, x_1, x_3	$\varphi(\omega)$
3.	$P_0 + P_1$	$x_0 + x_1, x_2, x_3$	$\varphi(\omega)$
4.	D	$x_1/x_0, x_2/x_0, x_3/x_0$	$x_0^{-k}\varphi(\omega)$
5.	$J_{01} + P_2$	$x_0^2 - x_1^2,$ $\ln(x_0 + x_1) - x_2, x_3$	$\exp\{\frac{1}{2}\gamma_0\gamma_1\cdot$ $\cdot\ln(x_0 + x_1)\}\varphi(\omega)$
6.	$J_{01} + J_{12}+$ $+\alpha D, (\alpha \neq 0)$	$\frac{x_0^2 - x_1^2 - x_2^2}{x_3^2}, \frac{x_0 - x_2}{x_3},$ $\alpha\frac{x_1}{x_0 - x_2} - \ln(x_0 - x_2)$	$(x_0 - x_2)^{-k}\exp\left\{\frac{1}{2\alpha}\gamma_1\cdot\right.$ $\left.\cdot(\gamma_2 - \gamma_0)\ln(x_0 - x_2)\right\}\cdot$ $\cdot\varphi(\omega)$
7.	$J_{01} + J_{12}$	$x_0 - x_2, x_3,$ $x_0^2 - x_1^2 - x_2^2$	$\exp\{\frac{x_1}{2(x_0 - x_2}\cdot$ $\cdot\gamma_1(\gamma_0 - \gamma_2)\}\varphi(\omega)$
8.	$J_{01} + J_{12}+$ $+\beta P_3, (\beta \neq 0)$	$x_0 - x_2, x_0^2 - x_1^2 - x_2^2,$ $\beta x_1 - x_3(x_0 - x_2)$	$\exp\{\frac{x_3}{2\beta}\gamma_1(\gamma_2 - \gamma_0)\}\cdot$ $\cdot\varphi(\omega)$
9.	$J_{01} + J_{12}-$ $-P_0 + \beta P_3$	$x_3 + \beta(x_0 - x_2),$ $2x_1 + (x_0 - x_2)^2,$ $3x_3 + 3x_1(x_0 - x_2)+$ $+(x_0 - x_2)^3$	$\exp\left\{\frac{1}{2}\gamma_1(\gamma_2 - \gamma_0)\cdot\right.$ $\left.\cdot(x_2 - x_0)\right\}\varphi(\omega)$
10.	$J_{23} + \alpha D$	$\frac{x_0}{x_1}, \ln(x_2^2 + x_3^2)+$ $+2\alpha\arctan\frac{x_2}{x_3},$ $(x_2^2 + x_3^2)/x_0 x_1$	$(x_2^2 + x_3^2)^{-k/2}\cdot$ $\cdot\exp\{-\frac{1}{2}\gamma_1\gamma_2\cdot$ $\cdot\arctan\frac{x_2}{x_3}\}\varphi(\omega)$
11.	$J_{01} + \beta J_{23}+$ $+\alpha D,$ $(\alpha \neq -1)$	$(x_0 + x_1)^{2\alpha}\cdot$ $\cdot(x_0^2 - x_1^2)^{-(\alpha+1)},$ $(x_0^2 - x_1^2)/(x_2^2 + x_3^2),$ $\beta\ln(x_2^2 + x_3^2)+$ $+2\alpha\arctan\frac{x_2}{x_3}$	$(x_0^2 - x_1^2)^{-k/2}\cdot$ $\cdot\exp\{\frac{1}{2}\gamma_0\gamma_1\cdot$ $\cdot\frac{\ln(x_0 + x_1)}{\alpha + 1} - \frac{1}{2}\gamma_2\gamma_3\cdot$ $\cdot\arctan\frac{x_2}{x_3}\}\varphi(\omega)$
12.	$\beta J_{23} + J_{01}-$ $-D$	$x_0 + x_1, \frac{(x_0^2 - x_1^2)}{(x_2^2 + x_3^2)},$ $\beta\ln(x_2^2 + x_3^2)-$ $-2\arctan\frac{x_2}{x_3},$	$(x_0^2 - x_1^2)^{-k/2}\exp\{-\frac{1}{4}\cdot$ $\cdot\gamma_0\gamma_1\ln(x_0 + x_1)-$ $-\frac{1}{2}\gamma_2\gamma_3\arctan\frac{x_2}{x_3}\}\varphi(\omega)$
13.	$J_{01} + \alpha J_{23}+$ $D + \beta P_0$	$[2(x_0 + x_1) + \beta]e^{2(x_1 - x_0)/\beta},$ $[2(x_0 + x_1) + \beta]\,(x_2^2 + x_3^2)^{-1},$ $\alpha\ln(x_2^2 + x_3^2)+$ $+2\arctan\frac{x_2}{x_3}$	$[2(x_0 + x_1) + \beta]^{-k/2}\cdot$ $\cdot\exp\{+\frac{1}{4}\gamma_0\gamma_1\cdot$ $\cdot\ln\,[2(x_0 + x_1) + \beta]-$ $-\frac{1}{2}\gamma_2\gamma_3\arctan\frac{x_2}{x_3}\}\varphi(\omega)$
14.	$J_{23} + \alpha_0 P_0+$ $+\alpha_1 P_1$	$x_2^2 + x_3^2, \arctan\frac{x_2}{x_3}+$ $+\beta_0 x_0 + \beta_1 x_1,$ $\alpha_1 x_0 + \alpha_0 x_1,$ $\alpha_1\beta_1 - \alpha_0\beta_0 = 1$	$\exp\{-\frac{1}{2}\gamma_2\gamma_3\cdot$ $\cdot\arctan\frac{x_2}{x_3}\}\varphi(\omega)$

In the same way as in the previous section we shall construct the $\widetilde{\mathrm{P}}(1,3)$-nonequivalent ansatze for spinor fields [109] ($\widetilde{\mathrm{P}}(1,3)$-nonequivalent one-dimensional subalgebras of $A\widetilde{\mathrm{P}}(1,3)$ are found in [109, 11*]; see also Appendix 2) and list them in the Table 2.2.1., in which $\alpha, \beta, \alpha_0, \beta_0, \alpha_1, \beta_1$ are arbitrary constants satisfying the indicated conditions.

Let us substitute ansatze 1–14 from Table 2.2.1 into Equation (2.2.1). Omitting some rather cumbersome calculations, we write down the resulting reduced system of PDEs for the function $\varphi(\omega)$:

$$(1)\ \gamma_a\varphi_a = iF\varphi, \tag{2.2.3}$$

$$(2)\ \gamma_0\varphi_1 + \gamma_2\varphi_2 + \gamma_3\varphi_3 + iF\varphi = 0,$$

$$(3)\ (\gamma_0 + \gamma_1)\varphi_1 + \gamma_2\varphi_2 + \gamma_3\varphi_3 = iF\varphi,$$

$$(4)\ -k\gamma_0\varphi + (\gamma_a - \gamma_0\omega_a)\varphi_a = iF\varphi$$

$$(5)\ \tfrac{1}{2}(\gamma_0+\gamma_1)\varphi+[(\gamma_0(\omega_1 + 1) + \gamma_1(\omega_1 - 1)]\,\varphi_1+(\gamma_0+\gamma_1-\gamma_2)\varphi_2+ +\gamma_3\varphi_3 = iF\varphi;$$

$$(6)\ (k\gamma_2 - \gamma_0)\varphi + \left[(\gamma_0 - \gamma_2)(\omega_1 + \alpha^{-2}\omega_2^2\omega_3^2) + (\gamma_0 + \gamma_2)\omega_2^2 - 2\alpha^{-1}\gamma_1\omega_3\omega_2^2 - \right.$$
$$\left. -2\gamma_3\omega_1\omega_2\right]\varphi_1 + \left[(\gamma_0 - \gamma_2)\omega_2 - \gamma_3\omega_2^2\right]\varphi_2 + [\alpha\gamma_1 + (\gamma_2 - \gamma_0)(\omega_3 + 1)]\varphi_3 = iF\varphi;$$

$$(7)\ \frac{1}{2\omega_1}(\gamma_0 - \gamma_2)\varphi + (\gamma_0 - \gamma_2)\varphi_1 + \gamma_3\varphi_2 +$$
$$+ \left[(\gamma_0 + \gamma_2)\omega_1 + (\gamma_0 - \gamma_2)\omega_3\omega_1^{-1}\right]\varphi_3 = iF\varphi;$$

$$(8)\ \frac{1}{2\beta}\gamma_4(\gamma_2 - \gamma_0)\varphi + (\gamma_0 - \gamma_2)\varphi_1 + \left[(\gamma_0 + \gamma_2)\omega_1 - 2\beta^{-1}\gamma_1\omega_3 + \right.$$
$$\left. +(\gamma_0 - \gamma_2)(\beta^{-2}\omega_3^2 + \omega_2)\omega_1^{-1}\right]\varphi_2 + (\beta\gamma_1 - \gamma_3\omega_1)\varphi_3 = iF\varphi;$$

$$(9)\ [\gamma_3 + \beta(\gamma_0 - \gamma_2)]\,\varphi_1 + 2\gamma_1\varphi_2 + \frac{3}{2}\left(2\gamma_2 + (\gamma_0 - \gamma_2)\omega_2\right)\varphi_3 = iF\varphi;$$

$$(10)\ \tfrac{1}{2}(1 - 2k)\gamma_3\varphi + \sqrt{\omega_1\omega_3}(\gamma_0 - \gamma_1\omega_1)\varphi_1 + 2(\gamma_3 + \alpha\gamma_2)\varphi_2 +$$
$$+ \left[2\gamma_3 - (\gamma_0 + \gamma_1\omega_1)\sqrt{\frac{\omega_3}{\omega_1}}\right]\omega_3\varphi_3 = iF\varphi;$$

$$(11)\ \left[-k\left(\gamma_0 \operatorname{ch}\ln\omega_1^{\frac{1}{2}(\alpha+1)} - \gamma_1 \operatorname{sh}\ln\omega_1^{\frac{1}{2}(\alpha+1)}\right) + \tfrac{1}{2}(\alpha+1)^{-1}(\gamma_0+\gamma_1)\omega_1^{-1/2(\alpha+1)}\right.$$
$$\left. +\tfrac{1}{2}\gamma_3\sqrt{\omega_2}\right]\varphi - 2(\alpha + 1)\omega_1\left(\gamma_0 \operatorname{ch}\ln\omega_1^{1/2(\alpha+1)} - \gamma_1 \operatorname{sh}\ln\omega_1^{\frac{1}{2}(\alpha+1)}\right)\varphi_1 +$$
$$+2\left[\gamma_0 \operatorname{ch}\ln\omega_1^{1/2(\alpha+1)} - \gamma_1 \operatorname{sh}\ln\omega_1^{1/2(\alpha+1)} - \gamma_3\sqrt{\omega_2}\right]\omega_2\varphi_2 +$$
$$+2(\alpha\gamma_2 + \beta\gamma_3)\sqrt{\omega_2}\varphi_3 = iF\varphi;$$

$$(12)\ \left[-k\left(\gamma_0 \operatorname{ch} \ln \sqrt{\omega_1} - \gamma_1 \operatorname{sh} \ln \sqrt{\omega_1}\right) + \tfrac{1}{4}(\gamma_0 - \gamma_1)\sqrt{\omega_1} + \tfrac{1}{2}\gamma_3\sqrt{\omega_2}\right]\varphi +$$
$$+(\gamma_0 + \gamma_1)\sqrt{\omega_1} + 2\omega_2(\gamma_0 \operatorname{ch} \ln \sqrt{\omega_1}\,\varphi_1 - \gamma_1 \operatorname{sh} \ln \sqrt{\omega_1} - 2\gamma_3\sqrt{\omega_2})\varphi_2 +$$
$$+2(\beta\gamma_3 - \gamma_2)\sqrt{\omega_2}\,\varphi_3 = iF\varphi;$$

$$(13)\ \tfrac{1}{2}\left[(1-2k)(\gamma_0 + \gamma_1) + \gamma_3\omega_2\right]\varphi + 2\left[(\beta - 1)\gamma_0 + (\beta + 1)\gamma_1\right]\omega_1\varphi_1 +$$
$$+2\omega_2\left(\gamma_0 + \gamma_1 - \sqrt{\omega_2}\gamma_3\right)\varphi_2 + 2(\gamma_2 + \alpha\gamma_3)\sqrt{\omega_2}\varphi_3 = iF\varphi;$$

$$(14)\ \frac{1}{2\sqrt{\omega_1}}\varphi + 2\sqrt{\omega_1}\,\gamma_3\varphi_1 + (\omega_1^{-1/2}\gamma_2 + \beta_0\gamma_0 + \beta_1\gamma_1)\varphi_2 +$$
$$+(\alpha_1\gamma_0 + \alpha_0\gamma_1)\varphi_3 = iF\varphi.$$

In these formulae $\varphi_a \equiv \partial\varphi/\partial\omega_a$; $F \equiv \lambda(\overline{\varphi}\varphi)^{1/2k}$; an equation for the number n, where $n = 1, 2, \ldots, 14$, is obtained by the ansatz with the same number n from Table 2.2.1.

Next we perform direct reduction everywhere possible of Equations (2.2.3) to ODEs or two-dimensional PDEs. This means that we will suppose that φ is dependent on one or two variables from $\omega_1, \omega_2, \omega_3$. Omitting cases similar to those considered in the previous section we have

$$(4)\ -k\gamma_0\varphi + (\gamma_a - \omega_a\gamma_0)\varphi_a = iF\varphi; \tag{2.2.4}$$

$$(6)\ k(\gamma_2 - \gamma_0)\varphi + \omega_2(\gamma_0 - \gamma_2 - \omega_2\gamma_3)\varphi_2 = iF\varphi;$$

$$(6')\ k(\gamma_2 - \gamma_0)\varphi + [(\gamma_2 - \gamma_0)(1 + \omega_3) + \alpha\gamma_1])\varphi_3 = iF\varphi;$$

$$(9)\ [\gamma_3 + \beta(\gamma_0 - \gamma_2)]\,\varphi_1 = iF\varphi;$$

$$(9')\ 2\gamma_1\varphi_2 = iF\varphi;$$

$$(9'')\ [\gamma_3 + \beta(\gamma_0 - \gamma_2)]\,\varphi_1 + 2\gamma_1\varphi_2 = iF\varphi;$$

$$(10)\ \tfrac{1}{2}(1 - 2k)\gamma_3\varphi + 2(\gamma_3 + \alpha\gamma_2)\varphi_2 = iF\varphi.$$

Here we have used the same notations as in (2.2.3).

The general solution of Equation (9′) from (2.2.4) has the form

$$\varphi(\omega_2) = \exp\left\{-\frac{i}{2}æ\gamma_1\omega_2\right\}\chi, \qquad æ \equiv \lambda\left(\overline{\chi}\chi\right)^{1/2k},$$

whence follows a solution of the initial Equation (2.2.1)

$$\psi(x) = \exp\left\{\frac{\gamma_1}{2}(\gamma_0 - \gamma_2)(x_0 - x_2)\right\} \times \tag{2.2.5}$$
$$\times \exp\left\{-\frac{i}{2}æ\gamma_1\left[2x_1 + (x_0 - x_2)^2\right]\right\}\chi$$

Next we consider Equation (10) from (2.2.4). Under $k = 1/2$ its general solution has the form

$$\varphi(\omega_2) = \exp\left\{-\tfrac{1}{2}\lambda(\overline{\chi}\chi)(1+\alpha^2)^{-1}(\gamma_3+\alpha\gamma_2)\omega_2\right\}\chi \tag{2.2.6}$$

Under $k \neq \frac{1}{2}$ and $\alpha \neq 0$ we did not succeed in integrating Equation (10) from (2.2.4). However, if $\alpha = 0$, then making a change of variables

$$\phi(\omega_2) = \exp\left\{\tfrac{1}{4}(2k-1)\omega_2\right\}\phi(\omega_2)$$

we obtain

$$2\exp\left\{\tfrac{1}{4}(1-2k)k^{-1}\omega_2\right\}\gamma_3\phi_2 = i\lambda(\overline{\phi}\phi)^{1/2k}\phi.$$

The general solution of the last equation has the form

$$\phi(\omega_2) = \exp\left\{\frac{2i\lambda k}{1-2k}(\overline{\chi}\chi)^{1/2k}\exp\left\{\frac{1}{4}\frac{2k-1}{k}\omega_2\gamma_3\right\}\right\}\chi.$$

Substituting the above results into ansatz N10 from Table 2.2.1, we obtain the following solutions of Equation (2.2.1): if $k = 1/2$:

$$\psi(x) = (x_2^2+x_2^2)^{-1/4}\exp\left\{-\tfrac{1}{2}\gamma_2\gamma_3\arctan\frac{x_2}{x_3}\right\}\times \tag{2.2.7}$$

$$\times\exp\left\{-\frac{i}{2}\lambda\overline{\chi}\chi(1+\alpha^2)^{-1}(\gamma_3+\alpha\gamma_2)\left[\ln(x_2^2+x_2^2)+2\alpha\arctan\frac{x_2}{x_3}\right]\right\}\chi;$$

if $k \neq 1/2$:

$$\psi(x) = (x_2^2+x_2^2)^{-1/4}\exp\left\{-\tfrac{1}{2}\gamma_2\gamma_3\arctan\frac{x_2}{x_3}\right\}\times \tag{2.2.8}$$

$$\times\exp\left\{\frac{2i\lambda k}{1-2k}(\overline{\chi}\chi)^{1/2k}(x_2^2+x_2^2)^{(2k-1)/4}\gamma_3\right\}\chi.$$

Next we consider Equation (9″) from (2.2.4). This two-dimensional PDE can be reduced to the two-dimensional Dirac equation. Having made a change of variables

$$z_1 = \omega_1, \quad z_2 = \tfrac{1}{2}\omega_2$$

and denoting

$$\Gamma_1 = \gamma_3 + \beta(\gamma_0 - \gamma_2), \qquad \Gamma_2 = \gamma_1,$$

we obtain

$$\Gamma_a\frac{\partial\varphi}{\partial z_a} = i\lambda(\overline{\varphi}\varphi)^{1/2k}\varphi; \qquad \Gamma_a\Gamma_b + \Gamma_b\Gamma_a = 2g_{ab}; \qquad a,b = 1,2. \tag{2.2.9}$$

We look for a solution of (2.2.9) in the form

$$\varphi(z) = [\Gamma_a z_a f(z_b z_b) + ig(z_b z_b)]\chi, \tag{2.2.10}$$

where f and g are unknown scalar functions. Substituting (2.2.10) into (2.2.9) gives the system of ODEs

$$f + \omega \dot{f} = \tfrac{1}{2}\lambda(\overline{\chi}\chi)^{1/2k}\left(g^2 - \omega f^2\right)^{1/2k} g,$$

$$\dot{g} = \tfrac{1}{2}\lambda(\overline{\chi}\chi)^{1/2k}\left(g^2 - \omega f^2\right)^{1/2k} f,$$

where dot means differentiation with respect to $\omega = z_a z_a$. A partial solution of this system is given by the formulae:

$k < 0$,

$$f \equiv f(\omega) = \sqrt{|k|}\left(\mp\frac{\left(k^2 + |k|\right)^{1/2}}{\lambda(\overline{\chi}\chi)^{1/2k}}\right)^{k} \omega^{-(k+1)/2}, \tag{2.2.11}$$

$$g \equiv g(\omega) = \mp\left(1 + |k|^{-1}\right)^{-1/2}\left(\mp\frac{\left(k^2 + |k|\right)^{1/2}}{\lambda(\overline{\chi}\chi)^{1/2k}}\right)^{k} \omega^{-(k/2)}$$

Since Equation (2.2.9) with $k = 1/2$ is conformally invariant with respect to AC(1,1) (see Paragraph 2.3) one can use the ansatz

$$\varphi(z) = \frac{\Gamma_a z_a}{z_b z_b}\phi(\omega), \qquad \omega = \frac{\beta_a z_a}{z_b z_b}; \qquad a, b = 1, 2, \tag{2.2.12}$$

which reduces (2.2.9) to the system of ODEs

$$\Gamma_a \beta_a \dot{\phi} = i\lambda(\overline{\phi}\phi)\phi,$$

whose general solution has the form

$$\phi(\omega) = \exp\left\{-i\lambda(\overline{\chi}\chi)(\beta_1^2 + \beta_2^2)(\Gamma_a\beta_a)\omega\right\}\chi \tag{2.2.13}$$

So, ansatz N9 from the Table 2.2.1 and formulae (2.2.10)–(2.2.13) give the following solutions of Equation (2.2.1): if $k < 0$,

$$\psi(x) = \exp\left\{\tfrac{1}{2}\gamma_1(\gamma_0 - \gamma_2)(x_0 - x_2)\right\}\Big\langle\left\{[\gamma_3 + \beta(\gamma_0 - \gamma_2)]\times\right. \tag{2.2.14}$$

$$\left.\times\left[x_3 + \beta(x_0 - x_2)\right] + \tfrac{1}{2}\gamma_1\left[2x_1 + (x_0 - x_2)^2\right\} f(\omega) + ig(\omega)\Big\rangle\chi;$$

if $k = \frac{1}{2}$,

$$\psi(x) = \exp\left\{\tfrac{1}{2}\gamma_1(\gamma_0 - \gamma_2)(x_0 - x_2)\right\}\Big\{[\gamma_3 + \beta(\gamma_0 - \gamma_2)]\times \tag{2.2.15}$$

$$\times\left[x_3 + \beta(x_0 - x_2)\right] + \tfrac{1}{2}\gamma_1\left[2x_1 + (x_0 - x_2)^2\right]\Big\}\omega^{-1}\times$$

$$\times \exp\Big\{-i\lambda(\overline{\chi}\chi)(\beta_1^2+\beta_2^2)^{-1}\Big\langle\beta_1\left[\gamma_3+\beta(\gamma_0-\gamma_2)\right]+\tfrac{1}{2}\beta_2\gamma_1\Big\rangle\times$$

$$\times\Big\langle\beta_1 x_3+\beta(x_0-x_2)\big]+\tfrac{1}{2}\beta_2\left[2x_1+(x_0-x_2)^2\right]\Big\rangle\omega^{-1}\Big\}\chi.$$

In (2.2.14) and (2.2.15) f, g are given in (2.2.11) and

$$\omega=\left[x_3+\beta(x_0-x_2)\right]^2+\tfrac{1}{4}\left[2x_1+(x_0-x_2)^2\right]^2, \tag{2.2.16}$$

Consider the ansatze [31*]

$$\psi(x)=\exp\left\{\tfrac{1}{2}\gamma_0\gamma_3\ln(x_0+x_3)\right\}\varphi(x_0^2-x_3^2). \tag{2.2.17}$$

$$\psi(x)=\exp\left\{\frac{x_1}{2(x_0+x_3)}(\gamma_0+\gamma_3)\gamma_1\right\}\exp\left\{\tfrac{1}{2}\gamma_0\gamma_3\ln(x_0+x_3)\right\}\varphi(x_0^2-x_1^2-x_2^2); \tag{2.2.18}$$

$$\psi(x)=\exp\left\{\frac{1}{2(x_0+x_3)}(\gamma_0+\gamma_3)(\gamma_1x_1+\gamma_2x_2\right\}\times$$

$$\times\exp\left\{\tfrac{1}{2}\gamma_0\gamma_3\ln(x_0+x_3)\right\}\varphi(x_\nu x^\nu) \tag{2.2.19}$$

They reduce Equation (2.2.1) to the systems of ODEs

$$4\omega\dot\varphi=-\left\{s(1+\gamma_0\gamma_3)-i\lambda(\overline{\varphi}\varphi)^{1/2k}\left[\omega(\gamma_0+\gamma_3)+\gamma_0-\gamma_3\right]\right\}\varphi, \tag{2.2.20}$$

with $s=1,2,3$; $\omega=\{x_0^2-x_3^2; x_0^2-x_1^2-x_2^2; x_\nu x^\nu\}$in accord with (2.2.17)-(2.2.19) and $\dot\varphi\equiv d\varphi/d\omega$. From (2.2.20) it follows that

$$4\omega\frac{d}{d\omega}(\overline{\varphi}\varphi)=-2s\,(\overline{\varphi}\varphi).$$

Hence

$$\overline{\varphi}\varphi=c\omega^{-s/2},\quad c=\text{const} \tag{2.2.21}$$

and thereby system (2.2.20) becomes the linear system

$$4\omega\dot\varphi=-\left\{s(1+\gamma_0\gamma_3)-i\lambda\left(c\omega^{-s/2}\right)^{1/2k}\left[\omega(\gamma_0+\gamma_3)+\gamma_0-\gamma_3\right]\right\}\varphi.$$

Writing it out we obtain

$$\begin{aligned}
&2\omega\dot\varphi^0=i\lambda'\omega^{\alpha+1}\varphi^2,\quad \lambda'=\lambda c^{1/2k},\ \alpha=-s/4k,\\
&2\omega\dot\varphi^1=-s\varphi^1+i\lambda'\omega^{\alpha}\varphi^3,\\
&2\omega\dot\varphi^2=-s\varphi^2+i\lambda'\omega^{\alpha}\varphi^0,\\
&2\omega\dot\varphi^3=is\omega^{\alpha+1}\varphi^1
\end{aligned}$$

whence follows

$$\omega^2\ddot{\varphi}^0 + \frac{s-2\alpha}{2}\omega\dot{\varphi}^0 + \frac{\lambda'^2}{4}\omega^{2\alpha+1}\varphi^0 = 0,$$
$$\omega^2\ddot{\varphi}^3 + \frac{s-2\alpha}{2}\omega\dot{\varphi}^3 + \frac{\lambda'^2}{4}\omega^{2\alpha+1}\varphi^3 = 0,$$
$$\varphi^2 = -\frac{2i}{\lambda'}\omega^{-\alpha}\dot{\varphi}^0, \tag{2.2.22}$$
$$\varphi^1 = -\frac{2i}{\lambda'}\omega^{-\alpha}\dot{\varphi}^3.$$

The general solution of Equations (2.2.22) has the form

$$\varphi^0 = \omega^{(1+\alpha-s/2)/2}\left(J_\ell(z)\chi^0 + Y_\ell(z)\chi^2\right), \tag{2.2.23}$$
$$\varphi^3 = \omega^{(1+\alpha-s/2)/2}\left(J_e(z)\chi^3 + Y_e(z)\chi^1\right),$$
$$\varphi^2 = -i\Big\{\omega^{-(1+\alpha-s/2)/2}(1+\alpha-s/2)(\lambda')^{-1}\left(J_\ell(z)\chi^0 + Y_\ell(z)\chi^2\right) +$$
$$+\omega^{(\alpha-s/2)/2}\left(\dot{J}_\ell(z)\chi^0 + \dot{Y}_\ell(z)\chi^1\right)\Big\},$$
$$\varphi^1 = -i\Big\{\omega^{-(1+\alpha+s/2)/2}(1+\alpha-s/2)(\lambda')^{-1}\left(J_\ell(z)\chi^3 + Y_\ell(z)\chi^1\right) +$$
$$+\omega^{(\alpha-s/2)/2}\left(\dot{J}_\ell(z)\chi^3 + \dot{Y}_\ell(z)\chi^1\right)\Big\},$$

where $J_\ell(z)$, $Y_\ell(z) = \frac{1}{\sin \ell\pi}\left[J_\ell(z)\cos\ell\pi - J_{-\ell}(z)\right]$ are Bessel functions, and

$$z = \frac{\lambda'}{2\alpha+1}\omega^{(2\alpha+1)/2}; \qquad \lambda' = \lambda c^{1/2k},$$
$$\alpha = -\frac{s}{4k}; \qquad l = \frac{1+\alpha-s/2}{2\alpha+1}. \tag{2.2.24}$$

Using (2.2.23) and the identity

$$J_\ell\dot{Y}_\ell - \dot{J}_\ell Y_\ell = \frac{2}{\pi z}$$

one can find the constant c from (2.2.21) explicitly.

Solutions (2.2.5)–(2.2.8) and (2.2.14)–(2.2.16), as well as (2.2.17)–(2.2.19) and (2.2.23) can be made $\widetilde{\mathrm{P}}(1,3)$-ungenerative by means of the formulae of GS from Table 2.1.2 and the formula of GS by the scale transformations

$$\psi_{II}(x) = e^{+k\theta}\psi_I(x'), \quad x'_\mu = e^\theta x_\mu, \quad \theta = \text{const}. \tag{2.2.25}$$

For example, solution (2.2.5) results in the following $\widetilde{\mathrm{P}}(1,3)$-ungenerative solution of Equation (2.2.1)

$$\psi(x) = \exp\left\{\frac{\theta}{2}\gamma a(\gamma d - \gamma b)(dy - by)\right\}\exp\Big\{-\tfrac{1}{2}i\lambda(\overline{\chi}\chi)^{1/2k}(\gamma a)\times$$
$$\times\left[2ay + \theta(dy - by)^2\right]\Big\}\chi. \qquad (2.2.26)$$

As another example, the $\widetilde{\mathrm{P}}(1,3)$-ungenerative solution of Equation (2.2.1) with $k = 1/2$ resulting from (2.2.7) is:

$$\psi(x) = \left[(by)^2 + (cy)^2\right]^{-1/4}\exp\left\{-\tfrac{1}{2}\gamma b\gamma c \arctan\frac{by}{cy}\right\}\times$$
$$\times\exp\Big\{-\frac{i\lambda}{1+\theta^2}(\overline{\chi}\chi)(\gamma c + \theta\gamma b)\times \qquad (2.2.27)$$
$$\times\left[\ln\left((by)^2 + (cy)^2\right) + 2\theta\arctan\frac{by}{cy}\right]\Big\}\chi;$$

In formulae (2.2.26) and (2.2.27) $y_\mu = x_\mu + \delta_\mu; \delta_\mu, \theta, a_\mu, b_\mu, c_\mu, d_\mu$ are arbitrary constants satisfying relations (2.1.27).

We do not list $\widetilde{\mathrm{P}}(1,3)$-ungenerative solutions of Equation (2.2.1) following from solutions (2.2.8), (2.2.14), (2.2.15),(2.2.17)–(2.2.19) because the astute reader will no doubt be able to do so alone. For the same reason, and for the sake of brevity, we do not present covariant counterparts of $\widetilde{\mathrm{P}}(1,3)$-nonequivalent ansatze for spinor fields, which can be constructed using the procedure of GS applied to the ansatze from Table 2.2.1.

2.3. *Conformal symmetry and formula of generating solutions for fields of arbitrary spin. C(1,3)-ungenerative ansätze.*

1. Conformal transformations are just a superposition of the inversion

$$x_\mu \to \widetilde{x}_\mu = \frac{x_\mu}{x^\nu x_\nu}, \qquad (2.3.1)$$

followed by translation and another inversion

$$x_\mu \to x'_\mu = \frac{\widetilde{x}_\mu - c_\mu}{(\widetilde{x}_\nu - c_\nu)(\widetilde{x}^\nu - c^\nu)} = \frac{x_\mu - c_\mu x^2}{\sigma(c,x)},$$
$$\sigma(c,x) = 1 - 2cx + c^2x^2; \qquad (c^2 \equiv c^\nu c_\nu,\ c_\mu = \text{const}). \qquad (2.3.2)$$

Transformations of inversion had been used in potential theory long ago to construct solutions of the Laplace equation (the so-called Kelvin transformation discovered in 1847 (see [205])). In 1909, Bateman and Cunningham [25, 43] had discovered that Maxwell's equations were conformally invariant. Later, it became clear that many important equations of mathematical and theoretical physics possessed such a symmetry.

The extended Poincare group supplemented by conformal transformations form the fifteen-parameter conformal group C(1,3). The general form of the basis elements of the corresponding Lie algebra AC(1,3) (up to nilpotent matrices which can be added to D and K_μ; see [145]) in the case of linear representation are as follows:

$$
\begin{aligned}
&P_0 = i\partial_0, \quad P_a = -i\partial_a, \quad (a = 1,2,3) \\
&J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu + S_{\mu\nu} \\
&D = x^\nu P_\nu + ik, \\
&K_\mu = 2x_\mu D - x^2 P_\mu + 2S_{\mu\nu}x^\nu,
\end{aligned}
\tag{2.3.3}
$$

where the constant k is the so-called conformal degree; $S_{\mu\nu} = -S_{\nu\mu}$ are matrices that provide a finite-dimensional representation of AO(1,3) (on representations of AO(1,3) see [111]).

The operators (2.3.3) satisfy the following commutation relations of AC(1,3):

$$
\begin{aligned}
&[P_\mu, P_\nu] = 0, [P_\sigma, J_{\mu\nu}] = i(g_{\sigma\mu}P_\nu - g_{\sigma\nu}P_\mu), \\
&[J_{\mu\nu}, J_{\rho\sigma}] = i(g_{\nu\rho}J_{\mu\sigma} + g_{\mu\sigma}J_{\nu\rho} - g_{\nu\sigma}J_{\mu\rho} - g_{\mu\rho}J_{\nu\sigma}), \\
&[P_\mu, D] = iP_\mu, \qquad [J_{\mu\nu}, D] = 0, \\
&[K_\mu, K_\nu] = 0, \qquad [K_\sigma, J_{\mu\nu}] = i(g_{\sigma\mu}K_\nu - g_{\sigma\nu}K_\mu), \\
&[K_\mu, D] = -iK_\mu, \\
&[K_\mu, P_\nu] = 2i(g_{\mu\nu}D - J_{\mu\nu}).
\end{aligned}
\tag{2.3.4}
$$

It will be noted that the algebra AC(1,3) is locally isomorphic to AO(2,4) (see, for example, [21]). The isomorphism is achieved by means of introducing additional "Lorentz" generators

$$J_{\mu 4} = \tfrac{1}{2}(P_\mu - K_\mu), \quad K_{\mu 5} = \tfrac{1}{2}(P_\mu + K_\mu), \quad J_{45} = -D,$$

which satisfy, together with $J_{\mu\nu}$, the commutation relations of AO(2,4)

$$[J_{AB}, J_{CD}] = i(g_{BC}J_{AD} + g_{AD}J_{BC} - g_{AC}J_{BD} - g_{BD}J_{AC}),$$

where $A, B, C, D = \overline{0,5}$, $g_{AB} = \{1, -1, -1, -1, -1, 1\}$. Besides that one can choose from AC(1,3) two Poincare algebras: the standard one, $\mathrm{AP}_1(1,3) \equiv \mathrm{AP}(1,3) = \{P_\mu, J_{\mu\nu}\}$ and $\mathrm{AP}_2(1,3) = \{K_\mu, J_{\mu\nu}\}$. Using subalgebraic classification of the Poincare algebra [163] one can construct, by analogy with Sections 2.1 and 2.2, new sets of AP_2(1,3)-inequivalent ansatze.

It can be seen from (2.3.3) that the algebra AC(1,3) is uniquely determined by the conformal degree k and the representation of AO(1,3), which is provided by the matrices $S_{\mu\nu}$. We point out these characteristics for some PDEs in the following table. As will become clear from what follows, the conformal degree may be considered a fundamental characteristic of a field, like spin.

Table 2.3.1. Conformally invariant PDEs and characteristics of their AC(1,3) (2.3.3)

N	Equation	Conformal degree	Matrix representaions
1	Wave $\Box u + \lambda u^3 = 0$	1	D(0,0)
2	Polywave $\Box^s u + \lambda u^{\frac{2+s}{2-s}} = 0$	$2-s$	D(0,0)
3	Eikonal $(\partial_\nu u)(\partial^\nu u) = 0$	0	D(0,0)
4	Continuity $\partial_\mu j^\mu = 0$	3	$D(1/2, 1/2)$
5	Weyl $i\sigma_\mu \partial^\mu \varphi = 0$	3/2	$D(0, 1/2)$, $D(1/2, 0)$
6	Dirac-Gursey $\left[i\gamma\partial + \lambda(\overline{\psi}\psi)^{1/3}\right]\psi = 0$	3/2	$D(0, 1/2) \oplus D(1/2, 0)$
7	Generalized Dirac-Gursey $\left[i\gamma\mathcal{D} + \lambda(\overline{\psi}\psi)^{1/2k}\right]\psi = 0,$ $\mathcal{D}_\mu = \partial_\mu + \frac{1}{2k}\left(\frac{3}{2} - k\right)\partial_\mu \ln(\overline{\psi}\psi)$	k	$D(0, 1/2) \oplus D(1/2, 0)$
8	Maxwell electromagnetic field $\partial_\mu F^{\mu\nu} = 0, \quad \partial_\mu \widetilde{F}^{\mu\nu},$ $\left(F^{0a} = E_a, \quad F^{ab} = \epsilon_{abc} H_c,\right.$ $\left.\widetilde{F}^{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}\right)$	2	$D(1, 0) \oplus D(0, 1)$
9	Generalized Maxwell equation for electromagnetic field $\mathcal{D}_\mu F^{\mu\nu} = 0, \quad \mathcal{D}_\mu \widetilde{F}^{\mu\nu} = 0$ $\mathcal{D}_\mu = \partial_\mu + \partial_\mu \ln\left(\mathcal{T}_1^\alpha \mathcal{T}_2^\beta\right),$ $\mathcal{T}_1 = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = \vec{E}^2 - \vec{H}^2,$ $\mathcal{T}_2 = \frac{1}{4}F_{\mu\nu}\widetilde{F}^{\mu\nu} = \vec{E}\cdot\vec{H}, \alpha + \beta = \frac{2-k}{2k},$	k	$D(1, 0) \oplus D(0, 1)$
10	Maxwell vector potential $\Box A_\mu - \partial_\mu \partial^\nu A_\nu = \lambda A_\mu A^\nu A_\nu$	1	$D(1/2, 1/2)$
11	Yang-Mills $\partial^\nu G^a_{\mu\nu} = e\epsilon_{abc} G^a_{\mu\nu} Y^{c\nu}$ $G^a_{\mu\nu} = \partial_\mu Y^a_\nu - \partial_\nu Y^a_\mu + e\epsilon_{abc} Y^b_\mu Y^c_\nu$	1	$\bigoplus_{i=1}^{3} D(1/2, 1/2)$

Remark 2.3.1. Eikonal (N3), Weyl (N5), and Maxwell's (N8) equations do not admit generalization by nonlinear addends constructed from the fields (without derivatives).

Remark 2.3.2. Of course, the equations listed in Table 2.3.1 do not exhaust all conformally invariant equations. There are many PDEs which are invariant under nonlinear representations of conformal algebra. For example, the relativistic Hamilton-Jacobi equation (1.2.1) is invariant under AC(1,4), with basis elements (1.2.2) containing nonlinear addends on u; see also Remark 1.3.1. (page 13). Another example of representation AC(1,3) different from (2.3.3) is given at the end of Section 2.8.

As distinct from transformations of the Poincare group (see Table 2.1.2) the special conformal transformations (2.3.2) are nonlinear. As will be shown below, this circumstance leads to highly nontrivial formulae of generating solutions [188].

To find an explicit form of transformations generated by the operator

$$\mathcal{K} = 2cxx\partial - x^2 c\partial - 2(iS_{\mu\nu}x^\nu - kx_\mu)c^\mu, \tag{2.3.5}$$

which is a linear combination of operators K_μ from (2.3.3), one has to solve the following Cauchy problem (the system of Lie equations)

$$\frac{\partial x'_\mu}{\partial c^\nu} = 2x'_\mu x'_\nu - x'_\sigma x'^5 g_{\mu\nu}, \qquad x'_\mu \Big|_{c^\nu=0} = x_\mu; \tag{2.3.6}$$

$$\frac{\partial \psi'(x')}{\partial c^\nu} = 2(iS_{\mu\nu}x'_\nu - kx'_\mu)\psi'(x'), \qquad \psi'(x')\Big|_{c^\nu=0} = \psi(x), \tag{2.3.7}$$

or calculate the operator expressions (see Paragraph 5.3)

$$\begin{aligned} x'_\mu &= \exp\left\{2cx\,x\partial - x^2 c\partial\right\} x_\mu \exp\left\{-(2cx\,x\partial - x^2 c\partial)\right\} \\ \psi'(x') &= \exp\left\{2cx\,x\partial - x^2 c\partial\right\} \exp\left\{-\mathcal{K}\right\} \psi(x) \end{aligned} \tag{2.3.8}$$

which are the formal solutions of the Lie equations (2.3.6) and (2.3.7). It is easy to confirm that formulae (2.3.2) give the solution of Equations (2.3.6). The solution of Equations (2.3.7) will be sought in the form

$$\psi'(x') = R(x,c)\psi(x), \tag{2.3.9}$$

where $R(x,c)$ is a nonsingular matrix, and $R(x,0) = 1$. Substituting (2.3.9) into (2.3.7) and using (2.3.2), we get

$$\begin{aligned} \frac{\partial R}{\partial c^\mu} &= 2(iS_{\mu\nu}x'^\nu - kx'_\mu)R = \\ &= \frac{2}{\sigma(c,x)}\left[(iS_{\mu\nu}(x^\nu - c^\nu x^2) - k(x_\mu - c_\mu x^2)\right]R. \end{aligned} \tag{2.3.10}$$

Since

$$\frac{\partial \sigma}{\partial c^\mu} = -2(x_\mu - c_\mu x^2),$$

(2.3.10) can be rewritten as follows:

$$\frac{\partial R}{\partial c^\mu} - \frac{k}{\sigma}\frac{\partial \sigma}{\partial c^\mu}R = 2iS_{\mu\nu}\frac{x^\nu - c^\nu x^2}{\sigma}R. \tag{2.3.11}$$

It can be seen from (2.3.11) that it is convenient to represent the matrix $R(x, c)$ in the form

$$R(x, c) = \sigma^k(x, c)T(x, c), \tag{2.3.12}$$

where $T(x, c)$ is a matrix. Substituting (2.3.12) into (2.3.11) gives the following for the matrix T:

$$\frac{\partial T}{\partial c^\mu} = 2iS_{\mu\nu}\frac{x^\nu - c^\nu x^2}{\sigma}T. \tag{2.3.13}$$

Multiplying (2.3.13) by c^μ and by x^μ we get, respectively

$$c^\mu\frac{\partial T}{\partial c^\mu} = 2i\sigma^{-1}S_{\mu\nu}c^\mu x^\nu T, \tag{2.3.14}$$

$$x^\mu\frac{\partial T}{\partial c^\mu} = 2i\sigma^{-1}x^2 S_{\mu\nu}c^\mu x^\nu T. \tag{2.3.15}$$

One can seek solutions of these equations in the form

$$T = \exp\left\{2iS_{\mu\nu}c^\mu x^\nu \tau(x, c)\right\}, \tag{2.3.16}$$

where $\tau(x, c)$ is a scalar differentiable function. Substituting (2.3.16) into (2.3.14), (2.3.15) gives

$$c^\mu\frac{\partial T}{\partial c^\mu} = 2iS_{\mu\nu}c^\mu x^\nu\left(\tau + c^\alpha\frac{\partial \tau}{\partial c^\alpha}\right)T = \frac{2i}{\sigma}S_{\mu\nu}c^\mu x^\nu T,$$

$$x^\mu\frac{\partial T}{\partial c^\mu} = 2iS_{\mu\nu}c^\mu x^\nu\left(x^\alpha\frac{\partial \tau}{\partial c^\alpha}\right)T = \frac{2i}{\sigma}x^2 S_{\mu\nu}c^\mu x^\nu T,$$

(here we used formulae (5.3.18), (5.3.20)), whence follows that the function $\tau(x, c)$ should satisfy the equations

$$\begin{aligned} \tau + c^\alpha\frac{\partial \tau}{\partial c^\alpha} &= \frac{1}{\sigma}, \\ x^\alpha\frac{\partial \tau}{\partial c^\alpha} &= \frac{x^2}{\sigma} \end{aligned} \tag{2.3.17}$$

Suppose that the function $\tau(x,c)$ depends, like $\sigma(x,c)$, only on the generalized variables

$$w_1 = cx, \qquad w_2 = c^2x^2. \tag{2.3.18}$$

In this case Equations (2.3.17) take the form

$$\tau + w_1\frac{\partial \tau}{\partial w_1} + w_2\frac{\partial \tau}{\partial w_2} = (1-2w_1+w_2)^{-1},$$

$$\frac{\partial \tau}{\partial w_1} + 2w_1\frac{\partial \tau}{\partial w_2} = (1-2w_1+w_2)^{-1}.$$

The general solution of these equations is given by the formula

$$\tau = \tfrac{1}{2}\left(w_1^2 - w_2\right)^{-1/2} \ln\left(\frac{1-w_1+\sqrt{w_1^2-w_2}}{1-w_1-\sqrt{w_1^2-w_2}}\right),$$

and, once again using (2.3.18), we find

$$\tau(x,c) = \tfrac{1}{2}\left[(cx)^2 - c^2x^2\right]^{-1/2} \ln\left(\frac{1-w_1+\sqrt{w_1^2-w_2}}{1-w_1-\sqrt{w_1^2-w_2}}\right), \tag{2.3.19}$$

Thereby we obtain the formula of final conformal transformations for fields of arbitrary spin (see [82], [173], [188]):

$$\psi'(x') = R(x,c)\psi(x) = \sigma^k \exp\{2iS_{\mu\nu}c^\mu x^\nu \tau\}\,\psi(x), \tag{2.3.20}$$

where σ and τ are given in (2.3.2) and (2.3.19).

Now it's easy to write the formula of generating solutions for field equations invariant under conformal transformations (2.3.2) and (2.3.20). According to (17) we find (see [188])

$$\psi_{II}(x) = \sigma^{-k} \exp\{-2iS_{\mu\nu}c^\mu x^\nu \tau\}\,\psi_I(x'), \tag{2.3.21}$$

where σ, x' and τ are determined in (2.3.2) and (2.3.19).

Let us apply formulae (2.3.20) and (2.3.21) to a spinor Dirac field. In this case we have

$$S_{\mu\nu} = \frac{i}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu), \qquad k = \frac{3}{2}, \tag{2.3.22}$$

where γ_μ are Dirac matrices (2.1.2). It is easy to calculate

$$2iS_{\mu\nu}c^\mu x^\nu = cx - \gamma c\gamma x, \tag{2.3.23}$$

$$(2iS_{\mu\nu}c^\mu x^\nu)^2 = (cx)^2 - c^2x^2 \overset{\text{def}}{=} \theta^2,$$

$$(2iS_{\mu\nu}c^\mu x^\nu)^3 = \theta^2(cx - \gamma c\gamma x),$$

. . . .

Hence

$$\exp\{2iS_{\mu\nu}c^\mu x^\nu \tau\} \equiv \exp\{(cx - \gamma c\gamma x)\tau\} =$$

$$= \operatorname{ch}\tau\theta + (cx - \gamma c\gamma x)\frac{\operatorname{sh}\tau\theta}{\theta} \qquad (2.3.24)$$

From (2.3.19), (2.3.24) we find

$$\operatorname{sh}\tau\theta = \frac{\theta}{\sqrt{\sigma}}, \qquad \operatorname{ch}\tau\theta = \frac{1 - cx}{\sqrt{\sigma}}. \qquad (2.3.25)$$

Thus, finally, we obtain the formula of conformal transformations for the Dirac spinor field

$$\psi'(x') = \sigma^{3/2}\exp\{(cx - \gamma c\gamma x)\tau\}\psi(x) = \sigma(1 - \gamma c\gamma x)\psi(x). \qquad (2.3.26)$$

Using (2.3.23)–(2.3.25) one can easily obtain from (2.3.21) the corresponding formula of generating solutions

$$\psi_{II}(x) = \frac{1 - \gamma x\gamma c}{\sigma^2}\psi_I(x'), \qquad (2.3.27)$$

where x' and σ are determined in (2.3.2).

Remark 2.3.3. In the case of a spinor field with arbitrary conformal degree k formulae (2.3.20) and (2.3.21), as follows from (2.3.22)–(2.3.25), have the form

$$\psi'(x') = \sigma^{k-1/2}(1 - \gamma c\gamma x)\psi(x), \qquad (2.3.28)$$

$$\psi_{II}(x) = \sigma^{-(k+1/2)}(1 - \gamma x\gamma c)\psi_I(x'), \qquad (2.3.29)$$

where x' and σ are given in (2.3.2).

Remark 2.3.4. In the case of n spacial variables, the conformally invariant Dirac equation, with nonlinearity of the Gursey type, has the form (see [93, 95])

$$\left[i\gamma\partial - \lambda(\bar{\psi}\psi)^{1/n}\right]\psi = 0 \qquad (2.3.30)$$

(here γ-matrices have appropriate structure, see [2*], §8.3). The conformal degree of its AC(1,n) is equal to $n/2$, the conformal transformations and corresponding formula of generating solutions having the form

$$\psi'(x') = \sigma^{(n-1)/2}(1 - \gamma c\gamma x)\psi(x), \qquad (2.3.31)$$

$$\psi_{II}(x) = \sigma^{-(n+1)/2}(1 - \gamma x\gamma c)\psi_I(x'), \tag{2.3.32}$$

where x' and σ are given in (2.3.2).

Now we write the explicit form of formulae (2.3.20) and (2.3.21) for scalar, vector, and tensor fields. For scalar fields u with $k = 1$ we have

$$u'(x') = \sigma u(x), \tag{2.3.33}$$

$$u_{II} = \sigma^{-1} u_I(x'). \tag{2.3.34}$$

For scalar fields u with $k = 2 - s$ we have

$$u'(x') = \sigma^{2-s} u(x), \tag{2.3.35}$$

$$u_{II}(x) = \sigma^{s-2} u_I(x'). \tag{2.3.36}$$

For vector fields A_μ with $k = 1$ we have [97]

$$A'_\mu(x') = \Big[g_{\mu\nu}\sigma + 2(x_\mu c_\nu - x_\nu c_\mu + +2cxc_\mu x_\nu - x^2 c_\mu c_\nu - c^2 x_\mu x_\nu)\Big] A^\nu(x), \tag{2.3.37}$$

$$A_{\mu II}(x) = \left[\frac{g_{\mu\nu}}{\sigma} + \frac{2}{\sigma^2}(c_\mu x_\nu - c_\nu x_\mu + +2cxx_\mu c_\nu - x^2 c_\mu c_\nu - c^2 x_\mu x_\nu)\right] A_I^\nu(x'). \tag{2.3.38}$$

For vector fields A_μ with arbitrary k we have

$$A'_\mu(x') = \Big[g_{\mu\nu}\sigma^k + 2\sigma^{k-1}(x_\mu c_\nu - x_\nu c_\mu + +2cxc_\mu x_\nu - x^2 c_\mu c_\nu - c^2 x_\mu x_\nu)\Big] A^\nu(x), \tag{2.3.39}$$

$$A_{\mu II}(x) = \Big[g_{\mu\nu}\sigma^{-k} + 2\sigma^{-(k+1)}(c_\mu x_\nu - c_\nu x_\mu + +2cxx_\mu c_\nu - x^2 c_\mu c_\nu - c^2 x_\mu x_\nu)\Big] A_I^\nu(x'). \tag{2.3.40}$$

For tensor fields $F_{\mu\nu}$ with $k = 2$ we have

$$F'_{\mu\nu}(x') = \sigma^2 F_{\mu\nu} + 2\sigma\Big\{\big[(2cx - 1)(c_\mu F_{\beta\nu} - c_\nu F_{\beta\mu}) - -c^2(x_\mu F_{\beta\nu} - x_\nu F_{\beta\mu})\big]x^\beta + \big[x_\mu F_{\alpha\nu} - x_\nu F_{\alpha\mu} - -x^2(c_\mu F_{\alpha\nu} - c_\nu F_{\alpha\mu}\big]c^\alpha + 2(c_\mu x_\nu - c_\nu x_\mu)F_{\alpha\beta}c^\alpha x^\beta\Big\} \tag{2.3.41}$$

$$F^{II}_{\mu\nu}(x) = \sigma^{-2}F^I_{\mu\nu}(x') + 2\sigma^{-3}\Big\{ \big[(2cx-1)\big(x_\mu F^I_{\beta\nu}(x') - \\ -x_\nu F^I_{\beta\mu}(x')\big) - x^2(c_\mu F^I_{\beta\nu}(x') - c_\nu F^I_{\beta\mu}(x')\big]c^\beta + \\ +\big[c_\mu F^I_{\alpha\nu}(x') - c_\nu F^I_{\alpha\mu}(x') - c^2\big(x_\mu F^I_{\alpha\nu}(x') - \\ -x_\nu F^I_{\alpha\mu}(x')\big)\big]x^\alpha + 2(x_\mu c_\nu - x_\nu c_\mu)x^\alpha c^\beta F^I_{\alpha\beta}(x')\Big\}. \tag{2.3.42}$$

From expressions (2.3.41) and (2.3.42) one can easy derive the explicit form of the conformal transformations and corresponding formula of generating solutions for electric $\vec{E}$ and magnetic $\vec{H}$ fields, satisfying Maxwell equations, by virtue of equalities

$$E_a = F^{0a}, \qquad H_a = \tfrac{1}{2}\epsilon_{abc}F^{bc}. \tag{2.3.43}$$

Let us write the conformal transformations and formula of generating solutions for the antisymmetric tensor $F_{\mu\nu}$ with arbitrary conformal degree k

$$F'_{\mu\nu}(x') = \sigma^k F_{\mu\nu}(x') + 2\sigma^{k-1}\Big\{ \big[(2cx-1)\big(c_\mu F_{\beta\nu} - \\ -c_\nu F_{\beta\mu}\big)\big) - c^2(x_\mu F_{\beta\nu} - x_\nu F_{\beta\mu})\big]x^\beta + \\ +\Big[x_\mu F_{\alpha\nu} - x_\nu F_{\alpha\mu} - x^2(c_\mu F_{\alpha\nu} - c_\nu F_{\alpha\mu})\Big]c^\alpha + \\ +2(c_\mu x_\nu - c_\nu x_\mu)F_{\alpha\beta}c^\alpha x^\beta\Big\}, \tag{2.3.44}$$

$$F^{II}_{\mu\nu}(x) = \sigma^{-k}F^I_{\mu\nu}(x') + 2\sigma^{-(k+1)}\Big\{ \big[(2cx-1)\times \\ \times\big(x_\mu F^I_{\beta\nu}(x') - x_\nu F^I_{\beta\mu}(x')\big) - x^2\big(c_\mu F^I_{\beta\nu}(x') - c_\nu F^I_{\beta\mu}(x')\big)\big]c^\beta + \\ +\big[c_\mu F^I_{\alpha\nu}(x') - c_\nu F^I_{\alpha\mu}(x') - c^2\big(x_\mu F^I_{\alpha\nu}(x') - x_\nu F^I_{\alpha\mu}(x')\big)\big]x^\alpha + \\ +2(x_\mu c_\nu - x_\nu c_\mu)x^\alpha c^\beta F^I_{\alpha\beta}(x')\Big\}. \tag{2.3.45}$$

In formulae (2.3.33)–(2.3.45) x' and σ are given in (2.3.2).

Remark 2.3.5. As we have already noted, the spinor field plays a crucial role in unified field theory because it allows us to construct fields with any spin. Here we would like to point out that to represent a given field via a spinor field ψ one must construct an appropriate combination from ψ and take ψ with the proper conformal degree. For example, scalar (u), vector (A_μ), and tensor ($F_{\mu\nu} = -F_{\nu\mu}$) fields with conformal degree $\widetilde{k}$ can be constructed as

$$u = \overline{\psi}\psi, \qquad A_\mu = \overline{\psi}\gamma_\mu\psi, \qquad F_{\mu\nu} = \frac{i}{2}\overline{\psi}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\psi \tag{2.3.46}$$

from the spinor field ψ with conformal degree $\tilde{k}/2$. Spinor field ψ with arbitrary conformal degree k can be constructed, in turn, by means of the Dirac spinor field ψ_D with conformal degree 3/2:

$$\psi = \left(\overline{\psi}_D\psi_D\right)^{-(\frac{3}{2}-k)/3}\psi_D, \qquad \overline{\psi}_D\psi_D \neq 0. \tag{2.3.47}$$

Using (2.3.46), (2.3.28), and (2.3.29) and what has been said above, we find another way of obtaining the formulae (2.3.33)–(2.3.45).

Now we shall construct conformally invariant ansatze. First we do this for a spinor field. Starting from the general expression (2.1.7), we determine the matrix $A(x)$ and the invariant variables ω as solutions of the equations

$$\begin{aligned}&(2cx\, x\partial)\omega(x) = 0,\\ &[2cx\, x\partial - x^2 c\partial + \gamma c\gamma x + (2k-1)cx]A(x) = 0\end{aligned} \tag{2.3.48}$$

(here we used (2.3.5) and (2.3.23)).

One can easy verify that the functions

$$\omega = \frac{\beta x}{x^\nu x_\nu}, \qquad A(x) = \gamma x \Big/ (x^\nu x_\nu)^{(2k+1)/2} \tag{2.3.49}$$

(β_ν are arbitrary constants, $\beta c = 0$) satisfy Equations (2.3.48). Hence a conformally invariant ansatz for a spinor field with conformal degree k has the form

$$\psi(x) = \left[\gamma x \Big/ (x^\nu x_\nu)^{(2k+1)/2}\right]\varphi(\omega), \qquad \omega = \frac{\beta x}{x^\nu x_\nu} \tag{2.3.50}$$

(β_ν are arbitrary constants, $\beta c = 0$)

In the particular case of the Dirac spinor field, when $k = 3/2$, from (2.3.50) we obtain the ansatz (see [93], [95])

$$\psi(x) = \frac{\gamma x}{(x^\nu x_\nu)^2}\varphi(\omega), \qquad \omega = \frac{\beta x}{x^\nu x_\nu} \tag{2.3.51}$$

Using (2.3.50) and (2.3.46) one can easily construct conformally invariant ansatze for scalar, vector, and tensor fields in the case of arbitrary conformal degree k. Without going into the details we simply list them:

$$u(x) = (x_\nu x^\nu)^{-k} v(\omega); \tag{2.3.52}$$

$$A_\mu(x) = \frac{B_\mu(\omega)}{(x^\nu x_\nu)^k} - 2x_\mu \frac{x^\nu B_\nu(\omega)}{(x_\alpha x^\alpha)^{k+1}}; \tag{2.3.53}$$

$$F_{\mu\nu}(x) = \frac{f_{\mu\nu}(\omega)}{(x^\alpha x_\alpha)^k} - 2\frac{(x_\mu f_{\beta\nu}(\omega) - x_\nu f_{\beta\mu}(\omega))\, x^\beta}{(x^\alpha x_\alpha)^{k+1}}. \tag{2.3.54}$$

In formulae (2.3.52)–(2.3.54), $\omega = \beta x/(x^\nu x_\nu)$.

Having applied transformations of translations to ansatzes (2.3.50)–(2.3.54), we obtain the C(1,3)-ungenerative ansatze. For example, from (2.3.51) we obtain the following C(1,3)-ungenerative ansatz for the Dirac spinor field:

$$\psi(x) = \frac{\gamma y}{(y_\nu y^\nu)^2}\varphi(\Omega), \qquad \Omega = \frac{\beta y}{y_\nu y^\nu} + æ \tag{2.3.55}$$

where $y_\nu = x_\nu + \delta_\nu$; $\delta_{\mu\nu}, \beta_\nu, æ$ are arbitrary constants.

Ungenerativity of ansatz (2.3.55) under transformations from the group $\widetilde{\mathrm{P}}(1,3)$ (the corresponding formulae are given in Table 2.1.2 and Equation (2.2.25)) is quite evident. Furthermore, one can confirm in a straight-forward way that applying the formula of GS (2.3.27) to (2.3.55) is the same as changing the parameters $\beta_\nu, \delta_\nu, æ$, and the function φ in (2.3.55) in such a manner that

$$\begin{aligned}
&\varphi \to \frac{1-\gamma c\gamma\delta}{\sigma^2(\delta,c)}\varphi, \quad \sigma(\delta,c) \equiv 1 - 2\delta c + \delta^2 c^2,\\
&\delta_\mu \to \delta'_\mu = \frac{\delta_\mu - c_\mu\delta^2}{\sigma(\delta,c)},\\
&\beta_\mu \to \beta'_\mu = \frac{\beta_\mu}{\sigma(\delta,c)} + \frac{2}{\sigma^2(\delta,c)}(\delta_\mu c_\nu - \delta_\nu c_\mu + 2\delta c c_\mu\delta_\nu - \delta^2 c_\mu c_\nu - c^2\delta_\mu\delta_\nu)\beta^\nu,\\
&æ \to æ' = \frac{\beta\delta c^2 - \beta c}{\sigma(\delta,c)}.
\end{aligned} \tag{2.3.56}$$

This means that the family of solutions like (2.3.55) is C(1,3)-ungenerative. Ungenerativity of ansatze constructed from (2.3.52)–(2.3.54), analogously to ansatz (2.3.55), is proved in much the same way.

It will be noted that any Lorentz-invariant solution of a conformally invariant equation can be made C(1,3)-ungenerative if formulae of GS by scale, conformal, and translational transformations are applied successively.

It will also be noted that the conformally invariant ansatz (2.3.51) (as well as ansatze (2.3.52)–(2.3.54)) can be obtained from a plane-wave ansatz $\psi(x) = \varphi(\beta x)$ by applying to it formula of GS (2.3.27) under $c_0 = 1$, $c_1 = c_2 = c_3 = 0$ and then making the translation transformation $x_0 \to x_0 + 1$.

In conclusion, let us show how to construct from a given solution of a linear PDE an infinite sequence of solutions (see [24*]). From the definition of invariance (see the Introduction) it follows that if operator Q is admitted by a linear system of PDEs, then

$$\psi_1 = a_1 Q\psi_0, \quad \psi_2 = a_2 Q\psi_1, \dots \psi_n = a_n Q\psi_{n-1} \cdots \tag{2.3.57}$$

(where $a_1, a_2, \dots$ are some normalization factors) will be solutions of this system as soon as $\psi_0(x)$ is a solution. For systems of PDEs invariant under

AC(1,3) (2.3.3) the procedure (2.3.57) makes nontrivial sense only for operators K_μ (as a matter of convenience we shall use operator $\mathcal{K}$ (2.3.5)). Below we construct, with the help of operator (2.3.5), sequences of solutions to the wave equation, the massless Dirac equation, and Maxwell's equations.

In the case of a scalar field with conformal degree $k = 1$, operator (2.3.5) takes the form

$$\mathcal{K} = 2cxx\partial - x^2\partial + 2cx. \tag{2.3.58}$$

Then, starting from $u_0 = 1$ and taking $a_n = 1/2n$ one obtains the sequence of solutions (2.3.57) of the wave equation

$$\Box u = 0 \tag{2.3.59}$$

in the form

$$\begin{aligned} u_0 &= 1 \\ u_1 &= cx, \\ u_2 &= (cx)^2 - \tfrac{1}{4}c^2x^2, \\ u_3 &= (cx)^3 - \tfrac{1}{2}c^2x^2, \\ u_4 &= (cx)^4 - \frac{3}{4}(cx)^2c^2x^2 - \frac{1}{16}(c^2x^2)^2, \end{aligned} \tag{2.3.60}$$

....

One can easily make sure that $u_1, u_2, u_3, \ldots$ satisfy Equation (2.3.59). It will be noted that $u_2, u_3, \ldots$ satisfy the following recurrence relations

$$u_n = \frac{1}{2n}\mathcal{K}u_{n-1} = (cx)u_{n-1} - \tfrac{1}{4}c^2x^2u_{n-2}, \quad n \geq 2. \tag{2.3.61}$$

One can make solutions (2.3.60) C(1,3)-ungenerative, generating them by conformal and translation transformations.

For the massless Dirac equation

$$i\gamma\partial\psi = 0 \tag{2.3.62}$$

we have

$$\mathcal{K} = 2cxx\partial - x^2c\partial + 2cx + \gamma c\gamma x \tag{2.3.63}$$

and starting from $\psi_0 = \chi$, where χ is a constant spinor, we obtain

$$\psi_0 = \chi$$

$$\psi_1 = (cx - \tfrac{1}{4}\gamma x\gamma c)\chi,$$

$$\psi_2 = \left[(cx)^2 - \frac{1}{6}c^2x^2 - \frac{1}{3}(cx)\gamma x\gamma c\right]\chi, \tag{2.3.64}$$

$$\psi_3 = \left[(cx)^3 - \frac{3}{8}c^2x^2(cx) + \left(\frac{1}{16}c^2x^2 - \frac{3}{8}(cx)^2\right)\gamma x\gamma c\right]\chi,$$

....

It is easy to show that functions (2.3.64) satisfy the Dirac equation (2.3.62). Solutions (2.3.64) can also be made C(1,3)-ungenerative if they are generated by conformal and translation transformations.

For Maxwell's equations for vector-potential

$$\Box A_\mu - \partial_\mu \partial^\nu A_\nu = 0 \tag{2.3.65}$$

we analogously find the sequence of solutions $A_\mu^{(1)}, A_\mu^{(2)}, \ldots$ starting from a solution $A_\mu^{(0)}$:

$$A_\mu^{(n)} = a_n\left[\left(2cx(x\partial + 1) - x^2c\partial\right)A_\mu^{(n-1)} + 2c_\mu(A_\nu^{(n-1)}x^\nu) - 2x_\mu(A_\nu^{(n-1)}c^\nu)\right]$$
$$n = 1, 2, \ldots \tag{2.3.66}$$

where a_n are some normalization constants. In particular, if we start with $A_\mu^{(0)} = a_n$ (a_n are arbitrary constants), then (2.3.66) gives

$$A_\mu^{(1)} = a_\mu cx + c_\mu ax - x_\mu ac,$$
$$A_\mu^{(2)} = a_\mu\left[(cx)^2 - \tfrac{1}{4}c^2x^2\right] + c_\mu\left(2(ax)(cx) - \tfrac{1}{2}acx^2\right) - \tag{2.3.67}$$
$$-x_\mu\left((ac)(cx) + \tfrac{1}{2}c^2ax\right),$$

....

Solutions (2.3.67) have the property $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = 0$. But if we start with $A_\mu^{(0)} = a_\mu bx$ (a_μ, b_ν are arbitrary constants) then we obtain from (2.3.66) another sequence of solutions of Equations (2.3.65):

$$A_\mu^{(1)} = a_\mu(cxbx - \tfrac{1}{4}x^2bc) + \tfrac{1}{2}bx(c_\mu ax - x_\mu ac), \tag{2.3.68}$$

....

and in this case $F_{\mu\nu} \neq 0$.

Lastly we write down formula (2.3.57) for the tensor of the electromagnetic field $F_{\mu\nu}$ (2.3.43). So, if $F_{\mu\nu}^{(0)}$ is a solution of Maxwell's equations

$$\partial_\mu F^{\mu\nu} = 0, \qquad \epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0 \tag{2.3.69}$$

then

$$F_{\mu\nu}^{(n+1)} = (c_\mu F_{\alpha\nu}^{(n)} - c_\nu F_{\alpha\mu}^{(n)})x^\alpha + (x_\nu F_{\alpha\mu}^{(n)} - x_\mu F_{\alpha\nu}^{(n)})c^\alpha + 2cxF_{\mu\nu}^{(n)} +$$
$$+(cxx^\alpha - \tfrac{1}{2}x^2c^\alpha)\partial_\alpha F_{\mu\nu}^{(n)}; \quad n = 0, 1, \ldots \tag{2.3.70}$$

will be also solutions of Maxwell's Equations (2.3.69).

2.4. *Conformally invariant nonlinear equations for spinor fields and their solutions*

1. Let us consider the following Poincare invariant nonlinear PDE for a spinor field * [101], [102]):

$$\Big\{\gamma^{\mu}\big[i\partial_{\mu} + (\overline{\psi}\gamma_{\mu}\psi)F_1 + (\overline{\psi}\gamma_5\gamma_{\mu}\psi)F_2 + (\overline{\psi}\gamma_{\mu}\psi)\gamma_5F_3 + (\overline{\psi}\gamma_5\gamma_{\mu}\psi)\gamma_5F_4\big]+$$
$$+(\overline{\psi}\sigma_{\mu\nu}\psi)\sigma^{\mu\nu}F_5 + (\overline{\psi}\sigma_{\mu\nu}\psi)\gamma_5\sigma^{\mu\nu}F_6 + F_7 + \gamma_5F_8\Big\}\psi = 0, \qquad (2.4.1)$$

where

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}; \qquad \sigma_{\mu\nu} = \frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu); \qquad (2.4.2)$$

$F_1, \ldots, F_8$ are arbitrary scalar functions depending on $\overline{\psi}\psi$ and $\overline{\psi}i\gamma_5\psi$. The rest of the notations are given in (2.1.1)–(2.1.3).

From the set of Equations (2.4.1) we select those which are invariant under scale transformations

$$x_\mu \to x'_\mu = e^{\theta}x_\mu; \quad \psi(x) \to \psi'(x') = e^{-\theta k}\psi(x) \qquad (2.4.3)$$

and under conformal transformations (2.3.2), (2.3.26).

Theorem 2.4.1. [101, 102]. *Equation* (2.4.1) *is invariant under the scale transformations* (2.4.3) *iff*

$$\begin{aligned} F_i &= (\overline{\psi}\psi)^{(1-2k)/2k}\phi_i, \quad i = \overline{1,6}; \\ F_j &= (\overline{\psi}\psi)^{1/2k}\phi_j, \quad j = 7,8, \end{aligned} \qquad (2.4.4)$$

where $\phi_1, \ldots, \phi_8$ *are arbitrary scalar functions depending on* $(\overline{\psi}\psi)/(\overline{\psi}i\gamma_5\psi)$.

Proof. The necessary and sufficient condition of the invariance of equation (2.4.1) under the transformation (2.4.3) is the fulfilment of the relations

* There is no nonlinear Poincare-invariant equation of such a type for a two-component spinor field φ (Weyl field) because it is impossible to add to the Weyl equation a nonlinear term constructed from φ and φ^+ and to conserve even Lorentz invariance. The same statement holds true for Maxwell's equations for electromagnetic fields in a vacuum.

$$\exp\{\theta(1-2k)\}\, F_i\left(\exp\{-2\theta k\}\overline{\psi}\psi, \exp\{-2\theta k\}\overline{\psi}i\gamma_5\psi\right) =$$
$$= F_i(\overline{\psi}\psi, \overline{\psi}i\gamma_5\psi); \quad i=\overline{1,6};$$
$$\exp\{\theta(1-k)\}\, F_j\left(\exp\{-2\theta k\}\overline{\psi}\psi, \exp\{-2\theta k\}\overline{\psi}i\gamma_5\psi\right) =$$
$$= F_j(\overline{\psi}\psi, \overline{\psi}i\gamma_5\psi); \quad j=7,8; \tag{2.4.5}$$

Differentiation of (2.4.5) with respect to θ followed by setting $\theta = 0$ gives

$$\begin{aligned} (1-2k)F_i &= 2ku\frac{\partial F_i}{\partial u} + 2kv\frac{\partial F_i}{\partial v}, \quad i=\overline{1,6} \\ (1-k)F_j &= 2ku\frac{\partial F_j}{\partial u} + 2kv\frac{\partial F_i}{\partial v}, \quad j=7,8, \end{aligned} \tag{2.4.6}$$

where $u = \overline{\psi}\psi$, $v = \overline{\psi}i\gamma_5\psi$.

After integrating (2.4.6) we get (2.4.4). One can directly confirm that Equation (2.4.1) with functions F (2.4.4) is invariant under the scale transformations (2.4.3). The theorem is proved.

Remark 2.4.1. Obviously, Equation (2.4.1) with functions

$$\begin{aligned} F_{i_1} &= \left[(\overline{\psi}\gamma_\mu\psi)(\overline{\psi}\gamma^\mu\psi)\right]^{(1-2k)/4k}\phi_{i_1}, \qquad i_1 = 1,3; \\ F_{i_2} &= \left[(\overline{\psi}\gamma_5\gamma_\mu\psi)(\overline{\psi}\gamma_5\gamma^\mu\psi)\right]^{(1-2k)/4k}\phi_{i_2}, \qquad i_2 = 2,4; \\ F_{i_3} &= \left[(\overline{\psi}\sigma_{\mu\nu}\psi)(\overline{\psi}\sigma^{\mu\nu}\psi)\right]^{(1-2k)/4k}\phi_{i_3}, \qquad i_3 = 5,6; \\ F_j &= \left(\overline{\psi}\psi\right)^{1/2k}\phi_j, \qquad j=7,8 \end{aligned} \tag{2.4.7}$$

where $\phi_1, \ldots, \phi_8$ are arbitrary scalar functions depending on $\dfrac{\overline{\psi}\psi}{\overline{\psi}i\gamma_5\psi}$, is also invariant under the scale transformations (2.4.3).

Remark 2.4.2. From four-component spinor fields ψ and $\overline{\psi}$ one can construct 16 real quadratic forms $\rho_j (j=\overline{1,16})$

$$\begin{aligned} &u = \overline{\psi}\psi && \text{— scalar,} \\ &v = \overline{\psi}i\gamma_5\psi && \text{— pseudoscalar,} \\ &j_\mu = \overline{\psi}\gamma_\mu\psi && \text{— vector,} \\ &n_\mu = \overline{\psi}\gamma_5\gamma_\mu\psi && \text{— pseudovector,} \\ &f_{\mu\nu} = \overline{\psi}\sigma_{\mu\nu}\psi && \text{— antisymmetric tensor.} \end{aligned} \tag{2.4.8}$$

These 16 bispinor densities are not independent since the spinor wave function ψ is composed of four independent complex functions. Furthermore, as the overall phase of the spinor has no effect on the bispinor densities ρ_j, we conclude that these 16 functions must satisfy a total of nine algebraic equations. These equations are known as identities of Fierz-Pauli and have the form (see [42])

$$j_\mu j^\mu = -n_\mu n^\mu = u^2 + v^2;$$
$$j_\mu n^\mu = 0; \tag{2.4.9}$$
$$f_{\mu\nu} = (u^2+v^2)^{-1}\left[u\epsilon_{\mu\nu\rho\sigma} j^\rho n^\sigma - v(j_\mu n_\nu - j_\nu n_\mu)\right], \qquad u^2+v^2 \neq 0.$$

From this one can conclude that Equations (2.4.1) with functions (2.4.4) and (2.4.7) are in a sense equivalent.

Theorem 2.4.2. (See [101]). *Equation* (2.4.1) *is invariant under the conformal group* C(1,3) *iff functions F have the form* (2.4.4) *and* $k = 3/2$.

Proof. Since the conformal group contains the extended Poincare group we can use the previous theorem. Then one can confirm that conformal transformations (2.3.2), (2.3.26) leave equation (2.4.1) with functions F from (2.4.4) invariant iff $k = 3/2$. The theorem is proved.

As a consequence of Theorem 2.4.2 we get the Dirac-Gursey Equation (2.1.5) and the following conformally invariant generalization of the Dirac-Heisenberg Equation (2.1.4)

$$\left\{ i\gamma\partial + \lambda \frac{(\overline{\psi}\gamma_\nu\psi)\gamma^\nu}{\left[(\overline{\psi}\gamma_\mu\psi)(\overline{\psi}\gamma^\mu\psi)\right]^{1/3}} + æ \frac{(\overline{\psi}\gamma_5\gamma_\mu\psi)\gamma_5\gamma^\mu}{\left[(\overline{\psi}\gamma_5\gamma_\nu\psi)(\overline{\psi}\gamma_5\gamma^\nu\psi)\right]^{1/3}} \right\} \chi \tag{2.4.10}$$

(λ, æ are arbitrary constants).

2. Now we describe some exact solutions of the nonlinear conformally invariant spinor wave Equations (2.1.5) and (2.4.10). For the first time exact solutions of the Dirac-Gursey Equation (2.1.5) were obtained by Kortel (see [132]) with the help of the Heisenberg ansatz (see (2.1.28) with $\omega = \sqrt{x_\nu x^\nu}$). Later these results, slightly generalized, were expounded in [5],[6], and [150]. We list the Kortel solutions as they are given in [150]:

$$\psi(x) = \left(\frac{4\alpha}{\lambda}\right)^{3/2} \frac{i\gamma x + \alpha}{(\omega^2+\alpha^2)^2}\chi, \tag{2.4.11}$$

$$\psi(x) = \frac{1}{4}\left(\frac{3}{\lambda}\right)^{3/2} \omega^{-5/2}(i\gamma x + \omega)\chi, \tag{2.4.12}$$

where $\omega = \sqrt{x_\nu x^\nu}$, α is a constant, and $\overline{\chi}\chi = 1$. Note that solutions (2.4.11) and (2.4.12) are nonanalytic in the coupling constant λ. New solutions of

Equation (2.1.5) can be obtained from (2.1.26) under $m = 0$ and $k = 1/3$. Let us list some of them:

$$\psi(x) = \left[(ax)^2 + (bx)^2\right]^{-1/4} \exp\{-\tfrac{1}{2}(\gamma a)(\gamma b)\times \tag{2.4.13}$$

$$\times \arctan \frac{ax}{bx}\} \exp\Big\{-\frac{3}{2} i\lambda(\overline{\chi}\chi)^{1/3}(\gamma b)\left[(ax)^2 + (bx)^2\right]^{2/3}\Big\}\chi;$$

$$\psi(x) = \exp\left\{-\tfrac{1}{4}(\gamma a + \gamma d)\gamma b(ax + dx)\right\} \times \tag{2.4.14}$$

$$\times \exp\left\{-i\lambda(\overline{\chi}\chi)^{1/3}\gamma b\left[bx + \tfrac{1}{4}(ax + dx)^2\right]\right\}\chi,$$

where a_ν, b_ν, d_ν are arbitrary constants satisfying (2.1.27).

One can apply the formulae of generating solutions (2.3.27) to obtain further solutions from (2.4.11)–(2.4.14). Having made in these solutions transformation of translations one obtains C(1,3)-ungenerative families of solutions. For example, solution (2.4.11) yields the following multiparameter C(1,3)-ungenerative family of solutions of Equation (2.1.5):

$$\psi(x) = \left(\frac{4\alpha}{\lambda}\right)^{3/2} \frac{i\gamma y + \alpha(1 - \gamma y \gamma c)}{\left[y^2\left(1 + \alpha^2 c^2\right) + \alpha^2(1 - 2cy)\right]^2}\chi, \tag{2.4.15}$$

where $y_\nu = x_\nu + \delta_\nu$; c_ν, δ_ν are arbitrary constants; $\overline{\chi}\chi = 1$.

The substitution of conformally invariant ansatz (2.3.51) into (2.1.5) yields the following ODE

$$\frac{d\varphi}{d\omega} = -i\lambda(\beta_\nu\beta^\nu)^{-1}(\overline{\varphi}\varphi)^{1/3}(\gamma\beta)\varphi$$

the general solution of which has the form

$$\varphi(\omega) = \exp\left\{-i\lambda(\beta_\nu\beta^\nu)^{-1}(\overline{\chi}\chi)^{1/3}(\gamma\beta)\omega\right\}\chi$$

This gives one more solution of Equation (2.1.5) (see [93, 95])

$$\psi(x) = \frac{\gamma x}{(x^\nu x_\nu)^2} \exp\left\{-i\lambda(\beta_\nu\beta^\nu)^{-1}(\overline{\chi}\chi)^{1/3}(\gamma\beta)\frac{\beta x}{x^\nu x_\nu}\right\}\chi. \tag{2.4.16}$$

The corresponding C(1,3)-ungenerative solution has the form

$$\psi(x) = \frac{\gamma y}{(y^\nu y_\nu)^2} \exp\left\{-i\lambda(\beta_\nu\beta^\nu)^{-1}(\overline{\chi}\chi)^{1/3}(\gamma\beta)\left(\frac{\beta y}{y^\nu y_\nu} + \alpha\right)\right\}\chi. \tag{2.4.17}$$

where $y_\nu = x_\nu + \delta_\nu$; δ_ν, α are arbitrary constants.

Following [101, 102] we construct solutions of the modified Dirac-Heisenberg Equation (2.4.10) with æ = 0

$$\left\{ i\gamma\partial - \lambda \frac{(\overline{\psi}\gamma_\mu\psi)\gamma^\mu}{\left[(\overline{\psi}\gamma_\nu\psi)(\overline{\psi}\gamma^\nu\psi)\right]^{1/3}} \right\} \psi = 0. \tag{2.4.18}$$

Ansatz (2.3.51) reduces (2.4.18) to the system of ODEs

$$i(\gamma\beta)\frac{d\varphi}{d\omega} - \lambda \frac{(\overline{\varphi}\gamma_\mu\varphi)\gamma^\mu\varphi}{\left[(\overline{\varphi}\gamma_\nu\varphi)(\overline{\varphi}\gamma^\nu\varphi)\right]^{1/3}} = 0. \tag{2.4.19}$$

Consider the case of a complex coupling constant, that is

$$\lambda = \lambda_1 + i\lambda_2, \qquad \text{where} \qquad \lambda_1 = \mathrm{Re}\lambda, \quad \lambda_2 = \mathrm{Im}\lambda.$$

Then we get the following solutions of Equation (2.4.19):

if $\lambda_2 = 0$, then $\varphi(\omega) = \exp\{i\lambda\omega\}\chi$;

if $\lambda_1 = 0$, then $\varphi(\omega) = \left(c + \frac{2}{3}\lambda_2\omega\right)^{-3/2} \chi$;

if $\lambda_1\lambda_2 \neq 0$, then $\varphi(\omega) = (f_1 + if_2)\chi$;

$$\begin{aligned} f_1 &= \pm\left[(w-2v)^{1/2} + (w+2v)^{1/2}\right], \\ f_2 &= \mp\left[(w-2v)^{1/2} - (w+2v)^{1/2}\right], \end{aligned} \tag{2.4.20}$$

$$w = \left(c_1 - 2\frac{\lambda_2}{\lambda_1}v^2\right)^{1/2}, \qquad \int \frac{dv}{w^{4/3}} = 2\lambda_2\omega + c_2.$$

In formulae (2.4.20)

$$\begin{aligned} \beta_\mu &= \frac{\overline{\chi}\gamma_\mu\chi}{\left[(\overline{\chi}\gamma_\nu\chi)(\overline{\chi}\gamma^\nu\chi)\right]^{1/3}}, \\ \omega &= \frac{\beta x}{x_\nu x_\nu}. \end{aligned} \tag{2.4.21}$$

The C(1,3)-ungenerative solution of Equation (2.4.18) with real coupling constant has the form

$$\psi(x) = \frac{\gamma y}{(y_\nu y^\nu)^2} \exp\left\{ i\lambda\left(\frac{\beta y}{y_\nu y^\nu} + æ\right)\right\}\chi, \tag{2.4.22}$$

where $y_\nu = x_\nu + \delta_\nu$; β_ν are given in (2.4.21); æ, δ_ν are arbitrary constants.

3. Consider a conformally invariant equation for a spinor field with arbitrary conformal degree k [149]

$$\begin{gathered}\left[i\gamma\mathcal{D}+\lambda(\overline{\psi}\psi)^{1/2k}\right]\psi=0,\\ \mathcal{D}_\mu=\partial_\mu+\frac{1}{2k}\left(\frac{3}{2}-k\right)\partial_\mu\ln\left(\overline{\psi}\psi\right)\end{gathered}\tag{2.4.23}$$

As we have already noted, spinor fields with arbitrary conformal degree play a fundamental role in unified field theory because they allow us to construct fields with any spin and with a given conformal degree (see Remark 2.3.5).

Equation (2.4.23) is merely the Dirac-Gursey Equation (2.1.5) when written in terms of the spinor

$$\psi_{\mathcal{D}}=(\overline{\psi}\psi)^{\frac{1}{2k}\left(\frac{3}{2}-k\right)}\psi,\tag{2.4.24}$$

where $\psi_{\mathcal{D}}$ denotes a Dirac spinor with conformal degree $k=3/2$, and ψ is a spinor field with arbitrary conformal degree k (the inverse formula expressing ψ via $\psi_{\mathcal{D}}$ is given in (2.3.47)).

In order to construct a scalar field with conformal degree 1 from a Dirac spinor field (it is this conformal degree which makes the scalar field satisfy the wave equation (see Table 2.3.1)) one must set $k=1/2$ in (2.3.47) and then use the corresponding formula from (2.3.46). This gives

$$u(x)=(\overline{\psi}_{\mathcal{D}}\psi_{\mathcal{D}})^{1/3}.\tag{2.4.25}$$

Let us substitute solutions (2.4.11) and (2.4.12) of the Dirac-Gursey Equation (2.1.5) in (2.4.25). The result is

$$u(x)=\frac{3}{2\lambda}(x_\nu x^\nu)^{-1/2},\tag{2.4.26}$$

$$u(x)=\frac{4}{\lambda}\frac{\alpha}{x_\nu x^\nu+\alpha^2}.\tag{2.4.27}$$

Functions (2.4.26) and (2.4.27), as one can easily check, satisfy the nonlinear wave equation

$$\Box u+\lambda_1 u^3=0\tag{2.4.28}$$

provided $\lambda=\frac{3}{2}\sqrt{\lambda_1}$ for (2.4.26) and $\lambda=\sqrt{2\lambda_1}$ for (2.4.27), $\lambda_1>0$. It is noteworthy [61*] that these solutions of Equation (2.4.28) lead by means of the t'Hooft-Corrigan-Fairlie-Wilczek ansatz to the well-known meron solution of de Alfaro-Fubini-Furlan [4*] and to the instanton solution of Belavin-Polyakov-Schwartz-Tyupkin [1*] of the SU(2) Yang-Mills field equations (see Section 2.10).

4. As we have considered how to construct other fields from the spinor field (see Remark 2.3.5), it is natural to consider the inverse problem: how to construct the spinor field itself by means of other fields.

It is obvious that any spinor wave function can be represented as an expansion in the basis of the 4 × 4 matrices

$$\psi(x) = e^{i\varphi}\Big(U - i\gamma_5 V + \gamma_\mu J^\mu - \gamma_5\gamma_\mu N^\mu + \tfrac{1}{2}\sigma_{\mu\nu}F^{\mu\nu}\Big)\chi \equiv e^{-i\varphi}\sum_{\alpha=1}^{16} R^\alpha \Gamma_\alpha \chi, \tag{2.4.29}$$

where

$$\varphi,\ R^\alpha = \{U, V, J^\mu, N^\mu, F^{\mu\nu}\} \tag{2.4.30}$$

are real scalar functions; χ is a constant spinor; and the rest of the notations are given in (2.1.2) and (2.4.2). Clearly, due to the Fierz-Pauli relations (2.4.9), which hold for any spinor field, the 16 functions R^α (2.4.30) are not independent. The set of functions R^α can always be chosen such that they satisfy the bispinor algebra (2.4.9), that is

$$\begin{aligned} &J_\mu J^\mu = -N_\mu N^\mu = U^2 + V^2; \\ &N_\mu J^\mu = 0; \\ &F_{\mu\nu} = \left(U^2 + V^2\right)^{-1}\left[U\epsilon_{\mu\nu\rho\sigma}J^\rho N^\sigma - V(J_\mu N_\nu - J_\nu N_\mu)\right], \qquad U^2 + V^2 \neq 0. \end{aligned} \tag{2.4.31}$$

This fact allows us to prove the factorization theorem (see [42]), according to which the general form of the spinor field (2.4.29) can be factored in the following manner:

$$\begin{aligned}\psi(x) = e^{-i\varphi}(U - i\gamma_5 V + \gamma_\mu J^\mu)\times \\ \times\left[1 - (U^2 + V^2)^{-1}(U + i\gamma_5 V)N_\nu\gamma_5\gamma^\nu\right]\chi.\end{aligned} \tag{2.4.32}$$

Let

$$\rho^\alpha = \{u, v, j_\mu, n_\mu, f_{\mu\nu} = -f_{\nu\mu}\}; \qquad j_0 > 0, \qquad u^2 + v^2 \neq 0 \tag{2.4.33}$$

be a given set of real functions forming a bispinor algebra (2.4.9). Then the following statement (the inversion theorem) holds true.

Theorem 2.4.3 [42]. *The spinor ψ which generates the given bispinor algebra* (2.4.33), (2.4.8), (2.4.9) *is determined by Equations* (2.4.29) *or* (2.4.32), *functions R^α* (2.4.30) *having the form*

$$R^\alpha = \rho^\alpha \,/\, (4J), \tag{2.4.34}$$

where

$$J^2 = \tfrac{1}{4}\overline{\chi}(u - i\gamma_5 v + j_\mu\gamma^\mu)\left[1 - (u^2+v^2)^{-1}(u + i\gamma_5 v)n_\nu\gamma_5\gamma^\nu\right]\chi \tag{2.4.35}$$

So, to construct a spinor field ψ one needs, in general, the set of 16 real functions (2.4.33) which represent themselves as scalar, pseudoscalar, vector, pseudovector, and antisymmetric tensor of rank-two fields satisfying relations (2.4.9). In such a case spinor ψ is determined by use of formulae (2.4.29) or (2.4.32), (2.4.34), and (2.4.35). If it is necessary to construct a spinor field with conformal degree k one has to take fields ρ^α (2.4.33) with conformal degree $2k$.

One can construct a spinor field in the following manner. Let U and J_α be scalar and vector fields. Then formulae

$$\begin{aligned} \psi_1(x) &= \left(\gamma^\mu \frac{\partial U}{\partial x^\mu}\right)\chi, \\ \psi_2(x) &= \gamma_\mu J^\mu \chi \end{aligned} \tag{2.4.36}$$

determine spinor fields (in sense of ansatze) ψ_1 and ψ_2. But in this case the connection between U and $u = \overline{\psi}_1\psi_1$, and between J_μ and $j_\mu = \overline{\psi}_2\gamma_\mu\psi_2$ is not so simple as it was in (2.4.34). In particular, for U and u we find

$$u = \overline{\psi}_1\psi_1 = \frac{\partial U}{\partial x^\nu}\frac{\partial U}{\partial x_\nu}(\overline{\chi}\chi) \tag{2.4.37}$$

It is clear that there are many ways of constructing spinor fields by use of formulae like (2.4.36). The Heisenberg ansatz (2.1.6) and its generalization (2.1.28) is an example of such a construction of a spinor field by means of three real scalar fields (functions) ω, f, g.

2.5. *Reduction and exact solutions of coupled nonlinear PDEs for spinor and scalar fields*

Consider the following system of PDEs

$$\begin{aligned} &\left\{i\gamma\partial - \left[\lambda_1|u|^{k_1} + \lambda_2(\overline{\psi}\psi)^{k_2}\right]\right\}\psi = 0, \\ &\Box u + \left[\mu_1|u|^{k_1} + \mu_2(\overline{\psi}\psi)^{k_2}\right]^2 u = 0, \end{aligned} \tag{2.5.1}$$

where $\psi = \psi(x)$ is a four-component spinor field; $u = u(x)$ is a complex scalar field; $|u| = \sqrt{u^*u}$; $x \in R(1,3)$; λ_i, μ_i, k_i $(i = 1,2)$ are arbitrary real parameters; γ_μ are Dirac matrices (2.1.2).

System (2.5.1) is invariant under the conformal group C(1,3) if $k_1 = 1$, $k_2 = 1/3$, conformal degrees being 3/2 for spinor fields and 1 for scalar fields; with arbitrary numbers k_1, k_2, system (2.5.1) is invariant under $\widetilde{\mathrm{AP}}(1,3)$.

Solutions of system (2.5.1) under $k_1 = 1$, $k_2 = 1/3$ are sought by using the conformally invariant ansatze (2.3.51) and (2.3.52). These ansatze reduce (2.5.1) to the following system of ODEs:

$$\begin{aligned} &i(\gamma\beta)\dot{\varphi} + \left[\lambda_1|v| + \lambda_2(\overline{\varphi}\varphi)^{1/3}\right]\varphi = 0, \\ &\beta^2\ddot{v} + \left[\mu_1|v|^{k_1} + \mu_2(\overline{\varphi}\varphi)^{1/3}\right]^2 v = 0, \end{aligned} \tag{2.5.2}$$

where dot indicates derivative with respect to $\omega = \dfrac{\beta x}{x^\nu x_\nu}$.

The simplest solutions of Equations (2.5.2) are as follows

$$\begin{aligned} &\varphi(\omega) = \exp\{i\text{æ}(\gamma\beta)\omega\}\,\chi, \\ &v(\omega) = \rho\exp\{i\omega\}, \end{aligned} \tag{2.5.3}$$

where æ, ρ, β_ν are arbitrary constants satisfying relations

$$\begin{aligned} &\text{æ}\beta^2 = \lambda_1\rho + \lambda_2(\overline{\chi}\chi)^{1/3} \\ &\beta^2 = \left[\mu_1\rho + \mu_2(\overline{\chi}\chi)^{1/3}\right]^2. \end{aligned} \tag{2.5.4}$$

Formulae (2.5.3), (2.3.51), (2.3.52) give solutions of Equations (2.5.1):

$$\begin{aligned} &\psi(x) = \frac{\gamma y}{(y^\nu y_\nu)^2}\exp\left\{i\text{æ}(\gamma\beta)\left(\frac{\beta y}{y^\nu y_\nu}\right\} + \alpha\right)\chi, \\ &u(x) = \frac{\rho}{y^\nu y_\nu}\exp\left\{i\left(\frac{\beta y}{y^\nu y_\nu}\right\} + \alpha\right), \end{aligned} \tag{2.5.5}$$

where the constants æ, ρ, β_ν satisfy relations (2.5.4). Having made in (2.5.5) translation transformations we obtain, according to Section 2.3, the multiparameter family of C(1,3)-ungenerative solutions of Equations (2.5.1):

$$\psi(x) = \frac{\gamma y}{(y^\nu y_\nu)^2}\exp\left\{i\text{æ}(\gamma\beta)\left(\frac{\beta y}{y_\nu y^\nu} + \alpha\right)\right\}\chi,$$

$$u(x) = \frac{\rho}{y_\nu y^\nu}\exp\left\{i\left(\frac{\beta y}{y_\nu y^\nu} + \alpha\right)\right\},$$

where $y_\nu = x_\nu + \delta_\nu$; α, δ are arbitrary real constants.

For finding solutions of Equations (2.5.1) with arbitrary k_1 and k_2 we use the ansatz (see [212], [13*])

$$\psi(x) = \left\{ig_1(\omega) + \gamma_4 g_2(\omega) - \left[if_1(\omega) + \gamma_4 f_2(\omega)\right]i\gamma\partial\omega\right\}\chi, \tag{2.5.6}$$

$$u(x) = v(\omega)$$

where g_1, g_2, f_1, f_2 are some real scalar functions; v is a complex scalar function; $\omega = \omega(x)$ are new independent variables determined in (2.1.31); $\gamma_4 = -i\gamma_5$, and γ_5 is determined in (2.4.2).

By substituting (2.5.6) into (2.5.1) and taking into account (2.1.30) we obtain

$$\begin{aligned}
&\epsilon\ddot{v} + \frac{N}{\omega}\dot{v} = -\left[\mu_1|v|^{k_1} + \widetilde{\mu}_2 F\right]^2 v,\\
&\epsilon\dot{f}_1 + \frac{N}{\omega}f_1 = \left[\lambda_1|v|^{k_1} + \widetilde{\lambda}_2 F\right] g_1,\\
&\dot{g}_1 = -\left[\lambda_1|v|^{k_1} + \widetilde{\lambda}_2 F\right] f_1,\\
&\dot{g}_2 = \left[\lambda_1|v|^{k_1} + \widetilde{\lambda}_2 F\right] f_2,\\
&\epsilon\dot{f}_2 + \frac{N}{\omega}f_2 = -\left[\lambda_1|v|^{k_1} + \widetilde{\lambda}_2 F\right] g_1,
\end{aligned} \tag{2.5.7}$$

where dot means derivative with respect to ω as defined in (2.1.31); $\widetilde{\lambda}_2 = \lambda_2(\overline{\chi}\chi)^{k_2}$, $\widetilde{\mu}_2 = \mu_2(\overline{\chi}\chi)^{k_2}$, and $F = \left[g_1^2 - g_2^2 + \epsilon(f_1^2 - f_2^2)\right]^{k_2}$.

We succeeded in constructing the general solution of system (2.5.7) only in the case where $N = 0$; in the rest of the cases particular solutions were found. Without going into details we cite the results obtained.

When $\epsilon = -1$, $N = -2, -1$ we have

$$\begin{aligned}
&f_j(\omega) = c_j\omega^{-1/2k_2}, \qquad j = 1, 2;\\
&g_j(\omega) = \mp(-1)^j(1 + 2k_2N)^{1/2}c_j\omega^{-1/2k_2};\\
&v(\omega) = E\omega^{-1/k_1}
\end{aligned} \tag{2.5.8}$$

and the conditions on arbitrary constants c_1, c_2 (real), and E (complex) hold

$$\begin{aligned}
&[(m-1)k_1 - 1]\,k_1^{-2} + \left\{\mu_1|E|^{k_1} + \widetilde{\mu}_2\left[2mk_2(c_1^2 - c_2^2)\right]^{k_2}\right\}^2 = 0,\\
&\pm(1 - 2mk_2)^{1/2} - 2k_2\left\{\lambda_1|E|^{k_1} + \widetilde{\lambda}_2\left[2mk_2(c_1^2 - c_2^2)\right]^{k_2}\right\} = 0,
\end{aligned} \tag{2.5.9}$$

with $k_2 < \dfrac{1}{2m}$, $\quad k_1 < \dfrac{1}{m-1}$; $\quad m \equiv -N$.

When $\epsilon = -1$, $N = 0$ we have

$$f_1 = c_1 \operatorname{sh}\left\{-\lambda_1\int |v(\omega)|^{k_1}d\omega + \widetilde{\lambda}_2(c_3^2 - c_1^2)^{k_2}\omega + c_2\right\},$$

$$f_2 = c_3 \operatorname{ch}\left\{\lambda_1\int |v(\omega)|^{k_1}d\omega + \widetilde{\lambda}_2(c_3^2 - c_1^2)^{k_2}\omega + c_4\right\},$$

$$g_1 = c_1 \operatorname{sh}\left\{-\lambda_1 \int |v(\omega)|^{k_1} d\omega + \widetilde{\lambda}_2 (c_3^2 - c_1^2)^{k_2}\omega + c_2\right\}, \tag{2.5.10}$$

$$g_2 = c_3 \operatorname{ch}\left\{\lambda_1 \int |v(\omega)|^{k_1} d\omega + \widetilde{\lambda}_2 (c_3^2 - c_1^2)^{k_2}\omega + c_4\right\},$$

$$v(\omega) = \rho(\omega)\exp\{i\theta(\omega)\},$$

where

$$\int^{\rho(\omega)} [a_-(z) + c_6]^{-1/2}\, dz = \omega + c_5,$$

$$\theta(\omega) = \int \rho^{-1/2}(\omega) d\omega + c_8,$$

$$a_-(z) = \frac{\mu_1^2}{k_1+1} z^{2(k_1+1)} + \widetilde{\mu}_2^2 (c_3^2 - c_1^2)^{2k_2} z^2 + \frac{4\mu_1\widetilde{\mu}_2}{k_1+2}(c_3^2 - c_1^2)^{k_2} z^{k_1+2} + 2c_5^2 z.$$

When $\epsilon = 1$, $N = 0$ we have

$$f_1 = c_1 \sin\left\{\lambda_1 \int |v(\omega)|^{k_1} d\omega - \widetilde{\lambda}_2 (c_3^2 - c_1^2)^{k_2}\omega + c_2\right\},$$

$$f_2 = c_3 \cos\left\{\lambda_1 \int |v(\omega)|^{k_1} d\omega - \widetilde{\lambda}_2 (c_3^2 - c_1^2)^{k_2}\omega + c_4\right\},$$

$$g_1 = c_1 \cos\left\{-\lambda_1 \int |v(\omega)|^{k_1} d\omega - \widetilde{\lambda}_2 (c_3^2 - c_1^2)^{k_2}\omega + c_2\right\}, \tag{2.5.11}$$

$$g_2 = c_3 \sin\left\{\lambda_1 \int |v(\omega)|^{k_1} d\omega - \widetilde{\lambda}_2 (c_3^2 - c_1^2)^{k_2}\omega + c_4\right\},$$

$$v = \rho(\omega)\exp\{i\theta(\omega)\},$$

where

$$\int^{\rho(\omega)} [a_+(z) + c_6]^{-1/2}\, dz = \omega + c_6,$$

$$\theta(\omega) = \int \sqrt{\rho(\omega)}\, d\omega + c_8,$$

$$a_+(z) = -\frac{\mu_1^2}{k_1+1} z^{2(k_1+1)} - \widetilde{\mu}_2^2 (c_1^2 - c_3^2)^{2k_2} z^2 - \frac{4\mu_1\widetilde{\mu}_2}{k_1+2}(c_1^2 - c_3^2)^{k_2} z^{k_1+2} + 2c_5 z.$$

When $\epsilon = 1$, $N = 1, 2, 3$ we have

$$f_j = c_j \omega^{-1/2k_2}, \quad j = 1, 2;$$

$$g_k = \mp(-1)^j (2Nk_2 - 1)^{1/2} c_j \omega^{-1/2k_2}; \tag{2.5.12}$$

$$v = E\omega^{-1/k_1},$$

where the arbitrary constants c_1, c_2 (real), and E (complex) satisfy the relations

$$(k_1+1)k_1^{-2} - Nk_1^{-1} + \left\{\mu_1|E|^{k_1} + \tilde{\mu}_2\left[2Nk_2(c_1^2-c_2^2)\right]^{k_2}\right\} = 0 \qquad (2.5.13)$$

$$\mp(2Nk_2-1)^{1/2} - 2k_2\left\{\lambda_1|E|^{k_1} + \tilde{\lambda}_2\left[2Nk_2(c_1^2-c_2^2)\right]^{k_2}\right\} = 0,$$

$$k_1 > \frac{1}{N-1}, \qquad k_2 > \frac{1}{2N}.$$

When $\epsilon = 1$; $N = 2,3$; $K_1 = 2/(N-1)$; $K_2 = 1/N$ we have

$$\begin{aligned} f_j &= (-1)^j\theta\omega g_j(\omega), \qquad j = 1,2; \\ g_j &= c_j(1+\theta^2 w)^{-(N+1)/2}; \\ v &= E(1+\theta^2\omega^2)^{(1-N)/2}, \end{aligned} \qquad (2.5.14)$$

where the arbitrary constants c_1, c_2, θ (real), and E (complex) satisfy the relations

$$\begin{aligned} \theta^2(N^2-1) &= \left[\mu_1|E|^{2/(N-1)} + \tilde{\mu}_2(c_1^2-c_2^2)^{1/N}\right]^2, \\ \theta(N+1) &= \lambda_1|E|^{2/(N-1)} + \tilde{\lambda}_2(c_1^2-c_2^2)^{1/N} \end{aligned} \qquad (2.5.15)$$

By substituting the explicit form of functions $f_1(\omega)$, $f_2(\omega)$, $g_1(\omega)$, and $g_2(\omega)$ given in (2.5.8)–(2.5.14) and the corresponding ω written in (2.1.31) into ansatz (2.5.6), we obtain solutions for Equations (2.5.1). Under $k_1 = 1$ and $k_2 = 1/3$ one can apply to these solutions formulae of generating solutions (2.3.27), (2.3.34) and in this way obtain new families of solutions of Equations (2.5.1) when $k_1 = 1$, $k_2 = 1/3$.

2.6. *Exact solutions of systems of nonlinear equations of quantum electrodynamics* *

Consider nonlinear coupled PDEs which describe interacting electron (Dirac spinor) and electromagnetic fields

$$\begin{aligned} &\left[\gamma^\mu(i\partial_\mu - eA_\mu) - m_1\right]\psi = 0, \\ &\Box A_\mu - \partial_\mu\partial^\nu A_\nu + m_2^2 A_\mu + \lambda A_\mu A^\nu A_\nu = e\overline{\psi}\gamma_\mu\psi, \end{aligned} \qquad (2.6.1)$$

where $\psi = \psi(x)$ is a four-component Dirac spinor; $A_\mu = A_\mu(x)$ is a four-vector potential of the electromagnetic field; $\mu, \nu = \overline{0,3}$; e, λ, m_1, m_2 are real

* Some results stated in this section were obtained in collaboration with R.Z. Zhdanov.

constants; γ_μ are Dirac matrices (2.1.2). When $\lambda = m_2 = 0$, system (2.6.1) coincides with the well-known standard equations of quantum electrodynamics [2, 31]. When $m_1 = m_2 = 0$, system (2.6.1) takes the form

$$\begin{aligned} &\gamma^\mu(i\partial_\mu - eA_\mu)\psi = 0, \\ &\Box A_\mu - \partial_\mu\partial^\nu A_\nu + \lambda A_\mu A^\nu A_\nu = e\bar{\psi}\gamma_\mu\psi, \end{aligned} \tag{2.6.2}$$

When $\lambda = 0$ system (2.6.1) takes the form

$$\begin{aligned} &\left[\gamma^\mu(i\partial_\mu - eA_\mu) - m_1\right]\psi = 0, \\ &\Box A_\mu - \partial_\mu\partial^\nu A_\nu + m_2^2 A_\mu = e\bar{\psi}\gamma_\mu\psi. \end{aligned} \tag{2.6.3}$$

And lastly, when $m_1 = m_2 = \lambda = 0$ system (2.6.1) takes the form

$$\begin{aligned} &\gamma^\mu(i\partial_\mu - eA_\mu)\psi = 0, \\ &\Box A_\mu - \partial_\mu\partial^\nu A_\nu = e\bar{\psi}\gamma_\mu\psi. \end{aligned} \tag{2.6.4}$$

Systems (2.6.1)–(2.6.4) are invariant under the following groups: (2.6.1) under P(1,3); (2.6.1) with $\lambda = m_2 = 0, m_1 \neq 0$ under P(1,3) $\oplus$ U(1); (2.6.2) under C(1,3); (2.6.3) under P(1,3); (2.6.4) under P(1,3) $\oplus$ U(1). Below we shall obtain exact solutions for these equations.

By substituting conformally invariant ansatze (2.3.51), (2.3.53) into (2.6.2) we get the system of ODEs

$$\begin{aligned} &i(\gamma\beta)\dot{\varphi} = e(\gamma B)\varphi \\ &\beta^2\ddot{B}_\mu - \beta_\mu\beta^\nu\ddot{B}_\nu + \lambda B_\mu B^\nu B_\nu = e\bar{\varphi}\gamma_\mu\varphi \end{aligned} \tag{2.6.5}$$

where dots indicate derivatives with respect to $\omega = \beta x/(x_\nu x^\nu)$. The simplest solution of Equations (2.6.5) is

$$\varphi(\omega) = \exp\{-ie\omega\}\chi, \qquad B_\mu(\omega) = \beta_\mu \tag{2.6.6}$$

where constant spinor χ and arbitrary constants β_μ satisfy the conditions

$$\lambda\beta^2\beta_\mu = e\bar{\chi}\gamma_\mu\chi. \tag{2.6.7}$$

From (2.6.6), returning to (2.3.51), (2.3.53), we get the following solution of Equations (2.6.2) (see [97], [98]):

$$\begin{aligned} \psi(x) &= \frac{\gamma x}{(x_\nu x^\nu)^2}\exp\left\{-ie\frac{\beta x}{x^\nu x_\nu}\right\}\chi, \\ A_\mu(x) &= \frac{\beta_\mu}{x_\nu x^\nu} - 2x_\mu\frac{\beta x}{(x^\nu x_\mu)^2} \equiv \partial_\mu\omega \end{aligned} \tag{2.6.8}$$

whence follows (see formulae (2.3.55), (2.3.56)) the C(1,3)-ungenerative solution of Equations (2.6.2)

$$\begin{aligned}\psi(x) &= \frac{\gamma y}{(y_\nu y^\nu)^2}\exp\left\{-ie\left(\frac{\beta y}{y^\nu y_\nu}+æ\right)\right\}\chi,\\ A_\mu(x) &= \frac{\beta_\mu}{y_\nu y^\nu}-2y_\mu\frac{\beta y}{(y^\nu y_\mu)^2}\end{aligned} \tag{2.6.9}$$

where $y_\nu = x_\nu + \delta_\nu$; $\delta_\nu, æ$ are arbitrary real constants; β_ν, χ satisfy relations (2.6.7).

Solutions (2.6.8), (2.6.9) have the properties: $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = 0$; and

$$\bar{\psi}\psi = \begin{cases}(x_\nu x^\nu)^{-3}(\bar{\chi}\chi) & \text{for (2.6.8)}\\ (y_\nu y^\nu)^{-3}(\bar{\chi}\chi) & \text{for (2.6.9)}\end{cases}$$

constant e (electron charge) tends to zero as soon as the coupling constant of self-interaction of the electromagnetic field $\lambda \to 0$.

Now we'll try to find solutions of system (2.6.2) with the help of the ansatz

$$\begin{aligned}\psi(x) &= (\gamma b)\exp\{if(ax)\},\\ A_\mu(x) &= b_\mu g_1(ax) + a_\mu g_2(ax); \qquad b^2 = 0\end{aligned} \tag{2.6.10}$$

where a_μ, b_μ are arbitrary real constants; f, g_1, g_2 real scalar functions to be determined; χ, as usual, is a constant spinor.

Substitution of (2.6.10) into (2.6.2) gives rise to the following system of ODEs:

$$\begin{aligned}&\dot{f} + eg_2 = 0,\\ &a^2\ddot{g}_1 = 2eb^\nu\bar{\chi}\gamma_\nu\chi - \lambda g_1\left(a^2g_2^2 + 2abg_1g_2\right),\\ &ab\ddot{g}_1 = \lambda g_2\left(a^2g_2^2 + 2abg_1g_2\right)\end{aligned} \tag{2.6.11}$$

where dot means differentiation with respect to $\omega = ax$.

System (2.6.11) can be integrated when $a^2 = 0$, $ab \neq 0$. In this case it takes the form

$$\begin{aligned}&\dot{f} + eg_2 = 0,\\ &eb^\nu\bar{\chi}\gamma_\nu\chi - \lambda abg_1^2g_2 = 0,\\ &\ddot{g}_1 - 2\lambda g_1g_2^2 = 0,\end{aligned}$$

whence follows

$$g_2 = \frac{æ}{\lambda}g_1^{-2}, \tag{2.6.12}$$

$$\ddot{g}_1 = \frac{2æ^2}{\lambda} g_1^{-3},$$

$$\dot{f} + eg_2 = 0,$$

where

$$æ = \frac{e}{ab} b^{\mu} \overline{\chi} \gamma_{\mu} \chi.$$

The general solution of system (2.6.12), depending on sign of λ is given by the formulae:

when $\lambda > 0, \quad c_1 \neq 0$ we have

$$g_1(\omega) = \pm c_1^{-1/2} \left[(c_1\omega + c_2)^2 + \frac{2æ^2}{\lambda} \right]^{1/2},$$

$$g_2(\omega) = \frac{æ c_1}{\lambda} \left[(c_1\omega + c_2)^2 + \frac{2æ^2}{\lambda} \right]^{-1}, \tag{2.6.13}$$

$$f(\omega) = -\frac{e}{\sqrt{2\lambda}} \arctan \left[\frac{\sqrt{\lambda}}{\sqrt{2}æ} (c_1\omega + c_2) \right];$$

when $\lambda < 0$, $c_1 \neq 0$ we have

$$g_1(\omega) = \pm c_1^{-1/2} \left[(c_1\omega + c_2)^2 + \frac{2æ^2}{\lambda} \right]^{1/2},$$

$$g_2(\omega) = \frac{æ c_1}{\lambda} \left[(c_1\omega + c_2)^2 + \frac{2æ^2}{\lambda} \right]^{-1}, \tag{2.6.14}$$

$$f(\omega) = -\frac{e}{2\sqrt{2|\lambda|}} \ln \left| \frac{\sqrt{|\lambda|}(c_1\omega + c_2) + \sqrt{2}æ}{\sqrt{|\lambda|}(c_1\omega + c_2) - \sqrt{2}æ} \right|;$$

when $c_1 = 0$, $\lambda < 0$ we have

$$g_1(\omega) = \pm \left[\frac{2\sqrt{2}æ}{\sqrt{|\lambda|}} + c_3 \right]^{1/2},$$

$$g_2(\omega) = \left[-2\sqrt{2|\lambda|}\omega + \frac{\lambda}{æ} c_3 \right]^{-1}, \tag{2.6.15}$$

$$f(\omega) = \frac{e}{2\sqrt{2|\lambda|}} \ln \left(2\sqrt{2|\lambda|}\omega - \frac{\lambda}{æ} c_3 \right);$$

where c_1, c_2, c_3 are arbitrary real constants. Inserting f, g_1, g_2 from (2.6.13)–(2.6.15) into (2.6.10) one obtains families of solutions of system (2.6.2):

$\lambda > 0, \quad c_1 \neq 0$:

$$\psi(x) = (\gamma b) \exp\left\{ -\frac{ie}{\sqrt{2\lambda}} \arctan\left[\frac{\sqrt{\lambda}}{\sqrt{2}æ} (c_1 ax + c_2) \right] \right\} \chi, \tag{2.6.16}$$

$$A_\mu(x) = \pm b_\mu c_1^{-1/2} \left[(c_1 ax + c_2)^2 + \frac{2æ^2}{\lambda} \right]^{1/2} + a_\mu æ c_1 \left[\lambda (c_1 ax + c_2)^2 + 2æ^2 \right]^{-1};$$

$\lambda < 0, \quad c_1 \neq 0$:

$$\psi(x) = (\gamma b) \exp\left\{ -\frac{ie}{2\sqrt{2|\lambda|}} \ln \left| \frac{\sqrt{|\lambda|}(c_1 ax + c_2) + \sqrt{2}æ}{\sqrt{|\lambda|}(c_1 ax + c_2) - \sqrt{2}æ} \right| \right\} \chi, \tag{2.6.17}$$

$$A_\mu(x) = \pm b_\mu c_1^{-1/2} \left[(c_1 ax + c_2)^2 + \frac{2æ^2}{\lambda} \right]^{1/2} + a_\mu æ c_1 \left[\lambda (c_1 ax + c_2)^2 + 2æ^2 \right]^{-1};$$

$\lambda < 0, \quad c_1 = 0$:

$$\psi(x) = (\gamma b) \exp\left\{ \frac{ie}{2\sqrt{2|\lambda|}} \ln \left(2\sqrt{|\lambda|} ax - \frac{\lambda}{æ} c_3 \right) \right\} \chi, \tag{2.6.18}$$

$$A_\mu(x) = \pm b_\mu \left[\frac{2\sqrt{2}æ}{\sqrt{|\lambda|}} ax + c_3 \right]^{1/2} + a_\mu \left[-2\sqrt{2|\lambda|} ax + \frac{\lambda}{æ} c_3 \right]^{-1}.$$

Let us recall that in (2.6.16)–(2.6.18) $b^2 = a^2 = 0$, $ab \neq 0$ (without loss of generality we can choose $ab = 1$);

$$æ = \frac{e}{ab} b^\nu (\bar{\chi} \gamma_\nu \chi);$$

c_1, c_2, c_3 are arbitrary real constants.

One can easily check that solutions (2.6.16)–(2.6.18) give $\overline{\psi}\psi = 0$ (due to the condition $b^2 = 0$); the tensor $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is given by the expressions, respectively

$$F_{\mu\nu} = \pm (b_\mu a_\nu - b_\nu a_\mu) \sqrt{c_1} (c_1 ax + c_2) \left[(c_1 ax + c_2)^2 + \frac{2æ^2}{\lambda} \right]^{-1/2};$$

$$F_{\mu\nu} = \pm (b_\mu a_\nu - b_\nu a_\mu) \sqrt{c_1} (c_1 ax + c_2) \left[(c_1 ax + c_2)^2 + \frac{2æ^2}{\lambda} \right]^{-1/2};$$

$$F_{\mu\nu} = \pm (b_\mu a_\nu - b_\nu a_\mu) \frac{æ}{\sqrt{2|\lambda|}} \left[\frac{2æ}{\sqrt{2|\lambda|}} ax + c_3 \right]^{-1/2}; \tag{2.6.19}$$

One can obtain from solutions (2.6.16)–(2.6.18) new families of solutions for Equations (2.6.2) if one applies to them the formulae of generating solutions (2.3.27) and (2.3.38).

Solutions for system (2.6.3) will be sought in the form

$$\begin{aligned} &\psi(x) = (\gamma b)\exp\{-im_1(\gamma a)\omega + if(bx)\}\chi, \\ &A_\mu(x) = b_\mu g(\omega), \qquad \omega = ax; \\ &b^2 = ab = 0, \qquad a^2 = -1. \end{aligned} \tag{2.6.20}$$

Substituting (2.6.20) into (2.6.3) gives rise to ODEs for the scalar function $g(\omega)$:

$$\ddot{g} - m_2^2 g + 2e(\theta_1 \operatorname{ch} 2m_1\omega + \theta_2 \operatorname{sh} 2m_1\omega) = 0, \tag{2.6.21}$$

where

$$\theta_1 = \bar{\chi}\gamma b\chi, \quad \theta_2 = i\bar{\chi}\gamma a\gamma b\chi \tag{2.6.22}$$

(one can easily confirm that θ_1 and θ_2 are real constants). With $m_2^2 > 0$, and $m_2^2 \neq 4m_1^2$ the general solution of Equation (2.6.21) has the form

$$g(\omega) = c_1 \operatorname{ch} m_2\omega + c_2 \operatorname{sh} m_2\omega + \frac{2e}{m_2^2 - 4m_1^2}(\theta_1 \operatorname{ch} 2m_1\omega + \theta_2 \operatorname{sh} 2m_1\omega). \tag{2.6.23}$$

With $m_2^2 = 4m_1^2$ the general solution of Equation (2.6.21) has the form

$$g(\omega) = \left(-\frac{e\theta_1}{2m_1}\omega + c_1\right) \operatorname{ch} 2m_1\omega + \left(-\frac{e\theta_2}{2m_1}\omega + c_2\right) \operatorname{sh} 2m_1\omega. \tag{2.6.24}$$

And, lastly, under $m_2^2 = -m^2 < 0$ the general solution of Equation (2.6.21) has the form

$$g(\omega) = c_1 \cos m\omega + c_2 \sin m\omega - \frac{2e}{m^2 + 4m_1^2}(\theta_1 \operatorname{ch} 2m_1\omega + \theta_2 \operatorname{sh} 2m_1\omega). \tag{2.6.25}$$

In formulae (2.6.23)–(2.6.25) c_1, c_2 are arbitrary real constants.

Inserting expressions (2.6.23)–(2.6.25) into ansatz (2.6.20) we obtain three families of solutions for Equations (2.6.3): when $m_2^2 > 0$, $m_2^2 \neq 4m_1^2$ we have

$$\begin{aligned} A_\mu(x) = b_\mu \Big[& c_1 \operatorname{ch} m_2ax + c_2 \operatorname{sh} m_2ax + \\ & + \frac{2e}{m_2^2 - 4m_1^2}(\theta_1 \operatorname{ch} 2m_1ax + \theta_2 \operatorname{sh} 2m_1ax)\Big]; \end{aligned} \tag{2.6.26}$$

when $m_2^2 = 4m_1^2$ we have

$$A_\mu(x) = b_\mu \left[\left(-\frac{e\theta_1}{2m_1} ax + c_1 \right) \operatorname{ch} 2m_1 ax + \left(-\frac{e\theta_2}{2m_1} ax + c_2 \right) \operatorname{sh} 2m_1 ax \right]; \tag{2.6.27}$$

when $m_2^2 = -m^2 < 0$ we have

$$A_\mu(x) = b_\mu \Big[c_1 \cos max + c_2 \sin max - \tag{2.6.28}$$

$$- \frac{2e}{m^2 + 4m_1^2} \left(\theta_1 \operatorname{ch} 2m_1 ax + \theta_2 \operatorname{sh} 2m_1 ax \right) \Big],$$

where θ_1, θ_2 are defined in (2.6.22); c_1, c_2, c_3 are arbitrary constants; for all three cases the spinor $\psi(x)$ has the form stated in (2.6.20) with the arbitrary differentiable function $f(x)$; $a^2 = -1$, $ab = b^2 = 0$. It will be noted that solution (2.6.26) has resonance nature when $m_2^2 \to 4m_1^2$.

Let us select from solutions (2.6.27) a subfamily

$$\begin{aligned} \psi(x) &= (\gamma b) \exp\{-im_1(\gamma a)ax\} \chi, \\ A_\mu(x) &= \frac{e\theta}{2m_1} b_\mu (ax)(\operatorname{sh} 2m_1 ax - \operatorname{ch} 2m_1 ax), \end{aligned} \tag{2.6.29}$$

where $\theta = \theta_1 = -\theta_2$. One can easily confirm that for these functions the following relations hold:

$$\begin{aligned} &\Box \psi + m_1^2 \psi = 0, \\ &\Box A_\mu + 4m_1 \left(m_1 - \frac{1}{ax} \right) A_\mu = 0 \end{aligned} \tag{2.6.30}$$

which apparently can be treated in such a manner. As a result of the interaction of the fields ψ and A_μ a particle comes into being. This particle has variable mass $M = M(x)$

$$M^2 = 4m_1 \left(m_1 - \frac{1}{ax} \right) \tag{2.6.31}$$

So $M^2 > 0$ when $(ax)^{-1} < m_1$, $M = 0$ when $(ax)^{-1} = m_1$, and lastly $M^2 < 0$ when $(ax)^{-1} > m$. In the case of non-resonance interactions such an effect does not take place.

Let us try to find solutions for system (2.6.3) with the help of the ansatz

$$\begin{aligned} \psi(x) &= (\gamma a) \exp\left\{ -ie \int^{\omega_3} v(\omega_0, \tau) d\tau \right\} \varphi(\omega_0, \omega_1, \omega_2, \omega_3), \\ A_\mu(x) &= a_\mu u(\omega_0, \omega_1, \omega_2, \omega_3) + d_\mu v(\omega_0, \omega_3) \end{aligned} \tag{2.6.32}$$

where

$$\begin{aligned} \omega_0 &= ax, & \omega_1 &= bx, \\ w_2 &= cx, & \omega_3 &= dx; \end{aligned} \tag{2.6.33}$$

$a_\mu, b_\mu, c_\mu, d_\mu$ are arbitrary real constants satisfying relations

$$\begin{aligned} &a^2 = d^2 = ab = bc = cd = ac = bd = 0, \\ &b^2 = c^2 = -1, \qquad ad = 2; \end{aligned} \tag{2.6.34}$$

where $u = u(\omega_0, \omega_1, \omega_2, \omega_3)$, $v = v(\omega_0, \omega_3)$ are scalar real functions, and $\varphi = \varphi(\omega_0, \omega_1, \omega_2, \omega_3)$ is a four-component spinor.

After substituting (2.6.32) into (2.6.3) we get the following system of PDEs

$$\begin{aligned} &i(\gamma b)\varphi_1 + i(\gamma c)\varphi_2 + m_1\varphi = 0, \\ &u_{11} + u_{22} = m_2^2 u - 2e\overline{\varphi}\gamma a\varphi + 4u_{03}, \\ &u_3 + v_0 = 0, \\ &4v_{03} + m_2^2 v = 0, \end{aligned} \tag{2.6.35}$$

where $\varphi_\mu \equiv \dfrac{\partial \varphi}{\partial \omega_\mu}$; $\quad u_\mu \equiv \dfrac{\partial u}{\partial \omega_\mu}$; $\quad \mu = \overline{0,3}$.

Since $v = v(\omega_0, \omega_3)$, equation $v_0 + u_3 = 0$ from (2.6.35) yields

$$u = U(\omega_0, \omega_3) + \widetilde{U}(\omega_0, \omega_1, \omega_2). \tag{2.6.36}$$

Inserting this expression into (2.6.35) we get system of PDEs

$$\begin{aligned} &1^\circ \quad i\gamma b\varphi_1 + i\gamma c\varphi_2 + m_1\varphi = 0, \\ &2^\circ \quad \widetilde{U}_{11} + \widetilde{U}_{22} = m_2^2\widetilde{U} - 2e\overline{\varphi}\gamma a\varphi, \\ &3^\circ \quad m_2^2 U + 4U_{03} = 0, \\ &4^\circ \quad m_2^2 v + 4v_{03} = 0, \\ &5^\circ \quad v_0 + U_3 = 0. \end{aligned} \tag{2.6.37}$$

System (2.6.37) is formally a nonlinear one, but if we solve the first equation 1° and insert the result into the second equation 2°, we will reduce it to a linear inhomogeneous system of PDEs which can be solved using the classical method of separation of variables.

Let us present two partial solutions of Equation 1° (2.6.37):

$$\varphi(\omega) = \exp\{-im_1(\gamma b)\omega_1\}\,\chi(w_0), \tag{2.6.38}$$

$$\varphi(\omega) = \left(\omega_1^2 + \omega_2^2\right)^{-1/4} \exp\left\{-\tfrac{1}{2}(\gamma b)(\gamma c)\arctan\frac{\omega_1}{\omega_2}\right\} \times \tag{2.6.39}$$

$$\times \exp\left\{-im_1(\gamma c)\left(\omega_1^2 + \omega_2^2\right)^{1/2}\right\}\chi(\omega_0),$$

where χ is an arbitrary spinor depending on $\omega_0 = ax$.

The general solution of Equation 2° (2.6.37) has the form

$$\widetilde{U} = \widetilde{U}^{(1)} + \widetilde{U}^{(2)}, \tag{2.6.40}$$

where $\widetilde{U}^{(1)}$ is the general solution of the Helmholtz equation

$$\widetilde{U}^{(1)}_{11} + \widetilde{U}^{(1)}_{22} = \widetilde{m}_2^2 U^{(1)}, \tag{2.6.41}$$

and $\widetilde{U}^{(2)}$ is a partial solution of Equation 2° (2.6.37). Under φ from (2.6.38) a partial solution of Equation 2° (2.6.37) is given by the expression

$$\widetilde{U}^{(2)} = \begin{cases} \dfrac{2e}{m_2^2 - 4m_1^2}\left(\theta_1 \operatorname{ch} 2m_1\omega_1 + \theta_2 \operatorname{sh} 2m_1\omega_1\right), & m_2^2 \neq 4m_1^2; \\ -\dfrac{e}{2m_1}\left(\theta_1\omega_1 \operatorname{sh} 2m_1\omega_1 + \theta_2\omega_1 \operatorname{ch} 2m_1\omega_1\right), & m_2^2 = 4m_1^2; \end{cases} \tag{2.6.42}$$

where

$$\begin{aligned} \theta_1 &= \overline{\chi}(\omega_0)\gamma a\chi(\omega_0), \\ \theta_2 &= i\overline{\chi}(\omega_0)\gamma a\gamma b\chi(\omega_0). \end{aligned} \tag{2.6.43}$$

Under φ from (2.6.39) a partial solution of Equation 2° (2.6.37) is expressed via the Bessel functions. We do not write it because it is rather cumbersome.

The general solution of Equations 3° – 5° (2.6.37) can easily be found by means of the method of separation of variables. It has the form

$$\begin{aligned} U &= c_1 \exp\left\{4c_2\omega_0 - \frac{m_2^2}{c_2}\omega_3\right\}, \\ v &= \frac{m_2^2 c_1}{4c_2^2} \exp\left\{4c_2\omega_0 - \frac{m_2^2}{c_2}\omega_3\right\}, \end{aligned} \tag{2.6.44}$$

where c_1, c_2 are arbitrary real constants, $c_2 \neq 0$.

Using (2.6.32)–(2.6.34), (2.6.36), (2.6.38), (2.6.40)–(2.6.44) it is not difficult to find corresponding solutions of system (2.6.3), but we do not do it here for the sake of brevity. Let us only note that these solutions, due to the term $\widetilde{U}^{(2)}$, (2.6.42) have the resonance nature analogous to that of solution (2.6.26).

Solutions for system (2.6.4) are sought in the form

$$\begin{aligned} \psi(x) &= (\gamma b) \exp\{if(bx)\}\chi, \\ A_\mu(x) &= b_\mu g(\omega), \qquad \omega = ax, \end{aligned} \tag{2.6.45}$$

where a_μ, b_μ are arbitrary constants and $a^2 = -1, \quad b^2 = ab = 0$. Substituting (2.6.45) into (2.6.4) gives rise to the ODE for a scalar real function

$$\ddot{g} + 2e\overline{\chi}\gamma b\chi = 0. \tag{2.6.46}$$

The general solution of Equation (2.6.46) is given by

$$g(\omega) = -e\overline{\chi}\gamma b\chi\omega^2 + c_1\omega + c_2$$

where c_1, c_2 are arbitrary constants. After substituting this expression into (2.6.45) we get a partial solution of system (2.6.4):

$$\begin{aligned} \psi(x) &= (\gamma b)\exp\{if(bx)\}\chi, \\ A_\mu(x) &= b_\mu\left[-e\overline{\chi}\gamma b\chi(ax)^2 + c_1 ax + c_2\right] \end{aligned} \tag{2.6.47}$$

where f is an arbitrary differentiable real function; a_μ, b_μ, c_1, c_2 are arbitrary constants, $a^2 = -1$, $b^2 = ab = 0$. Having applied the formulae of generating solutions (2.3.27) and (2.3.38) to (2.6.47) one obtains a new family of solutions for system (2.6.4).

Now we shall demonstrate how to do a partial linearization of the standard equations of quantum electrodynamics which coincide with (2.6.1) under $\lambda = m_2 = 0$ and thereby construct wide families of their solutions.

Let us write the standard equations of quantum electrodynamics

$$\begin{aligned} \left[\gamma^\mu(i\partial_\mu - eA_\mu) - m\right]\psi &= 0, \\ \Box A_\mu - \partial_\mu\partial^\nu A_\nu &= e\overline{\psi}\gamma_\mu\psi. \end{aligned} \tag{2.6.48}$$

Solutions of these equations are sought in the form

$$\begin{aligned} \psi(x) &= (\gamma\beta)\varphi(\omega), \\ A_\mu(x) &= \beta_\mu g(\omega), \end{aligned} \tag{2.6.49}$$

where $g(\omega)$ is a scalar real function, $\varphi(\omega)$ is a spinor to be determined; ω are three new variables

$$\begin{aligned} &\omega_1 = ax, \qquad \omega_2 = bx, \qquad \omega_3 = \beta x; \\ &a^2 = b^2 = -1, \qquad \beta^2 = ab = a\beta = b\beta = 0. \end{aligned} \tag{2.6.50}$$

By substituting (2.6.49) into (2.6.48) we get the system of PDEs

$$\begin{aligned} &(\gamma a)\varphi_1 + (\gamma b)\varphi_2 - im\varphi = 0, \\ &g_{11} + g_{22} + 2e\overline{\varphi}\gamma\beta\varphi = 0, \end{aligned} \tag{2.6.51}$$

where $\varphi_1 = \partial\varphi/\partial\omega_1$, $\varphi_2 = \partial\varphi/\partial\omega_2$ and so on.

Let $g = \widetilde{g}(\omega, \varphi)$ be a partial solution of the second equation of system (2.6.51). Then on making the change of variables

$$\begin{aligned} \phi(\omega) &= \varphi(\omega), \\ G(\omega) &= g(\omega) - \widetilde{g}(\omega, \varphi) \end{aligned} \tag{2.6.52}$$

we reduce Equations (2.6.51) (and thereby system (2.6.48)) to the linear system of PDEs

$$\begin{aligned}&(\gamma a)\phi_1 + (\gamma b)\phi_2 - im\phi = 0,\\ &G_{11} + G_{22} = 0.\end{aligned} \tag{2.6.53}$$

This allows us to construct wide classes of solutions for system (2.6.48).

Suppose that the function φ in (2.6.51) does not depend on ω_2. In such a case we have for $\varphi = \varphi(\omega_1, \omega_2)$ the ODE

$$(\gamma a)\varphi_1 - im\varphi = 0;$$

the general solution of which has the form

$$\varphi = \exp\{-im_1(\gamma a)\omega_1\}\,\chi(\omega_3). \tag{2.6.54}$$

Inserting (2.6.54) into the second equation of system (2.6.51) we get

$$g_{11} + g_{22} + 2e\,[\theta_1 \operatorname{ch} 2m\omega_1 + \theta_2 \operatorname{sh} 2m\omega_1] = 0, \tag{2.6.55}$$

where

$$\theta_1 = \overline{\chi}(\omega_3)\gamma\beta\chi(\omega_3), \qquad \theta_2 = i\overline{\chi}(\omega_3)\gamma a\gamma\beta\chi(\omega_3).$$

The general solution of Equation (2.6.55) can be written as

$$g(\omega) = f(z, \omega_3) + f(z^*, \omega_3) - \frac{e}{2m^2}(\theta_1 \operatorname{ch} 2m\omega_1 + \theta_2 \operatorname{sh} 2m\omega_1), \tag{2.6.56}$$

where $z = \omega_1 + i\omega_2$; f is an analytic function of z. By substituting (2.6.56) and (2.6.54) into (2.6.49) we obtain a family of exact solutions of system (2.6.48)

$$\begin{aligned}&\psi(x) = (\gamma\beta)\exp\{-im(\gamma a)ax\}\,\chi(\beta x),\\ &A_\mu(x) = \beta_\mu\Big[f(z, \beta x) + f(z^*, \beta x) - \frac{e}{2m^2}\,(\theta_1 \operatorname{ch} 2max + \theta_2 \operatorname{sh} 2max)\Big],\end{aligned} \tag{2.6.57}$$

where $z = ax + ibx$; θ_1, θ_2 are the same as in (2.6.55); a_μ, b_μ, β_μ are defined in (2.6.50).

It will be noted that solution (2.6.57) is analytic in the constant e and singular in the constant m.

2.7. *On the linearization and general solution of two-dimensional Dirac-Heisenberg-Thirring and quantum electrodynamics equations*

Following [106] and [212] we consider two two-dimensional nonlinear systems of PDEs: equations of the Dirac-Heisenberg-Thirring type

$$i\Gamma_\mu \partial_\mu \psi = \lambda(\bar{\psi}\Gamma_\mu\psi)\Gamma^\mu\psi, \tag{2.7.1}$$

where $\mu = 0, 1$;

$$\Gamma_0 = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}, \qquad \Gamma_1 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} \tag{2.7.2}$$

and the equations of quantum electrodynamics

$$\begin{gathered} \left[i\gamma_\mu \partial_\mu - e\gamma_\mu A_\mu + \lambda(\bar{\psi}\gamma_\mu\psi)\gamma^\mu\right]\psi, \\ \Box A_\mu - \partial_\mu \partial^\nu A_\nu = -e\bar{\psi}\gamma_\mu\psi \end{gathered} \tag{2.7.3}$$

Here $\psi = \psi(x_0, x_1)$ is a four-component spinor; γ_μ are Dirac matrices (2.1.2); $A_\mu = A_\mu(x_0, x_1)$ is a vector-potential of electromagentic field; σ_2, σ_3 are Pauli matrices (2.1.3); λ, e are arbitrary real constants; $\mu, \nu = 0, 1$.

Systems (2.7.1) and (2.7.3) are linearized by means of reversible nonlocal transformations. This allows us to construct the general solutions of the systems.

1. Let us rewrite system (2.7.1) by introducing the two-component spinors φ and χ such that $\psi = \text{column}(\varphi, \chi)$:

$$\begin{gathered} i(\sigma_2\varphi_0 + \sigma_3\varphi_1) = \lambda\left[i\left(|\varphi|^2 + |\chi|^2\right)\sigma_2 - i(\varphi^+\sigma_2\sigma_3\varphi + \chi^+\sigma_2\sigma_3\chi)\sigma_3\right]\varphi, \\ i(i\sigma_2\chi_0 + \sigma_3\chi_1) = \lambda\left[i\left(|\varphi|^2 + |\chi|^2\right)\sigma_2 - i(\varphi^+\sigma_2\sigma_3\varphi + \chi^+\sigma_2\sigma_3\chi)\sigma_3\right]\chi, \end{gathered} \tag{2.7.4}$$

In light-cone variables

$$\xi = x_0 - x_1, \qquad \eta = x_0 + x_1 \tag{2.7.5}$$

system (2.7.4) takes the form

$$\begin{aligned} i\varphi^0_\xi &= -\lambda\left(|\chi^1|^2 + |\varphi^1|^2\right)\varphi^0, \\ i\varphi^1_\eta &= \lambda\left(|\chi^0|^2 + |\varphi^0|^2\right)\varphi^1, \\ i\chi^0_\xi &= -\lambda\left(|\chi^1|^2 + |\varphi^1|^2\right)\chi^0, \\ i\chi^1_\eta &= \lambda\left(|\chi^0|^2 + |\varphi^0|^2\right)\chi^1, \end{aligned} \tag{2.7.6}$$

where superscripts mean components, and subscripts mean differentiation with respect to the corresponding index variable (ξ or η).

By means of the following nonlocal reversible change of variables

$$\begin{aligned} \varphi^0 &= v^0 \exp\left\{i\lambda \int \left(|u^1|^2 + |v^1|^2\right) d\xi\right\}, \\ \varphi^1 &= v^1 \exp\left\{-i\lambda \int \left(|u^0|^2 + |v^0|^2\right) d\eta\right\}, \end{aligned} \tag{2.7.7}$$

$$\chi^0 = u^0 \exp\left\{i\lambda \int (|u^1|^2 + |v^1|^2)\, d\xi\right\},$$

$$\chi^1 = u^1 \exp\left\{-i\lambda \int (|u^0|^2 + |v^0|^2)\, d\eta\right\},$$

where v^0, v^1, u^0, u^1 are complex scalar functions depending on ξ and η, system (2.7.6) is reduced to the decomposed linear system of PDEs

$$v^0_\xi = u^0_\xi = 0, \qquad v^1_\eta = u^1_\eta = 0. \tag{2.7.8}$$

The reader can easily confirm that substituting (2.7.7) into (2.7.6) results in (2.7.8). Therefore the problem of integrating Equations (2.7.1) is reduced to that of integrating Equations (2.7.8). The general solution of Equations (2.7.8) is not difficult to find. It has the form

$$\begin{aligned} u^0 &= F^0(\eta), \qquad & u^1 &= F^1(\xi), \\ v^0 &= G^0(\eta), \qquad & v^1 &= G^1(\xi), \end{aligned} \tag{2.7.9}$$

where F^0, G^0, F^1, G^1 are arbitrary complex differentiable functions. After substituting (2.7.9) and (2.7.5) into (2.7.7) we obtain the general solution of the system (2.7.1)

$$\begin{aligned}
\varphi^0 &= F^0(x_0 + x_1) \exp\left\{ i\lambda \int^{x_0 - x_1} (|F^1|^2 + |G^1|^2)\, d\xi \right\}, \\
\varphi^0 &= F^1(x_0 - x_1) \exp\left\{ -i\lambda \int^{x_0 + x_1} (|F^0|^2 + |G^0|^2)\, d\eta \right\}, \\
\chi^0 &= G^0(x_0 + x_1) \exp\left\{ i\lambda \int^{x_0 - x_1} (|F^1|^2 + |G^1|^2)\, d\xi \right\}, \\
\chi^1 &= G^1(x_0 - x_1) \exp\left\{ i\lambda \int^{x_0 + x_1} (|F^0|^2 + |G^0|^2)\, d\eta \right\},
\end{aligned} \tag{2.7.10}$$

Remark 2.7.1. Whether or not it is possible to linearize the Dirac-Heisenberg-Thirring system (2.7.1) is closely related to the fact that this system admits infinite-dimensional local groups of invariance. For that reason there is another way of obtaining the general solution (2.7.10). It consists of the following. One must find a partial solution to system (2.7.1) and then multiply it by means of formulae of generating solutions (which are constructed according to (17)) until it becomes ungenerative. This latter solution will be the general solution of (2.7.10).

Using Lie's algorithm one can prove that the maximal invariance group of Equations (2.7.4) is O(4)⊗O(4)⊗G_∞, where G_∞ is the infinite-dimensional group of transformations

$$x_0 \to x_0' = \tfrac{1}{2}\left[\int^{x_0-x_1} f_1^{-2}(\xi)d\xi + \int^{x_0+x_1} f_0^{-2}(\eta)d\eta\right],$$

$$x_1 \to x_1' = \tfrac{1}{2}\left[\int^{x_0+x_1} f_0^{-2}(\eta)d\eta - \int^{x_0-x_1} f_1^{-2}(\xi)d\xi\right],$$

$$\varphi^0 \to \varphi^{0\prime} = f_0(x_0+x_1)\varphi^0,$$

$$\chi^0 \to \chi^{0\prime} = f_0(x_0+x_1)\chi^0,$$

$$\varphi^1 \to \varphi^{1\prime} = f_1(x_0-x_1)\varphi^1,$$

$$\chi^1 \to \chi^{1\prime} = f_1(x_0-x_1)\chi^1,$$

where f_0, f_1 are arbitrary real functions.

Note that system (2.7.4) also admits the following nonlocal group of transformations

$$\chi^0 \to \chi^{0\prime} = a\chi^0 \exp\left\{i\lambda\int[(|b|^2-1)|\chi^1|^2 + (|d|^2-1)|\varphi^1|^2]d\xi\right\},$$

$$\chi^1 \to \chi^{1\prime} = b\chi^1 \exp\left\{-i\lambda\int[(|a|^2-1)|\chi^0|^2 + (|c|^2-1)|\varphi^0|^2]d\eta\right\},$$

$$\varphi^0 \to \varphi^{0\prime} = c\varphi^0 \exp\left\{i\lambda\int[(|b|^2-1)|\chi^1|^2 + (|d|^2-1)|\varphi^1|^2]d\xi\right\},$$

$$\varphi^1 \to \varphi^{1\prime} = b\varphi^1 \exp\left\{-i\lambda\int[(|a|^2-1)|\chi^0|^2 + (|c|^2-1)|\varphi^0|^2]d\eta\right\},$$

where a, b, c, d are arbitrary complex parameters.

2. Let us consider the two-dimensional equations of quantum electrodynamics (2.7.3). Rewriting (2.7.3) in terms of light-cone variables (2.7.5) and representing ψ as a column vector with four components $\psi^0, \psi^1, \psi^2, \psi^3$ we get the equivalent system of PDEs:

$$i\psi^0_\xi = -\left[\tfrac{1}{2}e(A^1-A^0) + \lambda\left(|\psi^1|^2+|\psi^3|^2\right)\right]\psi^0,$$

$$i\psi^1_\eta = \left[\tfrac{1}{2}e(A^1+A^0) + \lambda\left(|\psi^0|^2+|\psi^2|^2\right)\right]\psi^1,$$

$$i\psi^2_\xi = -\left[\tfrac{1}{2}e(A^1-A^0) + \lambda\left(|\psi^1|^2+|\psi^3|^2\right)\right]\psi^2, \qquad (2.7.11)$$

$$i\psi^3_\eta = \left[\tfrac{1}{2}e(A^1 + A^0) + \lambda\left(|\psi^0|^2 + |\psi^2|^2\right)\right]\psi^3,$$
$$\left(\partial_\xi - \partial_\eta\right)\left(A^0_\eta - A^0_\xi + A^1_\eta + A^1_\xi\right) = e\left(|\psi^0|^2 + |\psi^1|^2 + |\psi^2|^2 + |\psi^3|^2\right),$$
$$\left(\partial_\xi + \partial_\eta\right)\left(A^0_\eta - A^0_\xi + A^1_\eta + A^1_\xi\right) = e\left(-|\psi^0|^2 + |\psi^1|^2 - |\psi^2|^2 + |\psi^3|^2\right).$$

System (2.7.11) is linearized by means of the following nonlocal reversible change of variables:

$$\psi^0 = u^0 \exp\left\{i\lambda\int\left(|u^1|^2 + |u^3|^2\right)d\xi + \frac{ie}{2}\int(A^1 - A^0)d\xi\right\},$$
$$\psi^1 = u^1 \exp\left\{-i\lambda\int\left(|u^0|^2 + |u^2|^2\right)d\eta + \frac{ie}{2}\int(A^1 + A^0)d\eta\right\}, \qquad (2.7.12)$$
$$\psi^2 = u^2 \exp\left\{i\lambda\int\left(|u^1|^2 + |u^3|^2\right)d\xi + \frac{ie}{2}\int(A^1 - A^0)d\xi\right\},$$
$$\psi^3 = u^3 \exp\left\{-i\lambda\int\left(|u^0|^2 + |u^2|^2\right)d\eta + \frac{ie}{2}\int(A^1 + A^0)d\eta\right\},$$

where u^0, u^1, u^2, u^3 are complex functions. Substituting (2.7.12) into (2.7.11) gives rise to the equations for the functions $u^0, u^1, u^2, u^3, A^0, A^1$:

$$u^0_\xi = u^2_\xi = 0, \qquad u^1_\eta = u^3_\eta = 0,$$
$$\left(\partial_\xi - \partial_\eta\right)\left(A^0_\eta - A^0_\xi + A^1_\eta + A^1_\xi\right) = e\left(|u^0|^2 + |u^1|^2 + |u^2|^2 + |u^3|^2\right), \quad (2.7.13)$$
$$\left(\partial_\xi + \partial_\eta\right)\left(A^0_\eta - A^0_\xi + A^1_\eta + A^1_\xi\right) = e\left(-|u^0|^2 + |u^1|^2 - |u^2|^2 + |u^3|^2\right).$$

After integrating this system and substituting the result into (2.7.12), we obtain the general solution of system (2.7.3):

$$A^0 = e\int^{x_0+x_1}\int^{z}\left(|u^0|^2 + |u^2|^2\right)d\eta dz + \frac{\partial f}{\partial x_0},$$
$$A^0 = -e\int^{x_0-x_1}\int^{z}\left(|u^1|^2 + |u^3|^2\right)d\xi dz - \frac{\partial f}{\partial x_1},$$
$$\psi^0 = u^0(x_0 + x_1)\exp\Big\{i\int^{x_0-x_1}\left[\lambda(|u^1|^2 + |u^3|^2) + \frac{e}{2}\left(A^1 - A^0\right)\right]d\xi\Big\},$$
$$\psi^1 = u^1(x_0 - x_1)\exp\Big\{-i\int^{x_0+x_1}\left[\lambda(|u^0|^2 + |u^2|^2) + \frac{e}{2}\left(A^1 + A^0\right)\right]d\eta\Big\},$$
$$\psi^2 = u^2(x_0 + x_1)\exp\Big\{i\int^{x_0-x_1}\left[\lambda\left(|u^1|^2 + |u^3|^2\right) + \frac{e}{2}\left(A^1 + A^0\right)\right]d\eta\Big\},$$

$$u^3(x_0 - x_1)\exp\Big\{-i \int^{x_0+x_1} [\lambda(|u^0|^2 + |u^3|^2) + \frac{e}{2}(A^1 - A^0)]d\eta\Big\},$$

where u^0, u^1, u^2, u^3 are arbitrary complex functions, and $f(x_0, x_1)$ is an arbitrary real function.

2.8. *Symmetry analysis of nonlinear equations of classical electrodynamics*

In describing the electromagnetic field in various media, an investigator commonly uses Maxwell's equations in the form

$$\begin{aligned} \frac{\partial \vec{D}}{\partial t} &= \operatorname{rot} \vec{H}, \qquad \operatorname{div} \vec{D} = 0, \\ \frac{\partial \vec{B}}{\partial t} &= -\operatorname{rot} \vec{E}, \qquad \operatorname{div} \vec{B} = 0, \end{aligned} \tag{2.8.1}$$

where $\vec{E}$ and $\vec{H}$ are vectors of intensity and $\vec{D}$ and $\vec{B}$ are vectors of the induction of electric and magnetic fields. The system of Equations (2.8.1) is undetermined, and it is therefore necessary to add to it the constitutive equations (equations connecting $\vec{D}$, $\vec{B}$, $\vec{E}$, and $\vec{H}$) which reflect the properties of the medium. As a rule, the constitutive equations considerably restrict the symmetry of the whole system of field equations. From our point of view [107] one can use symmetry as a guide-line in constructing and selecting constitutive equations. Below we exploit this idea to describe constraints between $\vec{D}$, $\vec{B}$, $\vec{E}$, and $\vec{H}$ which conserve Poincare and conformal symmetry of the field equations (2.8.1). Note, that in [143*] it is proposed another approach to nonlinear generalization of Maxwell's equations. It is based on the idea of changing the speed of light c onto $v = f(\vec{E}^2 - \vec{H}^2, \vec{E}\cdot\vec{H})$, the latter being the speed of electromagnetic field energy transfer.

Symmetry properties of Maxwell's equations in vacua were investigated by Lorentz, Poincare, and Einstein, and in full detail by Bateman and Cunningham [25, 43]. The two latter authors established that Maxwell's equations possess only a 16-parameter (in Lie's sense, of course) group of invariance which contains as a subgroup the 15-parameter conformal group C(1,3). As mentioned above, the essential difference between Equations (2.8.1) and Maxwell's equations for an electromagnetic field in vacuum is the strong undetermination of the former. Therefore it is natural to expect system (2.8.1) to have wider symmetry than Maxwell's equations in vacuum. It is appropriate to point out that Maxwell's equations in vacuum represent an overdetermined system of eight equations for six unknown quantities. The symmetry properties of Equations (2.8.1) are summarized by the following statement [107].

Theorem 2.8.1. *Equations (2.8.1) are invariant under infinite-dimensional Lie algebra, with basis elements having the form*

$$X_1 = \xi^\mu(x)\partial_\mu + \eta_{\mu\nu}\partial_{F_{\mu\nu}} + \widetilde{\tau}_{\mu\nu}\partial_{H_{\mu\nu}}, \tag{2.8.2}$$

$$X_2 = \tfrac{1}{2}F_{\mu\nu}\partial_{F_{\mu\nu}},$$

$$X_3 = \tfrac{1}{2}\widetilde{H}_{\mu\nu}\partial_{\widetilde{H}_{\mu\nu}},$$

$$X_4 = \tfrac{1}{2}F_{\mu\nu}\partial_{\widetilde{H}_{\mu\nu}},$$

$$X_5 = \tfrac{1}{2}\widetilde{H}_{\mu\nu}\partial_{F_{\mu\nu}},$$

where $\xi^\mu(x)$ are arbitrary differentiable functions; $\mu, \nu = \overline{0,3}$;

$$\begin{aligned} &F_{\mu\nu} = -F_{\mu\nu}, \qquad F_{0a} = E_a, \qquad F_{ab} = \epsilon_{abc}B_c \\ &\widetilde{F}_{\mu\nu} = -\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}; \qquad H_{\mu\nu} = -H_{\nu\mu}, \end{aligned} \tag{2.8.3}$$

$$H_{0a} = D_a, \qquad H_{ab} = -\epsilon_{abc}H_c, \qquad \widetilde{H}_{\mu\nu} = -\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}H^{\rho\sigma};$$

$$\begin{aligned} &\eta_{\mu\nu} = -F_{\mu\alpha}\xi^\alpha_\nu - F_{\alpha\nu}\xi^\alpha_\mu \qquad \left(\xi^\alpha_\nu \equiv \frac{\partial\xi^\alpha}{\partial x^\nu}\right) \\ &\widetilde{\tau}_{\mu\nu} = -\widetilde{H}_{\mu\alpha}\xi^\alpha_\nu - \widetilde{H}_{\alpha\nu}\xi^\alpha_\mu. \end{aligned} \tag{2.8.4}$$

Proof. Using tensors $F_{\mu\nu}$ and $\widetilde{H}_{\mu\nu}$ from (2.8.3) we rewrite Equations (2.8.1) as follows:

$$\begin{aligned} L_1 &\equiv \partial_\alpha F_{\mu\nu} + \partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} = 0, \\ L_2 &\equiv \partial_\alpha \widetilde{H}_{\mu\nu} + \partial_\mu \widetilde{H}_{\nu\alpha} + \partial_\nu \widetilde{H}_{\alpha\mu} = 0. \end{aligned} \tag{2.8.5}$$

The proof of the theorem is reduced to the application of Lie's algorithm to system (2.8.5). This means that one has to construct all differential operators

$$X = \xi^\mu\partial_\mu + \eta^{\mu\nu}\partial_{F^{\mu\nu}} + \widetilde{\tau}^{\mu\nu}\partial_{\widetilde{H}^{\mu\nu}} \tag{2.8.6}$$

satisfying the conditions of invariance

$$\underset{1}{X}L_1\Big|_{\substack{L_1=0\\L_2=0}} = 0, \qquad \underset{1}{X}L_2\Big|_{\substack{L_1=0\\L_2=0}} = 0, \tag{2.8.7}$$

where the operator $\underset{1}{X}$ is constructed according to the prologation formulae (4).

From (2.8.7) we get the defining equations

$$\xi^{\mu}_{F_{\alpha\beta}} = \xi^{\mu}_{\widetilde{H}_{\alpha\beta}} = 0,$$

$$\partial_\alpha \eta_{\mu\nu} + \partial_\mu \eta_{\nu\alpha} + \partial_\nu \eta_{\alpha\mu} = 0,$$

$$\partial_\alpha \widetilde{\tau}_{\mu\nu} + \partial_\mu \widetilde{\tau}_{\nu\alpha} + \partial_\nu \widetilde{\tau}_{\alpha\mu} = 0, \tag{2.8.8}$$

$$\eta^{\mu\nu}_{F_{\rho\sigma}} = \widetilde{\tau}^{\mu\nu}_{\widetilde{H}_{\rho\sigma}} = \xi^{\rho}_{\nu}\delta_{\mu\sigma} - \xi^{\sigma}_{\nu}\delta_{\mu\rho} + \xi^{\sigma}_{\mu}\delta_{\nu\rho} - \xi^{\rho}_{\mu}\delta_{\nu\sigma},$$

$$\widetilde{\tau}^{\mu\nu}_{\widetilde{H}_{\mu\nu}} = c_1, \quad \eta^{\mu\nu}_{F_{\mu\nu}} = c_2, \quad \eta^{\mu\nu}_{\widetilde{H}_{\mu\nu}} = c_3, \quad \eta^{\mu\nu}_{\widetilde{H}_{\mu\nu}} = c_4 \qquad \text{(no sum over } \mu,\nu),$$

where c_1, c_2, c_3, c_4 are arbitrary constants.

The general solution of Equations (2.8.8) determines the maximal invariance algebra of system (2.8.1). The theorem is proved.

Theorem 2.8.2 [107]. *System* (2.8.5) *is invariant under 20-dimensional Lie algebra* AIGL(4,R) *which contains as subalgebras Poincare algebra* AP(1,3) *and Galilei algebra* AG(1.3).

Proof. According to the previous theorem, Equations (2.8.5) admit infinitesimal operators (2.8.6) with arbitrary functions $\xi^\mu(x)$. Consider $\xi^\mu(x) = c^{\mu\nu}x_\nu + a^\mu$ ($c^{\mu\nu}$, a^μ are arbitrary constants). If $c_{\mu\nu} = -c_{\nu\mu}$ then the operator X_1 (2.8.2) yields basis elements of AP(1,3)

$$\begin{aligned} P_0 &= i\partial_0, \qquad P_a = -\partial_a, \qquad a = 1,2,3, \\ J_{\mu\nu} &= x_\mu P_\nu - x_\nu P_\mu + (S_{\mu\nu}\psi)^l \partial_{\psi^l} \end{aligned} \tag{2.8.9}$$

where the summation is assumed over l from 1 to 12; ψ is a 12-component column $(E_1, \ldots, B_1, \ldots, D_1, \ldots, H_1, \ldots, H_3)$; $S_{\mu\nu} = -S_{\nu\mu}$ are matrices of the form

$$S_{ab} = \begin{pmatrix} \widehat{S}_{ab} & \widehat{0} & \widehat{0} & \widehat{0} \\ \widehat{0} & \widehat{S}_{ab} & \widehat{0} & \widehat{0} \\ \widehat{0} & \widehat{0} & \widehat{S}_{ab} & \widehat{0} \\ \widehat{0} & \widehat{0} & \widehat{0} & \widehat{S}_{ab} \end{pmatrix} \quad S_{0a} = \tfrac{1}{2}\epsilon_{abc} \begin{pmatrix} \widehat{0} & \widehat{0} & \widehat{0} & \widehat{S}_{bc} \\ \widehat{0} & \widehat{0} & -\widehat{S}_{bc} & \widehat{0} \\ \widehat{0} & \widehat{S}_{bc} & \widehat{0} & \widehat{0} \\ -\widehat{S}_{bc} & \widehat{0} & \widehat{0} & \widehat{0} \end{pmatrix}, \tag{2.8.10}$$

$$\widehat{S}_{12} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \widehat{S}_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \widehat{S}_{31} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix},$$

where $\widehat{0}$ is a three-row zero matrix. If we take $c_{0\mu} = 0$, $c_{ab} = -c_{ba}$ then the operator X_1 (2.8.2) yields basis elements of AG(1,3)

$$\begin{aligned} P_0 &= i\partial_0, \qquad P_a = -i\partial_a, \\ J_{ab} &= x_a P_b - x_b P_a + (S_{ab}\psi)^l \partial_{\psi^l} \\ G_a &= x_0 P_a + (M_a\psi)^l \, \partial_{\psi^l}, \end{aligned} \tag{2.8.11}$$

where

$$M_a = \tfrac{1}{2}\epsilon_{abc}\begin{pmatrix} \widehat{0} & \widehat{S}_{bc} & \widehat{0} & \widehat{0} \\ \widehat{0} & \widehat{0} & \widehat{0} & \widehat{0} \\ \widehat{0} & \widehat{0} & \widehat{0} & \widehat{0} \\ \widehat{0} & \widehat{0} & \widehat{S}_{bc} & \widehat{0} \end{pmatrix}$$

The theorem is proved.

In a similar way one can confirm that operator X_1 (2.8.2) also includes basis elements of AC(1,3) as well as basis elements of ASch(1,3). They have the form (2.8.9) and

$$\begin{aligned} D &= x^\nu P_\nu + 2i\psi^l \partial_{\psi^l}, \\ K_\mu &= 2x_\mu D - x^2 P_\mu + 2(x^\nu S_{\mu\nu}\psi)^l \partial_{\psi^l}; \end{aligned} \tag{2.8.12}$$

(2.8.11) and

$$\begin{aligned} D &= 2x_0 P_0 - x_a P_a + (\lambda_0\psi)^l \partial_{\psi^l}, \\ \Pi &= x_0^2 P_0 + x_0(\lambda_0\psi)^l \partial_{\psi^l} - x_a G_a, \end{aligned} \tag{2.8.13}$$

where

$$\lambda_0 = i\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \times I_3,$$

with I_3 the three-dimensional unit matrix.

The above results mean that system (2.8.1) without constraints (constitutive equations) satisfies both Lorentz-Poincare-Einstein and Galilei principles of relativity. As was indicated in [103] the same holds for the nonlinear Euler equation of an ideal fluid.

Let us write down the final Lorentz and Galilei transformations which are admitted by Equations (2.8.1) (they are generated by operators J_{0a} (2.8.9) and G_a (2.8.11), respectively):

$$\begin{aligned} & x_0' = \gamma(x_0 - \vec{x}\cdot\vec{v}) \qquad \vec{x}' = \vec{x} + \frac{\gamma-1}{v^2}(\vec{x}\cdot\vec{v})\vec{v} - \gamma\vec{v}x_0, \qquad \left(\gamma = \frac{1}{\sqrt{1-v^2}}\right) \\ & \vec{E}' = \gamma\vec{E} + \frac{1-\gamma}{v^2}\vec{v}(\vec{E}\cdot\vec{v}) - \gamma\vec{v}\times\vec{B}, \\ & \vec{B}' = \gamma\vec{B} + \frac{1-\gamma}{v^2}\vec{v}(\vec{B}\cdot\vec{v}) + \gamma\vec{v}\times\vec{E}, \\ & \vec{H}' = \gamma\vec{H} + \frac{1-\gamma}{v^2}\vec{v}(\vec{H}\cdot\vec{v}) + \gamma\vec{v}\times\vec{D},\ \vec{D}' = \gamma\vec{D} + \frac{1-\gamma}{v^2}\vec{v}(\vec{D}\cdot\vec{v}) - \gamma\vec{v}\times\vec{H}; \end{aligned} \tag{2.8.14a}$$

$$\begin{aligned} & x_0' = x_0, \qquad \vec{x}' = \vec{x} + \vec{v}x_0, \\ & \vec{E}' = \vec{E} - \vec{v}\times\vec{B}, \qquad \vec{B}' = \vec{B}, \end{aligned} \tag{2.8.14b}$$

$\vec{H}' = \vec{H} + \vec{v} \times \vec{D}, \qquad \vec{D}' = \vec{D}.$

Let us consider constitutive equations of the form

$$H_{\mu\nu} = \phi_{\mu\nu}(F_{01}, F_{02}, F_{03}, F_{12}, F_{23}, F_{31}) \tag{2.8.15}$$

where $\phi_{\mu\nu} = -\phi_{\nu\mu}$ are smooth functions of components of the tensor $F_{\mu\nu}$.

Theorem 2.8.3 [107]. *The system of Equations* (2.8.5), (2.8.15) *is invariant under the Poincare group if and only if*

$$H_{\mu\nu} = MF_{\mu\nu} + N\widetilde{F}_{\mu\nu} \tag{2.8.16}$$

where $M = M(J_1, J_2)$, $N = N(J_1, J_2)$ *are arbitrary differentiable functions of invariants of the electromagnetic field*

$$J_1 = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} = \vec{E}^2 - \vec{B}^2, \qquad J_2 = \tfrac{1}{4}\widetilde{F}_{\mu\nu}F^{\mu\nu} = \vec{B}\cdot\vec{E}. \tag{2.8.17}$$

Proof. Since Equations (2.8.15) do not contain field derivatives, then to prove the theorem it is sufficient to find the functions $\phi_{\mu\nu}$ which ensure the Poincare invariance of Equations (2.8.15). The conditions of Poincare invariance have the form

$$P_\rho\left[H_{\mu\nu} - \phi_{\mu\nu}(F)\right]\Big|_{H_{\mu\nu}=\phi_{\mu\nu}(F)} = 0 \tag{2.8.18}$$

$$J_{\alpha\beta}\left[H_{\mu\nu} - \phi_{\mu\nu}(F)\right]\Big|_{H_{\mu\nu}=\phi_{\mu\nu}(F)} = 0 \tag{2.8.19}$$

Using expressions (2.8.9), we rewrite (2.8.18) and (2.8.19) as follows:

$$(S_{\alpha\beta}\psi)^l\frac{\partial}{\partial\psi^l}\left[H_{\mu\nu} - \phi_{\mu\nu}(F)\right]\Big|_{H_{\mu\nu}=\phi_{\mu\nu}(F)} = 0 \tag{2.8.20}$$

The general solution of the determined Equations (2.8.20) is given by (2.8.16). The theorem is proved.

In terms of field strengths $\vec{D}$, $\vec{B}$, $\vec{E}$, and $\vec{H}$, the constitutive Equations (2.8.16) take the form

$$\vec{D} = M\vec{E} + N\vec{B}, \qquad \vec{H} = M\vec{B} - N\vec{E} \tag{2.8.21}$$

Note that if $M = \alpha$, $N = \beta$, ($\alpha.\beta$ constants) then constraints (2.8.21) together with (2.8.1) lead to Maxwell's equations in vacuum. If we take

$$\begin{gathered} m = L^{-1}, \qquad N = (\vec{B}\cdot\vec{E})L^{-1}, \\ L = \left[1 + (\vec{B}^2 - \vec{E}^2) - (\vec{B}\cdot\vec{E})^2\right]^{1/2} \end{gathered} \tag{2.8.22}$$

then Equations (2.8.1) together with (2.8.22) coincide with the nonlinear equations for the electromagnatic field suggested by Born and cited in the literature as Born-Infeld equations.

Let us present one more example of constitutive Equations (2.8.21). Letting $M = \epsilon$, $N = -\mu \vec{B} \cdot \vec{E}$ (ϵ, μ constants) one can resolve relations (2.8.21) with respect to $\vec{D}$ and $\vec{B}$ to obtain the following example of Poincare invariant constitutive equations:

$$\begin{aligned} \vec{D} &= \epsilon \left\{ 1 + \frac{\mu^2 (\vec{E} \cdot \vec{H})^2}{\epsilon^2 (\epsilon + \mu \vec{E}^2)} \right\} \vec{E} - \frac{\mu (\vec{E} \cdot \vec{H})}{\epsilon (\epsilon + \mu \vec{E}^2)} \vec{H}, \\ \vec{B} &= \frac{\vec{H}}{\epsilon} - \frac{\mu (EH)}{\epsilon (\epsilon + \mu \vec{E}^2)} \vec{E}. \end{aligned} \tag{2.8.23}$$

Consider constitutive equations of the form

$$\vec{D} = \epsilon(\vec{E}, \vec{H}) \vec{E}, \qquad \vec{B} = \mu(\vec{E}, \vec{H}) \vec{H}. \tag{2.8.24}$$

Such equations are widely used in describing the electromagnetic field in various real media. The following statements are consequences of Theorem 2.8.3.

Consequence 2.8.1. The system of Equations (2.8.1) and (2.8.24) is Poincare invariant if and only if

$$\epsilon(\vec{E}, \vec{H}) \cdot \mu(\vec{E}, \vec{H}) = 1. \tag{2.8.25}$$

Consequence 2.8.2. If $\vec{B} = \vec{\varphi}(\vec{H})$, $\vec{D} = \vec{f}(\vec{E}, \vec{H})$ then the requirement of Poincare invariance of Equations (2.8.1) together with these constraints results in a linear dependence of $\vec{D}$ and $\vec{B}$ on $\vec{E}$ and $\vec{H}$, that is

$$\vec{D} = \epsilon \vec{E}, \qquad \vec{B} = \frac{1}{\epsilon} \vec{H}. \tag{2.8.26}$$

Now we shall try to find which of the constitutive Equations (2.8.21) are conformally invariant. The answer is given by the following theorem.

Theorem 2.8.4 [107]. *The system of Equations* (2.8.1) *and* (2.8.21) *is invariant under the conformal group* C(1,3) *if and only if*

$$M = M\left(\frac{J_1}{J_2}\right), \quad N = N\left(\frac{J_1}{J_2}\right) \tag{2.8.27}$$

with arbitrary differentiable functions M and N depending on the ratio of invariants J_1 and J_2 determined in (2.8.17).

Proof. One can obtain the proof analogously to that of Theorem 2.8.3. So, it is easy to establish that the requirement of scale invariance restricts the form of constitutive Equations (2.8.21) or (2.8.16) as follows:

$$H_{\mu\nu} = M\left(\frac{J_1}{J_2}\right)F_{\mu\nu} + N\left(\frac{J_1}{J_2}\right)\widetilde{F}_{\mu\nu} \tag{2.8.28}$$

The next requirement, the invariance of (2.8.16) under the pure conformal transformations, doesn't impose any additional restrictions on (2.8.28). Therefore, the theorem is proved.

Letting

$$M = \mu\frac{\vec{E}^2 - \vec{B}^2}{\vec{E}\cdot\vec{B}}, \qquad N = 0$$

we get an example of conformally invariant constitutive equations in the resolved form

$$\vec{D} = \left(\frac{\mu\vec{H}^2}{\mu\vec{E}^2 - \vec{E}\cdot\vec{H}}\right)^{1/2}\vec{E}, \qquad \vec{B} = \left(\frac{\mu\vec{E}^2 - \vec{E}\cdot\vec{H}}{\mu\vec{H}^2}\right)^{1/2}\vec{H}. \tag{2.8.29}$$

Consequence 2.8.3. The nonlinear Born-Infeld Equations (2.8.1) and (2.8.22) are not conformally invariant.

Now we turn to equations of electrodynamics for a vector-potential. Consider a nonlinear system

$$\Box A_\mu - \partial_\mu\partial^\nu A_\nu = A_\nu F(A^\nu A_\nu), \tag{2.8.30}$$

Theorem 2.8.5. *The system of PDEs* (2.8.30) *is invariant under the conformal group* C(1,3) *if and only if*

$$F(A_\nu A^\nu) = \lambda A_\nu A^\nu, \qquad \lambda = \text{const.} \tag{2.8.31}$$

Proof. Equations (2.8.30) under $F = 0$ are conformally invariant, with generators of AC(1,3) determined in (2.3.3) and Table 2.3.1. In Lie's approach these operators can be written as follows:

$$\begin{aligned}
&P_0 = i\partial_0, \qquad P_a = -i\partial_a,\\
&J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu + A_\mu\mathcal{P}_\nu - A_\nu\mathcal{P}_\mu,\\
&D = x^\nu P_\nu - A^\nu\mathcal{P}_\nu,\\
&K_\mu = 2x_\mu D - x^2P_\mu + 2x^\nu(A_\mu\mathcal{P}_\nu - A_\nu\mathcal{P}_\mu),\\
&\qquad\qquad (\mathcal{P}_0 = i\partial/\partial A_0, \qquad \mathcal{P}_a = -i\partial/\partial A_a).
\end{aligned} \tag{2.8.32}$$

The relativistic invariance of Equations (2.8.30) with arbitrary function F is obvious. So the next step is to clear up what functions F do not break down scale invariance of the free Equations (2.8.30). The scale (dilatation) transformation

$$x'_m u = e^{\theta} x_{\mu}, \qquad A'_{\mu}(x') = e^{-\theta} A_{\mu}(x), \qquad \theta = \text{const} \qquad (2.8.33)$$

generated by operator D from (2.8.32), transforms system (2.8.31) as follows:

$$e^{-3\theta}(\Box A_{\mu} - \partial_{\mu}\partial^{\nu} A_{\nu}) = e^{-\theta} F\left(e^{-2\theta} A_{\nu} A^{\nu}\right) A_{\nu}$$

whence, if we require invariance, we get the law of transformation of the function F:

$$F(A_{\nu} A^{\nu}) = e^{2\theta} F\left(e^{-2\theta} A_{\nu} A^{\nu}\right) \qquad (2.8.34)$$

Differentiation of (2.8.34) with respect to θ followed by setting $\theta = 0$ gives rise to the determining equation

$$F - u\frac{\partial F}{\partial u} = 0, \qquad u \equiv A_{\nu} A^{\nu},$$

the general solution of which is given in (2.8.31).

Using infinitesimal conformal transformations

$$x'_{\mu} = (1 + 2cx)x_{\mu} - c_{\mu} x^{\nu} x_{\nu},$$

$$A'_{\mu}(x') = \left[(1 - 2cx)g_{\mu\nu} + 2(x_{\mu} c_{\nu} - x_{\nu} c_{\mu})\right] A^{\nu}(x)$$

it is not difficult to complete the proof.

Let us present some more statements which can be proved in much the same way as by means of Lie's algorithm.

Theorem 2.8.6. *The system of coupled equations*

$$\begin{aligned} (\partial_{\mu} - eA_{\mu})A^{\mu} &= 0, \\ \Box A_{\mu} - \partial_{\mu}\partial^{\nu} A_{\nu} &= 0, \end{aligned} \qquad (2.8.35)$$

where e is a constant (electron charge), is invariant under AC(1,3) *with infinitesimal generators P_{μ}, $J_{\mu\nu}$, and D given by* (2.8.32), *and*

$$\widetilde{K}_{\mu} = K_{\mu} + \frac{2}{e}\mathcal{P}_{\mu} \equiv 2x_{\mu} D - x^2 P_{\mu} + 2x^{\nu}(A_{\mu}\mathcal{P}_{\nu} - A_{\nu}\mathcal{P}_{\mu}) + \frac{2}{e}\mathcal{P}_{\mu}. \qquad (2.8.36)$$

It is to be pointed out that generators $\widetilde{K}_{\mu}$ (2.8.36) are non-analytic in the parameter of the nonlinearity e. One can make sure that under $e = 0$, system

(2.8.35) is not conformally invariant, although the first equation $\partial_\mu A_\mu = 0$, as well as the second one $\Box A_\mu - \partial_\nu \partial^\nu A_\nu = 0$ are separately conformally invariant. The point is that these equations have different conformal degrees: $k = 3$ for the first, and $k = 1$ for the second (see Table 2.3.1).

Another peculiarity of generators (2.8.36) is the impossibility of representing these operators as matrix-differential ones of type (7).

System (2.8.35) is a rather rare example of what is essentially an extension of the symmetry group of a linear system of PDEs by means of a nonlinear addend without derivatives.

Final transformations generated by operators (2.8.36) have the form (2.3.2) and

$$A'_\mu(x') = \left[g_{\mu\nu}\sigma + 2(x_\mu c_\nu - x_\nu c_\mu + 2cxc_\mu x_\nu - x^2 c_\mu c_\nu - c^2 x_\mu x_\nu)\right] A^\nu(x) + \\ + \frac{2}{e}\left[c_\mu(1-2cx) + x_\mu c^2\right] \quad (2.8.37)$$

(compare with (2.3.37)). Transformations (2.8.37), just as expected, are non-analytic in the parameter e.

In conclusion we show how to construct equations like Maxwell's equations for the tensor $\mathcal{F}_{\mu\nu}$ with arbitrary conformal degree k.

From the standard tensor $F_{\mu\nu}$

$$F^{0a} = E_a, \qquad F^{ab} = \epsilon_{abc} H_c \quad (2.8.38)$$

with conformal degree equal to 2 (see Table 2.3.1) one can construct the new tensor

$$\mathcal{F}^{\mu\nu} = \left(I_1^{\alpha_1} I_2^{\beta_1}\right) F^{\mu\nu}, \quad (2.8.39)$$

where

$$I_1 = -\tfrac{1}{2} F_{\mu\nu} F^{\mu\nu} = \vec{E}^2 - \vec{H}^2$$
$$I_2 = \frac{1}{8}\epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \equiv \tfrac{1}{4} F^{\mu\nu} \widetilde{F}_{\mu\nu} = \vec{E} \cdot \vec{H}, \quad (2.8.40)$$

α_1, β_1 are arbitrary constants, $\alpha_1 + \beta_1 = (k-2)/4$. It can be easily confirmed that tensor (2.8.39) possesses conformal degree k.

The inverse relation

$$F^{\mu\nu} = (J_1^\alpha J_2^\beta) \mathcal{F}^{\mu\nu} \quad (2.8.41)$$

where

$$J_1 = -\tfrac{1}{2}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}, \qquad J_2 = \frac{1}{8}\epsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\mu\nu}\widetilde{\mathcal{F}}^{\rho\sigma}, \qquad \alpha + \beta = \frac{2-k}{2k}, \quad (2.8.42)$$

expresses the tensor $F^{\mu\nu}$ in terms of the tensor $\mathcal{F}^{\mu\nu}$.

Inserting (2.8.41) into the Maxwell's equations we get

$$\partial_\mu F^{\mu\nu} = \partial_\mu \left[(J_1^\alpha J_2^\beta)\mathcal{F}^{\mu\nu}\right] = 0,$$
$$\partial_\mu \widetilde{F}^{\mu\nu} = \partial_\mu \left[(J_1^\alpha J_2^\beta)\widetilde{\mathcal{F}}^{\mu\nu}\right] = 0.$$

These equations can be rewritten in the form

$$D_\mu \mathcal{F}^{\mu\nu} = 0, \qquad D_\mu \widetilde{\mathcal{F}}^{\mu\nu} = 0 \tag{2.8.43}$$

with

$$D_\mu = \partial_\mu + \partial_\mu \ln (J_1^\alpha J_2^\beta), \qquad \alpha + \beta = \frac{2-k}{2k.} \tag{2.8.44}$$

So we obtain the system of nonlinear PDEs (2.8.43) which is invariant under AC(1,3) with arbitrary conformal degree k (see also Table 2.3.1, Equations N9).

2.9. *Solutions of nonlinear equations for vector fields*

Let us consider the system of PDEs

$$\Box A_\mu - \partial_\mu \partial^\nu A_\nu + m^2 A_\mu + \lambda A_\mu A^\nu A_\nu = 0, \tag{2.9.1}$$

where $A_\mu = A_\mu(x)$; $x \in$ R(1,3); $\mu, \nu = \overline{0,3}$; m, λ are arbitrary constants (to be definite we choose $\lambda > 0$). The maximal group (MG), in Lie's sense, of system (2.9.1) depending on m and λ is as follows: 1) $m = \lambda = 0$, MG = C(1,3) $\otimes$ U(1); 2) $m = 0$. $\lambda \neq 0$, MG = C(1,3); 3) $m \neq 0$, $\lambda = 0$, MG = P(1,3); 4) $m \neq 0$, $\lambda \neq 0$, MG = P(1,3).

To find exact solutions of system (2.9.1) we use the ansatze listed in the Table 2.1.4, preliminarily simplified.

Consider the ansatz

$$A_\mu(x) = a_\mu f(\omega_1), \qquad \omega_1 = bx, \tag{2.9.2}$$

where f is a scalar real function, and a_μ, b_μ are arbitrary real constants satisfying relations (2.1.27).

Substitution of Equation (2.9.2) into (2.9.1) gives rise to ODEs for the function $f = f(\omega_1)$

$$\ddot{f} - m^2 + \lambda f^3 = 0. \tag{2.9.3}$$

Solutions of Equation (2.9.3) are expressed by means of Jacobi elliptic functions (see Appendix 1):

$$f(\omega_1) = m\sqrt{\frac{2}{\lambda(2-k^2)}}\ \mathrm{dn}\left(\frac{m}{\sqrt{2-k^2}}\omega_1, k\right), \qquad 0 \leq k < \sqrt{2}. \tag{2.9.4}$$

Under $k = 0$ and $k = 1$ the elliptic function (2.9.4) degenerates:

$$f(\omega_1) = \frac{m}{\sqrt{\lambda}}, \qquad k = 0; \tag{2.9.5}$$

$$f(\omega_1) = \sqrt{\frac{2}{\lambda}} \frac{m}{\mathrm{ch}\,(m\omega_1)}, \qquad k = 1. \tag{2.9.6}$$

Consider the ansatz

$$A_\mu(x) = d_\mu f(\omega_2), \qquad \omega_2 = ax \tag{2.9.7}$$

Where the arbitrary real constants d_μ and a_μ, b_μ, c_μ (to be introduced soon) satisfy relations (2.1.27). After substituting (2.9.7) into (2.9.1) we get the ODE for the function

$$\ddot{f} - m^2 - \lambda f^3 = 0. \tag{2.9.8}$$

which has a solution

$$f(\omega_2) = m\sqrt{\frac{2}{\lambda(2-k^2)}}\ \mathrm{cs}\left(\frac{m}{\sqrt{2-k^2}}\omega_2, k\right), \qquad 0 < \sqrt{2}. \tag{2.9.9}$$

Under $k = 0$ and $k = 1$ the elliptic function (2.9.9) degenerates:

$$f(\omega_1) = \frac{m}{\sqrt{\lambda}} \,\mathrm{cotan}\left(\frac{m}{\sqrt{2}}\omega_2\right), \qquad k = 0; \tag{2.9.10}$$

$$f(\omega_2) = \sqrt{\frac{2}{\lambda}} \frac{m}{\mathrm{sh}\,(m\omega_2)}, \qquad k = 1. \tag{2.9.11}$$

Consider another ansatz

$$A_\mu(x) = a_\mu f(\omega_3), \qquad \omega_3 = dx. \tag{2.9.12}$$

It reduces system (2.9.1) to the following ODE for the function $f = f(\omega_3)$

$$\ddot{f} + m^2 f - \lambda f^3 = 0. \tag{2.9.13}$$

Equation (2.9.13) has the solution

$$f(\omega_3) = mk\sqrt{\frac{2}{\lambda(1+k^2)}}\ \mathrm{sn}\left(\frac{m}{\sqrt{1+k^2}}\omega_3, k\right), \tag{2.9.14}$$

Under $k = 0$ and $k = 1$ the elliptic function (2.9.14) degenerates:

$$f(\omega_3) = 0, \qquad k = 0;$$

$$f(\omega_3) = \frac{m}{\sqrt{\lambda}} \operatorname{th}\left(\frac{m}{\sqrt{2}}\omega_3\right), \qquad k = 1. \tag{2.9.15}$$

Solutions (2.9.6), (2.9.11), and (2.9.15) present themselves as solitary waves. Solution (2.9.15) is known as a *kink* (see, for example, [170]).

Consider the ansatz

$$A_\mu(x) = a_\mu f(\omega_4) + (d_\mu + b_\mu) g(\omega_4), \qquad \omega_4 = cx, \tag{2.9.16}$$

where f and g are scalar real functions. Substituting (2.9.16) into (2.9.1) gives rise to the following system of ODEs:

$$\begin{aligned} \ddot{f} - m^2 f + \lambda f^3 &= 0. \\ \ddot{g} - m^2 g + \lambda f^2 g &= 0. \end{aligned} \tag{2.9.17}$$

The first equation from (2.9.17) coincides with (2.9.3) and therefore one can use expression (2.9.4) for the function $f = f(\omega_4)$. Inserting this result into the second equation of system (2.9.17) we get for the function $g(\omega_4)$ the Lame equation, which has Lame functions [24] as solutions.

Consider another ansatz

$$\begin{aligned} A_\mu(x) &= (a_\mu bx - b_\mu ax) f(\omega_5) + (a_\mu ax + b_\mu bx) g(\omega_5), \\ \omega_5 &= \left[(ax)^2 + (bx)^2\right]^{1/2} \end{aligned} \tag{2.9.18}$$

After substituting (2.9.18) into (2.9.1) we get

$$\omega \ddot{f} + 3\dot{f} + \lambda \omega^2 f^3 - m^3 \omega f = 0, \qquad g = 0. \tag{2.9.19}$$

Under $m = 0$ Equation (2.9.19) has a solution

$$f(\omega_5) = \tfrac{1}{2}\sqrt{\frac{3}{\lambda w_5}} \tag{2.9.20}$$

Let us present solutions of the massless Equations (2.9.3), (2.9.8). For equation $\ddot{f} + \lambda f^3 = 0$ we have

$$\begin{aligned} f(\omega) &= \frac{1}{\sqrt{2}} \operatorname{cn}\left(\omega, \frac{1}{\sqrt{2}}\right), \\ f(\omega) &= \sqrt{\frac{2}{\lambda}} \operatorname{dn}\left(\omega, \sqrt{2}\right). \end{aligned} \tag{2.9.21}$$

For equation $\ddot{f} - \lambda f^3 = 0$ we obtain solutions

$$\begin{aligned} f(\omega) &= \sqrt{\frac{2}{\lambda}}\frac{1}{\omega}, \\ f(\omega) &= \frac{1}{\sqrt{\lambda}} \operatorname{nc}\left(\omega, \frac{1}{\sqrt{2}}\right), \\ f(\omega) &= \sqrt{\frac{2}{\lambda}} \operatorname{dn}\left(\omega, \sqrt{2}\right). \end{aligned} \tag{2.9.22}$$

Under $m = 0$ one can look for solutions of system (2.9.1) by means of the conformally invariant ansatz (2.3.53). Substituting (2.3.53) into (2.9.1) with $m = 0$ gives rise to the system of ODEs for $B_\mu = B_\mu(\omega_6)$, $\omega_6 = (\beta x)/(x_\nu x^\nu)$:

$$\beta^2 \ddot{B}_\mu - \beta_\mu \beta^\nu \ddot{B}_\nu + \lambda B_\mu B^\nu B_\nu = 0. \tag{2.9.23}$$

Letting $\beta_\mu = b_\mu$, $B_\mu = a_\mu f(\omega_6)$ we reduce (2.9.23) to the equation $\ddot{f} + \lambda f^3 = 0$, and letting $\beta_\mu = d_\mu$, $B_\mu = a_\mu f(\omega_6)$ we reduce it to $\ddot{f} - \lambda f^3 = 0$. Therefore in these cases one can make use of solutions given by (2.9.21) and (2.9.22). Letting $\beta_\nu = c_\nu$, $B_\mu = a_\mu f(\omega_6) + (d_\mu + b_\mu) g(\omega_6)$ we get (2.9.17) under $m = 0$

$$\begin{aligned} &\ddot{f} + \lambda f^3 = 0, \\ &\ddot{g} + \lambda f^2 g = 0. \end{aligned} \tag{2.9.24}$$

A simple solution of system (2.9.24) is obtained under $\lambda < 0$:

$$f(\omega) = \sqrt{\frac{2}{|\lambda|}\frac{1}{\omega}}, \qquad g(\omega) = \alpha_1 \omega^2 + \frac{\alpha_2}{\omega} \tag{2.9.25}$$

where α_1, α_2 are arbitrary constants, and $\omega = \omega_6$

Using ansatz (2.3.53) and the solutions of system (2.9.23) described above, we obtain conformally invariant solutions to the massless system (2.9.1). Let us write down an example of such solutions:

$$A_\mu(x) = \frac{1}{\sqrt{\lambda}} \left(\frac{a_\mu}{x_\nu x^\nu} - 2x_\mu \frac{ax}{(x^\nu x_\nu)^2} \right) \mathrm{cn} \left(\frac{bx}{x_\nu x^\nu}, \frac{1}{\sqrt{2}} \right). \tag{2.9.26}$$

Another way of obtaining new solutions of the massless system (2.9.1) is achieved by means of the formula of generating solutions (2.3.38).

In conclusion let us note that all of the solutions of system (2.9.1) described above are non-analytic in the coupling constant λ. Such solutions, as is known from [170], lead to many interesting consequences in quantum field theory.

2.10. *Some exact solutions of SU(2) Yang-Mills field theory*

Recently, non-Abelian gauge field theories took a central role in describing the interactions of elementary particles. In particular, many works are devoted to the simplest non-Abelian gauge field theory SU(2), Yang-Mills (YM) gauge theory. An excellent review of classical solutions of SU(2) YM field equations is given in [4].

In the present paragraph new multiparameter exact solutions of SU(2) YM equations are obtained. These solutions are real, non-Abelian, and C(1,3)-ungenerative.

1. First of all let us give a brief review of the foundations of the SU(2) YM gauge theory. The field equations have the form

$$\partial_\mu\partial^\mu\vec{Y}_\nu - \partial_\nu\partial^\mu\vec{Y}_\mu + e\left[(\partial^\mu\vec{Y}_\mu)\times\vec{Y}_\nu - 2(\partial^\mu\vec{Y}_\nu)\times\vec{Y}_\mu + (\partial^\nu\vec{Y}_\mu)\times\vec{Y}_\mu\right] + \\ + e^2\vec{Y}^\mu\times(\vec{Y}_\mu\times\vec{Y}_\nu) = 0, \qquad (2.10.1)$$

where $\vec{Y}_\mu = \{Y^1_\mu, Y^1_\mu, Y^1_\mu\}$ is the YM potential; $\mu,\nu = \overline{0,3}$; e is a constant. Equations (2.10.1) follow from the Lagrangian

$$\mathcal{L} = -\tfrac{1}{4}G^a_{\mu\nu}G^{a\mu\nu}, \qquad (2.10.2)$$

where $G^a_{\mu\nu}$ is the field strength tensor

$$G^a_{\mu\nu} = \partial_\mu Y^a_\nu - \partial_\nu Y^a_\mu + e\epsilon_{abc}Y^b_\mu Y^c_\nu. \qquad (2.10.3)$$

By means of the tensor (2.10.3), YM equations (2.10.1) are written as follows:

$$\partial^\nu G^a_{\mu\nu} = e\epsilon_{abc}G^b_{\mu\nu}Y^{c\nu}, \qquad (2.10.4)$$

The YM equations (2.10.1) possess rich symmetry. As shown in [182] their maximal Lie symmetry group is the group SU(2)⊗C(1,3). Basis elements of AC(1,3) are written in (2.3.3) and also

$$k = 1, \qquad S_{\mu\nu} = \widehat{S}_{\mu\nu}\otimes I_3, \qquad (2.10.5)$$

where $\widehat{S}_{\mu\nu}$ are 4×4 matrices realizing irreducible representation D(1/2,1/2) of AO(1,3); I_3 is the unit 3×3 matrix; and the notation ⊗ means direct matrix multiplication.

Generators of the SU(2) gauge group have the form [4,182]

$$X = \left[\epsilon_{abc}Y^b_\mu\theta^c(x) + \frac{1}{e}\partial_\mu\theta^a(x)\right]\frac{\partial}{\partial Y^a_\mu} \qquad (2.10.6)$$

with arbitrary differentiable functions $\theta^a(x)$. It follows from (2.10.5) that conformal transformations of the YM potential have the form (compare with (2.3.37)) [96]

$$Y^a_\mu(x) = [\sigma(x,c)\delta^\nu_\mu + 2(x_\mu c^\nu - x^\nu c_\mu + 2cxc_\mu x^\nu - x^2c_\mu c^\nu - c^2x_\mu x^\nu)]Y^a_\nu(x). \qquad (2.10.7)$$

The corresponding formula of generating solutions is [96]

$$Y^a_{\mu II}(x) = [\sigma^{-1}(x,c)\delta^\nu_\mu + 2\sigma^{-2}(x,c)(c_\mu x^\nu - x_\mu c^\nu + +2cx x_\mu c^\nu - c^2 x_\mu x^\nu - x^2 c_\mu c^\nu)]Y^a_{\nu I}(x'). \tag{2.10.8}$$

The gauge transformations generated by operators (2.10.6) have the form [4]

$$Y'^a_\mu(x) = \cos\theta\, Y^a_\mu + \sin\theta\, \epsilon_{abc} Y^b_\mu n^c + 2\sin^2\frac{\theta}{2}\, n^a(n^b Y^b_\mu) + + \frac{1}{e}\left[\tfrac{1}{2} n^a \partial_\mu \theta + \tfrac{1}{2}\sin\theta\, \partial_\mu n^a + \sin^2\frac{\theta}{2}\, \epsilon_{abc}(\partial_\mu n^b) n^c\right], \tag{2.10.9}$$

where $n^a = n^a(x)$ is a unit vector defined by

$$\theta^a(x) \equiv n^a(x)\theta(x) \tag{2.10.10}$$

It will be noted that the tensor of the YM field (2.10.3) is not invariant under gauge transformations unlike the tensor of the electromagnetic field. This is an important difference between Abelian and non-Abelian gauge theories. The tensor changes under gauge transformations (2.10.9) as follows:

$$G^a_{\mu\nu} \to G'^a_{\mu\nu} = G^a_{\mu\nu}\cos\theta + \left(\sin\theta \epsilon_{abc} n^c + 2\sin^2\frac{\theta}{2}\, n^a n^b\right) G^b_{\mu\nu}. \tag{2.10.11}$$

By analogy with classical electrodynamics, "electric" and "magnetic" YM fields

$$E^a_k = G^a_{0k}, \qquad B^a_k = -\tfrac{1}{2}\epsilon_{kij} G^a_{ij} \tag{2.10.12}$$

are introduced.

The energy-momentum tensor is defined by

$$\theta_{\mu\nu} = -G^a_{\mu\lambda} G^{a\lambda}_\nu + \tfrac{1}{4} g_{\mu\nu} G^a_{\alpha\beta} G^{a\alpha\beta} \tag{2.10.13}$$

The components of $\theta_{\mu\nu}$ are

$$\begin{aligned} \theta_{00} &= \tfrac{1}{2}(E^a_k E^a_k + B^a_k B^a_k) = \sum_i \theta_{ii}, \\ \theta_{0j} &= -\epsilon_{jmn} E^a_m B^a_n, \\ \theta_{ij} &= -E^a_i E^a_j - B^a_i B^a_j + \delta_{ij}\tfrac{1}{2}(E^a_k E^a_k + B^a_k B^a_k). \end{aligned} \tag{2.10.14}$$

One can confirm that the tensor $\theta_{\mu\nu}$ is gauge invariant.

It is appropriate to note some essential differences between the YM equations (2.10.1) and the classical electrodynamics equations

$$\partial^\mu F_{\mu\nu} \equiv \partial^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) = \Box A_\nu - \partial_\nu \partial^\mu A_\mu = 0.$$

First of all, YM equations are essentially nonlinear. Secondly, they contain both the field tensor $G^a_{\mu\nu}$ and the potential Y^a_μ (see writing (2.10.4)). It seems that the YM potential plays a more basic role than the potential in an Abelian gauge theory. In electromagnetism one can work exclusively with the field strengths $\vec{E}$ and $\vec{B}$. Things are not so simple when the gauge group is non-Abelian. Thirdly, in Abelian gauge theories the field strengths locally determine the gauge potential up to an arbitrary gauge transformation. The same is not true for non-Abelian gauge theories: two YM potentials that are gauge inequivalent can provide the same YM field strengths. This leads to an interesting problem, specifically, the determination of all possible gauge potentials that yield a given field-strength tensor (see [4] and literature cited therein).

We shall note one more peculiarity of SU(2) gauge field theory. If the potential Y^a_μ satisfies the so-called self-duality conditon

$$\widetilde{G}^a_{\mu\nu} \equiv \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}G^{a\rho\sigma} = \pm iG^a_{\mu\nu}, \tag{2.10.15}$$

or equivalently, in terms of field strengths $\vec{E}$ and $\vec{B}$ (2.10.12),

$$\pm iE^a_n = B^a_n \tag{2.10.16}$$

then it automatically satisfies the equations of motion (2.10.1). Note that the factor i in this definition of self-duality is unavoidable because we are working in Minkowski space where $\widetilde{\widetilde{G}}{}^a_{\mu\nu} = -G^a_{\mu\nu}$. Clearly, in Minkowski space any self-dual field configuration contains complex fields. (In Euclidean space E^4 the factor i is absent and self-dual fields can be real.)

One can try to make use of the property of self-dual fields to satisfy the equations of motion (2.10.1) by searching for solutions of the self-dual equations (2.10.15), which are first order, rather than trying to solve the second-order equations of motion. As is shown in [182], the self-dual equations (2.10.15) possess the same symmetry as the field equations (2.10.1), that is their maximal Lie group of invariance is SU(2)⊗C(1,3). Self-duality is a very special property which most solutions do not have. Nevertheless, interesting solutions can be found in this way.

Any self-dual solution in Minkowski space has a vanishing energy-momentum tensor $\theta_{\mu\nu}$ (2.10.13). Indeed, representing it as

$$\theta_{\mu\nu} = -\tfrac{1}{4}\left(G^a_{\mu\alpha} + i\widetilde{G}^a_{\mu\alpha}\right)\left(G^{a\alpha}_\nu - i\widetilde{G}^{a\alpha}_\nu\right)$$

we get $\theta_{\mu\nu} = 0$ for any field configuration which satisfies self-dual conditions (2.10.15) or (2.10.16).

2. By now the interest in classical solutions of the YM field equations has become so widespread that many workers are involved in the search for new

solutions. A largely complete review of the known classical solutions of SU(2) gauge theory until 1979 is given in [4].

A well-known ansatz is the 't Hooft-Corrigan-Fairlie-Wilczek ansatz

$$\begin{aligned} eY_0^a &= \pm i\partial_a \ln\varphi, \\ eY_j^a &= (\epsilon_{jak}\partial_k \pm \delta_{aj}\partial_0)\ln\varphi, \end{aligned} \tag{2.10.17}$$

where $\varphi = \varphi(x)$ is a scalar function. Ansatz (2.10.17) can also be written in the form

$$Y_\mu^a = \eta_{a\mu\nu}\partial^\nu \ln\varphi \tag{2.10.18}$$

with 't Hooft tensor

$$\eta_{a\mu\nu} = \epsilon_{0a\mu\nu} \mp i g_{a\mu}g_{0\nu} \pm i g_{a\nu}g_{0\mu}. \tag{2.10.19}$$

Ansatz (2.10.17) reduces system (2.10.1) to the equation

$$\partial_\mu \Box\varphi - (3/\varphi)(\partial_\mu\varphi)\Box\varphi \equiv \varphi^3\partial_\mu(\varphi^{-3}\Box\varphi) = 0. \tag{2.10.20}$$

Equation (2.10.20) yields

$$\Box\varphi + \lambda\varphi^3 = 0, \tag{2.10.21}$$

where λ is an arbitrary constant. Solutions of Equation (2.10.21) have been considered in Paragraph 5.1 (see also Appendix 1), and in (2.10.17) these lead automatically to explicit solutions of the YM equations (2.10.1).

As seen from (2.10.17) the potentials Y_μ^a are complex for real φ. This is an unfortunate property of the ansatz, as one would like to find real solutions of the SU(2) YM theory. But complex solutions are also interesting, and furthermore there exists the possibility that for a particular solution φ the SU(2) potential (2.10.17) can be made real by a suitable complex SU(2) gauge transformation. For arbitrary φ this is not possible.

Let us write down several quantities of interest that follow from ansatz (2.10.17). The YM field strengths are

$$\begin{aligned} eE_n^a \equiv eG_{0n}^a = \epsilon_{nam}\left(\frac{1}{\varphi}\partial_0\partial_m\varphi - \frac{2}{\varphi^2}\partial_0\varphi\partial_m\varphi\right) \pm i\delta_{na}\Bigg[\frac{1}{\varphi}\partial_0^2\varphi - \\ -\frac{1}{\varphi^2}(\partial_0\varphi\partial_0\varphi + \partial_m\varphi\partial_m\varphi)\Bigg] \mp i\left[\frac{1}{\varphi}\partial_n\partial_a\varphi - \frac{2}{\varphi^2}\partial_n\varphi\partial_a\varphi\right], \end{aligned} \tag{2.10.22}$$

$$B_n^a \equiv -\tfrac{1}{2}e\epsilon_{nij}G_{ij}^a = \pm ieE_n^a + \delta_{an}\left(\frac{1}{\varphi}\right)\Box\varphi$$

The self-duality condition (2.10.16) implies $\Box\varphi = 0$. The field strengths E_n^a and B_n^a are in general complex. Remarkably, however, their squares are both real, and this means that the energy and Lagrangian densities obtained from ansatz (2.10.17) are real, even though the potential is complex.

It is easy to show that the energy-momentum tensor (2.10.13) following from ansatz (2.10.17) has the form

$$e^2\theta_{\mu\nu} = \frac{\Box\varphi}{\varphi}\left[\frac{4}{\varphi^2}\partial_\mu\varphi\partial_\nu\varphi - \frac{2}{\varphi}\partial_\mu\partial_\nu\varphi + \right.$$

$$\left. + g_{\mu\nu}\left(\frac{1}{2\varphi}\Box\varphi - \frac{1}{\varphi^2}\partial^\alpha\varphi\partial_\alpha\varphi\right)\right]. \qquad (2.10.23)$$

A self-dual solution has $\theta_{\mu\nu} = 0$ because $\Box\varphi = 0$.

The total energy is

$$\mathcal{E} = \int \theta_{00} d^3x = -\frac{6\lambda}{e^2}\int\left[\tfrac{1}{2}(\partial_0\varphi)^2 + \tfrac{1}{2}(\vec{\nabla}\varphi)^2 + \tfrac{1}{4}\lambda\varphi^4\right]d^3x, \qquad (2.10.24)$$

where we have neglected surface terms at infinity.

The Euclidean counterpart of ansatz (2.10.17)

$$\begin{aligned} eY_0^a &= \mp\partial_a \ln\varphi, \\ eY_j^a &= (\epsilon_{jan}\partial_n \pm i\delta_{aj}\partial_0)\ln\varphi, \end{aligned} \qquad (2.10.25)$$

is connected with such fundamental objects of non-Abelian gauge theories as merons and instantons. Below we consider these solutions of YM equations in Euclidean space.

As was shown in paragraph 2.4 (see formulae (2.4.26), (2.4.27)) the functions

$$\varphi(x) = \frac{1}{\sqrt{\lambda x_\nu x^\nu}}, \qquad (2.10.26)$$

$$\varphi(x) = \sqrt{\frac{8}{\lambda}}\frac{\alpha}{x_\nu x^\nu + \alpha^2}. \qquad (2.10.27)$$

(α is an arbitrary constant) are solutions of Equation (2.10.21). In Euclidean space these solutions lead [61*], by means of ansatz (2.10.25), to one meron solution of deAlfaro-Fubini-Furlan [4*]

$$\begin{aligned} eY_0^a &= \pm\frac{x_a}{x^2}, \\ eY_j^a &= -\epsilon_{jan}\frac{x_n}{x^2} \mp \delta_{aj}\frac{x_0}{x^2} \end{aligned} \qquad (2.10.28)$$

and to Belavin-Polyakov-Schwartz-Tyupkin instanton solution [1*]

$$\begin{aligned} eY_0^a &= \mp\frac{2x_a}{x^2+\alpha^2}, \\ eY_j^a &= -\epsilon_{jan}\frac{2x_n}{x^2} \pm \delta_{aj}\frac{2x_0}{x^2+\alpha^2} \end{aligned} \qquad (2.10.29)$$

of the YM equations in Euclidean space, respectively. Here we would like to pay attention to the following point. Both solutions (2.10.28) and (2.10.29) follow from solutions (2.4.11) and (2.4.12) of the nonlinear Dirac-Gursey equation (2.1.5) by means of formulae (2.4.25) and (2.10.25). Therefore one can consider meron (2.10.28) and instanton (2.10.29) as objects generated by the spinor field ψ obeying the nonlinear Dirac-Gursey equation (2.1.5) [61*]. It is a principal point which can lead to a new and deeper understanding of YM field theory. The very names meron and instanton are connected with topological properties of the solutions.

In Euclidean gauge theory a useful four-vector is defined:

$$J_\mu = \tfrac{1}{4}\epsilon_{\mu\nu\rho\sigma}\left(Y^a_\nu \partial_\rho Y^a_\sigma + \frac{e}{3}\epsilon_{abc} Y^a_\nu Y^b_\rho Y^c_\sigma\right) \tag{2.10.30}$$

(let us remember that in Euclidean space one does not need to distinguish upper and lower indices). The identity

$$\partial_\mu J_\mu = \tfrac{1}{8}\widetilde{G}^a_{\mu\nu} G^a_{\mu\nu}. \tag{2.10.31}$$

can easily be confirmed.

The integral

$$q = \frac{e^2}{4\pi^2}\int d^4x \partial_\mu J_\mu \equiv \frac{e^2}{32\pi^2}\int d^4x \widetilde{G}^a_{\mu\nu} G^a_{\mu\nu} \tag{2.10.32}$$

defines the topological index or charge of the Euclidean field configuration.

A meron is a localized, singular solution of the Euclidean gauge theory with one-half unit of topological charge. The name meron derives from a Greek word meaning part (of unit of topological charge). The topological charge of the meron is concentrated at the point where the solution is singular. This is to be contrasted with the instanton's nonsingular cloud of topological charge. The charge of the instanton is one unit. The name instanton is derived from its localization in E^4 which corresponds to finite duration as well as spacial extension in Minkowski space, i.e., "event." All instantons are self-dual and meron solutions are not self-dual. An instanton is also called a pseudoparticle.

Solutions of Equation (2.10.21) expressed by means of Jacobi elliptic functions lead to elliptic solutions of the YM equations. The elliptic solution depends on a continuous parameter k (see Appendix 1), and for $k = 1$ it reduces to the one-instanton solution, and for $k = 0$ it becomes the two-meron solution. Note that the two-meron solution is obtained by means of the ansatz (2.10.25) under

$$\varphi(x) = \left[\frac{(a_\nu - b_\nu)(a_\nu - b_\nu)}{\lambda(x-a)^2(x-b)^2}\right]^{1/2} \tag{2.10.33}$$

with arbitrary constants a_ν, b_ν.

This solution of Equation (2.10.21) follows from solution (2.10.26) by means of the formulae of generating solutions (2.3.2) and (2.3.34), and then letting $x_\mu \to x_\mu - a_\mu$ and taking $c_\mu = (b_\mu - a_\mu)/(a-b)^2$ [61*].

The corresponding spinor field ψ which gives, via (2.4.25), the scalar field (2.10.33), can be easily obtained from (2.4.12) by means of formulae (2.3.2) and (2.3.27) and then letting $x_\mu \to x_\mu - a_\mu$ and taking $c_\mu = (b_\mu - a_\mu)/(a-b)^2$.

Note that the same procedure of generating solutions applied to (2.10.27) results in another solution of Equation (2.10.21):

$$\varphi(x) = \sqrt{\frac{8(a-b)^2}{\lambda}}\,\frac{1}{(x-a)^2+(x-b)^2} \tag{2.10.27a}$$

under $\alpha^2 = (a-b)^2$. For more detail see [61*].

3. Now we will construct real C(1,3)-ungenerative non-Abelian solutions (when $\vec{Y}_\mu \times \vec{Y}_\nu \neq 0$) of the YM equations (2.10.1). For this aim we use the conformally invariant ansatz (compare with (2.3.53))

$$Y^a_\nu(x) = (x_\alpha x^\alpha)^{-1}\varphi^a_\nu(\omega) - \frac{2x_\nu x^\sigma \varphi^a_\sigma(\omega)}{(x_\alpha x^\alpha)^2}, \qquad \omega = \frac{\beta x}{x^\nu x_\nu}, \tag{2.10.34}$$

where β_ν are arbitrary constants.

Substitution of the ansatz (2.10.34) into (2.10.1) gives rise to the following system of ODEs

$$\beta^2\ddot{\vec{\varphi}}_\nu - \beta_\nu\beta^\nu\ddot{\vec{\varphi}}_\mu + e\left[\beta^\mu\left(\dot{\vec{\varphi}} \times \vec{\varphi}_\nu - 2\dot{\vec{\varphi}}_\nu \times \vec{\varphi}_\mu\right) + \beta_\nu\dot{\vec{\varphi}}_\mu \times \vec{\varphi}^\mu\right] + \\ + e^2\vec{\varphi}^\mu \times (\vec{\varphi}_\mu \times \vec{\varphi}_\nu) = 0, \tag{2.10.35}$$

where dot means differentiation with respect to ω; $\vec{\varphi} = \{\varphi^a_\nu\}$. We look for solutions of the system (2.10.35) in the form

$$\vec{\varphi}(\omega) = \alpha_\nu \vec{n} f(\omega) + \rho_\nu \vec{p} g(\omega), \tag{2.10.36}$$

where f and g are scalar real functions to be defined; α_ν, ρ_ν are arbitrary constants satisfying relations

$$\alpha\rho = \alpha\beta = \rho\beta = 0, \tag{2.10.37}$$

$\vec{n}^2, \vec{p}$ are mutually orthonormal constant vectors of isospin

$$\vec{n}^2 = \vec{p}^2 = 1, \qquad \vec{n}\cdot\vec{p} = 0. \tag{2.10.38}$$

Inserting (2.10.36) into (2.10.35) we get coupled nonlinear ODE for two unknown functions f and g

$$\begin{aligned} &\beta^2\ddot{f} - e^2\rho^2 g^2 f = 0, \\ &\beta^2\ddot{g} - e^2\alpha^2 f^2 g = 0. \end{aligned} \tag{2.10.39}$$

The system (2.10.39) has nontrivial solutions only if $\beta^2 \neq 0$. Further, there are 6 different cases.

Under $\rho^2 = 0$, $\alpha^2 = \beta^2 = -1$ the system (2.10.39) takes the form

$$\begin{aligned} \ddot{f} &= 0, \\ \ddot{g} - e^2 f^2 g &= 0. \end{aligned} \tag{2.10.40}$$

Under $\alpha^2 = 0$, $\rho^2 = \beta^2 = -1$ the system (2.10.39) takes the form

$$\begin{aligned} \ddot{g} &= 0, \\ \ddot{f} - e^2 g^2 f &= 0. \end{aligned} \tag{2.10.41}$$

Under $\rho^2 = 1$, $\alpha^2 = \beta^2 = -1$ and under $\alpha^2 = 1$, $\rho^2 = \beta^2 = -1$ the system (2.10.39) takes the form

$$\begin{aligned} \ddot{f} \pm e^2 g^2 f &= 0, \\ \ddot{g} \mp e^2 f^2 g &= 0 \end{aligned} \tag{2.10.42}$$

respectively.

Under $\beta^2 = 1$, $\alpha^2 = \rho^2 = -1$ the system (2.10.39) takes the form

$$\begin{aligned} \ddot{f} + e^2 g^2 f &= 0, \\ \ddot{g} + e^2 f^2 g &= 0. \end{aligned} \tag{2.10.43}$$

And finally, under $\alpha^2 = \beta^2 = \rho^2 = -1$ the system (2.10.39) takes the form

$$\begin{aligned} \ddot{f} - e^2 g^2 f &= 0, \\ \ddot{g} - e^2 f^2 g &= 0. \end{aligned} \tag{2.10.44}$$

Now we try to look for solutions of systems (2.10.40)–(2.10.44). From (2.10.40) it is easy to find the general solution for function f:

$$f(\omega) = c_0\omega + c \tag{2.10.45}$$

(c_0, c are constants of integration) and after substituting it into the second equation of the system we get the linear ODE for function $g(\omega)$

$$\ddot{g} = e^2(c_0\omega + c)^2 g. \tag{2.10.46}$$

Under $c_0 = 0$ the general solution of Equation (2.10.46) is given by

$$g(\omega) = \begin{cases} c_1 \operatorname{sh} ec\omega + c_2 \operatorname{sh} ec\omega, & c \neq 0; \\ c_1\omega + c_2 \end{cases} \tag{2.10.47}$$

(c_1, c_2 are arbitrary constants). Under $c_0 \neq 0$ Equation (2.10.46) is reduced by the substitution

$$\omega \to y = \omega + \frac{c}{c_0}$$

to the ODE

$$\frac{d^2g}{dy^2} = e^2c_0^2y^2g,$$

whose solutions are expressed in terms of Bessel functions

$$g(y) = \sqrt{y}Z_{1/4}\left(\frac{i}{2}ec_0y^2\right).$$

By interchanging $f \leftrightarrow g$ the system (2.10.41) is transformed to the system (2.10.40).

Equations (2.10.42) have the first integral

$$\dot{f}^2 + \dot{g}^2 = E^2 = \text{const},$$

which allows the reduction of (2.10.42) to one second-order ODE. Letting

$$g = g(f)$$

we find

$$\dot{g} = \frac{dg}{df}\dot{f} \equiv g'\dot{f}, \quad \ddot{g} = g''\dot{f}^2 + g'\ddot{f}, \quad \dot{f}^2 = E^2[1 + (g')^2]^{-1}.$$

Inserting these expressions into (2.10.42) we get the ODE

$$g''\frac{E^2}{1 + (g')^2} \mp e^2gf(g'g + f) = 0, \quad g' \equiv \frac{dg}{df}.$$

Further consider the system (2.10.43) as well as the system (2.10.44). Letting $f = g$ we get Equations (2.9.3), (2.9.8) under $m = 0, \lambda = e^2$, respectively. Therefore one can use solutions (2.9.21), (2.9.22), whence follows solutions of the system (2.10.43)

$$\begin{aligned} f = g &= \frac{1}{e}\,\mathrm{cn}\left(\omega, \frac{1}{\sqrt{2}}\right), \\ f = g &= \frac{\sqrt{2}}{e}\,\mathrm{dn}\,(\omega, \sqrt{2}) \end{aligned} \tag{2.10.48}$$

and solutions of the system (2.10.44)

$$\begin{aligned} f = g &= \frac{\sqrt{2}}{e}\frac{1}{\omega}, \\ f = g &= \frac{1}{e}\,\mathrm{nc}\left(\omega, \frac{1}{\sqrt{2}}\right), \\ f = g &= \frac{\sqrt{2}}{e}\,\mathrm{nd}\,(\omega, \sqrt{2}). \end{aligned} \tag{2.10.49}$$

Under $g = 0$ we find from (2.10.43), (2.10.44)

$$f = c_1\omega + c_2, \tag{2.10.50}$$

where c_1, c_2 are arbitrary constants. Analogously, letting $f = 0$ one finds

$$g = c_1\omega + c_2, \tag{2.10.51}$$

Equations (2.10.43), (2.10.44) have the first integral (see also [169])

$$\dot{f}^2 + \dot{g}^2 + \epsilon e^2 f^2 g^2 = E, \qquad \epsilon = \pm 1 \tag{2.10.52}$$

which allows the reduction of systems (2.10.43), (2.10.44) to one second-order ODE. Letting

$$f = \frac{1}{e}F, \qquad g = \frac{1}{e}G(F) \tag{2.10.53}$$

we calculate

$$\begin{aligned} \dot{G} &= \frac{dG}{dF}\dot{F} \equiv G'\dot{F}, \\ \ddot{G} &= G''\dot{F}^2 + G'\ddot{F}, \end{aligned} \tag{2.10.54}$$

From (2.10.52)–(2.10.54) we have

$$\dot{F}^2 = \frac{E - \epsilon F^2 G^2}{1 + (G')^2}, \tag{2.10.55}$$

so

$$\int d\omega = \int \frac{dF}{\sqrt{(E - \epsilon F^2 G^2)/(1 + (G')^2)}}. \tag{2.10.56}$$

From (2.10.43), (2.10.44), (2.10.53)–(2.10.55) we get

$$(E - \epsilon F^2 G^2)G'' + \epsilon FG(F - GG')\left(1 + (G')^2\right) = 0. \tag{2.10.57}$$

Under $\epsilon = -1$ and $E = 0$, Equation (2.10.57) takes the form

$$F^2G^2G'' - (F^2G - FG^2G')\left(1 + (G')^2\right) = 0.$$

After the change of variables

$$G = Fx(\tau), \qquad \tau = \ln F$$

we obtain

$$x\frac{d^2x}{d\tau^2} - \left(\frac{dx}{d\tau}\right)^2 + x\left(x + \frac{dx}{d\tau}\right)^2 - 1 = 0. \tag{2.10.58}$$

Taking in (2.10.58) $\dfrac{dx}{d\tau}$ as a function of x, i.e.,

$$\frac{dx}{d\tau} = y(x), \qquad \frac{d^2x}{d\tau^2} = y\frac{dy}{dx}$$

we obtain the first-order equation on $y(x)$, namely the Abel equation of the second kind [130, 169]

$$xy\frac{dy}{dx} = -xy^3 + (1-3x^2)y^2 - 3x^3y + 1 - x^4. \tag{2.10.59}$$

Substituting solutions of the system (2.10.39) into (2.10.36), (2.10.34) we find solutions of the YM equations (2.10.1). Let us write down a couple of such solutions following from (2.10.47), (2.10.49):

$$\vec{Y}_\nu(x) = \vec{n}\frac{c}{x^2}\left(\alpha_\nu - 2x_\nu\frac{\alpha x}{x^2}\right) + \frac{\vec{p}}{x^2}\left(c_1 \operatorname{ch} ec\frac{\beta x}{x^2} + c_2 \operatorname{sh} ec\frac{\beta x}{x^2}\right) \times$$
$$\times\left(\rho_\nu - 2x_\nu\frac{\rho x}{x^2}\right), \tag{2.10.60}$$

$$\rho^2 = 0, \qquad \alpha^2 = \beta^2 = -1, \qquad \alpha\rho = \alpha\beta = \rho\beta = 0;$$

$$\vec{Y}_\nu(x) = \frac{\sqrt{2}}{e(\alpha x)}\left[\vec{n}\left(\alpha_\nu - 2x_\nu\frac{\alpha x}{x^2}\right) + \vec{p}\left(\rho_\nu - 2x_\nu\frac{\rho x}{x^2}\right)\right], \tag{2.10.61}$$

$$\alpha^2 = \beta^2 = \rho^2 = -1, \qquad \alpha\rho = \rho\beta = \alpha\beta = 0.$$

Replacing x_μ on $y_\mu = x_\mu + \delta_\mu$ (δ_μ are constants) we obtain from (2.10.60), (2.10.61) two families of C(1,3)-ungenerative solutions of YM system (2.10.1). Note, that solution (2.10.60) is analytic in constant e but the solution (2.10.61) is not.

It will be also noted that solutions of YM equations described in Point 2 can be generated by means of formulae (2.10.8), (2.3.2) and to obtain in such a way new families of solutions.

In conclusion we present the reduced system of ODEs which is obtained when using the ansatz

$$\vec{Y}_\nu(x) = (a_\nu bx - b_\nu ax)f(\omega)\vec{n} + c_\mu\vec{p}g(\omega) + d_\mu\vec{q}h(\omega), \tag{2.10.62}$$

where $\omega = \left[(ax)^2 + (bx)^2\right]^{1/2}$; $a_\nu, b_\nu, c_\nu, d_\nu$ are arbitrary constants satisfying relations (2.1.27); f, g, h are real scalar functions; $\vec{n}, \vec{p}, \vec{q}$ are mutually orthonormal isospin vectors. Substitution of the ansatz (2.10.62) into (2.10.1) gives rise to the ODE

$$\begin{aligned} &\omega\ddot{f} + 3\dot{f} + e^2\omega f(h^2 - g^2) = 0, \\ &\omega\ddot{g} + \dot{g} + e^2\omega g(h^2 - \omega^2 f^2) = 0, \\ &\omega\ddot{h} + \dot{h} - e^2\omega h(\omega^2 h^2 + g^2) = 0. \end{aligned} \tag{2.10.63}$$

Unfortunately we do not succeeded in finding solutions of system (2.10.63).

2.11. *On connection between solutions of Dirac and Maxwell equations, dual Poincare invariance and superalgebras of invariance of Dirac equation*

Consider the massless Dirac equation

$$i\gamma\partial\psi = 0 \tag{2.11.1}$$

(we use notation given in Section 2.1). There is a connection between solutions of Equations (2.11.1) and Maxwell's equations for vacuum

$$\begin{aligned} \dot{\vec{E}} &\equiv \frac{\partial \vec{E}}{\partial t} = \operatorname{rot}\vec{H}, \qquad \operatorname{div}\vec{E} = 0, \\ \dot{\vec{H}} &\equiv \frac{\partial \vec{H}}{\partial t} = -\operatorname{rot}\vec{E}, \qquad \operatorname{div}\vec{H} = 0, \end{aligned} \tag{2.11.2}$$

where $\vec{E} = (E_1, E_2, E_3)$, $\vec{H} = (H_1, H_2, H_3)$ are vectors of electric and magnetic fields. To establish this connection let us decompose an arbitrary solution of Equation (2.11.1) into real and imaginary parts using notation of Ljolje [67*]

$$\psi = \psi_{\text{real}} + \psi_{\text{imag}} = \begin{pmatrix} -D_1 \\ D_3 \\ -B_2 \\ -G \end{pmatrix} + i \begin{pmatrix} D_2 \\ -F \\ -B_1 \\ B_3 \end{pmatrix}. \tag{2.11.3}$$

Theorem 2.11.1 [65*,64*] *Let ψ, defined by* (2.11.3), *be an arbitrary solution of the massless Dirac equation* (2.11.1). *Then the functions*

$$\begin{aligned} \vec{E} &= \vec{D} + \vec{\nabla}\left(\int_{t_0}^{t} G(\tau,\vec{x})d\tau + \widetilde{G}(t_0,\vec{x})\right), \\ \vec{H} &= \vec{B} + \vec{\nabla}\left(\int_{t_0}^{t} F(\tau,\vec{x})d\tau + \widetilde{F}(t_0,\vec{x})\right), \end{aligned} \tag{2.11.4}$$

where $\widetilde{G}(t_0,\vec{x})$ and $\widetilde{F}(t_0,\vec{x})$ satisfy the Poisson equations

$$\Delta\widetilde{G}(t_0,\vec{x}) = \partial G(\tau,\vec{x})\tau\,\Big|_{\tau=t_0}, \qquad \Delta\widetilde{F}(t_0,\vec{x}) = \partial F(\tau,\vec{x})\tau\,\Big|_{\tau=t_0}, \tag{2.11.5}$$

t_0 is an arbitrary constant, are solutions of Maxwell equations (2.11.2).

Proof. First of all we note that after substitution of ψ (2.11.3) into (2.11.1) and separation into real and imaginary parts we get the following Maxwell equations with currents

$$\begin{aligned} \dot{\vec{D}} &= \operatorname{rot}\vec{B} - \vec{\nabla}G, \qquad \operatorname{div}\vec{D} = -\dot{G}, \\ \dot{\vec{B}} &+ \operatorname{rot}\vec{D} = -\vec{\nabla}F, \qquad \operatorname{div}\vec{B} = -\dot{F} \end{aligned} \tag{2.11.6}$$

where $\vec{D} = (D_1, D_2, D_3)$, $\vec{B} = (B_1, B_2, B_3)$, dot means differentiation with respect to t. So, the Dirac equation (2.11.1) and the system (2.11.6) are fully equivalent. Therefore, taking into account (2.11.6) and the well-known fact that every component of ψ-function (2.11.3) of the Dirac equation (2.11.1) satisfies wave equation $\Box\psi = 0$ (in particular, $\Delta G(\tau, \vec{x}) = \dfrac{\partial^2}{\partial\tau^2}G(\tau, \vec{x})$, we find after substitution of (2.11.4) into (2.11.2)

$$\dot{\vec{E}} - \operatorname{rot}\vec{H} = \dot{\vec{D}} + \vec{\nabla}G - \operatorname{rot}\vec{B} = 0,$$

$$\begin{aligned} \operatorname{div}\vec{E} &= \operatorname{div}\vec{D} + \int_{t_0}^{t} \Delta G(\tau, \vec{x})d\tau + \Delta\tilde{G}(t_0, \vec{x}) = \\ &= \operatorname{div}\vec{D} + \int_{t_0}^{t} \left[\frac{\partial^2}{\partial\tau^2}G(\tau, \vec{x})\right] d\tau + \Delta\tilde{G}(t_0, \vec{x}) = \\ &= \operatorname{div}\vec{D} + \dot{G} - \left.\frac{\partial G(\tau, \vec{x})}{\partial\tau)}\right|_{\tau = t_0} + \Delta\tilde{G}(t_0, \vec{x}) = 0. \end{aligned}$$

In the last equality we have used (2.11.5). In the same spirit one can prove the validity of the theorem for the second part of the Maxwell equations (2.11.2). Thus, the theorem is proved. And in addition, the inverse statement also holds.

Theorem 2.11.2 *Let there be given a solution $\vec{E}$, $\vec{H}$ of Maxwell's equations (2.11.2) and two solutions F and G of scalar wave equation*

$$\Box F = 0, \qquad \Box G = 0. \tag{2.11.7}$$

Then the ψ-function (2.11.3) with components F, G and

$$\begin{aligned} D_a &= E_a - \partial_a \left(\int_{t_0}^{t} G(\tau, \vec{x})d\tau + \tilde{G}(t_0, \vec{x}) \right), \\ B_a &= H_a - \partial_a \left(\int_{t_0}^{t} F(\tau, \vec{x})d\tau + \tilde{F}(t_0, \vec{x}) \right), \end{aligned} \tag{2.11.8}$$

where $a = 1, 2, 3$ *and* $\widetilde{G}(t_0, \vec{x})$, $\widetilde{F}(t_0, \vec{x})$ *are determined from* (2.11.5), *is a solution of the massless Dirac equation* (2.11.1).

Proof. Let us use the equivalence between the Dirac equation (2.11.1) and the system (2.11.6). Having substituted (2.11.8) into (2.11.6) and taking into account (2.11.2), (2.11.7), and (2.11.5), we get

$$\dot{\vec{D}} - \operatorname{rot}\vec{B} + \vec{\nabla}G = \dot{\vec{E}} - \vec{\nabla}G + \vec{\nabla}G - \operatorname{rot}\vec{H} = 0,$$

$$\operatorname{div}\vec{D} + \dot{G} = \operatorname{div}\vec{E} - \int_{t_0}^{t} \Delta G(\tau, \vec{x})d\tau - \Delta\widetilde{G}(t_0, \vec{x}) + \dot{G} = 0.$$

Analogously, one proves the theorem for the remaining equations of system (2.11.6).

Theorem 2.11.2 has an important corollary: choosing $F = G = 0$ we get from (2.11.8) $\vec{D} = \vec{E}$, $\vec{B} = \vec{H}$ and in this case the formula (2.11.3) takes especially simple form

$$\psi = \begin{pmatrix} -E_1 + iE_2 \\ E_3 \\ -H_2 - iH_1 \\ iH_3 \end{pmatrix} \tag{2.11.9}$$

So, if $\vec{E}$ and $\vec{H}$ satisfy Maxwell equations (2.11.2), then ψ given by (2.11.9) automatically satisfies Dirac equation (2.11.1), and one can consider relation (2.11.9) as representation of spinor field ψ by electromagnetic field $\vec{E}$, $\vec{H}$. It is appropriate to note that if $\vec{E}$ and $\vec{H}$ are transformed under Lorentz boost as electromagnetic field, the ψ-function (2.11.9) is not transformed like a Dirac spinor (this question will be discussed in detail below). It will also be noted that, according to Theorem 2.11.1, the procedure of obtaining solutions of the vacuum Maxwell equations (2.11.2) from those of the massless Dirac equations (2.11.1) and associated Poisson equations (2.11.5) is unique to within a gauge transformation, whereas the inverse procedure (Maxwell→Dirac) involves ambiguities due to the arbitrary choice of additional scalar fields F, G satisfying (2.11.7).

Consider an example. Let us take solutions of Maxwell equations (2.11.2) and wave equations (2.11.7) in the form

$$\vec{E} = \vec{\alpha} \times \vec{x}, \quad \vec{H} = -2\vec{\alpha}t, \quad F = G = 3t^2 + \vec{x}^2, \quad \vec{\alpha} = \text{const}.$$

Then, by means of (2.11.8), (2.11.3) one easily finds the following solution of the Dirac equation (2.1.1)

$$\psi = \begin{pmatrix} -[(\vec{\alpha} \times \vec{x})_1 - 2tx_1] + i[(\vec{\alpha} \times \vec{x})_2 - 2tx_2] \\ [(\vec{\alpha} \times \vec{x})_3 - 2tx_3] - i(3t^2 + \vec{x}^2) \\ 2t(\alpha_2 + x_2) + 2it(\alpha_1 + x_1) \\ -(3t^2 + \vec{x}^2) - 2it(\alpha_3 + x_3) \end{pmatrix}$$

In terms of $\vec{D}$, $\vec{B}$, F, G from (2.11.3)

$$\overline{\psi}\psi = \vec{D}^2 - \vec{B}^2 + F^2 - G^2 \tag{2.11.10}$$

and in the considered case we have

$$\overline{\psi}\psi = \vec{\alpha}^2\vec{x}^2 - (\vec{\alpha}\cdot\vec{x})^2 - 4t^2(\vec{\alpha}^2 + 2\vec{\alpha}\cdot\vec{x}).$$

Let us make a four-component ψ-function as

$$\psi = i\gamma\partial \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} \tag{2.11.11}$$

where $\varphi_0, \ldots, \varphi_3$ are arbitary solutions of the wave equation, that is $\Box\varphi_\mu = 0$. So, (2.11.11), (2.11.3), and (2.11.4) give the following chain of solutions: scalar wave equation→massless Dirac equation→vacuum Maxwell equations. Infinite series of solutions of scalar wave equation is given in (2.3.60).

For further analysis it is convenient to consider Dirac equation (2.11.1) together with its conjugation and to write it uniformly as

$$i\Gamma^\mu\partial_\mu\psi = 0, \tag{2.11.12}$$

where $\psi = \psi(x) = \text{column}(\psi, \widetilde{\psi})$, $\widetilde{\psi} = \gamma_0\psi^*$, Γ^μ are 8×8 matrices

$$\Gamma^\mu = \begin{pmatrix} \gamma^\mu & 0_4 \\ 0_4 & -(\gamma^\mu)^T \end{pmatrix} \tag{2.11.13}$$

where 0_4 is the 4×4 zero matrix.

Symmetry properties of Equation (2.11.12) were studied first by Dirac, who had shown that it is comformally invariant (see §2.3). Afterwards, Pauli and Touschek found that this equation admits as well the eight-parameter group G_8 of component transformations. And finally, Ibragimow [66*] had proved that the 23-parameter group $G_{23}=C(1,3)\oplus G_8$ is maximal in the sense of the Lie-invariance group of the equation. Relativistic invariance of Equation (2.11.12) is usually understood as invariance with respect to the spinor representation

$$D(\tfrac{1}{2},0)\oplus D(0,\tfrac{1}{2})\oplus D(\tfrac{1}{2},0)\oplus D(0,\tfrac{1}{2}) \tag{2.11.14}$$

of the Poincare group P(1,3) (it means that ψ is transformed as a spinor under the Lorentz boost). However, the invariance of Equation (2.11.12) under the Pauli-Touschek 8-parameter group allows for two additional representations of AP(1,3), which are realized on the set of solutions of the equation, namely

$$D(1,0)\oplus D(0,1)\oplus D(0,0)\oplus D(0,0) \tag{2.11.15}$$

and

$$D(\tfrac{1}{2},\tfrac{1}{2}) \oplus D(\tfrac{1}{2},\tfrac{1}{2}). \tag{2.11.16}$$

So, it is natural to call this dual Poincare invariance. The explicit form for the basis elements of AP(1,3) for representations (2.11.14)–(2.11.16) is

$$\mathrm{AP}^k(1,3) = \left\langle P_\mu = \frac{\partial}{\partial x^\mu}, \quad J_{\mu\nu}^{(k)} = x_\mu P_\nu - x_\nu P_\mu + S_{\mu\nu}^{(k)} \right\rangle, \tag{2.11.17}$$

where $k = 1, 2, 3$ corresponds to (2.11.14)–(2.11.16), respectively;

$$\begin{aligned} &S_{\mu\nu}^{(1)} = -\tfrac{1}{4}[\Gamma_\mu, \Gamma_\nu]; \\ &S_{\mu\nu}^{(2)} = S_{\mu\nu}^{(1)} + Q_{\mu\nu}; \\ &S_{01}^{(3)} = S_{01}^{(2)}, \qquad S_{02}^{(3)} = S_{02}^{(2)}, \qquad S_{03}^{(3)} = S_{03}^{(2)} - 2Q_{03} \\ &S_{12}^{(3)} = S_{12}^{(2)}, \qquad S_{13}^{(3)} = S_{13}^{(2)} - 2Q_{13}, \qquad S_{23}^{(3)} = S_{23}^{(2)} - 2Q_{23}; \end{aligned} \tag{2.11.18}$$

Γ_μ are given in (2.11.13); $Q_{\mu\nu}$ are six basis elements of the Pauli-Touschek algebra:

$$\begin{aligned} &Q_{01} = \frac{1}{2}\begin{pmatrix} 0_4 & -i\gamma_0\gamma_2 \\ i\gamma_0\gamma_2 & 0_4 \end{pmatrix}, \qquad Q_{02} = \frac{1}{2}\begin{pmatrix} 0_4 & -\gamma_0\gamma_2 \\ -\gamma_0\gamma_2 & 0_4 \end{pmatrix}, \\ &Q_{03} = \frac{1}{2}\begin{pmatrix} -\gamma_5 & 0_4 \\ 0_4 & \gamma_5 \end{pmatrix}, \qquad Q_{12} = \frac{i}{2}\begin{pmatrix} I_4 & 0_4 \\ 0_4 & -I_4 \end{pmatrix}, \\ &Q_{13} = \frac{1}{2}\begin{pmatrix} 0_4 & -\gamma_1\gamma_3 \\ -\gamma_1\gamma_3 & 0_4 \end{pmatrix}, \qquad Q_{23} = \frac{i}{2}\begin{pmatrix} 0_4 & \gamma_1\gamma_3 \\ -\gamma_1\gamma_3 & 0_4 \end{pmatrix}. \end{aligned} \tag{2.11.19}$$

It will be noted that the action of operators (2.11.17) is defined in the space of 8-component functions introduced in (2.11.12). Invariance of Equation (2.11.12) under $\mathrm{AP}^{(2)}(1,3)$ results in the possibility to represent it in the form (2.11.6), and invariance of (2.11.12) under $\mathrm{AP}^{(3)}(1,3)$ allows us to rewrite it as [67*]

$$\begin{gathered} \partial_\mu A_\nu - \partial_\nu A_\mu - \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\left(\partial^\rho B^\sigma - \partial^\sigma B^\rho\right) = 0, \\ \partial_\nu A^\nu = \partial_\nu B^\nu = 0, \end{gathered} \tag{2.11.20}$$

where

$$\psi = \psi_{\text{real}} + i\psi_{\text{imag}} = \begin{pmatrix} -A^2 \\ -B^0 \\ -B^1 \\ B^3 \end{pmatrix} + i\begin{pmatrix} -A^1 \\ A^3 \\ B^2 \\ -A^0 \end{pmatrix} \tag{2.11.21}$$

Now consider the following three states of symmetry operators of Equation (2.11.12)

$$\mathrm{SA}^{(k)} = \left\langle P_\mu, J_{\mu\nu}^{(k)}, \Gamma_4, I; Q_{\mu\nu} \right\rangle \tag{2.11.22}$$

where P_μ, $J_{\mu\nu}^{(k)}$ and $Q_{\mu\nu}$ are defined in (2.11.17) and (2.11.19), Γ_μ are given in (2.11.13), $\Gamma_4 = \Gamma_0\Gamma_1\Gamma_2\Gamma_3$. These sets of operators form Lie algebras as well as superalgebras [64*]. Operators P_μ, $J_{\mu\nu}^{(k)}$, Γ_4, I are even, and $Q_{\mu\nu}$ are odd in corresponding superalgebras. To prove this statement we write down commutation and anticommutation relations for these operators.

Operators P_μ and $J_{\mu\nu}^{(k)}$ satisfy standard commutation relations of the Poincare algebra (1.1.4); Γ_4 and I commute with all elements of $\mathrm{SA}^{(k)}$. Further, it is convenient to introduce notations

$$\begin{aligned} R_a &= Q_{0a}, \qquad & T_a &= \tfrac{1}{2}\epsilon_{abc}Q_{bc} \\ N_a^{(k)} &= J_{0a}^{(k)}, \qquad & M_a^{(k)} &= \tfrac{1}{2}\epsilon_{abc}J_{bc}^{(k)} \end{aligned} \tag{2.11.23}$$

It is easy to verify that

$$\begin{aligned} &\{R_a, R_b\} \equiv R_aR_b + R_bR_a = \tfrac{1}{2}\delta_{ab}I \\ &\{T_a, T_b\} = -\tfrac{1}{2}\delta_{ab}I, \qquad \{R_a, T_b\} = \delta_{ab}\Gamma_4 \end{aligned} \tag{2.11.24}$$

Operators R_a, T_a from $\mathrm{SA}^{(1)}$ commute with all even operators of $\mathrm{SA}^{(1)}$. For $\mathrm{SA}^{(2)}$ we have

$$\begin{aligned} &[P_\mu, R_a] = [P_\mu, T_a] = 0, \\ &[N_a^{(2)}, R_b] = [R_a, R_b] = \epsilon_{abc}T_c, \\ &[N_a^{(2)}, T_b] = [R_a, T_b] = -\epsilon_{abc}R_c, \\ &[M_a^{(2)}, R_b] = -\epsilon_{abc}R_c, \\ &[M_a^{(2)}, T_b] = [T_a, T_b] = -\epsilon_{abc}T_c, \end{aligned} \tag{2.11.25}$$

Superalgebra $\mathrm{SA}^{(3)}$ is isomorphic to $\mathrm{SA}^{(2)}$. The isomorphism is achieved by the transformation

$$R_3 \to R_3' = -R_3, \qquad T_1 \to T_1' = -T_1, \qquad T_2 \to T_2' = -T_2 \tag{2.11.26}$$

So, the structure of superalgebras (2.11.22) is fully described.

In conclusion, it will be noted that dual Poincare invariance analogous to that considered above has system of two Dirac equations with masses m and $-m$. For more detail see [64*].

In [64*] nonlinear generalizations of the Dirac system are constructed which possess dual Poincare invariance. Therein we construct the complete set of P(1,3)-inequivalent ansatze of codimension 1 for all representations of Poincare algebra discussed. These ansatze are used for reduction and for finding exact solutions of some nonlinear Dirac equations.

Chapter 3

Euclid and Galilei Groups and Nonlinear PDEs for Scalar Fields

In the present chapter we describe a wide class of nonlinear PDEs for scalar fields invariant under Euclid, Galilei, or larger groups. For some of such equations we construct multiparameter families of exact solutions.

3.1. *Second-order PDEs invariant under the extended Euclid group* $\widetilde{\mathrm{E}}(1, n-1)$

The aim of the present paragraph is to describe equations of the form

$$\Box u + F(u, \underset{1}{u}) u_0 = 0 \tag{3.1.1}$$

where $u = u(x)$, $x = (x_0, x_1, \ldots, x_{n-1})$, $\underset{1}{u} = \left\{ \dfrac{\partial u}{\partial x_a} \right\}$, $\mu = \overline{1, n-1}$. F is a differentiable function, which is invariant under the extended Euclid group $\widetilde{\mathrm{E}}(1, n-1)$, i.e., the Euclid group $\mathrm{E}(1, n-1)$ ($\mathrm{E}(1, n-1)$ contains translation and rotation in R^{n-1} added by time translation) augmented by the one-parameter group of scale transformation. The corresponding Lie algebra $A\widetilde{\mathrm{E}}(1, n-1)$ is defined by the following commutation relations

$$[P_0, P_a] = 0, \quad [P_0, J_a] = 0, \quad [P_a, J_b] = i\epsilon_{abc} P_c, \quad a, b = \overline{1, n-1},$$
$$[J_a, J_b] = i\epsilon_{abc} J_c, \quad [P_0, D] = iP_0, \quad [P_a, D] = iP_a. \quad [J_a, D] = 0 \tag{3.1.2}$$

Below we consider the standard representation of $A\tilde{E}(1, n-1)$

$$\begin{aligned} P_0 &= i\partial_0, \qquad P_a = -i\partial_a, \\ J_a &= (\vec{x} \times \vec{P})_a = \epsilon_{abc} x_b P_c. \end{aligned} \tag{3.1.3}$$

The most general form of the operator D which satisfies relations (3.1.2) is as follows

$$D = x_\nu P^\nu + i(au + æ)\partial_u \equiv x_0 P_0 - x_a P_a + i(au + æ)\partial_u, \tag{3.1.4}$$

where a, æ are arbitrary constants.

Theorem 3.1.1. [91] *Equation* (3.1.1) *is invariant with respect to the group* $\tilde{E}(1, n-1)$ *iff*

$$\begin{aligned} &1^\circ \quad F = u^k f\left(\frac{u_a u_a}{u^{2(k+1)}}\right), \\ &2^\circ \quad F = e^u f\left(\frac{u_a u_a}{e^{4u}}\right), \\ &3^\circ \quad F = \sqrt{u_a u_a} f(u), \end{aligned} \tag{3.1.5}$$

where k is an arbitrary constant, f is an arbitrary differentiable function.

Proof. In order to prove the theorem we use the Lie algorithm in much the same way as in §1.1. The infinitesimal operator may be written in the form

$$X = \xi^\mu(x)\partial_\mu + \eta(u)\partial_u = \alpha^\mu P_\mu + \beta^a J_a + æD, \qquad \mu = \overline{0, n-1}, \tag{3.1.6}$$

where α_μ, β_a, æ are arbitrary constants.

Using the condition of invariance (necessary and sufficient)

$$\underset{2}{X}(\Box u + F u_0)\Big|_{\Box u + F u_0 = 0} = 0,$$

where $\underset{2}{X}$ is the second extension of the operator X (3.1.6) which is defined according to formulae (1.1.7), we obtain

$$\begin{aligned} &u_b F_{u_a} - u_a F_{u_b} = 0, \\ &(a - æ) u_a F_{u_a} + \eta F_u + æF = 0. \end{aligned} \tag{3.1.7}$$

The first equation of (3.1.7) implies that

$$F(u, \underset{1}{u}) = F(u, w), \qquad w = u_a u_a,$$

which allows us to rewrite the second equation from (3.1.7) in the form

$$2(a - æ) w F_w + \eta F_u + æF = 0. \tag{3.1.8}$$

There are three possibilities

$$1^\circ \quad a \neq 0,\ æ = 0, \qquad 2^\circ \quad a = 0,\ æ \neq 0, \qquad 3^\circ \quad a = æ = 0.$$

According to them we have three different solutions of Equation (3.1.8) (see (3.1.5)) and obtain three different forms of the operator D (3.1.4)

$$\begin{aligned} &1^\circ \quad D = x^\nu P_\nu + \frac{i}{k}, \\ &2^\circ \quad D = x^\nu P_\nu - i\partial_u, \\ &3^\circ \quad D = x^\nu P_\nu. \end{aligned} \tag{3.1.9}$$

It is easy to verify that Equation (3.1.1) with F from (3.1.5) is invariant under $\widetilde{\mathrm{E}}(1, n-1)$. The invariance with respect to translations and rotations generated by operator (3.1.3) is obvious. The scale transformations generated by dilation operators D (3.1.9) have the form

$$\begin{aligned} &1^\circ \quad x'_\mu = e^\theta x_\mu, \qquad u'(x') = e^{-\theta/k} u(x), \qquad \theta = \text{const}, \\ &2^\circ \quad x'_\mu = e^\theta x_\mu, \qquad u'(x') = u(x) - \theta \\ &3^\circ \quad x'_\mu = e^\theta x_\mu, \qquad u'(x') = u(x). \end{aligned} \tag{3.1.10}$$

One can make sure that these transformations leave Equation (3.1.1) with corresponding function F invariant. The theorem is proved.

Remark. Equation (3.1.1) with function $F = F(u)$ is invariant under $\widetilde{\mathrm{E}}(1, n-1)$ iff

$$\begin{aligned} &1^\circ \qquad F = \lambda_1 u^k, \qquad D = x^\nu P_\nu + \frac{i}{k}, \\ &2^\circ \qquad F = \lambda_2 \exp(u), \quad D = x^\nu P_\nu - i\partial_u, \\ &3^\circ \qquad F = 0, \qquad D = x^\nu P_\nu. \end{aligned} \tag{3.1.11}$$

where λ_1, λ_2, k are arbitrary constants.

If $F = F(\underset{1}{u})$ (3.1.8) yields for $a = \left(1 - \frac{1}{k}\right) æ$

$$F = \lambda (u_a u_a)^{k/2}, \qquad D = x^\nu P_\nu + \left(1 - \frac{1}{k}\right) i, \qquad \lambda, k = \text{const}. \tag{3.1.12}$$

In addition, Equation (3.1.1) with function F (3.1.12) is also invariant with respect to the transformations

$$u(x) \to u'(x') = u(x) + \text{const}, \qquad x'_\mu = x_\mu$$

3.2. *Reduction and exact solutions of nonlinear PDEs of the type* $\Box u + F(u, \underset{1}{u})u_0 = 0$

In this paragraph we reduce and find exact solutions of $\widetilde{\mathrm{E}}(1,3)$-invariant nonlinear Equations (3.2.4), (3.2.42), (3.2.51), following the algorithm expounded in §1.4. With that end in view the complete set of $\widetilde{\mathrm{E}}(1,3)$-nonequivalent ansatze is built.

1. Invariants of the Euclid group $\widetilde{\mathrm{E}}(1,3)$.

To obtain invariants of E(1,3) group we act as in §1.4, where invariants of $\widetilde{\mathrm{P}}(1,2)$ are described. Let us proceed from the equation

$$\xi^{\mu}(x)\partial_{\mu}\omega(x) = 0, \tag{3.2.1}$$

where $\xi^{\mu}(x)$ are defined in (3.1.6), and go on to the equivalent characteristic equations

$$\frac{dx_0}{\xi^0} = \frac{dx_1}{\xi^1} = \frac{dx_2}{\xi^2} = \frac{dx_3}{\xi^3} = d\tau \tag{3.2.2}$$

which can be rewritten as system of ODEs

$$\frac{dx_{\mu}}{d\tau} = \xi^{\mu}(x(\tau)). \tag{3.2.3}$$

Since $\xi^{\mu}(x)$ are linear functions of x_{μ} (see (3.1.5)), so the system (3.2.3) is linear. The general solution of this system after eliminating τ by means of Equation (3.2.2) yields the unknown invariants. Depending on the correlation between coefficients in (3.1.5) we shall have several different cases.

Without going into details of lengthy though elementary calculations we list in Table 3.2.1 the invariants of $\widetilde{\mathrm{E}}(1,3)$.

2. Let us apply the above results for obtaining solutions of the equation

$$\Box u + \lambda u u_0 = 0 \tag{3.2.4}$$

This equation is used in field theory and gas dynamics. Solutions of two-dimensional Equation (3.2.4) are obtained in [172]. Following §1.4 [178] we seek solutions of Equation (3.2.4) in the form [61]

$$u(x) = f(x)\varphi(\omega), \tag{3.2.5}$$

where $\omega(x) = \{\omega_1(x), \omega_2(x), \omega_3(x)\}$ are given in the Table 3.2.1 and functions $f(x)$ are to be determined from the equation

$$\frac{du}{-\text{æ} u} = d\tau \tag{3.2.6}$$

and (3.2.2) or from the condition of splitting.

Table 3.2.1. The invariants of $\widetilde{\mathrm{E}}(1,3)$.

N	Invariant variables $\omega(x)=\{\omega_1(x),\omega_2(x),\omega_3(x)\}$	Conditions on the parameters
1.	$\frac{\alpha_a y_a}{y_0}$, $\frac{y_a y_a}{y_0^2}$, $y_0 \exp\left\{b_1 \arctan \frac{\gamma_a y_a}{\beta_a y_a}\right\}$	$\alpha_a^2=\alpha_a\alpha_a=\alpha^2\neq 0$, $\beta_a^2\neq 0, \gamma_a^2\neq 0$, $\alpha_a\beta_a=\alpha_a\gamma_a=\beta_a\gamma_a=0$, $(\beta_a y_a)^2+(\gamma_a y_a)^2=$ $=\frac{b_1^2}{\alpha^2}(\alpha^2\omega_2-w_1^2)$.
2.	$\frac{\alpha_a y_a}{y_0}$, $\frac{y_a y_a}{y_0^2}$, $\frac{\beta_a y_a}{\alpha_a y_a}-b_2\ln\alpha_a y_a$	$\beta_a^2\neq 0$, $\alpha_a^2=\alpha_a\beta_a=0$.
3.	$\alpha_a y_a/y_0$, $y_a y_a/y_0^2$, $\beta_a y_a/y_0^{b_3}$	$\alpha_a^2\neq 0$, $\beta_a^2=\alpha_a\beta_a=0$.
4.	$\frac{\alpha_a y_a}{y_0}$, $\frac{\sigma_a y_a}{y_0}$, $\delta_a x_a-b_4\ln y_0$	$\alpha_a^2\neq 0, \delta_a^2=\sigma_a^2=0$, $\alpha_a\delta_a=\alpha_a\sigma_a=0$, $\sigma_a\delta_a=a\neq 0$.
5.	$\beta_\nu x^\nu$, $\frac{(\alpha_a y_a)}{-\alpha_a^2}+y_a y_a$, $x_0-b_5\arctan \ell_a y_a/\delta_a y_a$	$\alpha_a^2=-\alpha_\nu\beta^\nu=\alpha^2\neq 0$, $\ell_a^2=\delta_a^2=\ell\neq 0$; $\alpha_a\ell_a=\alpha_a\delta_a=\ell_a\delta_a=\beta_\nu\delta^\nu=0$, $\beta_\nu\beta^\nu=\beta\neq 0$, $(\ell_a y_a)^2+(\delta_a y_a)^2=\ell\omega_2$.
6.	$\beta_\nu x^\nu$, $\frac{(\alpha_a y_a)^2}{-\alpha^2}+y_a y_a$, $x_0+b_6\ln\sigma_a z_a$	$\alpha_a^2=-\alpha_\nu\beta^\nu\neq 0$, $\beta_\nu\beta^\nu=\beta\neq 0$, $\alpha_a\sigma_a=\sigma_a^2=\beta_\nu\sigma^\nu=0$.
7.	$\beta_\nu x^\nu$, $\frac{(\alpha_a y_a)^2}{2}+b_7\ell_a y_a$, $\frac{1}{3}(\alpha_a y_a)^3+b_7(\alpha_a y_a)(\ell_a y_a)-b_7^2\delta_a y_a$	$\beta_\nu\beta^\nu\neq 0, \ell_a^2=\alpha_a\delta_a=\delta_a^2=$ $=\ell\neq 0$, $\alpha_a^2=\delta_a\ell_a=0$.
8.	$\alpha_a x_a$, $x_a x_a$, $x_0-b_8\arctan(\ell_a x_a/\delta_a x_a)$	$\alpha_a^2=\alpha^2\neq 0$, $\ell_a^2=\delta_a^2=\ell\neq 0$, $\alpha_a\ell_a=\alpha_a\delta_a=\ell_a\delta_a=0$.
9.	$\frac{y_1}{y_0}$, $\frac{y_2}{y_0}$, $\frac{y_3}{y_0}$.	
10.	$a_\nu x^\nu$, $\beta_a x_a$, $\gamma_a x_a$	$\alpha_\nu\alpha^\nu=\alpha^2\neq 0$, $\beta_a^2=\beta^2\neq 0$, $\gamma_a^2=\gamma^2\neq 0$, $\alpha_a\beta_a=b_9$, $\alpha_a\gamma_a=b_{10}$, $\beta_a\gamma_a=b_{11}$

Here $y_\nu=x_\nu+a_\nu$, $z_\nu=x_\nu+\frac{1}{2}a_\nu$; $a_\nu, b_\nu, c_a, \delta_a, \sigma_a, \alpha_\nu, \beta_\nu, \gamma_\nu$ are arbitrary constants satisfying conditions stated in the table.

So, we have 10 ansatze of the form (3.2.5)

$$u(x)=\begin{cases} y_0^{-1/2}\varphi(\omega), & \text{for } \omega \text{ N1–4} \\ \varphi(\omega), & \text{for } \omega \text{ N5–10} \end{cases} \tag{3.2.7}$$

The numeration of ansatze used here and below corresponds to the numeration of invariants of the Table 3.2.1.

Inserting ansatze (3.2.5) into the Equation (3.2.4) we obtain 10 reduced PDEs for function $\varphi(\omega)$

$$\begin{aligned}
&(1)\quad (\omega_1^2-\alpha^2)\varphi_{11}+4\omega_2(\omega_2-1)\varphi_{22}-\omega_3^2\left(1-\frac{\alpha^2 b_1^2}{\alpha^2\omega_2-\omega_1^2}\right)\varphi_{33}-\\
&\qquad -4\omega_1(1-\omega_1)\varphi_{12}-4\omega_2\omega_3\varphi_{23}+(10\omega_2-6)\varphi_2+4\omega_1\varphi_1+\\
&\qquad +\left(\frac{\alpha^2 b_1^2}{\alpha^2\omega_2-\omega_1^2}-2\right)\omega_3\varphi_3-\lambda\omega_1\varphi\varphi_1-2\lambda\omega_2\varphi\varphi_2+\lambda\omega_3\varphi\varphi_3+2\varphi-\lambda\varphi^2=0;\\
&(2)\quad \omega_1^2\varphi_{11}+4\omega_2(\omega_2-1)\varphi_{22}+\frac{b_2}{\omega_1^2}\varphi_{33}+4\omega_1(\omega_2-1)\varphi_{12}-4b_2\varphi_{23}+\\
&\qquad +4\omega_1\varphi_1+(10\omega_2-6)\varphi_2-\lambda\omega_1\varphi\varphi_1-2\lambda\omega_2\varphi\varphi_2+2\varphi-\lambda\varphi^2=0;\\
&(3)\quad (\omega_1^2+\alpha^2)\varphi_{11}+4\omega_2(\omega_2-1)\varphi_{22}+b_3^2\omega_3^2\varphi_{33}+4\omega_1(\omega_2-1)\varphi_{12}+\\
&\qquad +2b_3\omega_1\omega_3\varphi_{13}+4\omega_3(b_3\omega_2-1)\varphi_{23}+4\omega_1\varphi_1+(10\omega_2-6)\varphi_2+\\
&\qquad +b_3(b_3+3)\omega_3\varphi_3-\lambda\omega_1\varphi\varphi_1-\lambda\omega_2\varphi\varphi_2-\lambda b_3\varphi\varphi_3+2\varphi-\lambda\varphi^2=0;\\
&(4)\quad (\omega_1^2+\alpha^2)\varphi_{11}+4\omega_2^2\varphi_{22}+b_4^2\varphi_{33}+4\omega_1\omega_2\varphi_{12}+2b_4\varphi_{13}+11b_4\omega_2\varphi_{23}+\\
&\qquad +4\omega_1\varphi_1+10\omega_2\varphi_2+3b_4\varphi_3-\lambda\omega_1\varphi\varphi_1-2\lambda\omega_2\varphi\varphi_2-2b_4\varphi\varphi_3+2\varphi-\lambda\varphi^2=0;\\
&(5)\quad \beta\varphi_{11}+\omega_2\varphi_{22}+\left(1-\frac{b_5}{\omega_2}\right)\varphi_{33}+2\beta_0\varphi_{13}-4\varphi_2+\lambda\beta_0\varphi\varphi_1+\lambda\varphi\varphi_3=0;\\
&(6)\quad \beta\varphi_{11}+\omega_2\varphi_{22}+2\beta_0\varphi_{13}+\varphi_{33}-4b_6\varphi_{23}-4\varphi_2+\lambda\beta_0\varphi\varphi_1+\lambda\varphi\varphi_3=0;\\
&(7)\quad \beta\varphi_{11}-\ell b_7^2\varphi_{22}+2\ell b_7^2\left(\omega_2^2-b_7^2\right)\varphi_{33}-2\beta_7^2\ell^2\varphi_{13}+\lambda\beta_0\varphi\varphi_1=0;\qquad (3.2.8)\\
&(8)\quad -\alpha^2\varphi_{11}-4\omega_2\varphi_{22}+\left(1-\frac{b_8}{\omega_2}\right)\varphi_{33}+4\omega_1\varphi_{12}-6\varphi_2+\lambda\varphi\varphi_3=0;\\
&(9)\quad (\omega_1^2-1)\varphi_{11}+(\omega_2^2-1)\varphi_{22}+(\omega_3^2-1)\varphi_{33}+2(\omega_1\omega_2-1)\varphi_{12}+\\
&\qquad +2(\omega_1\omega_3-1)\varphi_{13}+2(\omega_2\omega_3-1)\varphi_{23}+4\omega_1\varphi_1+4\omega_2\varphi_2+4\omega_3\varphi_3-\\
&\qquad -\lambda\varphi(\omega_1\varphi_1+\omega_2\varphi_2+\omega_3\varphi_3)+2\varphi-\lambda\varphi^2=0;\\
&(10)\quad \alpha^2\varphi_{11}-\beta^2\varphi_{22}-\gamma^2\varphi_{33}+2b_9\varphi_{12}+2b_{10}\varphi_{13}+2b_{11}\varphi_{23}+\lambda\alpha_0\varphi\varphi_1=0.
\end{aligned}$$

Let in (3.2.8) $\varphi=\varphi(\omega_1)$, so we have the ODE

$$(\omega_1^2-\alpha^2)\varphi_{11}+4\omega_1\varphi_1-\lambda\omega_1\varphi\varphi_1+2\varphi-\lambda\varphi^2=0, \qquad (3.2.9)$$

which is reduced after first integration to the Riccati equation

$$\omega_1(\omega_1^2-\alpha^2)\varphi_1+(\omega_1^2+\alpha^2)\varphi-\frac{\lambda}{2}\omega_1^2\varphi^2=c_1,\quad c_1=\text{const} \qquad (3.2.10)$$

We seek solution of Equation (3.2.10) in the form

$$\varphi = c_1 \left[\omega_1 \psi(\omega_1) + \alpha^2\right]^{-1} \tag{3.2.11}$$

Inserting (3.2.11) into (3.2.10) we obtain the equation

$$\frac{d\psi}{\psi^2 - \alpha^2 + \frac{1}{2}\lambda c_1} = \frac{d\omega_1}{-(\omega_1^2 - \alpha^2)}$$

which, depending on the constant $\alpha^2 - \frac{1}{2}\lambda c_1$, has the following solutions

$$\psi = \begin{cases} \dfrac{(\alpha-\omega_1)^{a/\alpha} - c_2(\alpha+\omega_1)^{a/\alpha}}{(\alpha-\omega_1)^{a/\alpha} + c_2(\alpha+\omega_1)^{a/\alpha}}, & \alpha^2 - \frac{1}{2}\lambda c_1 = a^2 > 0; \\ 2\alpha\left(\ln\left|\frac{\alpha-\omega_1}{\alpha+\omega_1}\right| + c_2\right)^{-1}, & \alpha^2 - \frac{1}{2}\lambda c_1 = 0; \\ a\tan\left(\frac{a}{2\alpha}\ln\left|\frac{\alpha+\omega_1}{\alpha-\omega_1}\right| + c_2\right), & \alpha^2 - \frac{1}{2}\lambda c_1 = -a^2 < 0. \end{cases} \tag{3.2.12}$$

From (3.2.12) and (3.2.11) we have the following solutions of Equation (3.2.10)

$$\varphi = \begin{cases} c_1\left[a\omega_1\dfrac{(\alpha-\omega_1)^{a/\alpha} - c_2(\alpha+\omega_1)^{a/\alpha}}{(\alpha-\omega_1)^{a/\alpha} + c_2(\alpha+\omega_1)^{a/\alpha}} + \alpha^2\right]^{-1}, \\ c_1\left[2\alpha\omega_1\left(\ln\left|\frac{\alpha-\omega_1}{\alpha+\omega_1}\right| + c_2\right)^{-1} + \alpha^2\right]^{-1}, \\ c_1\left[a\omega_1\tan\left(\frac{a}{2\alpha}\ln\left|\frac{\alpha+\omega_1}{\alpha-\omega_1}\right| + c_2\right) + \alpha^2\right]^{-1}. \end{cases} \tag{3.2.13}$$

Equation (3.2.10) is reduced to Bernoulli one when $c_1 = 0$, and has the following general solution

$$\varphi = \left[c_2\frac{\omega_1^2 - \alpha^2}{\omega_1} - \frac{\lambda}{8\alpha}\frac{\omega_1^2 - \alpha^2}{\omega_1}\ln\left|\frac{\omega_1 - \alpha}{\omega_1 + \alpha}\right| + \frac{\lambda}{4}\right]^{-1}. \tag{3.2.14}$$

Functions (3.2.13) and (3.2.14) give the general solution of the Riccati equation (3.2.10).

The ansatz N1 from (3.2.7) and formulaes (3.2.13), (3.2.14) lead to the following solutions of Equation (3.2.4)

$$u(x) = c_1\left[a\alpha_a y_a\frac{(\alpha y_0 - \alpha_a y_a)^{a/\alpha} - c_2(\alpha y_0 + \alpha_a y_a)^{a/\alpha}}{(\alpha y_0 - \alpha_a y_a)^{a/\alpha} + c_2(\alpha y_0 + \alpha_a y_a)^{a/\alpha}} + \alpha^2 y_0\right]^{-1},$$

$$u(x) = c_1\left[2\alpha\alpha_a y_a\left(\ln\left|\frac{\alpha y_0 - \alpha_a y_a}{\alpha y_0 + \alpha_a y_a)}\right| + c_2\right)^{-1} + \alpha^2 y_0\right]^{-1}, \tag{3.2.15}$$

$$u(x) = c_1 \left[a\alpha_a y_a \tan\left(\frac{a}{2\alpha} \ln\left|\frac{\alpha y_0 + \alpha_a y_a}{\alpha y_0 - \alpha_a y_a)}\right| + c_2\right) + \alpha^2 y_0\right]^{-1},$$

$$u(x) = \left[c_2 \frac{(\alpha_a y_a)^2 - \alpha^2 y_0^2}{\alpha_a y_a} - \frac{\lambda}{8\alpha}\frac{(\alpha_a y_0)^2 - \alpha^2 y_0^2}{\alpha_a y_a} \ln\left|\frac{\alpha y_0 - \alpha_a y_a}{\alpha y_0 + \alpha_a y_a)}\right| + \frac{\lambda}{4} y_0\right]^{-1}.$$

If $\varphi = \varphi(\omega_2)$ then Equation (3.2.8(1)) takes the form

$$4\omega_2(\omega_2 - 1)\varphi_{22} + (10\omega_2 - 6)\varphi_2 - 2\lambda\omega_2\varphi\varphi_2 + 2\varphi - \lambda\varphi^2 = 0.$$

After integrating this ODE is reduced to the Riccati equation

$$4\omega_2(\omega_2 - 1)\varphi_2 + 2(\omega_2 - 1)\varphi - \lambda\omega_2\varphi^2 = c_1, \qquad c_1 = \text{const.} \tag{3.2.16}$$

The substitution

$$\psi(t) = t\varphi(t), \qquad t = \sqrt{\omega_2}$$

into Equation (3.2.16) yields the equation with separated variables. The general solution of the Riccati equation (3.2.16) is

$$\varphi = a\sqrt{\omega_2}\tan\left(\frac{\lambda a}{4}\ln\left|c_2\frac{1-\sqrt{\omega_2}}{1+\sqrt{\omega_2}}\right|\right), \qquad c_1 = \tfrac{1}{2}\lambda a^2;$$

$$\varphi = \frac{1}{\sqrt{\omega_2}}\left(-\frac{\lambda}{4}\ln\left|\frac{\sqrt{\omega_2}-1}{\sqrt{\omega_2}+1}\right| + c_2\right)^{-1}, \qquad c_1 = 0; \tag{3.2.17}$$

$$\varphi = \frac{a}{\sqrt{\omega_2}}\frac{c_2(1+\sqrt{\omega_2})^{\lambda a/2} - (1-\sqrt{\omega_2})^{\lambda a/2}}{c_2(1+\sqrt{\omega_2})^{\lambda a/2} + (1-\sqrt{\omega_2})^{\lambda a/2}}, \qquad c_1 = -\tfrac{1}{2}\lambda a^2.$$

The ansatz N1 from (3.2.7) and functions (3.2.17) give the following solution of Equation (3.2.4)

$$u(x) = a\sqrt{y_a y_a}\tan\left(\frac{\lambda a}{4}\ln\left|c_2\frac{y_0 - \sqrt{y_a y_a}}{y_0 + \sqrt{y_a y_a}}\right|\right),$$

$$u(x) = (y_a y_a)^{-\frac{1}{2}}\left(-\frac{\lambda}{4}\ln\left|\frac{y_0 - \sqrt{y_a y_a}}{y_0 + \sqrt{y_a y_a}}\right| + c_2\right)^{-1}, \tag{3.2.18}$$

$$u(x) = a(y_a y_a)^{-\frac{1}{2}}\frac{c_2(y_0 + \sqrt{y_a y_a})^{\lambda a/2} - (y_0 - \sqrt{y_a y_a})^{\alpha a/2}}{c_2(y_0 + \sqrt{y_a y_a})^{\lambda a/2} + (y_0 - \sqrt{y_a y_a})^{\alpha a/2}}.$$

If $\varphi = \varphi(\omega_1)$, then Equation (3.2.8(2)) takes the form

$$\omega_1^2\varphi_{11} + 4\omega_1\varphi_1 - \lambda\omega_1\varphi\varphi_1 + 2\varphi - \lambda\varphi^2 = 0. \tag{3.2.19}$$

With the help of the substitution

$$\varphi = t\psi(t), \qquad t = \frac{1}{\omega_1}, \tag{3.2.20}$$

the ODE (3.2.19) is reduced to

$$\psi_{tt} + \lambda\psi\psi_t = 0. \tag{3.2.21}$$

This latter equation is easily integrated and has the general solution as follows

$$\psi = c_1 \operatorname{th}\left(\frac{\lambda c_1}{2}t + c_2\right), \qquad \psi = c_1 \operatorname{cth}\left(\frac{\lambda c_1}{2}t + c_2\right),$$
$$\psi = -c_1 \tan\left(\frac{\lambda c_1}{2}t + c_2\right), \qquad \psi = \left(\frac{\lambda}{2}t + c_2\right)^{-1},$$

where c_1, c_2 are constants. Using these results and ansatz N2 from (3.2.7), we obtain the following solutions of Equation (3.2.4)

$$u(x) = \frac{c_1}{\alpha_a y_a} \operatorname{th}\left(\frac{\lambda c_1}{2}\frac{y_0}{\alpha_a y_a} + c_2\right), \quad u(x) = \frac{-c_1}{\alpha_a y_a} \tan\left(\frac{\lambda c_1}{2}\frac{y_0}{\alpha_a y_a} + c_2\right)$$
$$u(x) = \frac{c_1}{\alpha_a y_a} \operatorname{cth}\left(\frac{\lambda c_1}{2}\frac{y_0}{\alpha_a y_a} + c_2\right), \quad u(x) = \left(\frac{\lambda}{2}y_0 + c_2\alpha_a y_a\right)^{-1}. \tag{3.2.22}$$

If $u(x) = \varphi(\omega_3)h(x)$, $h(x) = (\alpha_a y_a)^{-1}$, ω_3 is from N2 Table 3.2.1 then Equation (3.2.4) is reduced to $\varphi_{\omega_3\omega_3} = 0$, i.e., $\varphi = c_1\omega_3 + c_2$, where c_1, c_2 are constants of integration. The corresponding solution of Equation (3.2.4) is

$$u(x) = c_1\left(\frac{\beta_a y_a}{(\alpha_a y_a)^2} - b_3 \ln\left|\frac{\beta_a y_a}{\alpha_a y_a}\right|\right) + c_2(\alpha_a y_a)^{-1}. \tag{3.2.23}$$

If $\varphi = \varphi(\omega_1)$, then Equation (3.2.8(3)) is reduced to the ODE

$$(\omega_1^2 + \alpha^2)\varphi_{11} + 4\omega_1\varphi_1 - \lambda\omega_1\varphi\varphi_1 + 2\varphi - \lambda\varphi^2 = 0. \tag{3.2.24}$$

Note, that Equation (3.2.24) coincides with Equation (3.2.9) to within sign, so taking into account (3.2.24) we can present at once its general solution

$$\varphi = c_1\left[a\omega_1 \operatorname{th}\left(\frac{a}{\alpha}\arctan\frac{\omega_1}{\alpha} + c_1\right) - \alpha^2\right]^{-1}, \quad \alpha^2 + \frac{\lambda c_1}{2} = -a^2 < 0;$$
$$\varphi = c_1\left[a\omega_1 \left(\arctan\frac{\omega_1}{\alpha} + c_2\right)^{-1} - \alpha^2\right]^{-1}, \quad \alpha^2 + \frac{\lambda c_1}{2} = 0; \tag{3.2.25}$$
$$\varphi = c_1\left[a\omega_1 \tan\left(-\frac{a}{\alpha}\arctan\frac{\omega_1}{\alpha} + c_2\right) - \alpha^2\right]^{-1}, \quad \alpha^2 + \frac{\lambda c_1}{2} = a^2 > 0;$$

$$\varphi = \left[c_2 \frac{\omega_1^2 + \alpha^2}{\omega_1} - \frac{\lambda}{4\alpha} \frac{\omega_1^2 + \alpha^2}{\omega_1} \arctan \frac{\omega_1}{\alpha} + \frac{\lambda}{4}\right]^{-1}, \quad c_1 = 0;$$

where c_1, c_2 are arbitrary constants.

Formulae (3.2.25) and ansatz N3 (3.2.7) give the following solutions of Equation (3.2.4)

$$\begin{aligned}
u(x) &= c_1 \left[a\alpha_a y_a \operatorname{th}\left(\frac{a}{\alpha} \arctan \frac{\alpha_a y_a}{\alpha y_0} + c_2\right) - \alpha^2\right]^{-1}, \\
u(x) &= c_1 \left[a\alpha_a y_a \tan\left(-\frac{a}{\alpha} \arctan \frac{\alpha_a y_a}{\alpha y_0} + c_2\right) - \alpha^2 y_0\right]^{-1}, \\
u(x) &= c_1 \left[\alpha\alpha_a y_a \left(\arctan \frac{\alpha_a y_a}{\alpha y_0} + c_2\right)^{-1} - \alpha^2 y_0\right]^{-1}, \\
u(x) &= \frac{\alpha_a y_a}{(\alpha_a y_a)^2 + \alpha^2 y_0^2} \left(\frac{\lambda}{\alpha} y_0 \frac{\alpha_a y_a}{(\alpha_a y_a)^2 + \alpha^2 y_0^2} - \frac{\lambda}{4\alpha} \arctan \frac{\alpha_a y_a}{\alpha y_0} + c_2\right)^{-1},
\end{aligned} \tag{3.2.26}$$

If $\varphi = \varphi(\omega_3)$, then Equation (3.2.8(3)) is reduced to the ODE

$$b_3^2 \omega_3^2 \varphi_{33} + b_3(b_3 + 3)\varphi_3 - \lambda b_3 \omega_3 \varphi \varphi_3 + 2\varphi - \lambda \varphi^2 = 0$$

which, in turn, by substitution

$$\omega_3 = z^{b_3} \tag{3.2.27}$$

is reduced to Equation (3.2.19). So we can use formulae (3.2.20), (3.2.19), (3.2.27) and then ansatz N3 (3.2.7) to obtain solutions of Equation (3.2.4)

$$\begin{aligned}
u(x) &= \frac{c_1}{(\beta_a y_a)^{1/b_3}} \operatorname{th}\left(\frac{\lambda c_1}{2} \frac{y_0}{(\beta_a y_a)^{1/b_3}} + c_2\right), \\
u(x) &= \frac{c_1}{(\beta_a y_a)^{1/b_3}} \operatorname{cth}\left(\frac{\lambda c_1}{2} \frac{y_0}{(\beta_a y_a)^{1/b_3}} + c_2\right), \\
u(x) &= \frac{-c_1}{(\beta_a y_a)^{1/b_3}} \tan\left(\frac{\lambda c_1}{2} \frac{y_0}{(\beta_a y_a)^{1/b_3}} + c_2\right), \\
u(x) &= \left(c_2 (\beta_a y_a)^{1/b_3} + \frac{\lambda}{y_0}\right)^{-1}.
\end{aligned} \tag{3.2.28}$$

If $\varphi = \varphi(\omega_3)$, then Equation (3.2.8(4)) is reduced to the ODE

$$b_4^2 \varphi_{33} + 3b_4 \varphi_3 - \lambda b_4 \varphi \varphi_3 + 2\varphi - \lambda \varphi^2 = 0,$$

which via the substitution

$$w_3 = \ln t \tag{3.2.29}$$

is reduced to the Equation (3.2.21). Using the general solution of Equation (3.2.21) and formulae (3.2.29) and ansatz N4 from (3.2.7) we get solutions of Equation (3.2.4)

$$\begin{gathered}
u(x) = c_1 \exp\left\{\frac{-\delta_a x_a}{b_4}\right\} \text{ th } \left(\frac{\lambda c_1}{2} y_0 \exp\left\{\frac{-\delta_a x_a}{b_4}\right\} + c_2\right), \\
u(x) = c_1 \exp\left\{\frac{-\delta_a x_a}{b_4}\right\} \text{ cth } \left(\frac{\lambda c_1}{2} y_0 \exp\left\{\frac{-\delta_a x_a}{b_4}\right\} + c_2\right), \\
u(x) = -c_1 \exp\left\{\frac{-\delta_a x_a}{b_4}\right\} \tan \left(\frac{\lambda c_1}{2} y_0 \exp\left\{\frac{-\delta_a x_a}{b_4}\right\} + c_2\right), \qquad (3.2.30) \\
u(x) = \left(c_2 \exp\left\{\frac{\delta_a x_a}{b_4}\right\} + \frac{\lambda}{2} y_0\right)^{-1}.
\end{gathered}$$

Below we list some more particular solutions of Equation (3.2.4) which are succeeded in finding solutions of the rest of Equations (3.2.8):

$$\begin{gathered}
u(x) = c_1 \text{ th } \left(\frac{\mu c_1}{2} \beta_\nu x^\nu + c_2\right) \\
u(x) = c_1 \text{ cth } \left(\frac{\mu c_1}{2} \beta_\nu x^\nu + c_2\right) \qquad (3.2.31) \\
u(x) = -c_1 \tan \left(\frac{\mu c_1}{2} \beta_\nu x^\nu + c_2\right) \\
u(x) = \left(\frac{\mu c_1}{2} \beta_\nu x^\nu + c_2\right)^{-1}
\end{gathered}$$

where $c_1,\ c_2$ are arbitrary constants $\mu = \dfrac{\lambda b_0}{b_5}$, $\beta_\nu \beta^\nu \neq 0$. ((3.2.7) N5)

$$\begin{gathered}
u(x) = c_1 \text{ th } \left[\frac{\lambda c_1}{2}(x_0 + b_6 \ln \sigma_a z_a) + c_2\right], \\
u(x) = c_1 \text{ cth } \left[\frac{\lambda c_1}{2}(x_0 + b_6 \ln \sigma_a z_a) + c_2\right], \\
u(x) = -c_1 \tan \left[\frac{\lambda c_1}{2}(x_0 + b_6 \ln \sigma_a z_a) + c_2\right], \qquad (3.2.32) \\
u(x) = \left[\frac{\lambda}{2}(x_0 + b_6 \ln \sigma_a z_a) + c_2\right]^{-1},
\end{gathered}$$

where $c_1,\ c_2$ are arbitrary constants, $\sigma_a^2 = 0$ ((3.2.7) N6).

Note that functions

$$u(x) = F(\beta_a y_a)\,\mathrm{th}\left(\frac{\lambda y_0}{2}F(\beta_a y_a) + c_2\right),$$

$$u(x) = F(\beta_a y_a)\,\mathrm{cth}\left(\frac{\lambda y_0}{2}F(\beta_a y_a) + c_2\right), \tag{3.2.33}$$

$$u(x) = -F(\beta_a y_a)\tan\left(\frac{\lambda y_0}{2}F(\beta_a y_a) + c_2\right),$$

$$u(x) = \left(F(\beta_a y_a) + \frac{\lambda y_0}{2}\right),$$

which are generalizations of solutions (3.2.28), (3.2.30) also satisfy Equation (3.2.4).

One more generalization. If in (3.2.32) $w_3 = \sigma_a z_a$ (see N6, Table 3.2.1) change into arbitrary differentiable function $G(\sigma_a z_a)$, then we obtain from (3.2.32) new family of solutions of Equation (3.2.4):

$$u(x) = c_1\,\mathrm{th}\left[\frac{\lambda c_1}{2}(x_0 + G(\sigma_a z_a)) + c_2\right],$$

$$u(x) = c_1\,\mathrm{cth}\left[\frac{\lambda c_1}{2}(x_0 + G(\sigma_a z_a)) + c_2\right],$$

$$u(x) = -c_1\tan\left[\frac{\lambda c_1}{2}(x_0 + G(\sigma_a z_a)) + c_2\right], \tag{3.2.34}$$

$$u(x) = \left[\frac{\lambda}{2}(x_0 + G(\sigma_a z_a)) + c_2\right]^{-1},$$

where $\sigma_a\sigma_a = 0$.

If $\varphi_{\omega_3} = 0$, then (3.2.8(5)) is reduced to PDE

$$\beta\varphi_{11} + \omega_3\varphi_{22} - 4\varphi_2 + \lambda\beta_0\varphi\varphi_1 = 0. \tag{3.2.35}$$

Using the substitution

$$y_0 = \omega_1, \qquad y_1 = 2\sqrt{-\beta\omega_2} \tag{3.2.36}$$

we transform Equation (3.2.35) to the canonical form

$$\varphi_{00} - \varphi_{11} - \frac{\varphi_1}{y_1} + \frac{\lambda\beta_0}{\beta}\varphi\varphi_0 = 0, \tag{3.2.37}$$

which is called the nonlinear Darboux equation.

One can prove with the help of Lie method the following statement.

Theorem 3.2.1. *The maximal local invariance group of nonlinear Darboux equation (3.2.37) is 2-parameter Lie group of scale and translation transformations. The basis elements of corresponding Lie algebra are*

$$P_0 = i\partial_0, \qquad D = i(y_0\partial_0 + y_1\partial_1 + i).$$

We seek solutions of Equation (3.2.37) in the form

$$\varphi = \Phi(w_1)g(y), \tag{3.2.38}$$

where $w_1 = \dfrac{z_1^2}{z_0}$, $g(y) = z_0^{-1}$, $z_0 = y_0 + b_0$, $z_1 = y_1$, $b_0 = \text{const}$, Inserting ansatz (3.2.38) into (3.2.37), we obtain ODE, which is reduced after first integration to Riccati one

$$4w_1(w_1 - 1)\Phi_1 + 2w_1\Phi - \frac{\lambda\beta_0}{\beta}w_1\Phi^2 = c, \qquad c = \text{const} \tag{3.2.39}$$

A particular solution of this equation is easily obtained when $c = 0$:

$$\Phi = \left[c_1\sqrt{|w_1 - 1|} + \frac{\lambda\beta_0}{\beta}\right]^{-1}.$$

Corresponding solution of Equation (3.2.35) is constructed by (3.2.38) and it has the form

$$\varphi = \left[c_1\sqrt{|y_0^2 - (y_0 + b_0)^2|} + \frac{\lambda\beta_0}{\beta}(y_0 + b_0)\right]^{-1}, \tag{3.2.40}$$

Using formulae (3.2.40), (3.2.36) and ansatz N5 from (3.2.7)) we find solution of Equation (3.2.4)

$$u(x) = \left[c_1\sqrt{4\beta\left(\frac{(\alpha_a y_a)^2}{-\alpha^2} + y_a y_a\right)^2 + (\beta_\nu x^\nu + b_0)^2 + \frac{\lambda\beta_0}{\beta}(\beta_\nu x^\nu + b_0)}\right]^{-1}. \tag{3.2.41}$$

In conclusion of this point it is worth while to note that above obtained solutions (3.2.15), (3.2.18), (3.2.22), (3.2.28), (3.2.30)–(3.2.34), (3.2.41) of Equation (3.2.4) one can easily generalize to the case of arbitrary number of independent variables.

3. Let us consider equation

$$\Box u + \lambda e^u u_0 = 0. \tag{3.2.42}$$

Since invariance algebra of Equation (3.2.42) is given by operators (3.1.2), 2° (3.1.8), its solutions we seek in the form [91]

$$u(x) = \begin{cases} \varphi(\omega) - \ln y_0, & \text{for } \omega \text{ N1--4} \\ \varphi(\omega), & \text{for } \omega \text{ N5--10} \end{cases} \tag{3.2.43}$$

Ansatz (3.2.43) is obtained from (3.2.5), (3.2.7) with the help of transformation (1.6.4).

Below we present the first six reduced equations which are obtained as a result of substitution of ansatz (3.2.43) into Equation (3.2.42)

$$\begin{aligned}
&(1)\quad (w_1^2 - \alpha^2)\varphi_{11} + 4w_2(w_2 - 1)\varphi_{22} + w_3^2\left(1 - \frac{\alpha^2 b_1^2}{\alpha^2 w_2 - \omega_1^2}\right)\varphi_{33} + \\
&\quad + 4\omega_1(\omega_2 - 1)\varphi_{12} - 4\omega_2\omega_3\varphi_{23} + 6(\omega_2 - 1)\varphi_2 + 2\omega_1\varphi_1 - \\
&\quad - \frac{\alpha^2 b_1^2 \omega_3}{\alpha^2\omega_2 - \omega_1^2}\varphi_3 + \lambda(-\omega_1\varphi_1 + 2\omega_2\varphi_2 + \omega_3\varphi_3 - 1)e^{\varphi} + 1 = 0; \\
&(2)\quad \omega_1^2\varphi_{11} + 4\omega_2(\omega_2 - 1)\varphi_{22} + \frac{b_2}{\omega_1^2}\varphi_{33} + 4\omega_1(\omega_2 - 1)\varphi_{12} - 4b_2\varphi_{23} + \\
&\quad + 2\omega_1\varphi_1 + 6(\omega_2 - 1)\varphi_2 - \lambda(\omega_1\varphi_1 + 2\omega_2\varphi_2 + 1)e^{\varphi} + 1 = 0; \\
&(3)\quad (\omega_1^2 + \alpha^2)\varphi_{11} + 4\omega_2(w_2 - 1)\varphi_{22} + b_3^2\omega_3^2\varphi_{33} + 4\omega_1(\omega_2 - 1)\varphi_{12} + \\
&\quad + 2b_3\omega_1\omega_3\varphi_{13} + 4\omega_3(b_3 w_2 - 1)\varphi_{23} + 2\omega_1\varphi_1 + 6(\omega_2 - 1)\varphi_2 + \\
&\quad + b_3(b_3 + 1)\omega_3\varphi_3 - \lambda(\omega_1\varphi_1 + 2\omega_2\varphi_2 + b_3\omega_3\varphi_3 + 1)e^{\varphi} + 1 = 0; \qquad (3.2.44) \\
&(4)\quad (\omega_1^2 + \alpha^2)\varphi_{11} + 4\omega_2^2\varphi_{22} + b_4^2\varphi_{33} + 4\omega_1\omega_2\varphi_{12} + 2b_4\omega_1\varphi_{13} + 4b_4\omega_2\varphi_{23} + \\
&\quad + 2\omega_1\varphi_1 + 6\omega_2\varphi_2 + b_4\varphi_3 - \lambda(\omega_1\varphi_1 + 2\omega_2\varphi_2 + b_4\varphi_3 + 1)e^{\varphi} + 1 = 0; \\
&(5)\quad \beta\varphi_{11} + \omega_2\varphi_{22} + \left(1 - \frac{b_5}{\omega_2}\right)\varphi_{33} + 2\beta_0\varphi_{13} - 4\varphi_2 + \lambda(\beta_0\varphi_1 + \varphi_3)e^{\varphi} = 0; \\
&(6)\quad \beta\varphi_{11} + \omega_2\varphi_{22} + 2\beta_0\varphi_{13} + \varphi_{33} - 4b_6\varphi_{23} - 4\varphi_2 + \lambda(\beta_0\varphi_1 + \varphi_3)e^{\varphi} = 0.
\end{aligned}$$

Now we put $\varphi = \varphi(\omega_1)$, then (3.2.44(1)) is reduced to the ODE

$$(\omega_1^2 - \alpha^2)\varphi_{11} - 2\omega_1\varphi_1 - \lambda(\omega_1\varphi_1 + 1)e^{\varphi} + 1 = 0, \tag{3.2.45}$$

which after first integration takes the form

$$(\omega_1^2 - \alpha^2)\varphi_1 - \lambda\omega_1 e^{\varphi} = -\omega_1 + c_1, \qquad c_1 = \text{const}$$

This latter equation is reduced by the substitution

$$\varphi = \ln v \tag{3.2.46}$$

to the Bernoulli equation

$$(\omega_1^2 - \alpha^2)v' + (\omega_1 - c_1)v - \lambda\omega_1 v^2 = 0,$$

which is easily integrated and has the general solution as follows

$$v = \left[(\omega_1 - \alpha)^{\frac{\alpha - c_1}{2\alpha}}(\omega_1 + \alpha)^{\frac{\alpha + c_1}{2\alpha}} \int \frac{-\lambda\omega_1 d\omega_1}{(w_1 - \alpha)^{(3\alpha - c_2)/(2\alpha)}(\omega_1 + \alpha)^{(3\alpha + c_2)/(2\alpha)}}\right]^{-1} \tag{3.2.47}$$

We get from (3.2.47), (3.2.46) two families of solutions (depending upon constant c_1) of Equation (3.2.45)

$$\begin{aligned} \varphi &= -\ln\left(\frac{\lambda}{4\alpha}(\omega_1 \pm \alpha)\ln \mid c_2\frac{\omega_1 + \alpha}{\omega_1 - \alpha} \mid + \frac{\lambda}{2}\right), \qquad c_1 = \pm\alpha; \\ \varphi &= -\ln\left(c_2\sqrt{\omega_1^2 - \alpha^2} + \lambda\right), \qquad c_1 = 0. \end{aligned} \tag{3.2.48}$$

In turn, formulae (3.2.48), (3.2.43) give solutions of Equation (3.2.42)

$$\begin{aligned} u(x) &= -\ln\left(\frac{\lambda}{4\alpha}(\alpha_a y_a \pm \alpha y_a)\ln\left| c_2\frac{\alpha_a y_a + \alpha y_a}{\alpha_a y_a - \alpha y_0}\right| + \frac{\lambda}{2}\right), \\ u(x) &= -\ln\left(c_2\sqrt{(\alpha_a y_a)^2 - \alpha^2 y_0^2} + \lambda\right), \qquad c_1 = 0. \end{aligned} \tag{3.2.49}$$

Below we present some more solutions of Equation (3.2.42), which were obtained in much the same way when considering the rest of reduced Equations (2)–(6) from (3.2.44)

$$\begin{aligned} u(x) &= -\ln\left[\frac{1}{c_1}(\beta_a y_a)^{1/b_3}(c_2\exp\left\{\frac{c_1 y_0}{(\beta_a y_a)^{1/b_3}}\right\} - \lambda\right], \quad \beta_a\beta_a = 0; \\ u(x) &= -\ln\left[c_2\sqrt{(\alpha_a y_a)^2 + \alpha^2 y_0^2}\,\exp\left\{-\frac{c_1}{\alpha}\arctan\frac{\alpha_a y_a}{a y_0}\right\} - \right. \\ &\qquad \left. - \frac{\lambda c_1}{c_1^2 + \alpha^2}\left(\alpha_a y_a - \frac{\alpha^2}{c_1}y_0\right)\right], \qquad \alpha^2 < 0; \\ u(x) &= -\ln\left[c_2\exp\{-c_1\beta_\nu x^\nu\} + \frac{\lambda\beta_0}{\beta c_1}\right], \qquad \beta = \beta_\nu\beta^\nu \neq 0; \\ u(x) &= -\ln\left[\frac{1}{c_1}\exp\left\{\frac{\delta_a x_a}{b_4}\right\}\left(c_2\exp\left\{\frac{c_1 b_4 y_0}{\delta_a x_a}\right\} - \lambda\right)\right], \qquad \delta_a\delta_a = 0; \\ u(x) &= -\ln\left[c_2(\sigma_a z_a)^{-c_1 b_6}\exp\{-c_1 x_0\} + \frac{\lambda}{c_1}\right], \qquad \sigma_a\sigma_a = 0; \end{aligned} \tag{3.2.50}$$

$$u(x) = -\ln\left[F(\beta_a y_a)(c_2 \exp\left\{\frac{y_0}{F(\beta_a y_a)}\right\} - \lambda)\right], \qquad \beta_a\beta_a = 0;$$

$$u(x) = -\ln\left[c_2\left(G(\sigma_a z_a)e^{x_0}\right)^{-c_1} + \frac{\lambda}{c_1}\right], \qquad \sigma_a\sigma_a = 0.$$

where F, G are arbitrary differentiable functions, β_a, α_a, δ_a, σ_a, c_1, c_2 are constants, satisfying conditions stated above.

4. Let us consider once more $\widetilde{\mathrm{E}}(1,3)$-invariant nonlinear PDEs

$$\Box u + \lambda\sqrt{u_a u_a}\, u_0 = 0. \tag{3.2.51}$$

As for as invariance algebra of Equation (3.2.51) is given by operators (3.1.3), we seek its solutions in the form [91]

$$u(x) = \varphi(\omega) \tag{3.2.52}$$

where new variables are written in Table 3.2.1.

Omitting cumbersome calculations connected with substitution of ansatze (3.2.52) into Equation (3.2.51), we present the final result: solutions of Equation (3.2.51)

$$u(x) = \frac{1}{ac_1}\arctan\frac{\alpha_a y_a}{ay_0} + c_2, \quad -\alpha^2 + \frac{\alpha\lambda}{2c_1} = a^2 > 0; \tag{3.2.53}$$

$$u(x) = \frac{1}{2ac_1}\ln\left|c_2\frac{\alpha_a y_a - ay_0}{\alpha_a y_a + ay_0}\right|, \qquad -\alpha^2 + \frac{\lambda\alpha}{2c_1} = -a^2 < 0, \quad \alpha_a^2 = \alpha^2 \neq 0;$$

$$u(x) = \frac{2}{\lambda}\int\frac{dt}{\ln\left((1-t^2)/c_1\right)}, \qquad t = \frac{y_0}{\sqrt{y_a y_a}};$$

$$u(x) = b\ln(\beta_a y_a)^{1/b_3} - \frac{c_1 y_0}{(\beta_a y_a)^{1/b_3}} + c_2, \qquad \beta_a\beta_a = 0;$$

$$u(x) = c_1 y_0 \exp\left\{-\frac{\delta_a x_a}{b_4}\right\}, \qquad \delta_a\delta_a = 0; \tag{3.2.54}$$

$$u(x) = \frac{-\beta^2}{\sqrt{\beta_a\beta_a}\lambda\beta_0}\ln\left|c_1\exp\{\beta_0 x_0\} + c_2\exp\{\beta_a x_a\}\right|, \qquad \beta_\nu\beta^\nu = \beta \neq 0;$$

$$u(x) = F(\beta_a y_a) + G(\beta_a y_a)x_0, \qquad \beta_a\beta_a = 0,$$

where F, G are arbitrary differentiable functions, α_a, β_ν, δ_a are arbitrary real constants satisfying conditions stated.

3.3. *PDEs admitting Galilei algebras*

One of the best known Galilei-invariant equation is the heat (or diffusion) equation

$$u_0 + \lambda \Delta u = 0, \tag{3.3.1}$$

where $u_0 = \dfrac{\partial u}{\partial x_0}$, $\Delta u = \dfrac{\partial^2 u}{\partial x_1^2} + \cdots + \dfrac{\partial^2 u}{\partial x_{n-1}^2}$.

In two-dimensional case the symmetry of Equation (3.3.1) had been studied by S.Lie. This result of Lie is easy generalized on n-dimensional case.

Theorem 3.3.1. *Maximal (in sense of Lie) invariance algebra of the heat equation* (3.3.1) *has the following basis elements:*

$$P_0 = i\partial_0, \qquad P_a = -i\partial_a, \qquad I = u\partial_u,$$

$$J_{ab} = x_a P_b - x_b P_a, \qquad G_a = x_0 P_a + \frac{i}{2\lambda} x_a,$$

$$D = 2x_0 P_0 - x_a P_a,$$

$$\Pi = x_0^2 P_0 - x_0 x_a P_a - \frac{in}{2} x_0 + \frac{i}{4\lambda} \vec{x}^2.$$

To prove this statement one can use Lie's method (see §1.1).

Unfortunately, most of real phenomena of diffusion and heat transmission are not described satisfactorily with the help of the linear Equation (3.3.1). A well-known nonlinear generalization of Equation (3.3.1)

$$u_0 + \vec{\nabla}(F(u)\vec{\nabla} u) = 0 \tag{3.3.2}$$

has an essential shortcoming: it is Galilei invariant iff $F(u) = \text{const}$, so that there is no one nonlinear equation of the form (3.3.2) which would satisfy relativistic principle of Galilei.

Below we describe Galilei-invariant nonlinear generalizations of the heat equation (3.3.1) and Schrödinger equation (3.3.24).

1. Consider equation

$$u_0 + F(x_0, \vec{x}, u, \underset{1}{u}, \underset{2}{u}) = 0, \tag{3.3.3}$$

where $\underset{1}{u} = (u_1, u_2, \ldots, u_{n-1})$, $\underset{2}{u} = (u_{11}, u_{12}, \ldots, u_{n-1\,n-1})$, are first and second derivatives of function u in spacial variables x_a. If function F does not depend on second derivatives $\underset{2}{u}$, so that Equation (3.3.3) is a first-order PDE, then the following statement holds true [181].

Theorem 3.3.2. *If equation*

$$u_0 + F(x_0, \vec{x}, u, \underset{1}{u}) = 0, \tag{3.3.4}$$

is invariant under Galilei algebra AG$(1, n-1)$ *with operators*

$$\begin{aligned} &P_0 = i\partial_0, \qquad P_a = -i\partial_a, \\ &J_{ab} = x_a P_b - x_b P_a, \\ &G_a = x_0 P_a + x_a a(x,u)\partial_u \end{aligned} \tag{3.3.5}$$

so it is locally equivalent to the Hamilton-Jacobi equation

$$u_0 + \frac{1}{2m}\left(\vec{\nabla} u\right)^2 = 0. \tag{3.3.6}$$

Proof. Coordinates ξ^0, ξ^a, η of infinitesimal operator

$$X = \xi^0\partial_0 + \xi^a\partial_a + \eta\partial_u$$

of AG$(1, n-1)$ (3.3.5) have the form

$$\begin{aligned} &\xi^0 = d_0, \qquad \xi^a = g_a x_0 + c_{ab}x_b + d_a, \\ &\eta = a(x,u)g_a x_a + c(x,u), \qquad a, b = \overline{1, n-1}, \end{aligned} \tag{3.3.7}$$

where $d_0, d_a, c_{ab} = -c_{ba}, g_a$ are parameters.

To meet the demands of translation and rotation invariance of Equation (3.3.4), it is obvious that function F should be

$$F(x_0, \vec{x}, u, \underset{1}{u}) = \Phi(u, (\vec{\nabla} u)^2). \tag{3.3.8}$$

Further, from invariance with respect to operator G_a we obtain

$$\begin{aligned} &\eta_0 = 0 \qquad 2\eta_a\Phi_2 = g_a, \\ &2\eta_u W_2\Phi_2 + \eta\Phi_1 - \eta_u\Phi = 0, \end{aligned} \tag{3.3.9}$$

where $\Phi_1 = \dfrac{\partial\Phi}{\partial W_1}$, $\Phi_2 = \dfrac{\partial\Phi}{\partial W_2}$, $W_1 = u$, $W_2 = (\vec{\nabla} u)^2$.

The general solution of the system (3.3.9) has the form

$$\begin{aligned} &\Phi = A'(u)\frac{1}{2m}(\vec{\nabla} u)^2 + \frac{\lambda_1}{A'(u)}, \\ &\eta = \frac{1}{A'(u)}(m g_a x_a + \lambda_2), \end{aligned}$$

where λ_a, λ_2, m are arbitrary constants, $A(u)$ is an arbitrary differentiable function.

So we see that Equation (3.3.4) to be invariant under AG$(1, n-1)$ (3.3.5) should have the form

$$u_0 + A'(u)\frac{1}{2m}(\vec{\nabla} u)^2 + \frac{\lambda_1}{A'(u)} = 0. \tag{3.3.10}$$

It is easy to show that Equation (3.3.10) is equivalent to the Hamilton-Jacobi equation (3.3.6) (to do it one has to perform the transformation $A(u) + \lambda_1 x_0 \to u$), so the theorem is proved.

Remark 3.3.1. The theorem yields that operators $G_a \subset$ AG$(1, n-1)$ have the form

$$G_a = x_0 P_a - imx_a \partial_u. \tag{3.3.11}$$

Below we use the following notations. Consider matrix A constructed from second derivatives u_{ab} of function u

$$A = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \dots & u_{nn} \end{pmatrix}$$

Let symbol $<K>$ denotes the sum of various minors of order k of main diagonal of the matrix A, so that

$$< 1 > = u_{11} + u_{22} + \cdots + u_{nn},$$

$$< 2 > = \begin{vmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{vmatrix} + \begin{vmatrix} u_{11} & u_{13} \\ u_{31} & u_{33} \end{vmatrix} + \cdots + \begin{vmatrix} u_{n-1\,n-1} & u_{n-1\,n} \\ u_{n\,n-1} & u_{nn} \end{vmatrix},$$

....

$$< n > = \begin{vmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \dots & u_{nn} \end{vmatrix} = \det A$$

Galilei algebra AG$(1, n)$ (3.3.5), (3.3.11) extended by generator of scale transformation

$$D = 2x_0 P_0 - x_a P_a \tag{3.3.12}$$

and generator of projective transformation

$$\Pi = x_0^2 P_0 - x_0 x_a P_a + \frac{im}{2}\vec{x}^2 \partial_u \tag{3.3.13}$$

we shall call extended Galilei algebra (denote $A\tilde{G}(1,n) = \{AG(1,n), D\}$) and Schrödinger algebra (denote $ASch(1,n) = \{A\tilde{G}(1,n), \Pi\}$).

Theorem 3.3.3. *Eqn. (3.3.3) is invariant with respect to algebras* $AG(1,n)$, $A\tilde{G}(1,n)$, $ASch(1,n)$, *iff it has the form*

$$u_0 + \frac{1}{2m}(\vec{\nabla}u)^2 + \Phi(<1>, <2>, \ldots, <n>) = 0, \tag{3.3.14}$$

$$u_0 + \frac{1}{2m}(\vec{\nabla}u)^2 + \Delta u \varphi(W_2, W_3, \ldots, W_n) = 0, \tag{3.3.15}$$

$$u_0 + \frac{1}{2m}(\vec{\nabla}u)^2 + \sqrt{(\Delta u)^2 + \frac{2n}{1-n} <2>}\, \psi(J_3, J_4, \ldots, J_n) = 0, \tag{3.3.16}$$

where Φ, φ, ψ *are arbitrary differentiable functions of corresponding arguments,* $W_k = \dfrac{<k>}{[<1>]^k}$, $k = \overline{2,n}$, $J_k, k = \overline{3,n}$ *are first integrals of the ODE*

$$\frac{dW_2}{2nW_2 - (n-1)} = \frac{dW_3}{3nW_3 - (n-1)W_2} = \cdots = \frac{dW_k}{nkW_k - (n-1)W_{k-1}} = \\ = \cdots \frac{dW_n}{n^2 W_n - (n-1)W_{n-1}}$$

Proof. From condition of invariance of Equation (3.3.3) under $AG(1,n)$ it follows that function F should satisfy the following system of PDEs

$$d_0 \frac{\partial F}{\partial x_0} = d_a \frac{\partial F}{\partial x_a} = d_{n+1} \frac{\partial F}{\partial u} = 0, \tag{3.3.17}$$

$$g_a \left(m \frac{\partial F}{\partial u_a} - u_a \right) = 0, \tag{3.3.18}$$

$$c_{ab} u_b \frac{\partial F}{\partial u_a} + (c_{bc} u_{ac} + c_{ac} u_{bc}) \frac{\partial F}{\partial u_{ab}} = 0, \tag{3.3.19}$$

$$æ \left(2u_{ab} \frac{\partial F}{\partial u_{ab}} + u_a \frac{\partial F}{\partial u_a} - 2F \right) = 0, \tag{3.3.20}$$

$$a \delta_{ab} \frac{\partial F}{\partial u_{ab}} = 0, \tag{3.3.21}$$

where $d_0, d_a, d_{n+1}, g_a, æ, c_{ab} = -c_{ba}$ are group parameters, δ_{ab} is the Kronecker symbol.

Function F, as it follows from (3.3.17), does not depend on variables x_0, x_a, u. Equation (3.3.18) yields

$$F = \frac{1}{2m}(\vec{\nabla}u)^2 + \Phi(\underset{2}{u}) \tag{3.3.22}$$

Substituting (3.3.22) into (3.3.18)–(3.3.21) we get

$$
\begin{aligned}
&(c_{ab}u_{ac} + c_{ac}u_{bc})\frac{\partial \Phi}{\partial u_{ab}} = 0, \\
&æ\left(u_{ab}\frac{\partial \Phi}{\partial u_{ab}} - \Phi\right) = 0, \\
&a\delta_{ab}\frac{\partial \Phi}{\partial u_{ab}} = 0.
\end{aligned}
\qquad (3.3.23)
$$

After solving Equations (3.3.23) we obtain formulae (3.3.14)–(3.3.16). The theorem is proved.

Remark 3.3.2. Equations (3.3.14)–(3.3.16) are nonlinear generalizations of Equation (3.3.1). It is obvious to make the transformation

$$u(x) \to \exp\left\{\frac{u}{2\lambda m}\right\}$$

which allows us to pass from Equation (3.3.1) to the equivalent one

$$u_0 + \frac{1}{2m}(\vec{\nabla} u)^2 + \lambda \Delta u = 0.$$

2. The Schrödinger equation for complex function $u(x_0, x)$ has the form

$$\left(P_0 + \frac{\vec{P}^2}{2m}\right) u = 0, \qquad (3.3.24)$$

where $P_0 = i\partial_0$, $\vec{P} = -i\vec{\nabla}$.

It is well-known that maximal Lie-invariant algebra of the Schrödinger equation is ASch$(1, n)$, basis elements having the form

$$
\begin{aligned}
&X_0 = i\partial_0, \qquad X_a = -i\partial_a, \qquad I = i(u\partial_u - u^*\partial_u), \\
&J_{ab} = x_a\partial_b - x_b\partial_a, \qquad G_a = x_0\partial_a - mx_aI \\
&D = 2x_0\partial_0 + x_a\partial_a - \frac{n}{2}(u\partial_u + u^*\partial_{u^*}), \\
&\Pi = x_0^2\partial_0 + x_0x_a\partial_a - \frac{m\vec{x}^2}{2}I - \frac{n}{2}x_0(u\partial_u + u^*\partial_{u^*}).
\end{aligned}
\qquad (3.3.25)
$$

Invariance algebra ASch$(1, 3)$ of three-dimensional Equation (3.3.24) has been established in [157].

Consider a quasilinear generalization of Equation (3.3.24)

$$iu_0 + f^{ab}(x_0, \vec{x}, u, u^*)u_{ab} + F(x_0, \vec{x}, u, u^*, \underset{1}{u}, \underset{1}{u}^*) = 0, \qquad (3.3.26)$$

where u^* is the complex conjugate to u.

The solution of the problem analogous to that considered above is the following statement.

Theorem 3.3.4. *Equation* (3.3.24) *is invariant under the algebras* $AG(1,n)$, $A\tilde{G}(1,n)$, $ASch(1,n)$ *iff it has the form*

$$\left(P_0 + \frac{\vec{p}^2}{2m}\right) u + f(|u|, (\vec{\nabla}|u|)^2) u = 0, \tag{3.3.27}$$

$$\left(P_0 + \frac{\vec{p}^2}{2m}\right) u + |u|^{-2/k} F_1 \left[|u|^{-2+2/k} (\vec{\nabla}|u|)^2\right] u = 0, \tag{3.3.28}$$

$$\left(P_0 + \frac{\vec{p}^2}{2m}\right) u + \left(\vec{\nabla}|u|\right)^2 F_2(|u|) u = 0, \tag{3.3.29}$$

$$\left(P_0 + \frac{\vec{p}^2}{2m}\right) u + |u|^{4/n} \varphi\left(|u|^{-2-4/n} \left(\vec{\nabla}|u|\right)^2\right) u = 0, \tag{3.3.30}$$

where f, F_1, F_2, φ *are arbitrary differentiable functions of corresponding arguments.*

One can obtain the proof of the theorem by proceeding in much the same way as in previous case (see Theorem 3.3.3).

Theorem 3.3.5. *The nonlinear Schrödinger equation*

$$\left(P_0 + \frac{\vec{p}^2}{2m}\right) u = F(u, u^*) \tag{3.3.31}$$

is invariant under $AG(1,n)$, $A\tilde{G}(1,n)$, $ASch(1,n)$ *if it has the form*

$$\left(P_0 + \frac{\vec{p}^2}{2m}\right) u = F(|u|) u \tag{3.3.32}$$

$$\left(P_0 + \frac{\vec{p}^2}{2m}\right) u = \lambda_1 |u|^k u \tag{3.3.33}$$

$$\left(P_0 + \frac{\vec{p}^2}{2m}\right) u = \lambda |u|^{4/n} u \tag{3.3.34}$$

respectively. The nonlinear equation

$$\left(P_0 + \frac{\vec{p}^2}{2m}\right) u = (\varphi(|u|) + i \ln u)\, u \tag{3.3.35}$$

is invariant under 11-dimensional Lie algebra with basis elements

$$P_0,\ P_a,\ J_{ab},\ G_a = e^{x_0}P_a + mx_aI,\\ I = e^{x_0}\big(u\partial_u - u^*\partial_{u^*}\big) \tag{3.3.36}$$

Theorem 3.3.6 [82*]. *The nonlinear Schrödinger equation for a particle with variable mass*

$$\left(P_0 + \frac{\vec{P}^2}{2M(x_0)}\right)u = \frac{M'}{M}\big(\varphi(x_0,|u|) + i\ln u\big)u \tag{3.3.37}$$

is invariant under the 10-dimensional Lie algebra with basis elements

$$P_a,\ J_{ab},\ G_a = x_0P_a + M(x_0)x_aI,\\ E = M(x_0)I. \tag{3.3.38}$$

3.4. *The Galilean relativistic principle and nonlinear PDEs*

We shall describe, following [71], nonlinear PDEs which are invariant with respect to the Galilean transformations

$$t \to t' = t, \qquad x_a \to x'_a = x_a + v_at \tag{3.4.1}$$

We consider two essentially different representations of Galilean transformations: the so-called projective Galilean transformations (PGT), when dependent variable is also transformed together with independent variables (for example, the heat equation (3.3.1) is invariant under transformations (3.4.1) iff $u(x) \to u'(x') = \exp\{-\frac{1}{2}v_a(x_a + v_at)\}u(x)$), and Galilean transformations (GT), when the dependent variable remains unchanged.

Theorem 3.4.1. *Equation*

$$F(t,x,u,\underset{1}{u},\underset{2}{u}) \equiv -\Delta u + A(t,x,u)u_t + B(t,x,u,\underset{1}{u}) \tag{3.4.2}$$

where $u = u(t,x)$, $\quad x \in R^n \quad \underset{1}{u} = \left\{\dfrac{\partial u}{\partial x_a}\right\}, \quad \underset{2}{u} = \left\{\dfrac{\partial^2 u}{\partial x_a \partial x_b}\right\}; \quad a,b = \overline{1,n};$

A, B are differentiable functions, is invariant under PGT iff

$$A(t,x,u) = f(t,w) \tag{3.4.3}$$

$$B(t,x,u,\underset{1}{u}) = ug(t,w_1,\dots,w_n) + (f(t,w)-\lambda)\left(\frac{x_a u_a}{t} + \frac{\lambda|x|^2}{4t^2}u\right), \quad (3.4.4)$$

where

$$w = u\exp\left\{\frac{\lambda|x|^2}{4t}\right\}, \qquad |x|^2 \equiv x_a x_a, \quad (3.4.5)$$

$$w_a = \left(u_a + \frac{\lambda}{2t}x_a u\right)\exp\left\{\frac{\lambda|x|^2}{4t}\right\} \quad (3.4.6)$$

f, g are arbitrary differentiable functions.

Proof. According to the Lie method, from condition of invariance

$$\underset{2}{X}F\Big|_{F=0} = 0,$$

where operator $\underset{2}{X}$ is constructed by formulae (1.1.7), and

$$\xi^0 = 0, \quad \xi^a = g^a t, \quad \eta = -\tfrac{1}{2}\lambda g_a x_a u$$

we obtain system to define functions A and B

$$t\frac{\partial A}{\partial x_a} - \tfrac{1}{2}\lambda x_a u \partial A u = 0, \quad (3.4.7)$$

$$\frac{2}{\lambda}t\frac{\partial B}{\partial x_a} - x_a u\frac{\partial B}{\partial u} - u\frac{\partial B}{\partial u_a} - x_a u_b\frac{\partial B}{\partial u_b} + x_a B - \frac{2}{\lambda}u_a(A-\lambda) = 0, \quad (3.4.8)$$

Rewriting (3.4.7) in equivalent form

$$\frac{dx_a}{t} = \frac{du}{-\frac{1}{2}\lambda x_a u}$$

we get invariants

$$w_0 = t, \qquad w = u\exp\left\{\frac{\lambda|x|^2}{4t}\right\},$$

which give the general solution of Equation (3.4.7) (see formula (3.4.3)).

Analogously, passing from (3.4.8) to corresponding characteristic system

$$-\frac{dx_a}{(2/\lambda)t} = \frac{du}{x_a u} = \frac{du_a}{u + x_a u_a} = \frac{du_1}{x_a u_1} = \cdots = \frac{du_{a-1}}{x_a u_{a-1}} = \frac{du_{a+1}}{x_a u_{a+1}} = \cdots =$$

$$= \frac{du_n}{x_a u_n} = \frac{dB}{x_a B + (2/\lambda)(\lambda - f(w_0,w))}, \quad a = \overline{1,n} \quad (3.4.9)$$

(there is no sum over repeated indices) we find the invariants

$$\begin{aligned} w &= u \exp\left\{\frac{\lambda|x|^2}{4t}\right\}, \quad w_0 = t \\ w_a &= \left(u_a + \frac{\lambda x_a}{2t} u\right) \exp\left\{\frac{\lambda|x|^2}{4t}\right\}, \\ I &= \left[B + (\lambda - f(w_0, w))\left(\frac{x_a u_a}{t} + \frac{\lambda|x|^2}{4t^2} u\right)\right] \exp\left\{\frac{\lambda|x|^2}{4t}\right\} \end{aligned} \tag{3.4.10}$$

and then, from the functional equation

$$\phi(w, w_0, w_1, \ldots, w_n, I) = 0$$

function B as in (3.4.4). The theorem is proved.

Consequence 3.4.1. Suppose the coefficient B in (3.4.2) to be independent on $\underset{1}{\psi}$, then equation

$$\Delta u = \lambda u_0 + u g(w, t) \tag{3.4.11}$$

is the most general one, invariant under PGT, g being arbitrary differentiable function.

Theorem 3.4.2. *Equation* (3.4.2) *is invariant under the algebra of operators*

$$\begin{aligned} G_a &= t\partial_a - \tfrac{1}{2}\lambda x_a u \partial_u, \qquad \left(\partial_a \equiv \frac{\partial}{\partial x_a}, \quad \partial_u \equiv \frac{\partial}{\partial u}\right) \\ J_{ab} &= x_a \partial_b - x_b \partial_a \end{aligned} \tag{3.4.12}$$

iff it has the form

$$\Delta u = f(w, t) u_t + u g(w, w_a w_a, t) + (f(w, t) - \lambda)\left(\frac{x_a x_a}{t} + \frac{\lambda|x|^2}{4t} u\right), \tag{3.4.13}$$

where

$$w_a w_a = \left[u_a u_a + \lambda x_a u_a \frac{u}{t} + \left(\frac{\lambda|x|u}{2t}\right)^2\right] \exp\left\{\frac{\lambda|x|^2}{2t}\right\}.$$

This theorem is proved in the same way as the first one.

It should be noted that Equation (3.4.11) can be obtained from (3.4.13) when function B in (3.4.2) is independent of $\underset{1}{u}$. Invariance under PGT automatically implies invariance under rotation group.

The futher restriction of the class of Equations (3.4.11) is achieved by the requirement for the equations to be invariant under the operator of scale transformations

$$D = 2t\partial_t + x_a \partial_a + k u \partial_u, \qquad k = \text{const} \tag{3.4.14}$$

and under the projective operator

$$\Pi = t^2\partial_t + tx_a\partial_a - \left(\tfrac{1}{4}\lambda|x|^2 + \tfrac{1}{2}nt\right) u\partial_u, \tag{3.4.15}$$

The following two theorems are proved in much the same way as previous ones.

Theorem 3.4.3. *Among Equations* (3.4.11) *the only one*

$$\Delta u = \lambda u_t + \frac{u}{t^2} g\left(t^{n/2}w\right) \tag{3.4.16}$$

admits the operator Π (3.4.15), g *being an arbitrary differentiable function.*

Theorem 3.4.4. *Among Equations* (3.4.11) *the only one*

$$\Delta u = \lambda u_t + \lambda_1 u t^{-2}\left(\frac{u}{\epsilon(t,x)}\right)^{\beta}, \qquad t^{n/2}w = \frac{u}{\epsilon}\cdot \text{const} \tag{3.4.17}$$

admits operators D (3.4.14) *with* $k = 2/\beta - n$ *and* Π (3.4.15),

$$\epsilon = \epsilon(t,x) = \left(\frac{\lambda}{\sqrt{2\pi t}}\right)^{n/2} \exp\left\{-\frac{\lambda|x|^2}{4t}\right\} \tag{3.4.18}$$

being the fundamental solution of the heat equation

$$\Delta v = \lambda v_t. \tag{3.4.19}$$

Note 3.4.1. Setting $\beta = 0$ in (3.4.17), we obtain equation

$$\Delta u = \lambda u_t + \lambda_1 t^{-2} u$$

which is reduced to (3.4.19) by means of local substitution

$$u = v(t,x)\exp\left\{\frac{\lambda_1}{\lambda t}\right\}, \qquad \lambda \neq 0.$$

Note 3.4.2. All equations considered above contain (explicitly or implicitly) the fundamental solution (3.4.18) of the heat equation (3.4.19). This is apparently because $\epsilon(t,x)$ is the general solution of the system

$$\begin{aligned} &\Delta u = \lambda u_t, \\ &G_a u \equiv t u_a + \tfrac{1}{2}\lambda x_a u = 0, \qquad a = \overline{1,n} \end{aligned} \tag{3.4.20}$$

2. Let us consider equations of the form

$$u_t = C(t, x, u)\Delta u + K\left(t, x, u, \underset{1}{u}\right) \tag{3.4.21}$$

where C and K are differentiable functions, and find out conditions which ensure invariance of (3.4.21) under GT generated by operators

$$\widetilde{G}_a = t\partial_a \tag{3.4.22}$$

Theorem 3.4.5. *Equation* (3.4.21) *admits operators* (3.4.22) *iff*

$$\begin{aligned} C(t, x, u) &= f(t, u) \\ K(t, x, u, \underset{1}{u}) &= g(t, x, u, \underset{1}{u}) - t^{-1}x_a u_a, \end{aligned} \tag{3.4.23}$$

where f and g are arbitrary differentiable functions.

To prove this theorem one has to repeat arguments used in proving previous theorems.

Let us present without proof some more statements dealing with Equations (3.4.21) which admit operators $\widetilde{G}_a$ (3.4.22), J_{ab} (3.4.12) and

$$\widetilde{\Pi} = t^2\partial_t + tx_a\partial_a \tag{3.4.24}$$

$$\widetilde{D} = 2t\partial_t + x_a\partial_a \tag{3.4.25}$$

Theorem 3.4.6. *Equations* (3.4.21) *are invariant under the operators* $\widetilde{G}_a$ *and* J_{ab} *iff they have the form*

$$u_t = f(t, u)\Delta u + g(t, u, w_{n+1}) - \frac{x_a u_a}{t}, \tag{3.4.26}$$

where f and g are arbitrary differentiable functions, $w_{n+1} = u_a u_a$.

Theorem 3.4.7. *Equation* (3.4.26) *admits operator* (3.4.24) *iff*

$$f(t, u) = f(u), \qquad g(t, u, w_{n+1}) = t^{-2}g(u, t^2 w_{n+1}) \tag{3.4.27}$$

Theorem 3.4.8. *Equation* (3.4.26) *admits operators* (3.4.24), (3.4.25) *iff it has the form*

$$u_t = f(u)\Delta u + u_a u_a g(u) - \frac{x_a u_a}{t} \tag{3.4.28}$$

Now consider a two-dimensional equation

$$F(x, u, u_0, u_1, u_{00}, u_{11}, u_{01}) = 0. \tag{3.4.29}$$

Theorem 3.4.9. *Amongst the set of Equations* (3.4.29) *only equations given by*

$$F(w^{(1)}, w^{(2)}, u, u_1, u_{11}) = 0 \tag{3.4.30}$$

are invariant under operators of GT (3.4.22).

In (3.4.30) we use the following notations:

$$w^{(1)} = \det \begin{pmatrix} u_0 & u_1 \\ u_{10} & u_{11} \end{pmatrix}, \qquad w^{(2)} = \det \begin{pmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{pmatrix} . \tag{3.4.31}$$

Theorem 3.4.9 can be easy generalized on the case of $(n+1)$-dimensional space.

Theorem 3.4.10. *Equation*

$$F\left(u, \underset{1}{u}, \underset{2}{u}\right) = 0 \tag{3.4.32}$$

is invariant under operators ∂_0, ∂_a *and* (3.4.22) *iff it has the form*

$$F_1\left(w^{(1)}, w^{(2)}, u, \underset{1}{u}, \underset{2}{u}\right) = 0 \tag{3.4.33}$$

where

$$\begin{aligned} w^{(1)} &= \det \begin{pmatrix} u_0 & u_1 & \dots & u_n \\ u_{10} & u_{11} & \dots & u_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n0} & u_{n1} & \dots & u_{nn} \end{pmatrix} \\ w^{(2)} &= \det \begin{pmatrix} u_{00} & u_{01} & \dots & u_{0n} \\ u_{10} & u_{11} & \dots & u_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n0} & u_{n1} & \dots & u_{nn} \end{pmatrix} \end{aligned} \tag{3.4.34}$$

Note 3.4.3. In the specific case when

$$F_1 = w^{(2)} = 0$$

it yields multi-dimensional Monge-Ampere equation, which has been studied in §1.10.

Note 3.4.4. The maximal invariance algebra of equation

$$w^{(1)} - \lambda = 0, \qquad \lambda = \text{const} \tag{3.4.35}$$

is infinite-dimensional and is given by operators

$$
\begin{aligned}
X &= \xi^{\mu}\partial_{\mu} + \eta\partial_{u}, \qquad \mu = \overline{0, n} \\
\xi^{0} &= c_{00}t + d_{0}, \qquad \xi^{a} = c_{ab}x_{b} + f_{a}(t), \\
\eta &= cu + d, \qquad c = (n+1)^{-1}\big(c_{00} + 2(c_{11} + \ldots + c_{nn})\big)
\end{aligned}
\tag{3.4.36}
$$

where c_{00}, c_{ab}, d_0, d are arbitrary constants, and $f_a(t)$ are arbitrary differentiable functions.

Note 3.4.5. It is easy to construct the general solution of the two-dimensional equation

$$
w^{(1)} = \det \begin{pmatrix} u_0 & u_1 \\ u_{01} & u_{11} \end{pmatrix} = 0 \tag{3.4.37}
$$

To do it we represent (3.4.37) as follows

$$
\frac{\partial}{\partial x_1}\left(\frac{u_1}{u_0}\right) = 0,
$$

whence the general solution is obtained

$$
u = F\big(x_1 + G(x_0)\big) \tag{3.4.38}
$$

(F and G are arbitrary differentiable functions). Direct verification shows that

$$
u = F\big(l_a x_a + G(x_0)\big), \qquad a = \overline{1, n}, \qquad l_a = \text{const}
$$

is a particular solution of $(n+1)$-dimensional Equation (3.4.35) with $\lambda = 0$.

Note 3.4.6. Equation

$$
w^{(1)} = \det \begin{pmatrix} u_0 & u_1 & \ldots & u_n \\ u_{01} & u_{11} & \ldots & u_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{0n} & u_{1n} & \ldots & u_{nn} \end{pmatrix} = F(u), \tag{3.4.39}
$$

where $F(u)$ is an arbitrary twice differentiable function, can be reduced to (3.4.35) at $\lambda = 1$ with the help of substitution

$$
u \to v = \int \frac{du}{\big[F(u)\big]^{1/(n+1)}}. \tag{3.4.40}
$$

In conclusion, we note that among Galilei invariant equations (3.4.33) one can distinguish a class of equations

$$
u_0 = \lambda(u, \underset{1}{u})\Delta u + Q(u, \underset{1}{u}) - w^{(3)} \,/\, w^{(2)},
$$

$$
\begin{aligned}
w^{(3)} &= \det \begin{pmatrix} 0 & u_1 & \dots & u_n \\ u_{10} & u_{11} & \dots & u_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n0} & u_{n1} & \dots & u_{nn} \end{pmatrix} \\
w^{(2)} &= \det \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \dots & u_{nn} \end{pmatrix}
\end{aligned} \tag{3.4.41}
$$

λ, Q being arbitrary functions, which are diffusive type nonlinear PDE with a strong nonlinearity. Note that in [112*] another approach is developed to describe diffusive processes with final rate.

3.5 *Reduction and exact solutions of nonlinear Schrödinger equation*

In this paragraph we consider a particular case of Sch(1,3)-invariant nonlinear generalization of Schrödinger equation, namely the equation

$$\left(i\frac{\partial}{\partial x_0} + \frac{\Delta}{2m}\right) u + \lambda|u|^{4/3}u = 0, \qquad |u| = \sqrt{u^* u}. \tag{3.5.1}$$

Solutions of Equation (3.5.1) we seek for in the form

$$u(x) = f(x)\varphi(\omega) \tag{3.5.2}$$

functions $f(x)$ and new variables $\omega = \{w_1(x), \omega_2(x), \omega_3(x)\}$ being determined with the help of method expounded in §1.4. So, it is necessary to make up a linear combination of symmetry operators (they are given in (3.3.25))

$$Q = \sum_{\ell=1}^{13} \theta_\ell Q_\ell = \xi^\mu(x)\partial_\mu + \eta(x)u\partial_u \tag{3.5.3}$$

and then, to find $f(x)$ and $\omega(x)$, to solve system of ODE

$$\frac{dx_0}{\xi^0} = \frac{dx_1}{\xi^1} = \dots = \frac{dx_3}{\xi^3} = \frac{du}{u\eta}. \tag{3.5.4}$$

Without going into details we present in Table 3.5.1 the final result: explicit form of Ansatze (3.5.2).

Table 3.5.1. Ansatze invariant under some subgroups of Sch(1,3)-group.

N	Invariants ω_1, ω_2, ω_3	Ansatze $u(x) = f(x)\varphi(x)$
1.	$\vec{\alpha}\cdot\vec{x} \,/\, \sqrt{1-x_0^2}$, $\vec{x}^2 \,/\, (1-x_0^2)$, $\operatorname{arcth} x_0 + \arctan(\vec{\beta}\cdot\vec{x}/\vec{\gamma}\cdot\vec{x})$	$u = (1-x_0^2)^{-3/4}\cdot\exp\{\frac{1}{2}im\cdot \cdot\left(x_0\vec{x}^2 \,/\, (1-x_0^2)\right)\}\varphi(\omega)$
2.	$\vec{\alpha}\cdot\vec{x}/x_0, \vec{x}^2/x_0^2$, $1/x_0 + \arctan\left(\vec{\beta}\cdot\vec{x} \,/\, \vec{\gamma}\cdot\vec{x}\right)$	$u = x_0^{-3/2}\cdot \cdot\exp\left\{-\frac{1}{2}im\vec{x}^2/x_0\right\}\varphi(\omega)$
3.	$\vec{\alpha}\cdot\vec{x} \,/\, (1+x_0^2)^{1/2}$, $\vec{x}^2/(1+x_0^2)$, $-\arctan x_0 + \arctan\left(\vec{\beta}\cdot\vec{x} \,/\, \vec{\gamma}\cdot\vec{x}\right)$	$u = (1+x_0^2)^{-3/4}\cdot \cdot\exp\left\{-\frac{1}{2}imx_0\vec{x}^2/(1+x_0^2)\right\}\varphi(\omega)$
4.	$\vec{\alpha}\cdot\vec{x} \,/\, \sqrt{x_0}$, $\vec{x}^2/x_0$, $-\ln x_0 + \arctan\vec{\beta}\cdot\vec{x}/\vec{\gamma}\cdot\vec{x}$	$u = x_0^{-3/4}\varphi(\omega)$
5.	$\vec{\alpha}\cdot\vec{x}/\sqrt{x_0}$, $\vec{\beta}\cdot\vec{x}/\sqrt{x_0}$, $\vec{\gamma}\cdot\vec{x}/\sqrt{x_0}$	$u = x_0^{-3/4}\varphi(\omega)$
6.	$\vec{\alpha}\cdot\vec{x}$, $\vec{x}^2$, $-x_0 + \arctan\vec{\beta}\cdot\vec{x}/\vec{\gamma}\cdot\vec{x}$	$u = \varphi(\omega)$
7.	$\vec{\alpha}\cdot\vec{x}$, $\vec{x}^2, x_0$	$u = \varphi(\omega)$
8.	$\vec{\alpha}\cdot\vec{x} + x_0\vec{\beta}\cdot\vec{x}$, $\vec{\alpha}\cdot\vec{x} + x_0\vec{\gamma}\cdot\vec{x}$, x_0	$u = \exp\{-\frac{1}{2}im(\vec{\alpha}\cdot\vec{x})^2/x_0\}\varphi(\omega)$
9.	$\vec{\alpha}\cdot\vec{x}$, $\vec{\beta}\cdot\vec{x}$, x_0	$u = \varphi(\omega)$

Note 3.5.1. Table 3.5.1 does not contain some cumbersome cases.

2. Having substituted Ansatze from Table 3.5.1 into Equation (3.5.1), we get reduced PDEs as follows:

(1) $L_2(\varphi) + 6\varphi_2 - 2im\varphi_3 + m^2\omega_2\varphi - 2\lambda m|\varphi|^{4/3}\varphi = 0$

(2) $L_2(\varphi) + 6\varphi_2 + 2im\varphi_3 - 2\lambda m|\varphi|^{4/3}\varphi = 0$

(3) $L_2(\varphi) + 6\varphi_2 + 2im\varphi_3 - m^2\omega_2\varphi - 2\lambda m|\varphi|^{4/3}\varphi = 0$

(4) $L_2(\varphi) + im\omega_1\varphi_1 + 2(im\omega_2 + 3)\varphi_2 + 2im\varphi_3 + \frac{3}{2}im\varphi - 2\lambda m|\varphi|^{4/3}\varphi = 0$

(5) $\Delta\varphi + im\omega_a\varphi_a + \left(\frac{3}{2}im - 2\lambda m|\varphi|^{4/3}\right)\varphi = 0$

(6) $L_2(\varphi) + 6\varphi_2 + 2im\varphi_3 - 2\lambda m|\varphi|^{4/3}\varphi = 0$

(7) $i\varphi_3 - \dfrac{1}{2m}(\varphi_{11} + 4\omega_1\varphi_{12} + 4\omega_2\varphi_{22}) + \lambda|\varphi|^{4/3}\varphi = 0$

(8) $(\omega_3^2 + 1)(\varphi_{11} + \varphi_{22}) + \varphi_{22} - \dfrac{2im}{\omega_3}(\omega_a\varphi_a + \frac{1}{2}\varphi) - 2\lambda m|\varphi|^{4/3}\varphi = 0$

(9) $i\varphi_3 - \dfrac{1}{2m}(\varphi_{11} + \varphi_{22}) + \lambda|\varphi|^{4/3}\varphi = 0$

where $L_2(\varphi) = \varphi_{11} + 4\omega_2\varphi_{22} + (\omega_2 - \omega_1^2)^{-1}\varphi_{33} + 4\omega_1\varphi_{12}$; $\varphi_a \equiv \dfrac{\partial\varphi}{\partial\omega_a}$

Having solved some equations from (3.5.5), we obtain solutions of Equation (3.5.1):

$$u(x) = x_0^{-3/2} \exp\left\{-\frac{im}{2}\frac{\vec{x}^2}{x_0}\right\}\varphi(\omega_1), \qquad \omega_1 = \frac{\vec{\alpha}\cdot\vec{x}}{x_0} \tag{3.5.6}$$

where function $\varphi(\omega_1)$ is determined from quadrature

$$\int_0^{\varphi} \frac{d\tau}{\sqrt{c_2 + \tau^{10/3}}} = \sqrt{\tfrac{6}{5}\lambda m}\,(\omega_1 + c_1) \tag{3.5.7}$$

c_1 and c_2 are arbitrary constants.

$$u(x) = x_0^{-3/2} \exp\left\{-\frac{im}{2}\frac{\vec{x}^2}{x_0}\right\}\varphi(\omega_2), \qquad \omega_2 = \frac{\vec{x}^2}{x_0} \tag{3.5.8}$$

where real function $\varphi(\omega_2)$ satisfies the Emden-Fauler equation

$$2\omega_2\varphi_{22} + 3\varphi_2 - \lambda m\varphi^{7/3} = 0 \tag{3.5.9}$$

If $\lambda = \frac{3}{2}i$ then

$$u(x) = (1 - x_0^2)^{-3/4} \exp\left\{\frac{im}{2}\frac{\vec{x}^2}{1 - x_0}\right\}.$$

Some more solutions of Equation (3.5.1):

$$u(x) = (\alpha_0 x_0 - \vec{\alpha}\cdot\vec{x})^{-3/2} \exp\left\{-\frac{im}{2}\frac{\vec{x}^2}{x_0}\right\}\varphi(\omega_2), \tag{3.5.10}$$

where α_0, α_a are arbitrary constants, and $\vec{\alpha}^2 = \frac{8}{15}\lambda m$.

$$u(x) = x_0^{-3/4} \exp\left\{-\frac{im}{2}\frac{\vec{x}^2 - \vec{\beta}\cdot\vec{x}}{x_0}\right\}, \tag{3.5.11}$$

where $\vec{\beta}$ are arbitrary constants, $\vec{\beta}^2 = -\frac{8\lambda}{m}$.

$$\begin{aligned} u(x) &= \left(\tfrac{8}{3}\lambda m\vec{x}^2\right)^{-3/4} \exp\left\{-\frac{im}{2}\frac{\vec{x}^2}{x_0}\right\}, \\ u(x) &= \left(\frac{c}{3\lambda}\right)^{3/4} \frac{1}{\sqrt{x_0}} \exp\left\{i\left(c x_0^{-1/3} - \frac{m}{2}\frac{(\vec{\alpha}\cdot\vec{x})^2}{x_0}\right)\right\}, \end{aligned} \tag{3.5.12}$$

where c, $\vec{\alpha}^2$ are arbitrary constants, $\vec{\alpha} = 1$.

$$u(x) = \left(\frac{3i}{4\lambda x_0}\right)^{3/4}, \qquad u(x) = \left(\frac{c}{\lambda}\right)^{3/4} \exp\{icx_0\} \tag{3.5.13}$$

$$u(x) = (\vec{\alpha}\cdot\vec{x})^{-3/2}, \qquad u(x) = \left(\tfrac{8}{3}\lambda m\vec{x}^2\right)^{-3/4}, \qquad u(x) = \exp\{i\vec{\beta}\cdot\vec{x}\} \tag{3.5.14}$$

where $\vec{\alpha}$, $\vec{\beta}$ are arbitrary constants, $\vec{\alpha}^2 = \frac{8}{15}\lambda m$, $\vec{\beta}^2 = -2\lambda m$.

Note 3.5.2. When $\lambda = 0$ (3.5.10), (3.5.11) give the fundamental solution of Schrödinger equation

$$u(x) = x_0^{-3/2} \exp\left\{-\frac{im}{2}\frac{\vec{x}^2}{x_0}\right\} \tag{3.5.15}$$

Now we present some soliton-like solutions of Equation (3.5.1) obtained in [70, 68*, 69*]. Let

$$u(x) = \exp\left\{\frac{ix_0}{\beta}\right\}\varphi(\omega), \qquad \omega = \alpha_a x_a \tag{3.5.16}$$

where $\varphi(\omega)$ is a real function, α_a, β are arbitrary real constants.

Having substituted (3.5.16) into (3.5.1) we get

$$\frac{\vec{\alpha}^2}{2m}\ddot{\varphi} + \frac{1}{\beta}\varphi + \lambda\varphi^{7/3} = 0 \tag{3.5.17}$$

Partial solutions of Equation (3.5.17) we seek for in the form

$$\varphi(\omega) = a(2\operatorname{sh} b\omega)^{k_1}(2\operatorname{ch} b\omega)^{k_2}, \tag{3.5.18}$$

where a, b, k_1, k_2 are constants to be defined. After substituting (3.5.18) into (3.5.17) we obtain

$$\frac{\alpha^2 b^2}{2m}\left[k_1 + k_2 + 2k_1k_2 + k_1(k_2-1)\left(\frac{\operatorname{ch} b\omega}{\operatorname{sh} b\omega}\right)^2 + \right. \tag{3.5.19}$$

$$\left. + k_2(k_2-1)\left(\frac{\operatorname{sh} b\omega}{\operatorname{ch} b\omega}\right)^2\right] + \lambda\left[a(2\operatorname{sh} b\omega)^{k_1}(2\operatorname{ch} b\omega)^{k_2}\right]^{4/3} = 0$$

In (3.5.19) constants a, b, k_1, k_2, can be selected so as to satisfy it identically. Let $k_1 = 0$, $k_2 = -3/2$ then (3.5.19) takes the form

$$\frac{1}{\beta} - \frac{3}{2}\alpha^2 b^2 + \frac{15}{4}\frac{\alpha^2}{2m}b^2\left(\frac{\operatorname{sh} b\omega}{\operatorname{ch} b\omega}\right)^2 + \frac{\lambda a^{4/3}}{4(\operatorname{ch} b\omega)^2} = 0$$

This equality holds true iff

$$a = \pm 2\sqrt{2}\left(\frac{5}{-3\lambda\beta}\right)^{3/4}, \qquad b = \pm\frac{2\sqrt{2m}}{3\sqrt{-\alpha^2\beta}}, \qquad \beta < 0 \quad \lambda > 0$$

It yields the solution of Equation (3.5.17)

$$\varphi(\omega) = a_-(\operatorname{ch} b\omega)^{-3/2}, \qquad a_- = \pm\left(\frac{-5}{3\lambda\beta}\right)^{3/4} \tag{3.5.20}$$

and then, returning to (3.5.16), the following solution of Equation (3.5.1)

$$u(x) = a_- \exp\left\{\frac{ix_0}{\beta}\right\}(\operatorname{ch} b\alpha_a x_a)^{-3/2} \tag{3.5.21}$$

where a_- is defined in (3.5.20).

The solution (3.5.21) is naturally to consider as soliton-like by analogy with that of [209]

$$\psi(t, x_1) = \left(-\frac{\lambda}{4}\right)^{1/4}\frac{\exp\left\{\frac{1}{2}iv\left[x_1 + \left(\frac{\epsilon}{2} + \frac{\lambda^2}{8v}\right)t\right]\right\}}{\operatorname{ch}\sqrt{-\frac{\lambda}{4}}(x_1 + vt)}$$

which is a solution of one-dimensional nonlinear Schrödinger equation

$$i\frac{\partial\Psi}{\partial x_0} + \frac{\partial^2\Psi}{\partial x_1^2} = \lambda(\Psi^*\Psi)\Psi$$

Setting in (3.5.19) $k_1 = -\frac{3}{2}$, $k_2 = 0$ we find

$$a = \pm\left(\frac{20}{3\lambda\beta}\right)^{3/4}, \qquad b = \pm\frac{2}{3}\sqrt{-\frac{\alpha^2\beta}{2m}}, \qquad \beta < 0 \quad \lambda < 0$$

The Corresponding solution of Equation (3.5.1) has the form

$$u(x) = a_+ \exp\left\{\frac{ix_0}{\beta}\right\}(\operatorname{sh} b\vec{\alpha}\cdot\vec{x})^{-3/2}, \qquad a_+ \equiv \pm\left(\frac{5}{3\lambda\beta}\right)^{3/4} \tag{3.5.22}$$

Using formulae of generating solutions (see Table 4.1.2) one can construct other solutions of Equation (3.5.1) from those presented here.

It will be noted that using ansatz (3.5.2) one can determine functions $f(x)$ and new variables $\omega_a(x)$ from splitting conditions (see §1.4). The substitution of (3.5.2) into (3.5.1) gives rise to the following equation

$$i\left(\frac{\partial f}{\partial t}\varphi+f\frac{\partial\omega_a}{\partial t}\frac{\partial\varphi}{\partial\omega_a}\right)+\frac{1}{2m}\left[\varphi\Delta f+2\frac{\partial f}{\partial x_a}\cdot\right. \tag{3.5.23}$$

$$\left.\cdot\frac{\partial\omega_j}{\partial x_a}\frac{\partial\varphi}{\partial\omega_j}+f\left(\frac{\partial\varphi}{\partial\omega_j}\Delta\omega_j+\frac{\partial\omega_j}{\partial x_a}\frac{\partial\omega_k}{\partial x_a}\frac{\partial^2\varphi}{\partial\omega_j\partial\omega_k}\right)\right]+\lambda(f^2\varphi^*\varphi)^{2/3}f\varphi=0,$$

whence follows the splitting conditions

$$\frac{1}{2m}\Delta f+i\frac{\partial f}{\partial t}=f^{7/3}F_0(\omega),$$

$$if\frac{\partial\omega_a}{\partial t}+\frac{1}{m}\left(\frac{\partial f}{\partial x_k}\frac{\partial\omega_a}{\partial x_k}+\tfrac{1}{2}f\Delta\omega_a\right)=f^{7/3}F_a(\omega), \tag{3.5.24}$$

$$\frac{\partial\omega_i}{\partial x_a}\frac{\partial\omega_j}{\partial x_a}=f^{7/3}G_{ij}(\omega),$$

where F_0, F_a, G_{ij} are smooth functions of ω_a. If conditions (3.5.24) are fulfilled, then Equation (3.5.1) (see also (3.5.23)) takes the form

$$\left[F_0(\omega)+\lambda(\varphi^*\varphi)^{2/3}\right]\varphi+F_a(\omega)\frac{\partial\varphi}{\partial\omega_a}+G_{ij}(\omega)\frac{\partial^2\varphi}{\partial\omega_i\partial\omega_j}=0. \tag{3.5.25}$$

Note that system (3.5.24) may have to possess wide symmetry, and it opens new ways of constructing solutions of the initial equation (3.5.1).

In conclusion we present, following [68*], in Table 3.5.2 the complete set of Sch(1,3)-inequivalent ansatze (3.5.2) for a Schrödinger-invariant equation (the basis elements of ASch(1,3) are given in (3.3.25)). In this table α_0, α, β are arbitrary real constants, $t\equiv x_0$. In [68*,69*] ansatze of Table 3.5.2 are used for reduction and finding exact solutions of some nonlinear Schrödinger equation.

As a concluding remark we note that the wave equation $\Box u=0$ is reduced to the heat equation $\varphi_\tau=\varphi_{11}+\varphi_{22}$ by means of the ansatz

$$u=\exp\{-(x_3+t)\}\,\varphi(\tau,x_1,x_2),\qquad \tau=\tfrac{1}{4}(x_3-t),$$

which is invariant under the operator

$$Q=\partial_t+\partial_3-2u\partial_u.$$

Table 3.5.2. Sch(1,3)-inequivalent ansatze for a complex scalar field.

N	Algebra	Invar. var. ω	Ansatze $u(x) = f(x)\varphi(\omega)$
1.	P_1	t, x_2, x_3	$f = 1$
2.	$X_0 + \alpha_0 I$	x_1, x_2, x_3	$f = \exp\{i\alpha_0 t\}$
3.	$J_{12} + \alpha I$	$t, x_1^2 + x_2^2, x_3$	$f = \exp\{i\alpha \arctan(x_2/x_1)\}$
4.	$J_{12} + G_3$	$t, x_1^2 + x_2^2,$ $x_3 - t\arctan(x_2/x_1)$	$f = \exp\{-imx_3^2/2t\}$
5.	$J_{12} - X_0$ $+\alpha_0 I$	$t + \arctan(x_2/x_1),$ $x_1^2 + x_2^2, x_3$	$f = \exp\{-i\alpha_0 t\}$
6.	$G_1 + X_2$	$t, x_1 - tx_2, x_3$	$f = \exp\{-imx_1^2/2t\}$
7.	$G_1 - X_0$	$2x_1 + t^2, x_2, x_3$	$f = \exp\{imt(x_1 + \frac{1}{3}t^2)\}$
8.	$J_{12} + \beta G_3-$ $-X_0$	$t + \arctan(x_2/x_1),$ $x_1^2 + x_2^2, 2x_3 + \beta t^2$	$f = \exp\{i\beta mt(x_3 + \frac{1}{3}\beta t^2)\}$
9.	$D + \alpha I$	$\frac{x_1}{\sqrt{t}}, \frac{x_2}{\sqrt{t}}, \frac{x_3}{\sqrt{t}}$	$f = t^{-3/4 + i\alpha m/2}$
10.	$X_0 + \Pi-$ $-\alpha I$	$\frac{x_1}{\sqrt{1+t^2}}, \frac{x_2}{\sqrt{1+t^2}},$ $\frac{x_3}{\sqrt{1+t^2}}$	$f = (t^2+1)^{-3/4} \exp\Big\{-\frac{1}{2}im\cdot$ $\cdot\Big(\frac{\vec{x}^2 t}{1+t^2} + 2\alpha\arctan t\Big)\Big\}$
11.	$J_{12} + \beta D$ $+\alpha I$	$\ln t + 2\beta\arctan(x_1/x_2),$ $\frac{x_1^2 + x_2^2}{t}, \frac{x_3}{\sqrt{t}}$	$f = t^{-3/4 + i\alpha m/2\beta}$
12.	$X_0 + \Pi-$ $-\beta J_{12} - \alpha I$	$\beta\arctan t - \arctan\left(\frac{x_1}{x_2}\right),$ $\frac{x_1^2 + x_2^2}{t^2+1}, \frac{x_3}{\sqrt{t^2+1}}$	$f = (t^2+1)^{-3/4} \exp\Big\{-\frac{1}{2}im\cdot$ $\cdot\Big(\frac{\vec{x}^2 t}{1+t^2} + 2\alpha\arctan t\Big)\Big\}$
13.	$X_0 + \Pi-$ $-J_{12}-$ $-\beta(G_1 + X_2)$	$\frac{tx_1 + x_2}{t^2+1} + \beta\arctan t,$ $\frac{tx_2 + x_1}{t^2+1}, \frac{x_3}{\sqrt{t^2+1}}$	$f = (t^2+1)^{-3/4} \exp\Big\{-\frac{1}{2}im\cdot$ $\cdot\Big(\frac{\vec{x}^2 t}{1+t^2}+$ $+2\beta\arctan\frac{t(tx_2 - x_1)}{t^2+1}\Big)\Big\}$

3.6 *Symmetry and some exact solutions of the Hamilton-Jacobi equation*

Here we investigate point and contact symmetry and construct exact solutions of Hamilton-Jacobi equation of free particle

$$u_t + \frac{1}{2m}\left(\vec{\nabla}u\right)^2 = 0, \qquad (3.6.1)$$

where function $u = u(t, \vec{x})$ means action, m is a constant (mass of particle).

Theorem 3.6.1. [36] *Maximal local (point) invariance group of the HJ equation* (3.6.1) *is the 21-parameter group, basis elements of corresponding Lie algebra having the form*

$$
\begin{aligned}
&p_0 = \partial_t, \qquad p_a = \partial_a, \qquad p_4 = \partial_u; \qquad a = 1,2,3\\
&J_{ab} = x_a p_b - x_b p_a\\
&G_a^{(1)} = t p_a + m x_a p_a, \qquad D^{(1)} = t p_0 + \tfrac{1}{2} x_a p_a,\\
&\Pi^{(1)} = t^2 p_0 + t x_a p_a + \frac{m}{2}\vec{x}^2 p_4,\\
&G_a^{(2)} = u p_a + m x_a p_0, \qquad D^{(2)} = u p_4 + \tfrac{1}{2} x_a p_a,\\
&\Pi^{(2)} = u^2 p_4 + u x_a p_a + \frac{m}{2}\vec{x}^2 p_0,\\
&K_a = 2x_a\big(D^{(1)} + D^{(2)}\big) + s^2 p_a, \qquad \big(s^2 \equiv \frac{2}{m} tu - \vec{x}^2\big).
\end{aligned}
\tag{3.6.2}
$$

Proof. We shall act in the same spirit as in §1.2 (see Theorem 1.2.1). From condition of invariance (3) we find the following defining system for coordinates of infinitesimal operator (2)

$$
\begin{aligned}
&\xi_u^t = 0, \qquad \xi_a^t = m\xi_u^a; \qquad a, b = 1,2,3\\
&\xi_b^a + \xi_a^b = 0, \qquad a \neq b;\\
&\eta_t = 0, \qquad \eta_a = m\xi_t^a,\\
&\eta_u + \xi_t^t = 2\xi_a^a \qquad \text{(no sum over } a).
\end{aligned}
\tag{3.6.3}
$$

The general solution of Equations (3.6.3) has the form

$$
\begin{aligned}
\xi^t &= 2t\vec{K}\cdot\vec{x} + a_1 t^2 + a_2\frac{m\vec{x}^2}{2} + b_1 t + g_{2a} m x_a + d_0,\\
\xi^a &= 2x_a\vec{K}\cdot\vec{x} + K_a s^2 + (a_1 t + a_2 u)x_a + g_{1a} t +\\
&\qquad + g_{2a} u + \tfrac{1}{2}(b_1 + b_2)x_a + c_{ab}x_b + d_a,\\
\eta &= 2u\vec{K}\cdot\vec{x} + a_2 u^2 + a_1\frac{m\vec{x}^2}{2} + b_2 u + g_{1a} m x_a + d_4,
\end{aligned}
\tag{3.6.4}
$$

K_a, a_1, a_2, b_1, b_2, g_{1a}, g_{2a}, $c_{ab} = -c_{ba}$, $d_0, \ldots, d_4$ are arbitrary constants. Thus follows (3.6.2). The theorem is proved.

Note 3.6.1. Looking closely at (3.6.2) one notices an interesting automorphism of the vector fields given by interchanging t and u. It allows us to pick out from algebra (3.6.2) the following subalgebras:

$$\begin{aligned}
&\mathrm{ASch}(1(t),3) = \left\{p_0,\ p_a,\ p_4,\ J_{ab},\ G_a^{(1)},\ D^{(1)},\ \Pi^{(1)}\right\};\\
&\mathrm{ASch}(1(u),3) = \left\{p_4,\ p_a,\ p_0,\ J_{ab},\ G_a^{(2)},\ D^{(1)},\ \Pi^{(2)}\right\};\\
&\mathrm{AG}(2,3) = \left\{p_0,\ p_4,\ p_a,\ J_{ab},\ G_a^{(1)},\ G_a^{(2)},\ \right\};\\
&\mathrm{A\widetilde{G}}(2,3) = \mathrm{AG}(2,3)\oplus\left\{D^{(1)}, D^{(2)}\right\};\\
&\mathrm{ASch}(2,3) = \{(3.6.2)\}.
\end{aligned}\tag{3.6.5}$$

Theorem 3.6.2. *Algebra* ASch(2,3) (3.6.2) *is locally isomorphic to conformal algebra* AC(1,4).

Proof. Due to symmetrical role of variables t and u in ASch(2,3), pointed out above, we can introduce the variables

$$x^0 = \frac{1}{\sqrt{2}}\left(t + \frac{u}{m}\right), \qquad x^4 = \frac{1}{\sqrt{2}}\left(t - \frac{u}{m}\right) \tag{3.6.6}$$

and the covariant notation $\alpha = 0, 1, \ldots, 4$, with the metric $g_{00} = -g_{11} = \ldots = -g_{44} = 1$; $g_{ab} = 0$ $(\alpha \neq \beta)$. Then upon introducing

$$\begin{aligned}
P_\alpha &= \frac{\partial}{\partial x^\alpha}, \qquad & J_{ab} &= x_\alpha P_\beta - x_\beta P_\alpha\\
D &= x^\alpha P_\alpha, \qquad & K_\alpha &= 2x_\alpha D - x^\beta x_\beta P_\alpha
\end{aligned}\tag{3.6.7}$$

we find that the generators

$$\begin{aligned}
&P_a = P_a, \qquad P_0 = \frac{1}{\sqrt{2}}(P_0 + mP_4)\\
&P_4 = \frac{1}{\sqrt{2}}(P_0 - mP_4)\\
&J_{ab} = J_{ab}, \qquad J_{04} = -D^{(1)} + D^{(2)},\\
&J_{0a} = -\frac{1}{\sqrt{2}}\left(G_a^{(1)} + \frac{1}{m}G_a^{(2)}\right),\\
&J_{4a} = -\frac{1}{\sqrt{2}}\left(G_a^{(1)} - \frac{1}{m}G_a^{(2)}\right),\\
&D = D^{(1)} + D^{(2)}
\end{aligned}\tag{3.6.8}$$

$$K_a = -\mathcal{K}_a,$$

$$K_0 = \sqrt{2}\left(\Pi^{(1)} + \frac{1}{m}\Pi^{(2)}\right),$$

$$K_4 = -\sqrt{2}\left(\Pi^{(1)} - \frac{1}{m}\Pi^{(2)}\right),$$

satisfy the commutation rules of AC(1,4) (1.2.3). The theorem is proved.

Now, to find the group action of Sch(2,3) group, we use the final transformations of C(1,4) group (see §2.3):

1) translations: generated by P_α

$$x'_\alpha = x_\alpha + a\alpha,$$

2) Lorentz transformations: generated by $J_{\alpha\beta}$

$$x'_\alpha = \Lambda^\beta_\alpha x_\beta, \qquad \Lambda^\beta_\alpha \in O(1,4)$$

3) dilatation: generated by D

$$x'_\alpha = e^\theta x_\alpha,$$

4) special conformal transformations: generated by $\mathcal{K}$

$$x'_\alpha = \frac{x_\alpha - c_\alpha x^\beta x_\beta}{1 - 2c^\beta x_\beta + (c^\beta c_\beta)(x^\beta x_\beta)}$$

Writing these transformations in terms of the light-cone variables t and u we obtain the symmetry group of the HJ equation:

$$\begin{aligned}
x'^a &= \Lambda^a_b x^b + \Lambda^a_+ t + \frac{1}{m}\Lambda^a_- u,\\
t' &= \Lambda^+_+ t + \frac{1}{m}\Lambda^+_- u + \Lambda^+_a x^a,\\
u' &= m\Lambda^-_+ t + \Lambda^-_- u + m\Lambda^-_a x^a;\\
x'_a &= x_a + a^a, \qquad t' = t + a^+, \qquad u' = u + a^-\\
x'_a &= e^\theta x_a, \qquad t' = e^\theta t \qquad u' = e^\theta u;\\
x'^a &= \sigma^{-1}\left(x^a - c^a\vec{x}^2 - \frac{2}{m}c^a tu\right),\\
t' &= \sigma^{-1}\left(t - c^+\left(\frac{2}{m}tu - \vec{x}^2\right)\right),\\
u' &= \sigma^{-1}\left(u - c^-\left(2tu - m\vec{x}^2\right)\right),
\end{aligned} \tag{3.6.9}$$

where

$$\sigma = \sigma(t, x^a, u) = 1 - 2c^+t - \frac{2}{m}c^-u + 2\vec{c}\vec{x} + \vec{c}^2\left(\frac{2}{m}tu - \vec{x}^2\right)$$

and the parameters indicated with + and − can easily be expressed in terms of their covariant counterparts.

Now consider contact symmetry of HJ equation.

Theorem 3.6.3. *Maximal invariance algebra of contact symmetry of HJ equation* (3.6.1) *is infinite-dimensional and is given by operators* (1.2.10) *with the characteristic function*

$$W = W(u_t, u_a, w_0, w_a), \qquad w_0 = 2u - x_a u_a, \qquad w_a = tu_a - mx_a \tag{3.6.10}$$

The proof is absolutely analogous to that of Theorem 1.2.2.

Note 3.6.2. Contact transformations of HJ equation described above contain homogeneous ones, so that transformations for which Lie equations have a Hamiltonian structure (see (1.2.13)). For example, when $W = W(w_a)$, we let

$$W = w_a w_a = t^2\left(\vec{\nabla}u\right)^2 - 2mtx_a u_a + m^2\vec{x}^2. \tag{3.6.11}$$

To find final transformations generated by infinitesimal operator (1.2.10) with characteristic function (3.6.11) one has to solve the following set of first-order ODEs (Lee equations, see (1.2.13)):

$$\dot{t}' = 0, \qquad t'(\theta = 0) = t$$

$$\dot{x}'_a = 2t'(mx'_a - t'u'_a), \qquad x'_a(\theta = 0) = x_a$$

$$\dot{u}' = 0, \qquad u'(\theta = 0) = u,$$

$$\dot{u}'_{t'} = 2(t'u'_a u'_a - mx'_a u'_a), \qquad u'_{t'}(\theta = 0) = u_t$$

$$\dot{u}'_a = 2m(mx'_a - t'u'_a), \qquad u'_a(\theta = 0) = u_a$$

where dot means differentiations with respect to parameter θ. After integrating this system we find

$$\begin{aligned} &t' = t, \qquad x'_a = x_a + 2\theta mt(x_a - u_a/m) \\ &u' = u, \qquad u'_a = u_a + 2m\theta(mx_a - tu_a) \\ &u'_t = u_t - 2\theta u_a(mx_a - tu_a) - 2m\theta^2(mx_a - tu_a)^2 \end{aligned} \tag{3.6.12}$$

One can make sure that transformations (3.6.12) leave Equation (3.6.1) invariant. Putting $u_a = mv_a$ and $u_t = -E$, where v_a and E are velocity and energy of particle, we can rewrite (3.6.12) as follows

$$t' = t, \qquad x'_a = x_a + 2\theta m t(x_a - v_a t)$$
$$v'_a = v_a + 2m\theta(x_a - v_a t) \tag{3.6.13}$$
$$E' = E + 2m^2\theta\vec{v}\cdot(\vec{x} - \vec{v}t) + 2m^3\theta^2(\vec{x} - \vec{v}t)^2$$

When $\vec{x} = \vec{v}t + \vec{x}^{(0)}$, $\vec{x}^{(0)}$ is a constant vector (uniform motion), (3.6.13) coincide with Galilean transformations. In other cases transformations (3.6.13) describe passing to uniformly accelerated frame of reference.

Before to construct solutions of the HJ equation (3.6.1) we note that the substitution

$$t = \frac{1}{\sqrt{2}}(x_0 + v), \qquad u = \frac{m}{\sqrt{2}}(x_0 - v) \tag{3.6.14}$$

transforms Equation (3.6.1) into the relativistic HJ equation (1.2.1)

$$\frac{4m}{(m + u_t)^2}\left(u_t + \frac{1}{2m}(\vec{\nabla}u)^2\right) = 0 \to 1 - v_\nu v^\nu = 0 \tag{3.6.15}$$

and vice versa, the substitution

$$x_0 = \frac{1}{\sqrt{2}}\left(t + \frac{u}{m}\right), \qquad v = \frac{1}{\sqrt{2}}\left(t - \frac{u}{m}\right) \tag{3.6.16}$$

transforms the relativistic HJ equation into nonrelativistic

$$\frac{m}{(1 + v_0)^2}(1 - v_\nu v^\nu) = 0 \to u_t + \frac{1}{2m}(\vec{\nabla}u)^2 = 0 \tag{3.6.17}$$

However, this equivalence, as it is seen from (3.6.15), (3.6.17), breaks down when

$$m + u_t = 0 \qquad \text{or} \qquad 1 + v_0 = 0.$$

It means that if to cast away from manifold of solutions of HJ equation (3.6.1) the solutions

$$u = -mt + \varphi(\vec{x}) \tag{3.6.18}$$

and from manifold of solutions of the relativistic HJ equation the solutions

$$v = -x_0 + \text{const} \tag{3.6.19}$$

then the remaining manifolds will be locally equivalent and this equivalence is given by (3.6.14), (3.6.16).

It will be noted that substitution (3.6.16) being applied to an arbitrary P(1,4)-invariant PDE results in at least Galilean invariant equation. For example, the Monge-Ampere equation is invariant under (3.6.16) (it is due to its IGL(4,R) invariance (see Theorem 1.10.1)). Another example: ELBI equation (1.1.17) having been applied (3.6.16) results in the equation

$$m^2 u_{tt} - 2m\left(u_t + \frac{1}{2m}(\vec{\nabla}u)^2\right)\Delta u + 2mu_a u_{ta} + u_a u_b u_{ab} = 0 \tag{3.6.20}$$

Theorem 3.6.4. *Maximal local invariance group of Equation* (3.6.20) *is the extended Galilei group* $\widetilde{G}(2,3)$, *basis elements of corresponding Lie algebra* $A\widetilde{G}(2,3)$ *written in* (3.6.5), (3.6.2). *The proof is analogous to that of Theorem 3.6.1.*

So, one can use Equation (3.6.20) as a nonrelativistic counterpart of ELBI equation to describe physical phenomena which satisfy Galilean relativistic principle.

Now let us list some exact solutions of HJ equation (3.6.1) obtained in [186]

$$\begin{aligned} &u = -\frac{t}{2m} + (\vec{a}\cdot\vec{x}),\\ &u = \frac{m}{2t}\vec{x}^2, \qquad u = \frac{m}{2t}(\vec{a}\cdot\vec{x})^2,\\ &u = \frac{4m}{3}\left(t^2 - \vec{a}\cdot\vec{x}\right)^{3/2} + 2mt\vec{a}\cdot\vec{x} - \frac{4m}{3}t^3, \end{aligned} \tag{3.6.21}$$

where $\vec{a} = \{a_1, a_2, a_3\}$ are arbitrary constants, $\vec{a}^2 = 1$.

One can use solutions (3.6.21) to generate with the help of generating formulae other solutions of Equation (3.6.1). Several such formulae are presented below:

$$\begin{aligned} &u_{II}(x) = u_I(t + a_0, \vec{x} + \vec{a}) + a_4,\\ &u_{II}(x) = u_I\left(\frac{t}{1-\theta t}, \frac{\vec{x}}{1-\theta t}\right) - \frac{m\vec{x}^2}{2}\frac{\theta}{1-\theta t},\\ &\frac{u_{II}(x)}{1-\theta u_{II}(x)} = u_I\left(t + \frac{m}{2}\vec{x}^2\frac{\theta}{1-\theta u_{II}(x)}, \frac{\vec{x}}{1-\theta u_{II}(x)}\right),\\ &u_{II}(x) = u_I(t, \vec{x} + \vec{v}t) - m\vec{v}\vec{x} - \frac{m\vec{v}^2}{2}t \end{aligned}$$

and so on; a_0, a_4, $\vec{a}$, $\vec{v}$, θ are arbitrary constants.

3.7 *Symmetry and some exact solutions of the Boussinesq equation*

Below we shall study symmetry and construct exact solutions of the Boussinesq equation

$$v_0 = \frac{\lambda}{2}\Delta v^2, \qquad \lambda = \text{const} \tag{3.7.1}$$

where $v = v(x)$, $x = \{x_0, \vec{x}\} \in R^{n+1}$. Solutions of the two-dimensional Boussinesq equation were obtained in [180].

For the sake of convenience we put

$$v^2 = u$$

and consider instead of (3.7.1) the equation

$$u_0 = \lambda\sqrt{u}\Delta u \tag{3.7.2}$$

and more general equation

$$u_0 = F(u)\Delta u \tag{3.7.3}$$

where $F(u)$ is arbitrary differentiable function.

Theorem 3.7.1. *The widest invariance algebra admitted by Equation* (3.7.3) *is* $A\tilde{E}(1,n) \oplus AC(n)$ *and it is achieved iff*

$$F(u) = \lambda u^{4/(2-n)} \tag{3.7.4}$$

basis elements having the form

$$P_0 = i\partial_0, \quad P_a = i\partial_a, \quad J_{ab} = x_a P_b - x_b P_a, \quad D_1 = 2x_0 P_0 - x_a P_a \tag{3.7.5}$$

$$D_2 = x_a P_a + \frac{2-n}{2} i, \tag{3.7.6}$$

$$K_a = 2x_a D_2 - \vec{x}^2 P_a \tag{3.7.7}$$

where n *is the number of spacial variables* $\vec{x}$, $a, b = \overline{1, n}$. *In other cases we have: if*

$$F(u) = \lambda u^k, \qquad k \neq 0, \ \frac{4}{2-n} \tag{3.7.8}$$

or

$$F(u) = \lambda e^u \tag{3.7.9}$$

then the maximal IA of Equation (3.7.3) *will be* $A\tilde{E}(1,n)$ *provided*

$$D_2 = x_a P_a + \frac{2}{k} i \tag{3.7.10}$$

for (3.7.8) *and*

$$D_2 = x_a P_a - 2i\frac{\partial}{\partial u} \tag{3.7.11}$$

for (3.7.9).

In the case of arbitrary function $F(u)$ *the maximal admitted IA is* $A\tilde{E}(1,n)$ *with basis elements* (3.7.5).

Proof. According to the Lie method, from condition of invariance (3) one can obtain the following system to define coordinates of infinitesimal symmetry operators (2):

$$\xi^0_u = \xi^a_u = \xi^0_a = \eta_{uu} = 0, \qquad \xi^a_b + \xi^b_a = 0, \qquad a \neq b \tag{3.7.12}$$

$$\xi^a_0 + F(2\eta_{ua} - \Delta\xi^a) = 0, \tag{3.7.13}$$

$$\eta\frac{F'}{F} + \xi^0_0 = 2\xi^1_1 = \ldots = 2\xi^n_n \tag{3.7.14}$$

If $F(u) \neq$ const, Equations (3.7.12)–(3.7.14) result in

$$\begin{aligned} \xi^0 &= 2æ_1 x_0 + d_0 \\ \xi^a &= b_a\vec{x}^2 - 2x_a\vec{b}\cdot\vec{x} + (æ_1 - æ_2)x_a + c_{ab}x_b + d_a \\ \eta &= a(x)u + b(x) = \big((n-2)\vec{b}\cdot\vec{x} + c_1\big)u + b(\vec{x}) \end{aligned} \tag{3.7.15}$$

where c_1, $æ_1$, $æ_2$, $c_{ab} = -c_{ba}$, b_a, d_0, d_a are group parameters, and function $b(\vec{x})$ is a solution of Laplace equation $\Delta b = 0$. For function $F(u)$ we have the ODE

$$\frac{F'}{F}\big(a(x)u + b(x)\big) = 2\xi^1_1 - \xi^0_0 \tag{3.7.16}$$

Here we have three irreducible cases

1) $a(\vec{x}) \neq 0, \qquad b(\vec{x}) = 0$

2) $a(\vec{x}) = 0, \qquad b(\vec{x}) \neq 0$

3) $a(\vec{x}) = b(\vec{x}) = 0$

Let us consider the first case. Substituting (3.7.15) into (3.7.16) we get

$$u\frac{F'}{F} = \frac{-4b(\vec{x}) - 2æ}{(n-2)b(\vec{x}) + c_1} = k = \text{const.} \tag{3.7.17}$$

If $b \neq 0$, then equality (3.7.17) holds true when $c_1 = \frac{1}{2}(n-2)æ_2$, $k = 4/(2-n)$, and after integrating it results in (3.7.4). In turn (3.7.15) gives (3.7.5)–(3.7.7). If $b = 0$, (3.7.17) yields $k \neq 0$, $c_1 = \frac{2}{k}æ_2$, $F(u) = \lambda u^k$.

In the second case (3.7.16) takes the form

$$\frac{F'}{F} = -2æ_2/b(\vec{x})$$

and it makes sense only if $b(\vec{x}) =$ const. Putting $b(\vec{x}) = -2æ_2$ we get (3.7.8), (3.7.11).

Supposing $a = b = 0$ we find that (3.7.16) holds true with arbitrary function $F(u)$ provided

$$2\xi^1_1 = \xi^0_0 \tag{3.7.18}$$

It follows that $b = æ_2 = 0$ and, hence, Equation (3.7.3) is invariant under algebra (3.7.5). The theorem is proved.

Remark 3.7.1. Theorem 3.7.1 is proved on the assumption that $F(u) \neq \text{const}$. Otherwise, Equation (3.7.3) coincides with the linear heat equation and symmetry of this latter one is well-known.

Consequence 3.7.1. The maximal IA of the Boussinesq equation (3.7.2) is the extended Euclid algebra with basis elements (3.7.5), (3.7.10).

Solutions of the Boussinesq equation we seek in the form (3.2.6), where invariant variables ω are given in Table 3.2.1 and

$$f(x) = \begin{cases} x_0^2, & \text{N } 1\text{–}4 \\ 1, & \text{N } 5\text{–}10 \end{cases}$$

Below we present reduction equations which succeed in integrating. In the 6*th* case (the enumeration corresponds to that of Table 3.2.1), on the assumption $\varphi_{\omega_2} = 0$, we have

$$\beta_0\varphi_{11} + \varphi_3 = \lambda\sqrt{\varphi}\beta^2\varphi_{11} \tag{3.7.19}$$

If $\beta_0 = 0$, $\beta^2 = -i/12\lambda$ then one can separate variables ω_1, ω_3 so that

$$\varphi = A(\omega_1)B(\omega_3) \tag{3.7.20}$$

Substituting (3.7.20) into (3.7.19) we find

$$\frac{A''}{A} = -\frac{6B'}{B^{3/2}} = k = \text{const}$$

whence $A = \left(\frac{k}{12}\right)^2(\omega_1 + c_1)^4$, $A = \left(\frac{12}{k}\right)^2(\omega_3 + c_3)^{-2}$; c_1, c_2 are constants. It follows the solution of Equation (3.7.2)

$$u(x) = \frac{(\vec{\beta}\cdot\vec{x} + c_1)^4}{(x_0 + c_2\ln\vec{\sigma}\vec{x} + c_3)^2}, \tag{3.7.21}$$

where c_1, c_2, c_3, $\vec{\beta}, \vec{\sigma}$ are arbitrary constants, $\beta^2 = -1/12\lambda$, $\vec{\beta}\cdot\vec{\sigma} = \vec{\sigma}^2 = 0$.

Solution (3.7.21) can be generalized as follows

$$u(x) = \frac{(\vec{\beta}\cdot\vec{x} + c_1)^4}{[x_0 + F(\vec{\sigma}\cdot\vec{x})]^2}, \tag{3.7.22}$$

where F is an arbitrary differentiable function.

Let $\varphi_3 = 0$, then (3.7.19) takes the form

$$\lambda\beta^2\varphi_{11}\sqrt{\varphi} = \beta_0\varphi_1$$

and has the general solution

$$\sqrt{\varphi} + c_2 \ln\left(\sqrt{\varphi} - c_2\right) = a(\omega_1 + c_1)$$

where c_1, c_2, $a = \beta_0/\lambda\beta^2$ are arbitrary constants. So we have one more solution of Equation (3.7.2)

$$\sqrt{u} + c_2 \ln\left(\sqrt{u} - c_2\right) = a(\beta_\nu x^\nu + c_1) \tag{3.7.23}$$

Let $\varphi_1 = 0$ then the $4th$ ansatz reduces (3.7.2) to the ODE

$$-2\omega_2\varphi_2 + \varphi_3 + 2\varphi = 2\lambda a\sqrt{\varphi}\varphi_{23} \tag{3.7.24}$$

We seek for solutions of Equation (3.7.24) in the form

$$\varphi = A(\omega_3)\omega_2^2 \tag{3.7.25}$$

The substitution (3.7.25) into (3.7.24) gives

$$A'(1 - 4\lambda a\sqrt{A}) = 2A$$

or

$$A\exp\{-4\lambda\sqrt{A}a\} = c_1\exp\{\omega_3\} \tag{3.7.26}$$

whence we find solution of Equation (3.7.2)

$$\begin{gathered}\ln\sqrt{u} - 4\frac{\vec{\sigma}\cdot\vec{\delta}}{\vec{\sigma}\cdot\vec{x}}x_0\sqrt{u} = \vec{\delta}\cdot\vec{x} - \ln\vec{\sigma}\vec{x},\\ \vec{\sigma}^2 = \vec{\delta}^2 = 0, \qquad \vec{\sigma}\cdot\vec{\delta} \neq 0.\end{gathered} \tag{3.7.27}$$

Two more solutions of the Boussinesq equation (3.7.2) obtained with the help of ansatz N8:

$$\sqrt{u} = -\frac{(\vec{\alpha}\cdot\vec{x})^2}{6\lambda\alpha^2 x_0} + c_1 x_0^{-1/3} \tag{3.7.28}$$

$$\sqrt{u} = -\frac{\vec{x}^2}{10\lambda x_0} + c_2 x_0^{-5/3}, \tag{3.7.29}$$

where c_1, c_2, $\vec{\alpha}$ are arbitrary constants.

3.8 *Symmetry properties of Fokker-Planck equations*

Following [76*], we consider symmetry properties of one- and two-dimensional Fokker-Planck (FP) equations.

The one-dimensional FP equation has the form [77*]

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left[A(x,t)u\right] + \tfrac{1}{2}\frac{\partial^2}{\partial x^2}\left[B(x,t)u\right], \tag{3.8.1}$$

where $u = u(x,t)$ is the probability density; A and B are differentiable functions. This is the basic equation in the theory of continuous Markovian processes. The following FP equations are of special interest [77*,78*].

(a) diffusion in a gravitational field

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(gu) + \tfrac{1}{2}D\frac{\partial^2 u}{\partial x^2}; \tag{3.8.2}$$

(b) the Ornstein-Uhlenbeck process

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(kxu) + \tfrac{1}{2}D\frac{\partial^2 u}{\partial x^2}; \tag{3.8.3}$$

(c) the Rayleigh-type process

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[\left(\gamma x - \frac{\mu}{x}\right)u\right] + \tfrac{1}{2}\mu\frac{\partial^2 u}{\partial x^2}; \tag{3.8.4}$$

(d) models in population genetics [78*]

$$\frac{\partial u}{\partial t} = \frac{\alpha}{2}\frac{\partial^2}{\partial x^2}[(x-c)^2 u] + \beta\frac{\partial}{\partial x}[(x-c)u], \tag{3.8.5}$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}[(1-x^2)^2 u], \tag{3.8.6}$$

$$\frac{\partial u}{\partial t} = \frac{\alpha}{2}\frac{\partial^2}{\partial x^2}[x^2(1-x^2)^2 u]; \tag{3.8.7}$$

(e) the Rayleigh process

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[\left(\gamma x - \frac{\mu}{x}\right)u\right] + \mu\frac{\partial^2 u}{\partial x^2}, \tag{3.8.8}$$

where D, g, k, γ, α, b, c are arbitrary constants.

Using Lie's method one can make sure that the maximal invariance group of Equations (3.8.2)–(3.8.7) is a six-parameter one. The same dimension has the invariance group of the heat equation. Although these six-parameter groups are different, they are locally isomorphic. That is why one can reduce Equations (3.8.2)–(3.8.7) to the heat equation.

Theorem 3.8.1. *The change of variables*

$$u(x,t) = f(x,t)w\big(y(x,t), \tau(x,t)\big), \tag{3.8.9}$$

where the function f and new independent variables y and τ are as follows:

$$f = \exp\left\{-\frac{g}{D}x - \frac{g^2}{2D}t\right\}, \qquad y = x, \qquad \tau = \frac{D}{2}t; \tag{3.8.10}$$

$$f = \exp\{kt\}, \qquad y = \exp\{kt\}x, \qquad \tau = \frac{D}{4k}\exp\{2kt\} \tag{3.8.11}$$

$$f = \exp\{2\gamma t\}x, \qquad y = \exp\{\gamma t\}x, \qquad \tau = \frac{\mu}{4\gamma}\exp\{2\gamma t\} \tag{3.8.12}$$

$$\begin{gathered} f = \exp\left\{-\left(\frac{\beta^2}{2\alpha} + \frac{\beta}{2} + \frac{\alpha}{8}\right)t\right\}(x-c)^{-(3/2+\beta/\alpha)} \\ y = \sqrt{\tfrac{2}{\alpha}}\ln(x-c), \qquad \tau = t; \end{gathered} \tag{3.8.13}$$

$$f = \exp\{-t\}(1-x^2)^{-3/2}, \qquad y = \tfrac{1}{2}\ln\frac{1+x}{1-x}, \qquad \tau = t; \tag{3.8.14}$$

$$f = \exp\left\{-\frac{\alpha t}{8}\right\}[x(1-x)]^{-3/2}, \qquad y = \ln\frac{x}{1-x} \qquad \tau = \frac{\alpha t}{2} \tag{3.8.15}$$

reduce Equations (3.8.2) – (3.8.7), *correspondingly, to the heat equation*

$$w_\tau = w_{yy} \tag{3.8.16}$$

The proof can be easily obtained by inspection.

Remark 3.8.1. One can prove a more general statement. Equation (3.8.1) with coefficients

$$A(x,t) = A(x), \qquad B(x,t) = B = \text{const} \tag{3.8.17}$$

is reduced to the heat equation if and only if

$$B\frac{\partial A}{\partial x} + A^2 = c_2x^2 + c_1x + c_0, \tag{3.8.18}$$

where c_0, c_1, c_2 are arbitrary constants. Note that Equations (3.8.2)–(3.8.4) satisfy condition (3.8.18) and Equation (3.8.8) does not. The general solution of Equation (3.8.18), which is a Riccati one, cannot be obtained in quadrature [130]. It will be also noted that FP equation (3.8.1) with coefficients

$$A(x,t) = A(x), \qquad B(x,t) = B(x) \tag{3.8.20}$$

is reduced to the case (3.8.17) by means of the change of variables (3.8.9), where

$$f = \frac{1}{\sqrt{B(x)}}, \qquad y = \int \frac{dx}{\sqrt{B(x)}}, \qquad \tau = t. \tag{3.8.21}$$

Remark 3.8.2. The FP equation of the form

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\Big[\big(a(t)x + b(t)\big)u\Big] + c(t)\frac{\partial^2 u}{\partial x^2} \tag{3.8.22}$$

was considered in [28*,79*], where a class of solutions of it was obtained by means of rather complicated algebraic method. This result can be easily obtained if we note that Equation (3.8.22) is reduced to the heat Equation (3.8.16) by the substitution (3.8.9) with

$$\begin{aligned} f &= \exp\Big\{-\int_0^t a(s)ds\Big\}, \\ y &= \exp\Big\{-\int_0^t a(s)ds\Big\}x - \int_0^t b(s)\exp\Big\{-\int_0^s a(\xi)d\xi\Big\}ds, \\ \tau &= \int_0^t c(s)\exp\Big\{-2\int_0^s a(\xi)d\xi\Big\}ds. \end{aligned} \tag{3.8.23}$$

Some group-theoretic aspects of FP equations are considered in [27,80*].

Now consider the two-dimensional FP equation which describes the motion of a particle in a fluctuating medium (so-called Brownian movement)

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}(yu) + \frac{\partial}{\partial y}\big(V'(x)u\big) + \gamma\frac{\partial}{\partial y}\left(yu + \frac{\partial u}{\partial y}\right), \tag{3.8.24}$$

where $u = u(t,x,y)$, γ is a constant and $V(x)$ is the potential (its gradient defines the exterior force). Equation (3.8.24) is known as the *Kramers* equation [77*].

Theorem 3.8.2. [76*] *The maximal invariance group of the free Kramers equation*

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}(yu) + \gamma\frac{\partial}{\partial y}\left(yu + \frac{\partial u}{\partial y}\right) \tag{3.8.25}$$

is a six-dimensional Lie group generated by the following operators

$$\begin{aligned} &P_0 = \partial_t, \qquad P_1 = \partial_x, \qquad I, \\ &G_1 = t\partial_x + \partial_y + \tfrac{1}{2}(y + \gamma x)I, \\ &S_1 = e^{\gamma t}\left(\frac{1}{\gamma}\partial_x + \partial_y + yI\right), \qquad T_1 = e^{-\gamma t}\left(\frac{1}{\gamma}\partial_x - \partial_y\right), \end{aligned} \tag{3.8.26}$$

which satisfy the commutation relations

$$\begin{aligned}
&[P_0, G_1] = P_1, \qquad [P_0, S_1] = \gamma S_1, \\
&[P_0, T_1] = -\gamma T_1, \qquad [P_1, G_1] = \tfrac{1}{2}\gamma I, \qquad [T_1, S_1] = -I
\end{aligned} \tag{3.8.27}$$

(the rest of the commutators are equal to zero). The proof can be obtained by Lie's method.

Remark 3.8.3. One can prove a more general statement: the widest symmetry group of Equation (3.8.24) is achieved when $V'(x) = c_1 x + c$ (c_1 and c are arbitrary constants) and it is a six-parameter group.

Remark 3.8.4. The change of variables

$$u = w(\tau, \xi, \eta), \qquad \tau = t, \qquad \xi = x - \frac{c}{\gamma}t, \qquad \eta = y - \frac{c}{\gamma} \tag{3.8.28}$$

reduces Equation (3.8.24) with $V'(x) = c$ to the free Kramers equation (3.8.25).

Let us write down the final transformations generated by operators (3.8.26). Operators P_0 and P_1 generate translations on variables t and x; I generates the identical transfomation;

G_1 generates

$$\begin{aligned}
&t' = t, \qquad x' = x + at, \qquad y' = y + a, \\
&u'(x') = \exp\left\{-\frac{1}{2}\left[ay + \frac{a^2}{2}(1 + \gamma t) + \gamma a x\right]\right\} u(x);
\end{aligned} \tag{3.8.29}$$

S_1 generates

$$\begin{aligned}
&t' = t, \qquad x' = x + \frac{b}{\gamma}e^{\gamma t}, \qquad y' = y + be^{\gamma t}, \\
&u'(x') = \exp\left\{bye^{\gamma t} - \frac{b^2}{2}e^{2\gamma t}\right\} u(x);
\end{aligned} \tag{3.8.30}$$

T_1 generates

$$\begin{aligned}
&t' = t, \qquad x' = x + \frac{\theta}{\gamma}e^{-\gamma t}, \qquad y' = y - \theta e^{-\gamma t}, \\
&u'(x') = u(x);
\end{aligned} \tag{3.8.31}$$

where a, b, θ are group parameters. It is appropriate to write the corresponding formulae of generating solutions which follow from (3.8.29)–(3.8.31):

$$u_{II}(t, x, y) = \exp\left\{\frac{a}{2}\left[y + \frac{a}{2}(1 + \gamma t) + \gamma x\right]\right\} u_I(t', x', y') \tag{3.8.32}$$

$$u_{II}(t,x,y) = \exp\left\{-bye^{\gamma t} + \frac{b^2}{2}e^{2\gamma t}\right\} u_I(t',x',y') \tag{3.8.33}$$

$$u_{II}(t,x,y) = u_I(t',x',y') \tag{3.8.34}$$

where t', x', y' are given in (3.8.29)–(3.8.31) respectively.

It will be noted that transformations (3.8.29) are just the Galilean ones as soon as the variable y in the Kramers equation is taken to be the velocity of the particle.

A well-known solution of the Kramers equation (3.8.24) is the Boltzmann distribution

$$u(x,y) = \mathcal{N} \exp\left\{-V(x) - \tfrac{1}{2}y^2\right\} \tag{3.8.35}$$

($\mathcal{N}$ is a normalization constant). It is a stationary solution. Applying to (3.8.35) with $V = 0$ formulae (3.8.32)–(3.8.34), one can easily obtain new non-stationary solutions of Equation (3.8.25).

Let us consider the ansatz invariant with respect to the operator S_1 from (3.8.26)

$$u(t,x,y) = \exp\left\{-\frac{y^2}{2}\right\} \varphi(\omega_1,\omega_2), \qquad \omega_1 = t, \qquad \omega_2 = \gamma x - y \tag{3.8.36}$$

Substituion of (3.8.36) into (3.8.25) gives rise to the heat equation

$$\frac{\partial \varphi}{\partial \omega_1} - \gamma \frac{\partial^2 \varphi}{\partial \omega_2^2} = 0 \tag{3.8.37}$$

The simplest solution of (3.8.37) is $\varphi = \text{const}$, but it is the solution that leads, together with the ansatz (3.8.36), to the Boltzmann distribution (3.8.35). It is clear that by using solutions of the heat equation (3.8.37) and the ansatz (3.8.36) one can construct many partial solutions of Equation (3.8.25). For example, the fundamental solution of (3.8.37) and (3.8.36) results in the following solution of Equation (3.8.25)

$$u(t,x,y) = \frac{1}{\sqrt{4\pi\gamma t}} \exp\left\{-\frac{y^2}{2} - \frac{(\gamma x - y)^2}{4\gamma t}\right\} \tag{3.8.38}$$

The operator T_1 from (3.8.26) leads to the ansatz

$$u = \widetilde{\varphi}(\omega_1,\omega_2), \qquad \omega_1 = t, \qquad \omega_2 = \gamma x + y \tag{3.8.39}$$

which reduces Equation (3.8.25) to the heat equation (3.8.37), where, $\varphi = \exp\{\gamma\omega_1\}\widetilde{\varphi}(\omega_1,\omega_2)$.

A great number of partial solutions of Equation (3.8.25) can be found by means of the method described in [24*] (see (2.3.58)). For example, starting from $u_0 = e^{\gamma t}$, we find

$$\begin{aligned} u_1 &= 2G_1 e^{\gamma t} = e^{\gamma t}(\gamma x + y), \\ u_2 &= G_1 u_1 = e^{\gamma t}[(\gamma t + 1) + \tfrac{1}{2}(\gamma x + y)^2], \ldots \end{aligned} \tag{3.8.40}$$

Analogously, by means of the operator T_1 from (3.8.26), we find, starting from $u_0 = \exp\{-y^2/2\}$,

$$
\begin{aligned}
u_1 &= T_1 u_0 = y \exp\left\{-\left(\gamma t + \frac{y^2}{2}\right)\right\}, \\
u_2 &= T_1 u_1 = (y^2 - 1) \exp\left\{-\left(2\gamma t + \frac{y^2}{2}\right)\right\}, \ldots
\end{aligned}
\tag{3.8.41}
$$

Solutions (3.8.38), (3.8.40), and (3.8.41) can be multiplied by the formulae of generating solutions (3.8.32)–(3.8.34).

Chapter 4

Systems of PDEs Invariant Under Galilei Group

In the present chapter we consider linear and nonlinear systems of PDEs invariant under various representations of the Galilei group and its generalizations (such as the extended Galilei group, the Schrödinger group). Sets of Sch(1,3)- and G(1,3)-nonequivalent ansatze are constructed. A wide class of linear and nonlinear Sch(1,3)-invariant systems of PDEs is described. Lame equations are studied: superalgebra of symmetry is found and a Galilei-invariant generalization is constructed. Gas dynamics and Navier-Stokes equations are considered. Exact solutions of some enumerated above equations are found.

4.1. *The Schrödinger group* Sch(1,3): *nonequivalent ansatze and final transformations for fields of arbitrary spin*

The maximal group of point transformations of the Schrodinger equation

$$\check{S}\psi \equiv \left(P_0 - \frac{\vec{P}^2}{2m} \right) \psi = 0, \tag{4.1.1}$$

where $\psi = \psi(x_0, \ldots, x_3)$ is complex wave function, is called the Schrödinger group. It is 13-dimensional and containes the Galilei group, scale and projective transformations. The basis elements of the corresponding algebra ASch(1,3) are written in (3.3.25). Further, we will consider the case of the multi-component ψ-function.

From the group-theoretic point of view an analogy between relativistic and nonrelativistic mechanics can be treated as follows:

Basic symmetry group of	
relativistic mechanics	nonrelativistics mechanics
Poincare group $P(1,3)$	Galilei group $G(1,3)$
extended Poincare group $\widetilde{\mathrm{P}}(1,3)=\{P(1,3),\ D\}$	extented Galilei group $\widetilde{\mathrm{G}}(1,3)=\{G(1,3),D\}$
conformal group $C(1,3)=\{\widetilde{\mathrm{P}}(1,3),K_\mu\}$	Schrödinger group $\mathrm{Sch}\,(1,3)=\{\widetilde{\mathrm{G}}(1,3),\Pi\}$

The most general form of basis operators realizing linear representation of the Schrödinger algebra ASch(1,3) is [78, 87*]:

$$\begin{aligned}
&P_0=i\partial_0, \qquad P_a=-i\partial_a, \qquad M=im,\\
&J_a=(\vec{x}\times\vec{p})_a+\widehat{S}_a, \qquad G_a=x_0P_a-mx_a+\lambda_a,\\
&D=2x_0P_0-\vec{x}\cdot\vec{P}+\lambda_0,\\
&\Pi=x_0D-x_0^2P_0+\tfrac{1}{2}m\vec{x}^2-\vec{\lambda}\cdot\vec{x},
\end{aligned} \tag{4.1.2}$$

where m is a constant; $\widehat{S}$, λ_a, λ_0 are numerical matrices. Operators (4.1.2) satisfy commutation rules

$$\begin{aligned}
&[P_0,P_a]=[P_0,J_a]=[P_a,P_b]=[G_a,G_b]=[D,J_a]=0\\
&[\Pi,J_a]=[\Pi,G_a]=[P_0,M]=[P_a,M]=[G_a,M]=0\\
&[D,M]=[\Pi,M]=0\\
&[J_a,J_b]=i\epsilon_{abc}J_c, \qquad [P_a,J_b]=i\epsilon_{abc}P_c,\\
&[J_a,G_b]=i\epsilon_{abc}G_c, \qquad [P_a,G_b]=M\delta_{ab},\\
&[P_0,G_a]=iP_a, \qquad [P_0,D]=2iP_0, \qquad [P_a,D]=iP_a,\\
&[D,G_a]=iG_a \qquad [P_0,\Pi]=iD, \qquad [P_a,\Pi]=iG_a, \qquad [D,\Pi]=2i\Pi.
\end{aligned} \tag{4.1.3}$$

When $\lambda_0=\frac{3}{2}i$, $\lambda_a=0$ then (4.1.2), (4.1.3) coincide with the maximal IA of the Schrödinger equation (4.1.1) (2.3.25).

As it follows from (4.1.3) matrices $\widehat{S}_a$, λ_0, λ_a should satisfy the commutation relations

$$\begin{aligned}
&[\widehat{S}_a,\widehat{S}_b]=i\epsilon_{abc}\widehat{S}_c,\\
&[\lambda_a,\lambda_b]=0, \qquad [\lambda_a,\widehat{S}_b]=i\epsilon_{abc}\widehat{S}_c,\\
&[\lambda_0,\widehat{S}_a]=0, \qquad [\lambda_0,\lambda_a]=i\lambda_a
\end{aligned} \tag{4.1.4}$$

About finite-dimensional representations of algebra (4.1.4) see the following paragraph.

Solutions invariant under algebra (4.1.2) one can seek for in the form (2.1.7), matrices $A(x)$ and invariant variables $\omega(x)$ being determined from the equations like (2.1.12), (2.1.13). Without going into details of these calculations we present in the Table 4.1.1 ansatze invariant under one-dimensional subalgebras of ASch(1,3). It will be noted that these subalgebras are Sch(1,3)-nonequivalent on the field of complex numbers.

Table 4.1.1. One-dimensional subalgebras of ASch(1,3) and corresponding ansätze.

N	Algebra	Invariant variables ω	Ansatze $\psi(x) =$
1.	M	x_0, x_1, x_2, x_3	0
2.	P_1	x_0, x_2, x_3	$\varphi(\omega)$
3.	D	$\dfrac{x_1}{\sqrt{x_0}}, \dfrac{x_2}{\sqrt{x_0}}, \dfrac{x_3}{\sqrt{x_0}}$	$x_0^{i\lambda_0/2}\varphi(\omega)$
4.	J_3	$x_1, (x_1^2+x_2^2)^{1/2}, x_3$	$\exp\{i\widehat{S}_3 \arctan(x_1/x_2)\}\varphi(\omega)$
5.	$J_3+\alpha M$	$x_0, (x_1^2+x_2^2)^{1/2}, x_3$	$\exp\left\{i(\widehat{S}_3+\alpha m)\cdot \cdot \arctan\dfrac{x_1}{x_2}\right\}\varphi(\omega)$
6.	$J_3+\alpha P_3$	$x_0, (x_1^2+x_2^2)^{1/2}, x_3-\alpha\arctan\dfrac{x_1}{x_2}$	$\exp\left\{i\widehat{S}_3 \arctan(x_1/x_2)\right\}\varphi(\omega)$
7.	$J_3+\alpha D$	$\ln(x_1^2+x_2^2)-2\alpha\arctan\dfrac{x_1}{x_2}, \dfrac{x_1^2+x_2^2}{\sqrt{x_0}\,x_3}, \dfrac{x_3}{\sqrt{x_0}}$	$(x_1^2+x_2^2)^{i\lambda_0/2}\exp\{i\widehat{S}_3\cdot \cdot\arctan(x_1/x_2)\}\varphi(\omega)$
8.	$J_3+\alpha D+$ $+\beta M$	$\ln(x_1^2+x_2^2)-2\alpha\arctan\dfrac{x_1}{x_2}, \dfrac{x_1^2+x_2^2}{\sqrt{x_0}\,x_3}, \dfrac{x_3}{\sqrt{x_0}}$	$x_0^{i\beta m/2}(x_1^2+x_2^2)^{i\lambda_0/2}\exp\{i\widehat{S}_3\cdot \cdot\arctan(x_1/x_2)\}\varphi(\omega)$
9.	$M+\beta D$	$\dfrac{x_1}{\sqrt{x_0}}, \dfrac{x_2}{\sqrt{x_0}}, \dfrac{x_3}{\sqrt{x_0}}$	$x_0^{\frac{1}{2}(i\lambda_0+\beta m)}\varphi(\omega)$
10.	$G_1+\alpha P_2$	$x_0, \alpha x_1 - x_0x_2, x_3$	$\exp\left\{im\left(\dfrac{x_1^2}{2x_0}-\dfrac{x_2}{\alpha m}\lambda_1\right)\right\}\cdot \cdot\varphi(\omega)$

Here $\alpha > 0$, $\beta \neq 0$ are arbitrary constants; an entry x_0 means, for example, if

$$\lambda_0 = \frac{i}{2}\begin{pmatrix} n & 0 \\ 0 & k \end{pmatrix},$$

where n and k are some numbers, then

$$x_0^{i\lambda_0/2} = \begin{pmatrix} x_0^{-n/4} & 0 \\ 0 & x_0^{-k/4} \end{pmatrix}$$

Transformations generated by operators (4.1.2) can be used for obtaining other solutions of Sch(1,3)-invariant PDEs in the same spirit as it was done in Paragraphs 1.4, 2.3. In Table 4.1.2 we present corresponding forulae.

Table 4.1.2. Final transformations of Sch(1,3) group and formulae of generating solutions.

N	Operator	Transformations		Formulae of GS
		$x \to x'$	$\psi(x) \to \psi'(x') =$	$\psi_{II} =$
1-4	$P_\mu = \{P_0, P_a\}$	$x'_\mu = x_\mu + a_\mu$	$= \psi(x)$	$= \psi_I(x')$
5-7	$J = \alpha_a J_a$	$x'_0 = x_0$, $\vec{x}' = \vec{x}\cos\alpha + {}+\frac{\vec{x}\times\vec{\alpha}}{\alpha}\sin\alpha + {}+\frac{\vec{\alpha}(\vec{\alpha}\cdot\vec{x})}{\alpha^2}\cdot {}\cdot(1-\cos\alpha)$	$= \exp\left\{-i\vec{\alpha}\cdot\vec{S}\right\}\cdot {}\cdot\psi(x)$	$= \exp\left\{i\vec{\alpha}\vec{S}\right\}\cdot {}\cdot\psi_I(x')$
8-10	$G = v_a G_a$	$x'_0 = x_0$, $\vec{x}' = \vec{x} + \vec{v}x_0$	$= \exp\{im\cdot {}\cdot(\vec{v}\vec{x} + \frac{1}{2}\vec{v}^2 x_0) - {}-i\vec{\lambda}\vec{v}\}\psi(x)$	$= \exp\{-im\cdot {}\cdot(\vec{v}\vec{x} + \frac{1}{2}\vec{v}^2 x_0) + {}+\vec{\lambda}\vec{v}\}\psi_I(x')$
11	M	$x'_\mu = x_\mu$	$= \exp\{im\theta\}\psi(x)$	$= \exp\{-im\theta\}\cdot {}\cdot\psi_I(x')$
12	D	$x'_0 = e^{2\beta}x_0$, $\vec{x}' = e^{\beta}$	$= \exp\{i\beta\lambda_0\}\psi(x)$	$= \exp\{-i\beta\lambda_0\}\cdot {}\cdot\psi_I(x')$
13	Π	$x'_0 = \frac{x_0}{1-\theta x_0}$, $\vec{x}' = \frac{\vec{x}}{1-\theta x_0}$	$= (1-\theta x_0)^{-i\lambda_0}\cdot {}\cdot\exp\left\{\frac{i\theta}{1-\theta x_0}\cdot {}\cdot\left(\frac{m\vec{x}^2}{2} - \vec{\lambda}\cdot\vec{x}\right)\right\}\cdot {}\cdot\psi(x)$	$= \exp\left\{-\frac{i\theta}{1-\theta x_0}\cdot {}\cdot\left(\frac{m\vec{x}^2}{2} - \vec{\lambda}\cdot\vec{x}\right)\right\}\cdot {}\cdot(1-\theta x_0)^{i\lambda_0}\cdot {}\cdot\psi_I(x')$

Here a_μ, $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, $\vec{v}$, β, θ are arbitrary real constants.

One can make sure in validity of the final transformations formulae of Table 4.1.2 by straightforward verification of corresponding Lie equations. In particular, Lie equations for G_a (see (4.1.2)) have the form

$$\frac{\partial x'_b}{\partial v_a} = x'_0\delta_{ab}, \qquad \frac{\partial x'_0}{\partial v_a} = 0; \qquad x'_0\Big|_{v_a=0} = x_0, \qquad x'_a\Big|_{v_b=0} = x_a;$$

$$\frac{\partial\psi'(x')}{\partial v_a} = (imx'_a - i\lambda_a)\psi'(x'), \qquad \psi'(x')\Big|_{v_a=0} = \psi(x)$$

and it is easy to see that they are identically satisfied by x'_a, $\psi'(x')$ N8–10 of

Table 4.1.2. Indeed,

$$\frac{\partial}{\partial v_a}(x_b + v_b x_0) \equiv \delta_{ab} x_0, \qquad x'_0 = x_0;$$

$$\frac{\partial}{\partial v_a}\psi'(x') \equiv \frac{\partial}{\partial v_a}\left[\exp\left\{im\left(\vec{v}\cdot\vec{x} + \tfrac{1}{2}\vec{v}^2 x_0\right) - i\vec{\lambda}\vec{v}\right\}\psi(x)\right] =$$

$$= [im(x_a + v_a x_0) - i\lambda_a]\psi'(x') \equiv i(mx'_a - \lambda_a)\psi'(x').$$

In the case of operator Π Lie equations have the form

$$\frac{dx'_0}{d\theta} = x_0'^2, \qquad \frac{d\vec{x}'}{d\theta} = x'_0\vec{x}'; \qquad x'_0\Big|_{\theta=0} = x_0, \qquad \vec{x}'\Big|_{\theta=0} = \vec{x};$$

$$\frac{\psi'(x')}{d\theta} = i\left(x'_0\lambda_0 + \frac{m\vec{x}'^2}{2} - \vec{\lambda}\cdot\vec{x}'\right)\psi'(x'), \qquad \psi'(x')\Big|_{\theta=0} = \psi(x).$$

Let us make sure that x', $\psi'(x')$ N13 of Table 4.1.2 are solutions of these equations. So, we calculate

$$\frac{d}{d\theta}x'_0 \equiv \frac{d}{d\theta}\left(\frac{x_0}{1-\theta x_0}\right) = \left(\frac{x_0}{1-\theta x_0}\right)^2 \equiv x_0'^2; \qquad \frac{d}{d\theta}\vec{x}' \equiv \frac{d}{d\theta}\left(\frac{\vec{x}}{1-\theta x_0}\right) =$$

$$= \frac{x_0}{1-\theta x_0}\cdot\frac{\vec{x}}{1-\theta x_0} \equiv x'_0\vec{x}';$$

$$\frac{d}{d\theta}\psi'(x') \equiv \frac{d}{d\theta}\left[(1-\theta x_0)^{-i\lambda_0}\exp\left\{\frac{i\theta}{1-\theta x_0}\left(\frac{m\vec{x}^2}{2} - \vec{\lambda}\cdot\vec{x}\right)\right\}\psi(x)\right] =$$

$$= (1-\theta x_0)^{-i\lambda_0}\left[i\lambda_0\frac{x_0}{1-\theta x_0} + \frac{i}{(1-\theta x_0)^2}\left(\frac{m\vec{x}^2}{2} - \vec{\lambda}\cdot\vec{x}\right)\right]\cdot$$

$$\cdot\exp\left\{\frac{i\theta}{1-\theta x_0}\left(\frac{m\vec{x}^2}{2} - \vec{\lambda}\cdot\vec{x}\right)\right\} \equiv i\left(\lambda_0 x'_0 + \frac{m\vec{x}'^2}{2} - \vec{\lambda}\cdot\vec{x}'\right)\psi'(x')$$

Here we have used the equality ($\lambda_0 \neq 0$)

$$(1-\theta x_0)^{-i\lambda_0-1}(\vec{\lambda}\cdot\vec{x}) = (\vec{\lambda}\cdot\vec{x})(1-\theta x_0)^{-i\lambda_0}.$$

Now we show a simple way of constructing Sch(1,3)-ungenerative solutions. Let us start from a projective invariant ansatz

$$\psi(x) = x_0^{i\lambda_0}\exp\left\{i\left(\frac{m\vec{x}^2}{2x_0} - \vec{\lambda}\vec{x}\right)\right\}\varphi(\omega), \qquad \omega = \frac{\vec{\beta}\cdot\vec{x}}{x_0} + \beta_0, \tag{4.1.5}$$

which results from the equations

$$\Pi A(x) \equiv \left(x_0^2 P_0 - x_0\vec{x}\cdot\vec{P} + \frac{m\vec{x}^2}{2} + \lambda_0 x_0 - \vec{\lambda}\vec{x}\right)A(x) = 0,$$

$$(x_0^2 P_0 - x_0 \vec{x} \cdot \vec{P})\omega(x) = 0; \qquad \psi(x) = A(x)\varphi(\omega).$$

Having done gauge and translation transformations (see Table 4.1.2, N11 and N1–4) on the ansatz (4.1.5) we obtain as a result a Sch(1,3)-ungenerative ansatz

$$\psi(x) = (x_0 + a_0)^{i\lambda_0} \exp\left\{ im \left(\frac{(\vec{x} + \vec{a})^2}{2(x_0 + a_0)} + \alpha \right) - i\vec{\lambda}(\vec{x} + \vec{a}) \right\} \varphi(y),$$

$$y = \frac{\vec{\beta} \cdot (\vec{x} + \vec{a})}{x_0 + a_0} + \beta_0; \qquad \beta_0,\ \beta_a,\ a_0,\ a_a,\ \alpha \text{ are constants.} \tag{4.1.6}$$

Ungenerativity of (4.1.6) with the help of GS formulae N1–12 of Table 4.1.2 is quite obvious. One can easy prove ungenerativity of (4.1.6) with respect to GS formulae N13 if use is made of the identity

$$\exp\left\{ i\frac{\theta}{1 - \theta x_0}(\vec{\lambda} \cdot \vec{x}) \right\} (1 - \theta x_0)^{i\lambda_0} = (1 - \theta x_0)^{i\lambda_0} \exp\{ i\theta \vec{\lambda} \cdot \vec{x} \}$$

and to note that this procedure of GS is reduced to transformations in the space of parameters:

$$\vec{\beta} \to \vec{\beta}' = \frac{\vec{\beta}}{1 - \theta a_0}, \qquad \beta_0 \to \beta_0' = \beta_0 - \frac{\theta}{1 - \theta a_0}(\vec{\beta} \cdot \vec{a}),$$

$$\vec{a} \to \vec{a}' = \frac{\vec{a}}{1 - \theta a_0}, \qquad a_0 \to a_0' = \frac{a_0}{1 - \theta a_0},$$

$$\alpha \to \alpha' = \alpha + \frac{\theta}{1 - \theta a_0} \cdot \frac{\vec{a}^2}{2},$$

$$\varphi \to \varphi' = (1 - \theta a_0)^{i\lambda_0}\varphi.$$

In conclusion, we present in Table 4.1.3 G(1,3)-nonequivalent ansatze. One dimensional subalgebras of AG(1,3) are found in [12].

4.2. *Linear and nonlinear systems of PDEs invariant under the Schrödinger group* Sch(1,3).

Following [94] we shall describe linear and nonlinear systems of PDEs for complex multi-component wave function ψ invariant with respect to ASch(1,3)

Table 4.1.3. One-dimensional subalgebras of AG(1,3) and corresponding ansatze.

N	Algebra	Invariant variables $\omega=\{\omega_1,\omega_2,\omega_3\}$	Ansatze $\psi(x)=$
1.	P_0	x_1, x_2, x_3	$=\varphi(\omega)$
2.	$P_0+\beta m$	x_1, x_2, x_3	$=\exp\{im\beta x_0\}\varphi(\omega)$
3.	P_1	x_0, x_2, x_3	$=\varphi(\omega)$
4.	G_1	x_0, x_2, x_3	$=\exp\left\{\frac{i}{x_0}(\frac{1}{2}mx_1^2-\lambda_1 x_1)\right\}\varphi(\omega)$
5.	$G_1+\alpha P_0$	$2\alpha x_1+x_0^2, x_2, x_3$	$=\exp\left\{-\frac{im}{\alpha}(x_0x_1+\frac{1}{3\alpha}x_0^3-\right.$ $\left.-\frac{1}{m}\lambda_1 x_0)\right\}\varphi(\omega)$
6.	$G_1+\alpha P_2$	$\alpha x_1-x_0x_2, x_0, x_3$	$=\exp\left\{im\left(\frac{x_1^2}{2x_0}-\frac{\lambda_1 x_2}{\alpha m}\right)\right\}\varphi(\omega)$
7.	J_3	$(x_1^2+x_2^2)^{1/2}, x_0, x_3$	$=\exp\left\{i\widehat{S}_3\arctan\frac{x_1}{x_2}\right\}\varphi(\omega)$
8.	$J_3+\alpha m$	$(x_1^2+x_2^2)^{1/2}, x_0, x_3$	$=\exp\left\{i(\widehat{S}_3+\alpha m)\cdot\right.$ $\left.\cdot\arctan\frac{x_1}{x_2}\right\}\varphi(\omega)$
9.	$J_3+\alpha P_0$	$(x_1^2+x_2^2)^{1/2}, x_3,$ $x_0+\alpha\arctan\frac{x_1}{x_2}$	$=\exp\left\{i\widehat{S}_3\arctan\frac{x_1}{x_2}\right\}\varphi(\omega)$
10.	$J_3+\alpha P_0+$ $+\beta m$	$(x_1^2+x_2^2)^{1/2}, x_3,$ $x_0+\alpha\arctan\frac{x_1}{x_2}$	$=\exp\left\{i(\widehat{S}_3+\beta m)\arctan\frac{x_1}{x_2}\right\}\varphi(\omega)$
11.	$J_3+\alpha P_3$	$(x_1^2+x_2^2)^{1/2}, x_0,$ $x_3-\alpha\arctan\frac{x_1}{x_2}$	$=\exp\left\{i\widehat{S}_3\arctan\frac{x_1}{x_2}\right\}\varphi(\omega)$
12.	$J_3+\alpha G_3$	$(x_1^2+x_2^2)^{1/2}, x_0,$ $x_3-\alpha x_0\arctan\frac{x_1}{x_2}$	$=\exp\left\{i(\widehat{S}_3+\alpha\lambda_3)\arctan\frac{x_1}{x_2}+\right.$ $\left.+imx_3^2/2x_0\right\}\varphi(\omega)$
13.	$J_3+\alpha P_0+$ $+\beta G_3$	$(x_1^2+x_2^2)^{1/2},$ $\beta x_0^2+2\alpha x_3,$ $x_0+\alpha\arctan\frac{x_1}{x_2}$	$=\exp\left\{i(\widehat{S}_3+\beta\lambda_3)\arctan\frac{x_1}{x_2}-\right.$ $\left.-\frac{im\beta}{\alpha}\left(x_0x_3+\frac{\beta x_0^3}{3\alpha}\right)\right\}\varphi(\omega)$

(4.1.2) and satisfying the Schrödinger Equation (4.1.1). In other words, we shall find additional conditions having been putted on ψ, select from the manifold of solutions of the Schrödinger equation (4.1.1) submanifold on which it is realized a representation of Sch(1,3)-group with nonzero mass m and spin s.

Theorem 4.2.1. [94]. *System of Equations* (4.1.1) *for multi-component complex function* ψ *is invariant under* ASch(1,3) (4.1.2) *iff*

$$L\psi \equiv \left(\lambda_0 - \frac{3}{2}i - \frac{1}{m}\vec{\lambda}\cdot\vec{P}\right)\psi = 0 \tag{4.2.1}$$

$$\Lambda\psi \equiv \vec{\lambda}^2\psi \equiv (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)\psi = 0 \tag{4.2.2}$$

Proof. Necessity. Let us use condition of invariance in the form (10), (11), so that

$$[\overset{\vee}{S}, Q]\psi \Big|_{\overset{\vee}{S}\psi=0} = 0 \tag{4.2.3}$$

where Q is any operator from (4.1.2), $\overset{\vee}{S}$ is the Schrödinger operator (4.1.1). As a result we have

$$\begin{aligned} &[\overset{\vee}{S}, D] = 2i\overset{\vee}{S},\\ &[\overset{\vee}{S}, \Pi] = 2ix_0\overset{\vee}{S} + iL,\\ &[\overset{\vee}{S}, Q_\ell] = 0, \qquad Q_\ell = \{P_0,\, P_a,\, J_a,\, G_a,\, M\}, \end{aligned} \tag{4.2.4}$$

where operator L is given in (4.2.1), whence it follows (4.2.1).

Sufficiency. One has to calculate commutators of operator L (4.2.1) with operators Q (4.1.2). It yields

$$\begin{aligned} &[L, Q_s] = 0, \qquad Q_s = \{P_0,\, P_a,\, J_a, G_a,\, M, D\},\\ &[L, \Pi] = \frac{1}{im}\Lambda, \qquad \Lambda = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \end{aligned}$$

and to complete the proof it remains to make sure that

$$\begin{aligned} &[\Lambda, Q_s] = 0, \qquad Q_\ell = \{P_0,\, P_a,\, J_a,\, G_a,\, M\},\\ &[\Lambda, D] = -2i\Lambda, \qquad [\Lambda, \Pi] = -2ix_0\Lambda \end{aligned}$$

So, the system of PDEs (4.1.1), (4.2.1), (4.2.2), which for the sake of convenience we write down separately

$$\begin{aligned} &\overset{\vee}{S}\psi \equiv \left(P_0 - \frac{\vec{P}^2}{2m}\right)\psi = 0,\\ &L\psi \equiv \left(\lambda_0 - \frac{3}{2}i - \frac{1}{m}\vec{\lambda}\cdot\vec{P}\right)\psi = 0\\ &\Lambda\psi \equiv (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)\psi = 0 \end{aligned} \tag{4.2.5}$$

is invariant under ASch(1,3) (4.1.2), that is, on the manifold of solutions of the system (4.2.5) a representation of Sch(1,3) group with nonzero mass m and spin s is realized. In other words, Schrödinger equation (4.1.1) for multicomponent function is conditionally invariant under ASch(1,3) and conditions ensuring this invariance are (4.2.1), (4.2.2). (About conditional symmetry see §5.7).

It is clear from the stated above that the problem of describing explicit form of systems (4.2.5) is reduced to the problem of describing finite-dimensional representations of AE(1,3) (4.1.4), which in turn is reduced to the problem of describing finite-dimensional representations of AE(3):

$$\begin{aligned} &[\widehat{S}_a, \widehat{S}_b] = i\epsilon_{abc}\widehat{S}_c, \\ &[\lambda_a, \lambda_b] = 0, \qquad [\lambda_a, \widehat{S}_b] = i\epsilon_{abc}\widehat{S}_c \end{aligned} \tag{4.2.6}$$

Algebra $AE(3)$ (4.2.6) has two invariant operators [78]

$$C_1 = \widehat{S}_a\lambda_a, \qquad C_2 = \Lambda \equiv \lambda_a\lambda_a, \tag{4.2.7}$$

which in the case of finite-dimensional representations are nilpotent matrices, that is

$$C_1^{N_1} = 0, \qquad C_2^{N_2} = 0 \tag{4.2.8}$$

where N_1, N_2 are some integers.

Following [78] we shall consider only those representations of algebra (4.2.6) which include not more than two non-equivalent representations when reduced on AO(3). In this case matrices $\widehat{S}_a$ can be taken in the form

$$\widehat{S}_a \equiv \widehat{S}_a^{(n\,m)} = \begin{pmatrix} I_n \otimes S_a & \widehat{0}^{+} \\ \widehat{0} & I_m \otimes \Sigma_a \end{pmatrix} \tag{4.2.9}$$

where S_a and Σ_a are generators of irreducible representations of AO(3), that is matrices satisfying relations

$$\begin{aligned} &[S_a, S_b] = i\epsilon_{abc}S_c, \qquad S_aS_a = s(s+1), \\ &[\Sigma_a, \Sigma_b] = i\epsilon_{abc}\Sigma_c, \qquad \Sigma_a\Sigma_a = s'(s'+1); \end{aligned} \tag{4.2.10}$$

I_n and I_m are unit matrices of dimension $n\times n$, $m\times m$, and $\widehat{0}$ is the zero-matrix of dimension $m \times n$; symbol $\otimes$ means direct (Kronecker) product of matrices.

Theorem 4.2.2. [78]. *Any (to within equivalence) indecomposable finite-dimensional representation of* AE(3) (4.2.6) *including not more than two non-equivalent representations of subalgebra* AO(3) *and satisfying additional condition such that invariant operator* C_1 (4.2.7) *has not more than two orthogonal* *eigensubspaces, can be enumerated by the set of integers* (n, m, α), *provided* $\alpha = 1, 2$; $n \leq 4$, $m \leq 4$; $|n - m| \leq 2$, $n \cdot m \neq 9$.

Explicit form of corresponding matrices $\widehat{S}_a$ *and* λ_a *is given by*

$$\widehat{S}_a = \widehat{S}_a^{(n\,m\,\alpha)} = \widehat{S}_a^{(n\,m)}; \qquad \lambda_a = \lambda_a^{(n\,m\,\alpha)} \tag{4.2.11}$$

$$\lambda_a^{(n\,m\,1)} = \frac{1}{2s}\begin{pmatrix} a_1^{n\,m} \otimes S_a & a_2^{n\,m} \otimes K_a \\ a_3^{n\,m} \otimes K_a^+ & a_4^{n\,m} \otimes \Sigma_a \end{pmatrix}; \qquad \lambda_a^{(n\,m\,2)} = \left[\lambda_a^{(n\,m\,2)}\right]^+,$$

where matrices $\widehat{S}_a^{(n\,m)}$ *are determined in* (4.2.9); $s' = s - 1$; K_a *are matrices of dimension* $(2s - 1) \times (2s + 1)$ *satisfying relations*

$$\begin{aligned} K_a S_b - \Sigma_b K_a &= i\epsilon_{abc} K_c, \\ S_a S_b + K_a^+ K_b &= i\epsilon_{abc} S_c + s^2 \delta_{ab}; \end{aligned} \tag{4.2.12}$$

$a_{1\text{-}4}^{n\,m}$ *are matrices with matrix elements*

$$\left(a_1^{n\,m}\right)_{ij} = \begin{cases} \delta_{i-1\,j}, & n \geq m, \quad i, j \leq n \\ \dfrac{s-1}{s+1}\delta_{i-1\,j}, & n < m \end{cases}$$

$$\left(a_2^{n\,m}\right)_{ij} = \begin{cases} -(2s-1)^{-1/2}\delta_{i-2\,j}, & n > m, \quad i \leq n, \quad j \leq m \\ (2s+1)^{-1/2}\delta_{i\,j}, & n < m \\ k\delta_{i\,j+1}, & n = m \end{cases} \tag{4.2.13}$$

$$\left(a_3^{n\,m}\right)_{ij} = \begin{cases} (2s-1)^{-1/2}\delta_{i\,j}, & n > m, \quad i \leq m, \quad j \leq n \\ (2s+1)^{-1/2}\delta_{i-2\,j}, & n \leq m \end{cases}$$

$$\left(a_4^{n\,m}\right)_{ij} = \begin{cases} \dfrac{s+1}{s-1}\delta_{i-1\,j}, & n \geq m \\ \delta_{i-1\,j}, & n < m \end{cases}$$

where k *is an arbitrary parameter. Representations* $D(n,\, m,\, \alpha = 1)$ *and* $D(n,\, m,\, \alpha = 2)$ *are equivalent if and only if* $|n - m| = 1$.

Proof. One can show that (4.2.11) defines the general form of matrices λ_a satisfying commutations relations of AE(3) (4.2.6) when matrices $\widehat{S}_a$ have the form (4.2.9). From the commutativity of matrices λ_a and making use of relations

$$\begin{aligned} K_a S_b - K_b S_a &= i(s+1)\epsilon_{abc} K_c, \\ \Sigma_a K_b - \Sigma_b K_a &= i(1-s)\epsilon_{abc} K_c, \\ K_a K_b^+ - K_b K_a^+ &= -i(2s+1)\epsilon_{abc} \Sigma_c, \\ K_a^+ K_b - K_b^+ K_a &= i(2s-1)\epsilon_{abc} S_c, \end{aligned} \tag{4.2.14}$$

we find the following system to define matrices $a_{1-4}^{n\,m}$:

$$\begin{aligned}(a_1^{n\,m})^2 + (2s-1)a_2^{n\,m}a_3^{n\,m} &= 0,\\ (s+1)a_1^{n\,m}a_2^{n\,m} - (s-1)a_2^{n\,m}a_4^{n\,m} &= 0,\\ (s+1)a_3^{n\,m}a_1^{n\,m} - (s-1)a_4^{n\,m}a_3^{n\,m} &= 0,\\ (a_4^{n\,m})^2 - (2s+1)a_3^{n\,m}a_2^{n\,m} &= 0.\end{aligned} \tag{4.2.15}$$

The condition concerning the operator C_1 (4.2.7) (that is, C_1 should have not more than two invariant eigensubspaces) means that matrices $a_1^{n\,m}$, $a_3^{n\,m}$ should not be reducible.

All nonequivalent solution of Equations (4.2.15) are given in (4.2.13), the greater number from the set (n, m) coinciding with the index of nilpotentness N_1 of the operator C_1 (4.2.7). Below we present the explicit form of matrices S_a and K_a in the basis $|s, s_3>$, that is in the basis in which operators $S^2 \equiv S_aS_a$ and S_3 are diagonal [120, 78, 87*]:

$$\begin{aligned}&S^2|s,s_3>= s(s+1)|s,s_3>, \qquad S_3|s,s_3>= s_3|s,s_3>,\\ &S_1|s,s_3>= a^s_{s_3\,s_3+1}|s,s_3+1> + a^s_{s_3\,s_3-1}|s,s_3-1>,\\ &S_2|s,s_3>= ia^s_{s_3\,s_3+1}|s,s_3+1> - ia^s_{s_3\,s_3-1}|s,s_3-1>,\\ &K_1|s,s_3>= æ_s^{s\,s-1}|s-1,s_3> - iæ^{s\,s-1}_{s_3\,s_3-2}|s-1,s_3-2>,\\ &K_2|s,s_3>= iæ_{s_3}^{s\,s-1}|s-1,s_3> - iæ^{s\,s-1}_{s_3\,s_3-2}|s-1,s_3-2>,\\ &K_3|s,s_3>= f_{s_3}^{s\,s-1}|s-1,s_3>\end{aligned} \tag{4.2.16}$$

where:

$$\begin{aligned}&s_3 = -s, -s+1, \dots, s; \qquad a^s_{s_3\,s_3\pm1} = \tfrac{1}{2}\sqrt{s_3(s_3\pm1) - s(s+1)},\\ &f_{s_3}^{s\,s-1} = \sqrt{s_3(2s-s_3)}; \qquad æ_{s_3}^{s\,s-1} = \tfrac{1}{2}\sqrt{(2s-s_3)(2s+1-s_3)},\\ &æ^{s\,s-1}_{s_3\,s_3-2} = \tfrac{1}{2}\sqrt{s_3(s_3+1)},\end{aligned} \tag{4.2.17}$$

Explicit form of matrices Σ_a can be obtained from (4.2.16), (4.2.17) by replacement $s \to s' = s-1$.

Theorems 4.2.1, 4.2.2 allow us to write down explicitely systems (4.2.5). Below we present some of such systems. It is convenient to enumerate these systems by indicating representation which is realized by matrices $\widehat{S}_a$ (4.2.9). Below, φ and χ denote, unless otherwise stated, $2s+1$ and $(2s-1)$-component column functions.

1) Representation $D(s) \oplus D(s)$:

$$\widehat{S}_a = \begin{pmatrix} S_a & 0 \\ 0 & S_a \end{pmatrix}, \qquad \lambda_a = \begin{pmatrix} 0 & 0 \\ S_a & 0 \end{pmatrix}, \qquad \lambda_0 = \frac{i}{2}\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$$

$$\check{S}\psi = 0, \qquad \psi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \tag{4.2.18}$$

$$im\varphi_2 - \big(\vec{S}\cdot\vec{P}\big)\varphi_1 = 0$$

In particular, for spin $s = 1/2$ we have $S_a = \frac{1}{2}\sigma_a$, σ_a are the Pauli matrices, and Equations (4.2.18) take the form

$$\check{S}\psi = 0, \qquad \psi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}; \text{ where } \varphi_1,\ \varphi_2 \text{ are two-component functions,}$$

$$2im\varphi_2 - (\vec{\sigma}\cdot\vec{P})\varphi_1 = 0$$

or equivalently

$$\begin{aligned} P_0\varphi_1 - i(\vec{\sigma}\cdot\vec{P})\varphi_2 &= 0, \\ 2im\varphi_2 - (\vec{\sigma}\cdot\vec{P})\varphi_1 &= 0 \end{aligned} \tag{4.2.19}$$

Equations (4.2.19) are known as Levi-Leblond ones for nonrelativistic particle with spin $s = 1/2$ [144].

2) Representation $D(s) \oplus D(s-1)$:

a) $\hat{S}_a = \begin{pmatrix} S_a & 0 \\ 0 & \Sigma_a \end{pmatrix}, \qquad \lambda_a = \begin{pmatrix} 0 & 0 \\ K_a & 0 \end{pmatrix}, \qquad \lambda_0 = \frac{i}{2}\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$

$$\check{S}\psi = 0, \qquad \psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \tag{4.2.20}$$

$$im\chi - \big(\vec{K}\cdot\vec{P}\big)\varphi = 0$$

In particular, for spin $s = 1$ we have $(S_a)_{bc} = i\epsilon_{abc}$:

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$K_1 = (i\,0\,0), \qquad K_2 = (0\,i\,0), \qquad K_1 = (0\,0\,i) \tag{4.2.21}$$

and Equations (4.2.20) take the form

$$\begin{aligned} \hat{S}\psi &= 0, \\ m\chi - (\vec{P}\cdot\vec{\varphi}) &= 0 \end{aligned} \tag{4.2.22}$$

$\psi =$column$(\vec{\varphi}, \chi)$, χ is a scalar function. Equations (4.2.22) can be considered as the Galilean counterpart of the Proca equations [78].

b) $\hat{S}_a = \begin{pmatrix} S_a & 0 \\ 0 & \Sigma_a \end{pmatrix}, \qquad \lambda_a = \begin{pmatrix} 0 & K_a^+ \\ 0 & 0 \end{pmatrix}, \qquad \lambda_0 = \frac{i}{2}\begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$

$$\overset{\vee}{S}\psi = 0, \qquad \psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \tag{4.2.23}$$

$$im\varphi - (\vec{K}^{+} \cdot \vec{P})\chi = 0$$

3) Representation $D(s) \oplus D(s) \oplus D(s-1)$:

a)

$$\widehat{S}_a = \begin{pmatrix} S_a & 0 & 0 \\ 0 & S_a & 0 \\ 0 & 0 & \Sigma_a \end{pmatrix}, \quad \lambda_a = -\frac{i}{2s}\begin{pmatrix} 0 & 0 & 0 \\ S_a & 0 & 0 \\ K_a & 0 & 0 \end{pmatrix}, \quad \lambda_0 = \frac{i}{2}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix},$$

$$\overset{\vee}{S}\psi = 0, \qquad \psi = \text{column}(\varphi_1\ \varphi_2\ \chi)$$

$$2ms\varphi_2 + (\vec{S} \cdot \vec{P})\varphi_1 = 0 \tag{4.2.24}$$

$$2ms\chi + (\vec{K} \cdot \vec{P})\varphi_1 = 0$$

Making use (4.2.12) one can rewrite Equations (4.2.24) as follows

$$sP_0\varphi_1 + (\vec{S} \cdot \vec{P})\varphi_2 + (\vec{K}^{+}\vec{P})\chi = 0$$

$$2ms\varphi_2 + (\vec{S} \cdot \vec{P})\varphi_1 = 0 \tag{4.2.25}$$

$$2ms\chi + (\vec{K} \cdot \vec{P})\varphi_1 = 0$$

System (4.2.25) is known as Hagen-Herley equations [120] and describes a nonrelativistic particle with mass m and arbitrary spin s.

b)
$$\widehat{S}_a = \begin{pmatrix} S_a & 0 & 0 \\ 0 & S_a & 0 \\ 0 & 0 & \Sigma_a \end{pmatrix}, \quad \lambda_a = \frac{i}{2s}\begin{pmatrix} 0 & S_a & K_a^{+} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_0 = \frac{i}{2}\begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

$$\overset{\vee}{S}\psi = 0, \qquad \psi = \text{column}(\varphi_1\ \varphi_2\ \chi) \tag{4.2.26}$$

$$2ms\varphi_1 - (\vec{S} \cdot \vec{P})\varphi_2 + (\vec{K}_a^{+}\vec{P})\chi = 0.$$

4) Representation $D(s) \oplus D(s) \oplus D(s) \oplus D(s-1)$:

a)
$$\widehat{S}_a = \begin{pmatrix} S_a & 0 & 0 & 0 \\ 0 & S_a & 0 & 0 \\ 0 & 0 & S_a & 0 \\ 0 & 0 & 0 & \Sigma_a \end{pmatrix}, \quad \lambda_0 = \frac{i}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\lambda_a = \begin{pmatrix} 0 & 0 & 0 & 0 \\ S_a & 0 & 0 & 0 \\ 0 & S_a & -K_a^+/\sqrt{2s-1} & 0 \\ K_a/\sqrt{2s-1} & 0 & 0 & 0 \end{pmatrix} \tag{4.2.27}$$

In this case $\Lambda = \lambda_a\lambda_a \neq 0$. To find Λ one has to use the relations which follow from (4.2.12), (4.2.14) [94]:

$$\begin{aligned} &K_a^+K_a = s(2s-1), \qquad K_aK_a^+ = s(2s+1) \\ &\Sigma_a\Sigma_b + K_aK_b^+ = -is\epsilon_{abc}\Sigma_c + s^2\delta_{ab} \\ &\Sigma_aK_a = 0, \qquad K_aS_a = 0 \\ &K_aK_b^+ = -\frac{1}{2}(\Sigma_a\Sigma_b + \Sigma_b\Sigma_a) - \frac{i}{2}(2s+1)\epsilon_{abc}\Sigma_c + s^2\delta_{ab} \\ &K_a^+K_b = -\frac{1}{2}(S_aS_b + S_bS_a) + \frac{i}{2}(2s-1)\epsilon_{abc}\Sigma_c + s^2\delta_{ab} \end{aligned} \tag{4.2.28}$$

Having used (4.2.28) we find Λ for λ_a from (4.2.27):

$$\Lambda = \lambda_a\lambda_a = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ s^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.2.29}$$

Finally, system (4.2.5) in this case takes the form

$$\begin{aligned} &\check{S}\psi = 0, \qquad \psi = \text{column}(\varphi_1\,\varphi_2\,\varphi_3,\,\chi) \\ &im\varphi_3 - (\vec{S}\cdot\vec{P})\varphi_2 + \frac{1}{\sqrt{2s-1}}(\vec{K}^+\cdot\vec{P})\chi = 0 \\ &\varphi_1 = 0 \end{aligned} \tag{4.2.30}$$

In much the same way one can continue constructing Sch(1,3)-invariant systems of PDEs of the type (4.2.5).

Now, let us consider the following problem: describe the most general form of functions F, G, R depending on ψ and ψ^+ so that the system of PDEs

$$\begin{aligned} &\check{S}\psi \equiv \left(P_0 - \frac{1}{2m}\vec{P}^2\right)\psi = F \\ &L\psi \equiv \left(\lambda_0 - \frac{3}{2}i - \frac{1}{m}\vec{\lambda}\cdot\vec{P}\right)\psi = G \\ &\Lambda\psi \equiv (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)\psi = R \end{aligned} \tag{4.2.31}$$

would be invariant under the Schrödinger group Sch(1,3). To solve this problem we shall use the final transformations of Sch(1,3) group written in Table 4.1.2. We restrict ourselves by considering several particular cases.

Theorem 4.2.3. [185, 94]. *Equations*

$$\begin{aligned} \check{S}\varphi_1 = F_1, \qquad \check{S}\varphi_2 = F_2 \\ im\varphi_2 - (\vec{S}\cdot\vec{P})\varphi_1 = G \end{aligned} \tag{4.2.32}$$

where $\varphi_1 = \varphi_1(x)$, $\varphi_2 = \varphi_2(x)$ *are* $(2s+1)$*-component functions;* F_1, F_2, G *are* $(2s+1)$*-component functions depending on* φ_1, φ_1^+, φ_2, φ_2^+ *are invariant under the Schrödinger group* Sch(1,3) *iff*

$$\begin{aligned} &F_1 = æF\left(\frac{\omega_1}{\omega_2}\right)\omega_2^2\varphi_1, \qquad æ = \text{const} \\ &F_2 = æF\left(\frac{\omega_1}{\omega_2}\right)\omega_2\varphi_2, \\ &G = 0, \end{aligned} \tag{4.2.33}$$

where F *is an arbitrary smooth function;*

$$\omega_1 = (\varphi_1^+\varphi_2 + \varphi_2^+\varphi_1)^{1/4}, \qquad \omega_2 = (\varphi_1^+\varphi_1)^{1/3}. \tag{4.2.34}$$

Proof. First of all we note that system (4.2.32) is a nonlinear generalization of Equations (4.2.18) and, as easy to see, it admits operators $\{P_0, P_a, J_a, M\}$ from ASch(1,3) (4.1.2) if functions F_1, F_2, G depend on $\varphi_1^+\varphi_1$, $\varphi_2^+\varphi_2$, $\varphi_1^+\varphi_2$, $\varphi_2^+\varphi_1$. Hence, to prove the theorem it is sufficient to consider the case with operator Π only.

Using formulae N8 – 10, 13 from Table 4.1.2 we write down Galilean and projective transformations for functions φ_1 and φ_2:

$$\begin{aligned} &\varphi_1'(x') = \exp\left\{im\left(\vec{v}\cdot\vec{x} + \tfrac{1}{2}\vec{v}^2x_0\right)\right\}\varphi_1(x), \\ &\varphi_2'(x') = \exp\left\{im\left(\vec{v}\cdot\vec{x} + \tfrac{1}{2}\vec{v}^2x_0\right)\right\}\left(\varphi_2(x) - i(\vec{S}\cdot\vec{v})\varphi_1(x)\right), \\ &\qquad x_0' = x_0 \qquad \vec{x}' = \vec{x} + \vec{v}x_0; \end{aligned} \tag{4.2.35}$$

$$\begin{aligned} &\varphi_1'(x') = (1-\theta x_0)^{3/2}\exp\left\{\frac{im\vec{x}^2}{2}\frac{\theta}{1-\theta x_0}\right\}\varphi_1(x), \\ &\varphi_1'(x') = (1-\theta x_0)^{5/2}\exp\left\{\frac{im\vec{x}^2}{2}\frac{\theta}{1-\theta x_0}\right\}\left(\varphi_2(x) - i\frac{\theta}{1-\theta x_0}(\vec{S}\cdot\vec{x})\varphi_1(x)\right), \\ &\qquad x_0' = \frac{x_0}{1-\theta x_0}, \qquad \vec{x}' = \frac{\vec{x}}{1-\theta x_0}. \end{aligned} \tag{4.2.36}$$

Under the influence of projective transformations we find that

$$P_0 \to P_0' \equiv i\frac{\partial}{\partial x_0'} = (1-\theta x_0)^2 P_0 + \theta(1-\theta x_0)\vec{x}\cdot\vec{P}$$

$$\vec{P} \to \vec{P}' = -i\vec{\nabla}' = (1-\theta x_0)\vec{P} \tag{4.2.37}$$

and system (4.2.32) takes the form

$$(1-\theta x_0)^{7/2} e^{if} \check{S}\varphi_1 = F_1', \qquad f \stackrel{\text{def}}{=} \frac{m\vec{x}^2}{2}\frac{\theta}{1-\theta x_0};$$

$$(1-\theta x_0)^{9/2} e^{if}\left[\check{S}\varphi_2 - \frac{\theta}{1-\theta x_0}\frac{1}{m}\Big(im\varphi_2 - (\vec{S}\cdot\vec{P})\varphi_1 + im(\vec{S}\cdot\vec{x})\check{S}\varphi_1\Big)\right] = F_2',$$

$$(1-\theta x_0)^{5/2} e^{if}\left[im\varphi_2 - (\vec{S}\cdot\vec{P})\varphi_1\right] = G'$$

whence it follows that

$$F_1' = (1-\theta x_0)^{7/2} e^{if} F_1, \qquad f \equiv \frac{m\vec{x}^2}{2}\frac{\theta}{1-\theta x_0};$$

$$F_2' = (1-\theta x_0)^{9/2} e^{if}\left[F_2 - i\frac{\theta}{1-\theta x_0}(\vec{S}\cdot\vec{x})F_1 - \frac{\theta}{m(1-\theta x_0)}G\right] \tag{4.2.38}$$

$$G' = (1-\theta x_0)^{5/2} e^{if} G$$

Comparing (4.2.38) with (4.2.36) one can conclude that functions F_1, F_2, G should have the structure

$$F_1 = \phi\varphi_1, \qquad F_2 = \phi\varphi_2, \qquad G = 0 \tag{4.2.39}$$

where ϕ is a scalar function which is transformed as follows

$$\phi' = (1-\theta x_0)^2\phi \tag{4.2.40}$$

From (4.2.36) it is easy to see that function ϕ can be constructed only from two arguments $\varphi_1^+\varphi_1$ and $\varphi_1^+\varphi_2 + \varphi_2^+\varphi_1$. Since

$$(\varphi_1^+\varphi_1)' = (1-\theta x_0)^3(\varphi_1^+\varphi_1), \qquad (\varphi_1^+\varphi_2+\varphi_2^+\varphi_1)' = (1-\theta x_0)^4(\varphi_1^+\varphi_2+\varphi_2^+\varphi_1)$$

function ϕ should have the form

$$\phi = æ F\left(\frac{\omega_1}{\omega_2}\right)\omega_2^2$$

where æ is a constant; ω_1, ω_2 are given in (4.2.34), F is an arbitrary smooth function. The theorem is proved.

Below we present some more Sch(1,3)-invariant nonlinear systems of PDEs constructed in the same spirit as (4.2.32) [94]:

$$\begin{aligned} P_0\varphi_1 - i(\vec{\sigma}\cdot\vec{P})\varphi_2 &= æF\omega_2\varphi_2 + \mu G\omega_2^2\varphi_1 \\ 2im\varphi_2 - (\vec{\sigma}\cdot\vec{P})\varphi_1 &= iæF\omega_2\varphi_1 \end{aligned} \tag{4.2.41}$$

where æ, μ are constants; φ_1 and φ_2 are two-component functions, $\vec{\sigma}$ are Pauli matrices, F and G are arbitrary smooth functions of ω_1/ω_2; ω_1 and ω_2 are given in (4.2.34). When $\mu = æ = 0$ Equations (4.2.41) coincide with those of Levi-Leblond (4.2.19).

System

$$\begin{aligned} &\check{S}\psi = æ(\varphi^+\varphi)^{2/3}\psi, \qquad \psi = \begin{pmatrix}\varphi\\ \chi\end{pmatrix} \\ &im\chi - (\vec{K}\cdot\vec{P})\varphi = 0 \end{aligned} \tag{4.2.42}$$

coincides with (4.2.20) when $æ = 0$.

System

$$\begin{aligned} &\check{S}\psi = æF\left(\frac{(\varphi_1^+\varphi_2 - \varphi_2^+\varphi_1)^{1/2}}{(\varphi_1^+\varphi_1)^{2/3}}\right)(\varphi_1^+\varphi_1)^{2/3}\psi, \quad \psi = \begin{pmatrix}\varphi_1\\ \varphi_2\\ \chi\end{pmatrix} \\ &2ms\varphi_2 + (\vec{S}\cdot\vec{P})\varphi_1 = 0 \\ &2ms\xi + (\vec{K}\cdot\vec{P})\varphi_1 = 0 \end{aligned} \tag{4.2.43}$$

coincides with (4.2.24) when $æ = 0$.

System

$$\begin{aligned} &sP_0\varphi_1 + (\vec{S}\cdot\vec{P})\varphi_2 + (\vec{K}^+\vec{P})\chi = æF\left(\frac{(\varphi_1^+\varphi_2 - \varphi_2^+\varphi_1)^{1/2}}{(\varphi_1^+\varphi_1)^{2/3}}\right)(\varphi_1^+\varphi_1)^{2/3}\psi, \\ &2ms\varphi_2 + (\vec{S}\cdot\vec{P})\varphi_1 = 0, \\ &2ms\chi + (\vec{K}\cdot\vec{P})\varphi_1 = 0 \end{aligned} \tag{4.2.44}$$

coincides with the equations of Hagen-Herley (4.2.25) when $æ = 0$.

4.3. *Systems of second-order PDEs invariant under the Galilei group*

Following [78] we shall obtain two classes of Galilean invariant systems of second-order PDEs. The essential difference between these equations and those studied in the previous paragraph consists in the following. Equations to be considered give more complete description of interection of a particle having spin with an external electromagnetic field. Equations of §4.2 describe spin-orbit coupling but fail to take into account quadrupole and Darwin interections. It is generally accepted to think that such interections are truly relativistic effects, and, for instance, if the particle spin $s = 1/2$, only the Dirac relativistic equation describes them adequately. Equations considered below refute this widespread opinion.

As was stated above (see §4.1) the Schrödinger equation (4.1.1) is invariant under the Galilei group and describes a free spinless particle with mass m. Naturally the question arises: are there equations of the form

$$P_0\psi \equiv i\frac{\partial}{\partial x_0}\psi = H_s(\vec{P})\psi, \qquad (4.3.1)$$

where $H_s(\vec{P})$ is a differential operator, $\psi = \psi(x_0, \vec{x})$ is a complex multi-component function, on the manifold of solutions of which a representation of the Galilei group G(1,3) with nonzero spin would be realized?

We seek for Galilean-invariant equations (4.3.1) in the space of $2(2s+1)$-component quadratic integrable function

$$\psi = \{\psi_1(x_0, \vec{x}), \ldots, \psi_{2(2s+1)}(x_0, \vec{x})\} \qquad (4.3.2)$$

The problem of description of such equations we solve in two, generally speaking, nonequivalent approaches. In the first one the problem is to find (within equivalence) all operators H_s^I satisfying the conditions

$$[P_0 - H_s, Q_A]\psi = 0, \qquad (4.3.3)$$

where $Q_A = \{P_0, P_a, J_a, G_a, M\}$ are given in (4.1.2), provided

$$\widehat{S}_a = \begin{pmatrix} S_a & 0 \\ 0 & S_a \end{pmatrix}, \qquad \lambda_a = k(\widehat{\sigma}_1 + i\widehat{\sigma}_2)S_a, \qquad (4.3.4)$$

S_a are matrices of irreducible representation $D(s)$ of AO(3); $\widehat{\sigma}_a$ are $2(2s+1)$-dimensional Pauli matrices

$$\widehat{\sigma}_0 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \quad \widehat{\sigma}_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad \widehat{\sigma}_2 = i\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad \widehat{\sigma}_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad (4.3.5)$$

I and 0 are unit and zero matrices of dimension $(2s+1) \times (2s+1)$; k is an arbitrary complex parameter.

Having substituted operators from AG(1,3) (4.1.2) into (4.3.3) we find that it is fulfilled if

$$[H_s^I, P_a] = [H_s^I, J_a] = 0 \qquad (4.3.6)$$

$$[H_s^I, G_a] = iP_a \tag{4.3.7}$$

The second approach to the problem is to find all operators H_a^{II} so that the operators

$$\begin{aligned} P_0^{II} &= H_s^{II}, \qquad P_a^{II} = P_a = -i\partial_a, \qquad M^{II} = \widehat{\sigma}_1 m \\ J_a^{II} &= J_a = (\vec{x}\times\vec{P})_a + \widehat{S}_a, \\ G_a^{II} &= x_0 P_a - M x_a + \lambda_a^{II} \end{aligned} \tag{4.3.8}$$

will be the generators of AG(1,3). In (4.3.8) λ_a^{II} are operators to be found; $\widehat{S}_a$ are given in (4.3.4).

We require the operators (4.3.8) to be Hermitian with respect to the usual scalar product

$$(\psi_1, \psi_2) = \int \psi_a^+(x_0, \vec{x})\psi_2(x_0, \vec{x}) d^3x. \tag{4.3.9}$$

As distinguished from (4.1.2) operators H_a^I, G_a^I are non-Hermitian with respect to (4.3.9) but Hermitian with respect to

$$(\psi_1, \psi_2) = \int \psi_a^+(x_0, \vec{x}) M \psi_2(x_0, \vec{x}) d^3x. \tag{4.3.10}$$

where M is a positively defined operator which will be found below. We require the Hamiltonian H_a^{II} to satisfy the conditon

$$\left(H_s^{II}\right)^2 = \left(m + \frac{\vec{P}^2}{2m}\right)^2 \tag{4.3.11}$$

It means that the intrinsic energy of particle (the eigenvalues of the invariant operator $C_1 = 2mP_0 - P_a P_a$) coincides with its mass.

Theorem 4.3.1. [78]. *The most general form of the Hamiltonian H_s^I satisfying together with the generators of* AG(1,3) (4.1.2), (4.3.4) *commutation relations* (4.3.6), (4.3.7) *is as follows*

$$H_s^I = \widehat{\sigma}_1 m a + 2ik\widehat{\sigma}_3 \widehat{S}_a P_a + \frac{1}{2m} C_{ab} P_a P_b \tag{4.3.12}$$

$$\widetilde{H}_s^I = \frac{a}{2}(\widehat{\sigma}_1 - i\widehat{\sigma}_2)m + \widehat{\sigma}_3 \widetilde{a} m + 2\widetilde{a}k(\widehat{\sigma}_2 - i\widehat{\sigma}_1)\widehat{S}_a P_a + \frac{1}{2m} C_{ab} P_a P_b \tag{4.3.13}$$

where

$$C_{ab} = \delta_{ab} - 2ak^2(\widehat{\sigma}_1 + i\widehat{\sigma}_2)(S_a S_b + S_b S_a) \tag{4.3.14}$$

a, $\widetilde{a}$, k are arbitrary parameters.

Proof. It is convenient to seek for H_s^I in a representation where $\lambda_a = 0$. It is achieved by the transformation

$$H_s^I \to \left(H_s^I\right)' = VH_s^I V^{-1}, \qquad P_a \to P_a' = VP_aV^{-1} = P_a, \tag{4.3.15}$$

$$J_a^I \to \left(J_a^I\right)' = VJ_a^I V^{-1} = J_a^I = J_a, \qquad \left(G_a^I\right)' = VG_aV^{-1} = x_0P_a - mx_a,$$

where

$$V = \exp\left\{\frac{i}{m}\vec{\lambda}\cdot\vec{P}\right\} = 1 + \frac{i}{m}\vec{\lambda}\cdot\vec{P} \tag{4.3.16}$$

From (4.3.6), (4.3.7), (4.3.15) we easy find the general form of operator $\left(H_s^I\right)'$

$$\left(H_s^I\right)' = \frac{\vec{P}^2}{2m} + A, \qquad A = \widehat{\sigma}_\mu a^\mu m, \tag{4.3.17}$$

where a^μ are arbitrary complex coefficients.

So, Equation (4.3.1) has in the representation (4.3.15) the form

$$P_0\psi' = \left(\frac{\vec{P}^2}{2m} + m\widehat{\sigma}_\mu a^\mu\right)\psi', \qquad \psi' = V\psi \tag{4.3.18}$$

Now we show that matrix A from (4.3.17) can be reduced to

$$A = \widehat{\sigma}_3\widetilde{a}m + \frac{a}{2}\left(\widehat{\sigma}_1 - i\widehat{\sigma}_2\right)m, \tag{4.3.19}$$

or

$$A = \widehat{\sigma}_1 am, \tag{4.3.20}$$

where a, $\widetilde{a}$ are arbitrary coefficients. Indeed, one can always turn coefficient a_0 into zero:

$$\begin{aligned}\left(H_s^I\right)' \to \exp\{ia_0mx_0\}\left(H_s^I\right)'\exp\{-ia_0mx_0\} + \\ + \exp\{ia_0mx_0\}P_0\exp\{-ia_0mx_0\} = \left(H_s^I\right)' - a_0m\end{aligned} \tag{4.3.21}$$

Further, there are three possibilities

$$A \equiv 0, \qquad a_b = 0; \tag{4.3.22}$$

$$A^2 = a_1^2 + a_2^2 + a_3^2 = 0, \qquad a_b \neq 0; \tag{4.3.23}$$

$$A^2 = a_1^2 + a_2^2 + a_3^2 = a^2 \neq 0; \tag{4.3.24}$$

The first case gives (4.3.19) provided $a = \widetilde{a} = 0$. The second case (4.3.23) corresponds to non-unitary representation of the Galilei group because the invariant operator $C_1 = 2mP_0 - P_aP_a = 2m^2A$ is a nilpotent matrix, but such cases we do not consider. Let in (4.3.24) $a_1^2 + a_2^2 \neq 0$. Having made transformation

$$
\begin{aligned}
&A \to V_1 A V_1^{-1},\\
&V_1 = b + i\hat{\sigma}_3 c + (\hat{\sigma}_1 + i\hat{\sigma}_2)d; \qquad b = \cos\varphi, \quad c = \sin\varphi \qquad (4.3.25)\\
&V_1^{-1} = b - i\hat{\sigma}_3 - (\hat{\sigma}_1 + i\hat{\sigma}_2)d.
\end{aligned}
$$

$$
\varphi = \tfrac{1}{2}\arctan\left(\frac{a_1 + 2d^2}{a_2 - 2id}\right), \qquad d = \sqrt{\frac{d_3^2(a_1^2 - ia^2)}{4a(a_1^2 + a_2^2)}} \qquad (4.3.26)
$$

we obtain (4.3.20). If $a_1^2 + a_2^2 = 0$, then matrix A is transformed by virtue of the operator

$$
\begin{aligned}
&V_2 = 1 + (\hat{\sigma}_1 + i\hat{\sigma}_2)\cdot f/2, \qquad V_2^{-1} = 1 - (\hat{\sigma}_1 + i\hat{\sigma}_2)\cdot f/2\\
&f = \begin{cases} a_1/a_3, & a_2 = ia_1\\ 0, & a_2 = -ia_1 \end{cases}
\end{aligned} \qquad (4.3.27)
$$

to (4.3.19). Operators (4.3.25), (4.3.27) satisfy conditions

$$
V_\alpha \lambda_a V_\alpha^{-1} = æ_\alpha \lambda_a, \qquad \alpha = 1, 2 \qquad (4.3.28)
$$

where matrices λ_a are given in (4.3.4); $æ_1 = \exp\{2i\varphi\}$, $æ_2 = 1$, parameter φ is determined in (4.3.26). One can make sure that there is no operator satisfying (4.3.28) and transforming (4.3.19) to (4.3.20).

Acting on (4.3.17), (4.3.19), (4.3.20) by transformation inverse to (4.3.15) we get Hamiltonians (4.3.12), (4.3.13) which evidently satisfy conditions (4.3.6), (4.3.7). The theorem is proved.

One can easy make sure that Equations (4.3.1), (4.3.12), (4.3.13) are Galilean invariant using transformations (see Table 4.1.2, N8–10)

$$
\begin{aligned}
&x_0' = x_0, \qquad \vec{x}' = \vec{x} + \vec{v}x_0\\
&\psi'(x') = \exp\left\{im\left(\vec{v}\cdot\vec{x} + \frac{\vec{v}^2}{2}x_0\right)\right\}(1 - \vec{\lambda}\cdot\vec{v})\psi(x)
\end{aligned} \qquad (4.3.29)
$$

generated by G_a (4.1.2), (4.3.4). Since

$$
P_0' = P_0 + \vec{v}\cdot\vec{P}, \qquad \vec{P}' = \vec{P} \qquad (4.3.30)
$$

it follows that equation

$$
P_0'\psi'(x') - H_s^I(\vec{P}')\psi'(x') = \left(P_0 + \vec{v}\cdot\vec{P} - H_s^I(\vec{P})\right)\psi'(x') = 0 \qquad (4.3.31)
$$

are identically satisfied on sets of solutions of Equations (4.3.1), (4.3.12), (4.3.13).

It is not difficult to calculate (the most simple it may be done in the representation (4.3.15)) that Casimir operators of AG(1,3) (4.1.2), (4.3.4) have the following eigenvalues

$$C_1 = 2MP_0 - P_aP_a = \pm m; \qquad C_2 = M = m$$
$$C_3 = W_aW_a = (MJ_a - \epsilon_{abc}P_bJ_c)^2 = m^2 s(s+1). \tag{4.3.32}$$

It allows us to interpret Equations (4.3.1), (4.3.12), (4.3.13) as those which describe a free nonrelativistic particle of mass m and spin s and intrinsic energy $\pm m$.

Now we shall find Hamiltonians H_s^{II} which are Hermitian in the metrics (4.3.9) and satisfy together with generators (4.3.8) relations (4.1.3), (4.3.11).

Theorem 4.3.2. [78]. *The most general form of the Hamiltonian H_s^{II} (within equivalence) Hermitian in the metrics* (4.3.9) *and satisfying relations* (4.3.8), (4.1.3), (4.3.11) *is as follows*

$$\begin{aligned} H_s^{II} =&\hat{\sigma}_1\left(m + \frac{\vec{P}^2}{2m} - \frac{\vec{\hat{S}}\cdot\vec{P}}{s^2 m}\sin^2\theta_s\right) + \hat{\sigma}_2\frac{\sqrt{2}\sin\theta_s}{s}\left(\vec{\hat{S}}\cdot\vec{P}\right) - \\ &- \hat{\sigma}_3\left(a_s\frac{\vec{P}^2}{2m} + b_s\left(\vec{\hat{S}}\cdot\vec{P}\right)^2 \Big/ 2ms^2\right), \end{aligned} \tag{4.3.33}$$

where

$$a_{1/2} = \sin 2\theta_{1/2}, \qquad b_{1/2} = 0; \qquad a_1 = 1, \qquad b_1 = \sin 2\theta_1; \tag{4.3.34}$$

$$a_{3/2} = b_{3/2} - \tfrac{5}{4}\sin\theta_{3/2} = -\tfrac{1}{8}\sin 2\theta_{3/2} - \tfrac{3}{4}\sin\theta_{3/2}\left(1 - \tfrac{1}{9}\sin\theta_{3/2}\right)^{1/2};$$

$a_s = b_s = \theta_s = 0$, $s > 3/2$ *and* $\theta_{1/2}, \theta_1, \theta_{3/2}$ *are arbitrary parameters.*

Proof. First of all, we prove that H_s^{II} does not include differential operators of more than second order. To do it we assume that $H_s^{II} = \sum_{i=0}^{N} H_i$, where H_i contains derivatives of ith order only. Then (4.3.11) results in

$$H_N H_N = H_N^{+} H_N = 0, \qquad H_N = 0, \quad \text{or} \quad N > 2 \tag{4.3.35}$$

It is convenient to represent operator H_s^{II} as an expansion on spin matrices $\hat{S}_a$ (4.3.4) and $2(2s+1)$-dimensional Pauli matrices (4.3.5)

$$H_s^{II} = \left(a_\mu m + b_\mu\frac{\vec{P}^2}{2m} + c_\mu\left(\vec{\hat{S}}\cdot\vec{P}\right) + d_\mu\frac{\left(\vec{\hat{S}}\cdot\vec{P}\right)^2}{2m}\right)\hat{\sigma}^\mu \tag{4.3.36}$$

where a_μ, b_μ, c_μ, d_μ are arbitrary real coefficients.

If to introduce the operators of orthogonal projection

$$\Lambda_r = \prod_{r \neq r'} \frac{(\vec{\hat{S}} \cdot \vec{P})/P - r'}{r - r'}; \qquad r, r' = -s, -s+1, \ldots, s$$

which satisfy conditions of orthonormality and completeness

$$\Lambda_r \Lambda_{r'} = \delta_{rr'} \Lambda_r, \qquad \sum_r \Lambda_r = 1, \qquad \sum_r r^k \Lambda_r = \left(\frac{\vec{\hat{S}} \cdot \vec{P}}{P} \right)^k$$

then (4.3.36) can be rewritten as follows

$$H_s^{II} = \sum_{r=-s'}^{r=+s'} \left(a_\mu m + (b_\mu + r^2 d_\mu) \frac{\vec{P}^2}{2m} + rpc_\mu \right) \hat{\sigma}^\mu \Lambda_r \tag{4.3.37}$$

It is obvious that operator H_s^{II} (4.3.37) satisfies commutation relations (4.3.6) and we require (4.3.37) to satisfy (4.3.11). Having substituted (4.3.37) into (4.3.11) and using properties of Λ_r we find after equating independent addends that coefficients a_μ, b_μ, c_μ, d_μ should satisfy one of the following systems of algebraic equations

$$\begin{aligned}
&\sum_{i=1}^{3} a_i^2 = 1; \qquad \sum_{i=1}^{3} \left[r^2 c_i^2 + a_i (b_i + r^2 d_i) \right] = 1; \\
&\sum_{i=1}^{3} c_i r (b_i + r^2 d_i) = 0; \qquad \sum_{i=1}^{3} r c_i a_i = 0; \qquad \sum_{i=1}^{3} (b_i + r^2 d_i)^2 = 1;
\end{aligned} \tag{4.3.38}$$

or

$$a_0 = b_0 = 1; \qquad d_0 = c_0 = a_i = b_i = c_i = d_i = 0; \qquad i = 1, 2, 3 \tag{4.3.39}$$

The general solution of Equations (4.3.38) has the form (within linear transformations of equivalence):

$$\begin{aligned}
&a_1 = 1, \qquad a_0 = a_2 = a_3 = 0; \\
&b_1 = 1; \qquad b_3 = a_s; \qquad b_0 = b_2 = 0; \\
&c_2 = \frac{\sqrt{2}}{s^2} \sin\theta_s, \qquad c_0 = c_1 = c_3 = 0 \\
&d_1 = -c_2^2, \qquad d_3 = b_s/s^2, \qquad d_0 = d_2 = 0
\end{aligned} \tag{4.3.40}$$

where a_s, b_s, θ_s are given in (4.3.34). One can make sure that Equations (4.3.39) are inconsistent with (4.1.2); (4.3.40), (4.3.36) result in Hamiltonian (4.3.33).

To conclude the proof it is sufficient to show the explicit form of operators λ_a^{II} which ensure fulfilment of relations of AG(1,3) (4.1.3) by operators (4.3.8). The simplest way of obtaining λ_a^{II} is as follows:

$$\lambda_a^{II} = [U, \hat{\sigma}_1 x_a m] U^\dagger, \tag{4.3.41}$$

where operator

$$U = \frac{E + H_s^{II}\hat{\sigma}_1}{\sqrt{2E\left[2E - \left(\frac{Pr}{m_s}\sin\theta_s\right)^2\right]}}\Lambda_r; \qquad E = m + \frac{\vec{P}^2}{2m} \tag{4.3.42}$$

diagonalizes Hamiltonian (4.3.33) and generators (4.3.8):

$$\begin{gathered} U^\dagger H_s^{II} U = \hat{\sigma}_1 E, \qquad U^\dagger G_a^{II} U = x_0 P_a - \hat{\sigma}_1 m x_a, \\ U^\dagger J_a^{II} U = J_a^{II}, \qquad U^\dagger P_a^{II} U = P_a^{II} \end{gathered} \tag{4.3.43}$$

The theorem is proved.

Now we shall find out whether the Equations (4.3.1) are invariant under the scale and projective transformations. For the equations of the first type (this is (4.3.12), (4.3.13)) it can be done quite simple in the representation (4.3.15)–(4.3.17). So, we calculate

$$\begin{aligned} D' = VDV^{-1} &= \left(1 + \frac{i}{m}\vec{\lambda}\cdot\vec{P}\right)(2x_0P_0 - \vec{x}\cdot\vec{P} + \lambda_0)\left(1 - \frac{i}{m}\vec{\lambda}\cdot\vec{P}\right) = \\ &= 2x_0P_0 - \vec{x}\cdot\vec{P} + \lambda_0 + \frac{1}{m}\vec{\lambda}\cdot\vec{P} = D + \frac{1}{m}\vec{\lambda}\cdot\vec{P} \end{aligned} \tag{4.3.44}$$

Note that matrix λ_0 satisfying commutation relations (4.1.4) with matrices λ_a (4.3.4) has the form

$$\lambda_0 = (i/2)\hat{\sigma}_3 \tag{4.3.45}$$

We find

$$\begin{aligned} [P_0 - (H_s^I)', D'] &= \left[P_0 - \frac{\vec{P}^2}{2m} - \hat{\sigma}_\mu a^\mu m, 2x_0P_0 - \vec{x}\vec{P} + \lambda_0 + \frac{1}{m}\vec{\lambda}\cdot\vec{P}\right] = \\ &= 2i\left(P_0 - \frac{\vec{P}^2}{2m}\right) - \left[\hat{\sigma}_\mu, \lambda_0 + \frac{1}{m}\vec{\lambda}\cdot\vec{P}\right]a^\mu m \neq 2i\left(P_0 - (H_s^I)'\right) \end{aligned}$$

Hence Equations (4.3.1): (4.3.12), (4.3.13) do not admit a Lie generator of scale transformations D (4.1.2). It follows in turn that these equations are non-invariant with respect to the local projective transformations generated by Π (see (4.1.2)). One can make sure in the same way that Equation (4.3.1): (4.3.33) is also non-invariant under the local scale and projective transformations. Nevertheless, there is a possibility to extend invariance algebra of these equations from AG(1,3) to ASch(1,3) with the help of the operators

$$D = \frac{1}{2m}(G_a P_a + P_a G_a), \qquad \Pi = \frac{1}{2m} G_a G_a \tag{4.3.46}$$

which, however, are differential operators of the second and third order and generate nonlocal transformations which are to be calculated according to (5.3.6).

It will be noted that in the case when $s = 1/2$, $\theta_{1/2} = \pi/4$, $k = -i$, $a = 1$ Equations (4.3.1): (4.3.12) and (4.3.33) have the form

$$\left(\gamma_\mu P^\mu - m - i\gamma_5 \frac{\vec{P}^2}{2m}\right)\psi = 0 \tag{4.3.47}$$

and

$$\left[\gamma_\mu P^\mu - m + (1 + \gamma_5 - \gamma_0)\frac{\vec{P}^2}{2m}\right]\psi = 0 \tag{4.3.48}$$

where γ_μ are Dirac matrices (2.1.2), $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. Equations (4.3.47), (4.3.48) differ from the standard Dirac equation in terms with $\vec{P}^2$ but it is these addends which change the Poincare invariance of the Dirac equation for Galilean one (local for (4.3.48) and nonlocal for (4.3.47)).

Let us write down Equation (4.3.1): (4.3.12) for spinning particle when $s = 1/2, 1$ at large: $s = 1/2$, $\psi = \text{column}(\varphi, \chi)$; φ, χ are two- component functions

$$\begin{aligned} &\left(P_0 - \frac{\vec{P}^2}{2m}\right)\varphi = am\chi + iak(\vec{\sigma}\cdot\vec{P})\varphi - \frac{a^2k^2}{2m}\vec{P}^2\chi; \\ &\left(P_0 - \frac{\vec{P}^2}{2m}\right)\chi = am\varphi - iak(\vec{\sigma}\cdot\vec{P})\chi. \end{aligned} \tag{4.3.49}$$

$s = 1$, $\psi = \text{column}(\vec{\varphi}\,\vec{\chi})$; $\vec{\varphi}$, $\vec{\chi}$ are three-component functions

$$\begin{aligned} &\left(P_0 - \frac{\vec{P}^2}{2m}\right)\vec{\varphi} = am\vec{\chi} - 2ak\vec{P}\times\vec{\varphi} - \frac{ak^2}{m}(\vec{P}^2\vec{\chi} - \vec{P}(\vec{P}\cdot\vec{\chi})); \\ &\left(P_0 - \frac{\vec{P}^2}{2m}\right)\vec{\chi} = am\vec{\varphi} + 2ak\vec{P}\times\vec{\chi} \end{aligned} \tag{4.3.50}$$

matrices S_a are given in (4.2.21).

Equation (4.3.48) can be rewritten as follows

$$P_0\psi = H\psi,$$
$$H = \widehat{\sigma}_3 m + \widehat{\sigma}_1(\vec{\sigma}\cdot\vec{P}) + \left[1 - (\widehat{\sigma}_3 + i\widehat{\sigma}_2)\right]\frac{\vec{P}^2}{2m} \tag{4.3.51}$$

It is rewarding to study its maximal Lie symmetry.

Theorem 4.3.3. [184]. *The maximal group of point transformations admitted by Equation* (4.3.51) *is the Galilei group* G(1,3), *basis elements of corresponding* AG(1,3) *having the form*

$$P_0 = i\frac{\partial}{\partial x_0}, \qquad P_a = -\frac{\partial}{\partial x_a}, \qquad M = m$$
$$J_a = (\vec{x}\times\vec{P})_a + \widehat{S}_a, \tag{4.3.52}$$
$$G_a = x_0 P_a - m x_a - \frac{1}{2}(\widehat{\sigma}_3 + i\widehat{\sigma}_2)\sigma_a$$

where

$$\widehat{S}_a = \frac{1}{2}\begin{pmatrix}\sigma_a & 0\\ 0 & \sigma_a\end{pmatrix} \tag{4.3.53}$$

Proof. Use of the standard Lie alogrithm requires some cumbersome calculations. To avoid this we shall use the criterion of invariance in the form (10) (see also §5.3) so that

$$[P_0 - H,\, Q]\psi = 0 \tag{4.3.54}$$

where Q is the first-order differential operator

$$Q = \xi^\mu(x)\partial_\mu + \eta(x) \tag{4.3.55}$$

ξ^μ are scalar functions; $\eta(x)$ is a matrix of dimension 4×4. So we find

$$\begin{aligned}R \equiv [P_0 - H, Q] &= i(\xi_0^a P_a + \eta_0) - m[\widehat{\sigma}_3, \eta] -\\ &- [\widehat{\sigma}_1\sigma_a, \eta]P_a + i\widehat{\sigma}_1\sigma_b(\xi_b^a P_a + \eta_b) + \frac{i}{m}(\xi_a^b P_a P_b + \eta_a P_a) +\\ &+ \frac{1}{2m}((\Delta\xi^a)P_a + \Delta\eta)\Gamma - [\Gamma, \eta]\frac{\vec{P}^2}{2m} +\\ &+ i\left(\xi_0^0 + \widehat{\sigma}_1\sigma_a\xi_a^0 + \frac{1}{m}\xi_a^0 P_a\Gamma - \frac{i}{2m}(\Delta\xi^0)\Gamma\right)P_0,\end{aligned} \tag{4.3.56}$$

where $\xi^{\mu}_{\nu} = \partial\xi^{\mu}/\partial x_{\nu}$, $\eta_{\mu} = \partial\eta/\partial x_{\nu}$; $\mu, \nu = \overline{1,3}$; $\Gamma = 1 - (\widehat{\sigma}_3 + i\widehat{\sigma}_2)$. The transition on the manifold of solutions of Equation (4.3.51) in (4.3.54) can be done as follows

$$[P_0 - H, Q]\Big|_{P_0=H} \equiv R\Big|_{P_0=H} = 0 \tag{4.3.57}$$

that is in the expression (4.3.56) P_0 should be replaced by H (4.3.51) everywhere. Equating the coefficients of the identity (4.3.57) yields a set of equations to determine $\xi^{\mu}(x)$ and $\eta(x)$:

$$\xi^a_0 + \xi^0_0\widehat{\sigma}_1\sigma_a + i[\widehat{\sigma}_1\sigma_a, \eta] + \widehat{\sigma}_1\sigma_a\xi^a_b + \frac{1}{m}\eta_a\Gamma - \frac{i}{2m}(\Delta\xi^a)\Gamma = 0;$$

$$\widehat{\sigma}_3 m\xi^0_0 + \eta_0 + im[\widehat{\sigma}_3, \eta] + \widehat{\sigma}_1\sigma_a\eta_a - \frac{i}{2m}(\Delta\eta)\Gamma = 0;$$

$$(\xi^0_0 + 2\xi^a_a)\Gamma + i[\Gamma, \eta] = 0, \text{ (no sum over } a)$$

$$\xi^b_a + \xi^a_b = 0, \qquad a \neq b.$$

The general solution of this system has the form

$$\begin{aligned} \xi^0 &= c^0, \qquad \xi^a = c^{ab}x_b + g^a x_0 + c^a, \\ \eta(x) &= -\frac{1}{4}\epsilon_{abc}\sigma_a c_{bc} - mg_a x_a - \frac{i}{2}(\widehat{\sigma}_3 + i\widehat{\sigma}_2)\sigma_a g_a + c_4 \end{aligned} \tag{4.3.58}$$

where c^0, c^a, c_4, $c_{ab} = -c_{bc}$, g_a are arbitrary constants. So, the theorem is proved.

4.4. *Solutions of the nonlinear Levi-Leblond equation*

Here we shall obtain some exact solutions of Sch(1,3)-invariant coupled nonlinear equations (see §4.2, (4.2.41))

$$\begin{aligned} P_0\varphi - i(\vec{\sigma}\cdot\vec{P})\chi &= æF\left(\frac{\omega_1}{\omega_2}\right)\omega_2\chi + \mu G\left(\frac{\omega_1}{\omega_2}\right)\omega_1^2\varphi, \\ 2im\chi - (\vec{\sigma}\cdot\vec{P})\varphi &= iæF\left(\frac{\omega_1}{\omega_2}\right)\omega_2\varphi. \end{aligned} \tag{4.4.1}$$

For this aim we use the projective invariant ansatz (4.1.5) which in this case takes the form

$$\begin{aligned} \varphi(x) &= x_0^{-3/2}\exp\left\{i\frac{m\vec{x}^2}{2x_0}\right\}\widetilde{\phi}(\omega), \qquad \omega = \frac{\vec{\beta}\vec{x}}{x_0}, \\ \chi(x) &= x_0^{-5/2}\exp\left\{i\frac{m\vec{x}^2}{2x_0}\right\}\left(\widetilde{\psi}(\omega) - \frac{i}{2}(\vec{\sigma}\cdot\vec{x})\widetilde{\phi}(\omega)\right) \end{aligned} \tag{4.4.2}$$

Having substituted (4.4.2) into (4.4.1) we obtain the following system of ODEs

$$2m\tilde{\psi} + (\vec{\sigma}\cdot\vec{\beta})\dot{\tilde{\phi}} = æF\delta_2\tilde{\phi},$$
$$-(\vec{\sigma}\cdot\vec{\beta})\dot{\tilde{\psi}} = æF\delta_2\tilde{\psi} + \mu G\delta_2^2\tilde{\phi}$$

where F, G are arbitrary smooth functions of δ_1/δ_2, $\delta_1 = (\tilde{\phi}^\dagger\tilde{\psi} + \tilde{\psi}^\dagger\tilde{\phi})^{1/4}$, $\delta_2 = (\phi^\dagger\phi)^{1/3}$. It is easy to check that this system has a solution

$$\begin{aligned}\tilde{\phi}(\omega) &= \exp\left\{im(\vec{\sigma}\cdot\vec{\beta})\omega\right\}\phi,\\ \tilde{\psi}(\omega) &= \exp\left\{im(\vec{\sigma}\cdot\vec{\beta})\omega\right\}\chi,\end{aligned} \tag{4.4.3}$$

where ϕ, χ are constant spinors satisfying the conditions

$$\begin{aligned}-(m\vec{\beta}^2)^2 &= (æF\delta_2)^2 + 2\mu mG\delta_2^2,\\ 2m\chi &= (æF\delta_2 - im\vec{\beta}^2)\phi.\end{aligned} \tag{4.4.4}$$

From (4.4.4)–(4.4.2) we obtain a projective invariant solution of Equation (4.4.1)

$$\begin{aligned}\varphi(x) &= x_0^{-3/2}\exp\left\{im\frac{\vec{x}^2}{2x_0}\right\}\exp\left\{im\omega(\vec{\sigma}\cdot\vec{\beta})\right\}\phi,\\ \chi(x) &= x_0^{-5/2}\exp\left\{im\left(\frac{\vec{x}^2}{2x_0} + \omega(\vec{\sigma}\cdot\vec{\beta})\right)\right\}\cdot\\ &\quad\cdot\left(\chi - \frac{i}{2}\exp\left\{-im\omega(\vec{\sigma}\cdot\vec{\beta})\right\}(\vec{\sigma}\cdot\vec{x})\exp\left\{im\omega(\vec{\sigma}\cdot\vec{\beta})\right\}\phi\right)\end{aligned} \tag{4.4.5}$$

and with the help of (4.1.6) from here it follows a Sch(1,3)-ungenerative solution

$$\begin{aligned}\varphi(x) &= (x_0 + a_0)^{-3/2}\exp\left\{im\left[\frac{(\vec{x}+\vec{a})^2}{2(x_0+a_0)} + y(\vec{\sigma}\cdot\vec{\beta}) + \alpha\right]\right\}\phi,\\ \chi(x) &= (x_0 + a_0)^{-5/2}\exp\left\{im\left[\frac{(\vec{x}+\vec{a})^2}{2(x_0+a_0)} + y(\vec{\sigma}\cdot\vec{\beta}) + \alpha\right]\right\}\cdot\\ &\quad\cdot\left(\chi - \exp\left\{-imy(\vec{\sigma}\cdot\vec{\beta})\right\}(\vec{\sigma}\cdot\vec{x})\exp\left\{imy(\vec{\sigma}\cdot\vec{\beta})\right\}\phi\right);\end{aligned} \tag{4.4.6}$$

$$y = \frac{\vec{\beta}\cdot\vec{x} + \vec{\beta}\vec{a}}{x_0 + a_0} + \beta_0$$

Galilean-invariant ansatz

$$\begin{aligned}\varphi(x) &= \exp\left\{im\frac{\vec{x}^2}{2x_0}\right\}\tilde{\phi}(\omega), \qquad \omega = \vec{k}\cdot\vec{x} + k_0x_0\\ \chi(x) &= \exp\left\{im\frac{\vec{x}^2}{2x_0}\right\}\left(\tilde{\psi}(\omega) - \frac{i}{2x_0}(\vec{\sigma}\cdot\vec{x})\tilde{\phi}(\omega)\right)\end{aligned} \tag{4.4.7}$$

reduces (4.4.1) to the following system of ODEs

$$2m\tilde{\psi} + (\vec{\sigma}\cdot\vec{k})\dot{\tilde{\phi}} = æF\delta_2\tilde{\phi}, \qquad \delta_1 = (\tilde{\phi}^\dagger\tilde{\psi} + \tilde{\psi}^\dagger\tilde{\phi})^{1/4},$$

$$-(\vec{\sigma}\cdot\vec{k})\dot{\tilde{\psi}} = æF\delta_2\tilde{\phi} + \mu G\delta_2\tilde{\phi}, \qquad \delta_2 = \left(\tilde{\phi}^\dagger\tilde{\phi}\right)^{1/3}$$

$$\omega_1\dot{\tilde{\phi}} + \tfrac{3}{2}\tilde{\phi} = 0,$$

which has the solution

$$\tilde{\phi}(\omega) = \omega^{-3/2}\phi, \qquad \tilde{\psi}(\omega) = \omega^{-5/2}\chi \tag{4.4.8}$$

where ϕ and χ are two-component constant columns satisfying conditions

$$\left[æF\delta_2(\vec{\sigma}\cdot\vec{k}) + \tfrac{15}{4}k^2 - (æF\delta_2)^2 - 2\mu mG\delta_2^2\right]\phi = 0,$$

$$(4.4.9)$$

$$2m\chi = \left(æF\delta_2 + \tfrac{3}{2}(\vec{\sigma}\cdot\vec{k})\right)\phi.$$

Formulae (4.4.7)–(4.4.9) result in a Galilean-invariant solution of the system (4.4.1):

$$\varphi(x) = (\vec{k}\cdot\vec{x} + k_0x_0)^{-3/2}\exp\left\{\frac{im\vec{x}^2}{2x_0}\right\}\phi,$$

$$(4.4.10)$$

$$\chi(x) = (\vec{k}\cdot\vec{x} + k_0x_0)^{-5/2}\exp\left\{\frac{im\vec{x}^2}{2x_0}\right\}\left(\chi - \frac{i}{2x_0}(\vec{k}\cdot\vec{x} + k_0x_0)(\vec{\sigma}\cdot\vec{x})\phi\right).$$

Consider another version of nonlinear equation for Galilean particle with spin $s = 1/2$, namely Equation [29*, 31*]

$$\left[(i\gamma_0 + \gamma_5)\partial_0 + i\gamma_a\partial_a + \lambda(\overline{\psi}\psi)^k\right]\psi = 0, \tag{4.4.11}$$

where $\psi = \psi(x)$ is a four-component complex function, γ-matrices are defined in (2.1.2), (2.4.2), λ, k are arbitrary real constants.

Theorem 4.4.1. [29*, 31*]. *The maximal point IA of the system* (4.4.11) *is given by the following basis IFO:*

Under $k = 1/3$,

$$P_0 = i\partial_0, \qquad P_a = -i\partial_a$$

$$J_{ab} = x_aP_b - x_bP_a + \tfrac{i}{4}[\gamma_a, \gamma_b], \tag{4.4.12}$$

$$G = \phi^a(x_0)P_a - \tfrac{1}{2}\dot{\phi}^a(x_0)\gamma_a(i\gamma_0 + \gamma_5),$$

$$\tilde{\Pi} = \phi^0(x_0)P_0 - \dot{\phi}^0(x_0)x^a P_a + \tfrac{3}{2}\ddot{\phi}^0(x_0)\gamma_a x_a(\gamma_0 - i\gamma_5).$$

Under $k \neq 1/3$, $k \neq 0$

$$\begin{aligned} &P_0 = i\partial_0, \qquad P_a = -i\partial_a \\ &J_{ab} = x_a P_b - x_b P_a + \tfrac{i}{4}[\gamma_a, \gamma_b], \\ &G = \phi^a(x_0)P_a - \tfrac{1}{2}\dot{\phi}^a(x_0)\gamma_a(i\gamma_0 + \gamma_5), \\ &D = x^\mu P_\mu + 2i/k, \end{aligned} \tag{4.4.13}$$

where ϕ^μ, $\mu = \overline{0,3}$ are arbitrary smooth functions of x_0, $\dot{\phi}^\mu \equiv d\phi^\mu/dx_0$.

One can prove the theorem by means of Lie's method. Note that under $k = 1/3$ Equation (4.4.11) is invariant under the Schrödinger algebra ASch(1,3), basis elements having the form

$$\begin{aligned} &P_0 = i\partial_0, \qquad P_a = -i\partial_a \\ &J_{ab} = x_a P_b - x_b P_a + \tfrac{i}{4}[\gamma_a, \gamma_b], \\ &G_a = x_0 P_a - \tfrac{1}{2}\gamma_a(i\gamma_0 + \gamma_5), \\ &D = x^\mu P_\mu + \tfrac{3}{2}i \\ &\Pi = x_0^2 P_0 - 2x_0 x_a P_a + 3ix_0 + \gamma_a x_a(\gamma_0 + i\gamma_5). \end{aligned} \tag{4.4.14}$$

The above described symmetry of Equation (4.4.11) can be used for constructing nontrivial formulae of generating solutions. In particular, we have ($k = 1/3$):

$$\psi_{II}(x) = \phi_0^{3/2}\exp\Big\{-\tfrac{1}{2}\left[\dot{\phi}_a\gamma_a + \dot{\phi}_0\phi_0^{-1}\gamma_a(x_a + \phi_a)\right]\cdot \cdot(\gamma_0 - i\gamma_5)\Big\}\psi_I\left(\int \phi_0 dx_0,\ (x_a + \phi_a)\phi_0^{-1}\right). \tag{4.4.15}$$

Taking as $\psi_I(x)$ the following partial solution of the Equation (4.4.11) under $k = 1/3$

$$\psi_I(x) = \exp\left\{-i\lambda(\gamma_a\theta_a)(\theta_b x_b)(\bar{\chi}\chi)^{1/3}\right\}\chi, \tag{4.4.16}$$

(θ_a are arbitrary constants) we get by means of (4.4.15) another family of solutions of the equation in question

$$\psi(x) = \phi_0^{3/2} \exp\{-\tfrac{1}{2}\Big(\gamma_a\dot{\phi}_a + \dot{\phi}_0\phi_0^{-1}\cdot \\ \cdot \gamma_a(x_a + \phi_a)\Big)(\gamma_0 - i\gamma_5)\} \exp\{-i\lambda(\gamma_a\theta_a)\cdot \\ \cdot (\theta_b x_b + \theta_b\phi_b)\phi_0^{-1}(\overline{\chi}\chi)^{1/3}\}\chi. \tag{4.4.17}$$

It is interesting to note that Equation (4.4.11) under $k = 1/3$ can be considered as nonrelativistic counterpart of the nonlinear Dirac-Gursey equation (2.1.5).

4.5. *Symmetry analysis of gas dynamics equations*

The aim of this paragraph is to study the symmetry of the basic equations of gas dynamics

$$\frac{\partial \vec{u}}{\partial x_0} + \left(\vec{u}\cdot\vec{\nabla}\right)\vec{u} + \frac{1}{\rho}\vec{\nabla}P = 0, \\ \frac{\partial \rho}{\partial x_0} + \operatorname{div}(\rho\vec{u}) = 0, \\ P = f(\rho), \tag{4.5.1}$$

where $\vec{u} = \vec{u}(x) = \{u^1, \ldots, u^n\}$ is the velocity vector of the gas spreading; $x = (x_0, \vec{x}) = (x_0, x_1, \ldots, x_n) \in R^{n+1}$; $\rho = \rho(x)$ and $P = P(x)$ are the density and pressure of the gas, and to obtain their exact solutions.

At the beginning we establish that the one-dimensional Equations (4.5.1) in the special case of isentropic and polytropic gas possess an infinite-dimensional group of point transformations. It allows us to construct the general solution of the one-dimensional Equations (4.5.1) [90]. We also study symmetry of multi-dimensional Equations (4.5.1) in some special cases of gas motion and construct exact solutions of these equations [90,91].

1. One-dimensional isentropic gas motion.

When number of spatial variables $n = 1$, Equations (4.5.1) take the form

$$\frac{\partial \rho}{\partial x_0} + \rho\frac{\partial u}{\partial x_1} + u\frac{\partial \rho}{\partial x_1} = 0, \\ \rho\frac{\partial u}{\partial x_0} + \rho u\frac{\partial u}{\partial x_1} + c^2(\rho)\frac{\partial \rho}{\partial x_1} = 0, \tag{4.5.2}$$

where $c^2(\rho)$ is called sound velocity and is determined by the formula

$$c(\rho) = \left[\frac{\partial f(\rho)}{\partial \rho}\right]^{1/2} > 0,$$

$p = f(\rho)$ is a given monotonic function.

Equations (4.5.2) were studied as far back as the end of the 17th century. In 1809, Poisson obtained a solution of these equations in the form of ordinary wave

$$u = F(x - (u + c)t)$$

where F is arbitrary differentiable function. Challis [37] noted that Equations (4.5.2) have not always a unique solution for velocity u. To obtain the unique solution Stokes suggested [197] a criterion of a breakdown of the velocity u (when the derivative of u aims at the infinity) and obtained two conditions on the breakdown. Earnshaw [47] found a solution of Equations (4.5.2) in the form of ordinary wave for arbitrary dependence $p = f(\rho)$. Independently, Riemann [171] developed theory of ordinary wave and gave the general solution of the Equations (4.5.2) by virtue of Riemann invariants.

A great contribution to the theory of gas dynamics was made by Hugoniot [122], Sedov, Hristianovich, Stanyukovich [196], Zeldovich, Landau and Lifshits [140].

Kurant [137] had shown that Equations (4.5.2) can be linearized by means of hodograph transformation

$$\begin{gathered} V^{\mu} = V^{\mu}(y_0, y_1), \qquad V^{\mu} = x_{\mu}, \qquad y_{\nu} = u^{\nu}, \qquad V^{\mu}_{\nu} = \frac{\partial V^{\mu}}{\partial y_{\nu}}, \\ x_0 = t, \qquad x_1 = x, \qquad u^0 = \rho, \qquad u^1 = u; \qquad \mu, \nu = 0, 1. \end{gathered} \tag{4.5.3}$$

As a result of application of (4.5.3) to (4.5.2) one obtains the system of linear equations

$$\begin{aligned} V^1_1 + y_0 V^0_0 - y_1 V^0_1 &= 0, \\ V^1_0 - y_1 V^0_0 + h(y_0) V^0_1 &= 0, \end{aligned} \tag{4.5.4}$$

where $h(y_0) = h(\rho) = \frac{1}{2}c^2(\rho)$.

In case of polytropic gas, when $p = A\rho^{\gamma}$, with A, γ constants and $\gamma = \dfrac{2N+1}{2N-1}$, $N = 0, 1, 2, \ldots$, Equations (4.5.4) are reduced to the following second-order system of PDEs

$$V^0_{rs} + N\frac{V^0_r + V^0_s}{r + s} = 0, \tag{4.5.5}$$

$$\begin{aligned} V^1_r &= \left(r - s - \frac{r+s}{2N-1}\right) V^0_r, \\ V^1_s &= \left(r - s + \frac{r+s}{2N-1}\right) V^0_s, \end{aligned} \tag{4.5.6}$$

where

$$\begin{aligned} r &= \tfrac{1}{2}\left(y_1 + \sqrt{A(4N^2-1)}\, y_0^{1/(2N-1)}\right), \\ s &= \tfrac{1}{2}\left(-y_1 + \sqrt{A(4N^2-1)}\, y_0^{1/(2N-1)}\right) \end{aligned} \tag{4.5.7}$$

are Riemannian invariants.

One can easy recognize in (4.5.5) the Darboux equation. Its general solution has the form

$$V^0 = \varphi(r) + g(s), \qquad N = 0 \tag{4.5.8}$$

$$V^0 = k + \frac{\partial^{N-1}}{\partial r^{N-1}}\left(\frac{\varphi(r)}{(r+s)^N}\right) + \frac{\partial^{N-1}}{\partial s^{N-1}}\left(\frac{g(s)}{(r+s)^N}\right), \qquad N \geq 1 \tag{4.5.9}$$

where φ, g are arbitrary differentiable functions, and k is a constant.

If one succeeds in finding the general solution of Equations (4.5.6), thereby making use of the inverse hodograph transformation, one constructs the general solution of Equations (4.5.2). When $\gamma = 3$ ($N = 1$) it is succeeded to do and in such a case the general solution of Equations (4.5.2) has the form (see [196])

$$\begin{aligned} x_1 - (u^1 + 3Au^0)x_0 &= F^1(u^1 + 3Au^0), \\ x_1 - (u^1 - 3Au^0)x_0 &= F^2(u^1 - 3Au^0), \end{aligned} \tag{4.5.10}$$

where F^1, F^2 are arbitrary differentiable functions.

It will be noted that under $\gamma = -1$ ($N = 0$) the explicit form of solution of the system (4.5.2) can be also easy found. By substituting (4.5.8) into (4.5.6) and denoting $\varphi(r) = F'(r)$, $g(s) = G'(s)$ we get the general solution of Equations (4.5.6) as follows

$$V^1 = rF'(r) - F(r) - sG'(s) + x. \tag{4.5.11}$$

Then by means of the inverse hodograph transformation we obtain from (4.5.8), (4.5.11) the general solution of Equations (4.5.2)

$$\begin{aligned} &t \equiv x_0 = F'\big(u - \sqrt{-A}\rho^{-1}\big) + G'\big(u + \sqrt{-A}\rho^{-1}\big), \\ &x \equiv x_1 = \big(u - \sqrt{-A}\rho^{-1}\big)F'\big(u - \sqrt{-A}\rho^{-1}\big) - F\big(u - \sqrt{-A}\rho^{-1}\big) - \\ &\quad - \big(u + \sqrt{-A}\rho^{-1}\big)G'\big(u + \sqrt{-A}\rho^{-1}\big) + G\big(u + \sqrt{-A}\rho^{-1}\big) + k \end{aligned} \tag{4.5.12}$$

Group properties of gas-dynamics Equations (4.5.1) had been studied by Ovsyannikov and summarized in [162]. Although this investigation was made for arbitrary number of independent variables, the two-dimensional case needs individual consideration. In this case the symmetry group of gas dynamics equations is infinite-dimensional. This result established in [90] does not follow from [162].

Theorem 4.5.1. [90]. *One-dimensional equations of gas dynamics* (4.5.2) *under* $p = A\rho^\gamma$, $\dfrac{2N+1}{2N-1}$, $N = 0, 1, 2, \ldots$, *that is*

$$\gamma = -1, 3, \tfrac{5}{3}, \tfrac{7}{5}, \ldots \tag{4.5.13}$$

are invariant with respect to the infinite-dimensional Lie group, coordinates ξ^0, ξ^1, η *of corresponding IFO*

$$X = \xi^0 \frac{\partial}{\partial x_0} + \xi^1 \frac{\partial}{\partial x_1} + \eta \frac{\partial}{\partial u} \tag{4.5.14}$$

are solutions of Equations (4.5.5), (4.5.6).

Proof. Using notations (4.5.3) Equations (4.5.2) can be rewritten as follows

$$\begin{aligned} u_0^0 + u^1 u_1^0 + u^0 u_1^1 &= 0, \\ u_0^1 + u^1 u_1^1 + h(u^0) u_1^0 &= 0. \end{aligned} \tag{4.5.15}$$

Applying to (4.5.15) the criterion of invariance (3) one obtains as a result the determining equations for functions ξ^0, ξ^1, η:

$$\begin{aligned} \eta_0^0 + u^1 \eta_1^0 + u^0 \eta_1^1 &= 0, \\ \eta_0^1 + u^1 \eta_1^1 + h(u^0) \eta_1^0 &= 0. \end{aligned} \tag{4.5.16}$$

$$\begin{aligned} \eta^1 &= -u^1(\xi_0^0 - \xi_1^1 + u^1 \xi_1^0) - h(u^0)(u^0 \xi_1^0 + \eta_{u^1}^0) + \xi_0^1 + u^0 \eta_{u^0}^1, \\ \eta^1 &= -u^1(\xi_0^0 - \xi_1^1 + u^1 \xi_1^0) - h(u^0)(u^0 \xi_1^0 - \eta_{u^1}^0) + \xi_0^1 - u^0 \eta_{u^0}^1; \end{aligned} \tag{4.5.17}$$

$$\begin{aligned} \eta^0 &= -u^0(\xi_0^0 - \xi_1^1 + 2u^1 \xi_1^0 + \eta_{u^1}^1 - \eta_{u^0}^0), \\ \eta^0 &= -\frac{h(u^0)}{h'(u^0)}(\xi_0^0 - \xi_1^1 + 2u^1 \xi_1^0 - \eta_{u^1}^1 + \eta_{u^0}^0); \end{aligned} \tag{4.5.18}$$

$$\begin{aligned} \xi_{u^0}^1 &= u^1 \xi_{u^0}^0 - h(u^0) \xi_{u^1}^0, \\ \xi_{u^1}^1 &= u^1 \xi_{u^1}^0 - u^0 \xi_{u^0}^0. \end{aligned} \tag{4.5.19}$$

Compatibility condition of Equations (4.5.19) yields

$$\xi_{u^0 u^0}^0 - \frac{h(u^0)}{u^0} \xi_{u^1 u^1}^0 + 2 \frac{\xi_{u^0}^0}{u^0} = 0. \tag{4.5.20}$$

1. Consider at first the case $p = A\rho^3$. Then (4.5.20) results in the Darboux equation for

$$\xi_{u^0 u^0}^0 - \lambda^2 \xi_{u^1 u^1}^0 + \frac{2}{u^0} \xi_{u^0}^0 = 0, \qquad (\lambda^2 = 3A) \tag{4.5.21}$$

The general solution of (4.5.21) has the form

$$\xi^0 = \frac{F(x, u^1 + \lambda u^0) + G(x, u^1 - \lambda u^0)}{2\lambda u^0}, \tag{4.5.22}$$

where F and G are arbitrary differentiable functions.

It is not difficult to find function ξ^1 from (4.5.22), (4.5.19)

$$\xi^1 = \frac{(u^1 - \lambda u^0)F(x, u^1 + \lambda u^0) + (u^1 + \lambda u^0)G(x, u^1 - \lambda u^0)}{2\lambda u^0} + H(x), \quad (4.5.23)$$

where $H(x)$ is an arbitrary differentiable function.

Since $p = A\rho^3$ results in $h(u^0) = \lambda^2 u^0$, we get from (4.5.18), (4.5.19)

$$\eta^0_{u^1} = \eta^1_{u^0}, \qquad \eta^0_{u^0} = \eta^1_{u^1}; \quad (4.5.24)$$

$$\begin{aligned} \eta^0 &= -\lambda u^0(\xi^0_0 - \xi^1_1 + 2u^1\xi^0_1), \\ \eta^1 &= -u^1(\xi^0_0 - \xi^1_1 + u^1\xi^0_1) - (\lambda u^0)^2\xi^0_1 + \xi^1_0. \end{aligned} \quad (4.5.25)$$

Following from (4.5.16), (4.5.23), (4.5.25) we find that functions F, G, and H should satisfy the equations

$$\begin{aligned} &F_{00} + 2(u^1 + \lambda u^0)F_{01} + (u^1 + \lambda u^0)^2 F_{11} = 0, \\ &G_{00} + 2(u^1 - \lambda u^0)G_{01} + (u^1 - \lambda u^0)^2 G_{11} = 0, \\ &H_{00} = H_{11} = H_{01} = 0. \end{aligned} \quad (4.5.26)$$

The general solution of system (4.5.26) has the form

$$\begin{aligned} &F = x_0 F^0\big(x_0(u^1 + \lambda u^0)) - x_1\big) + G^0\big(x_0(u^1 + \lambda u^0)) - x_1\big) \\ &G = x_0 F^1\big(x_0(u^1 - \lambda u^0)) - x_1\big) + G^1\big(x_0(u^1 - \lambda u^0)) - x_1\big) \\ &H = c_\nu x_\nu + d, \end{aligned} \quad (4.5.27)$$

where F^ν, G^ν are arbitrary differentiable functions, c_ν and d are arbitrary constants, $\nu = 0, 1$.

Now, using formulae (4.5.27), (4.5.25), (4.5.22), (4.5.23) we obtain

$$\begin{aligned} \xi^0 &= (2\lambda u^0)^{-1}(x_0F^0 + G^0 + x_0F^1 + G^1), \\ \xi^1 &= (2\lambda u^0)^{-1}\big[(u^1 - \lambda u^0)(x_0F^0 + G^0) + (u^1 + \lambda u^0)(x_0F^1 + G^1)\big] + c_\nu x_\nu + d, \\ \eta^0 &= -\tfrac{1}{2}(F^0 + F^1) + \lambda c_1 u^0, \\ \eta^1 &= -\tfrac{1}{2}(F^0 - F^1) + c_1 u^1 + c_0, \end{aligned} \quad (4.5.28)$$

where F^ν, G^ν are the same functions as in (4.5.27).

Since ξ^μ and η^μ depend on arbitrary functions F^ν, G^ν, it means that the symmetry group of Equations (4.5.15) under $\gamma = 3$ is infinite-dimensional, the infinitesimal transformations having the form

$$\begin{aligned} x'_\mu &= x_\mu + \epsilon\xi^\mu + O(\epsilon^2), \\ u'^\nu &= u^\nu + \epsilon\eta^\nu + O(\epsilon^2), \end{aligned} \quad (4.5.29)$$

where ξ^μ, η^ν are given by (4.5.28). So, case 1 of the theorem is proved.

Under $\gamma = \dfrac{2N+1}{2N-1}$, $N = 0, 2, 3, 4, \ldots$ the determining Equations (4.5.19), (4.5.20) coincide with (4.5.5), (4.5.6) and by means of hodograph transformation they are reduced to the initial system (4.5.15); the general solution of this latter, as we have shown above, contain arbitrary functions. This fact accomplishes the proof.

Remark 4.5.1. In a particular case when

$$F^0 = a\omega^0, \qquad F^1 = -a\omega^1, \qquad G^0 = b\omega^0, \qquad G^1 = -b\omega^1,$$
$$\omega^0 = x_0(u^1 + \lambda u^0) - x_1, \qquad \omega^1 = x_0(u^1 - \lambda u^0) - x_1$$

the above proved theorem gives rise to the result obtained under $\gamma = 3$ in [162].

2. In this item we consider another way of constructing the general solution of the two-dimensional equations of gas dynamics with $\gamma = 3$ (i.e., $p = A\rho^3$).

Let us note, that arbitrary functions which are contained in the general solution of Equations (4.5.2) depend on the same arguments as arbitrary functions contained by coordinates of IFO do. So, it is naturally to pass on these arguments in system (4.5.15) ($h(u^0) = \lambda(u^0)^2$, $\gamma = 3$):

$$\begin{aligned} V^0 &= (u^1 + \lambda u^0)x_0 - x_1, \\ V^1 &= (u^1 - \lambda u^0)x_0 - x_1. \end{aligned} \tag{4.5.30}$$

This change of variables results in decomposed PDEs

$$\begin{aligned} x_0 V^0_0 + (V^0 + x_1)V^0_1 &= 0, \\ x_0 V^1_0 + (V^1 + x_1)V^1_1 &= 0. \end{aligned} \tag{4.5.31}$$

the general solution of which has the form

$$\begin{aligned} V^0 &= x_0\varphi^0(V^0) - x_1, \\ V^1 &= x_0\varphi^1(V^1) - x_1. \end{aligned} \tag{4.5.32}$$

(φ^0, φ^1 are arbitrary differentiable functions), or

$$\begin{aligned} u^1 + \lambda u^0 &= \varphi^0\big(x_0(u^1 + \lambda u^0) - x_1\big), \\ u^1 - \lambda u^0 &= \varphi^1\big(x_0(u^1 - \lambda u^0) - x_1\big). \end{aligned} \tag{4.5.33}$$

This latter coincides with (4.5.10).

3. In this item the symmetry of multidimensional equations of gas dynamics is briefly discussed and then some exact solutions of these equations are obtained.

As we have already noted the symmetry properties of multidimensional gas dynamics equations were studied in [162]. In particular, there is established 13-parameter maximal invariance group G_{13} of these equations in the case $n = 3$ and

$$p = \lambda\rho^{5/3}. \tag{4.5.34}$$

The basis elements of corresponding Lie algebra have the form

$$\partial_\mu = \frac{\partial}{\partial x_\mu}; \qquad \mu = \overline{0,3} \tag{4.5.35}$$

$$J_{ab} = x_a\partial_b - x_b\partial_a + u^a\partial_{u^b} - u^b\partial_{u^a}; \qquad a, b = \overline{1,3}; \tag{4.5.36}$$

$$G_a = x_0\partial_a + \partial_{u^a}; \tag{4.5.37}$$

$$\mathcal{D}_0 = x_0\partial_0 + x_a\partial_a; \tag{4.5.38}$$

$$\mathcal{D}_1 = x_0\partial_0 - 3\rho\partial_\rho - u^a\partial_{u^a}; \tag{4.5.39}$$

$$\Pi = x_0(x_0\partial_0 + x_a\partial_a - 3\rho\partial_\rho - u^a\partial_{u^a}) + x_a\partial_{u^a}. \tag{4.5.40}$$

It is the widest symmetry group of Equations (4.5.1). In other cases relation (4.5.34) does not hold, the maximal symmetry group of Equations (4.5.1) is smaller than the above G_{13}. If

$$p = \lambda\rho^\gamma, \tag{4.5.41}$$

where λ and γ are arbitrary constants, $\gamma \neq \frac{5}{3}$, then the maximal IA of Equations (4.5.1) is 12-dimensional and determined by operators (4.5.35)–(4.5.38) and the operator

$$\mathcal{D}_1 = x_0\partial_0 - \frac{2}{\gamma - 1}\rho\partial_\rho - u^a\partial_{u^a}. \tag{4.5.42}$$

If

$$p = f(\rho) \tag{4.5.43}$$

where $f(\rho)$ is an arbitrary differentiable function, $f(\rho) \neq \lambda\rho^\gamma$, then the maximal IA of Equations (4.5.1) is 11-dimensional and determined by operators (4.5.35)–(4.5.38).

Using the symmetry of Equations (4.5.1) we find their exact solutions. At first consider the polytropic gas (that is relation (4.5.41) holds). In this case Equations (4.5.1) take the form

$$\begin{aligned} &\vec{u}_0 + (\vec{u}\vec{\nabla})\vec{u} + \lambda\rho^{\gamma-2}\vec{\nabla}\rho = 0,\\ &\rho_0 + (\vec{u}\vec{\nabla})\rho + \rho\,\mathrm{div}\,\vec{u} = 0. \end{aligned} \tag{4.5.44}$$

Since Equations (4.5.44) are invariant under AG(1,3) (4.5.35)–(4.5.40) we look for solutions in the form

$$\begin{aligned} &\rho \equiv u^0 = x_0^k\varphi^0(\omega),\\ &\vec{u} = \vec{M}(x_0)\varphi^1(\omega) + \vec{N}(x_0)\varphi^2(\omega) + \vec{L}(x_0)\varphi^3(\omega) + \vec{B}(x_0). \end{aligned} \tag{4.5.45}$$

One can use invariant variables ω obtained in §3.6. Without going into details we present below corresponding functions $\vec{M}$, $\vec{N}$, $\vec{L}$, $\vec{B}$ and indicate the constant k:

1. $\vec{M}(x_0) = \vec{\alpha} x_0^{-1/2}, \quad \vec{N}(x_0) = \vec{\beta} x_0^{-1/2}, \quad \vec{L}(x_0) = \vec{\gamma} x_0^{-1/2},$

$\vec{B}(x_0) = \vec{V}, \quad k = (1-\gamma)^{-1},$

where $\vec{\alpha}\vec{V} = -b, \quad \vec{\beta}\vec{V} = c\vec{\beta}\vec{\delta} - \alpha\vec{\gamma}\vec{\delta}, \quad \vec{\gamma}\vec{V} = c\gamma\delta + \alpha\vec{\beta}\vec{\delta};$

2. $\vec{M}(x_0) = \vec{\alpha} x_0^{-1/2}, \quad \vec{N}(x_0) = \vec{\beta} x_0^{-1/2}, \quad \vec{L}(x_0) = \vec{\gamma} x_0^{-1/2},$

$\vec{B}(x_0) = \vec{V}, \quad k = (1-\gamma)^{-1},$

where $\vec{\alpha}\vec{V} = \vec{\alpha}\vec{\delta}, \quad \vec{\beta}\vec{V} = \vec{\beta}\vec{\delta} - a\vec{\alpha}\vec{\delta}, \quad \vec{\gamma}\vec{V} = \vec{\gamma}\vec{\delta} - 2a\vec{\beta}\vec{\delta} + 2a^2\vec{\alpha}\vec{\delta};$

3. $\vec{M}(x_0) = \vec{\alpha} x_0^{-1/2}, \quad \vec{N}(x_0) = \vec{\beta} x_0^{-1/2}, \quad \vec{L}(x_0) = \vec{\gamma} x_0^{-1/2},$

$\vec{B}(x_0) = \vec{V}, \quad k = (1-\gamma)^{-1},$

where $\vec{\alpha}\vec{V} = -b, \quad \vec{\beta}\vec{V} = -b_+, \quad \vec{\gamma}\vec{V} = -b_-;$

4. $\vec{M}(x_0) = \vec{\alpha} x_0^{-1/2}, \quad \vec{N}(x_0) = \vec{\gamma} x_0^{-1}, \quad \vec{L}(x_0) = \vec{\beta},$

$$\vec{B}(x_0) = -\frac{\vec{\beta} b_4}{\beta^2} \ln x_0 - \frac{\vec{\gamma} b_2}{\beta^2}, \quad k = (1-\gamma)^{-1},$$

5. $\vec{M}(x_0) = \vec{\alpha}, \quad \vec{N}(x_0) = \vec{\beta}, \quad \vec{L}(x_0) = \vec{\gamma},$

$$\vec{B}(x_0) = -\vec{\alpha}\frac{2a}{\alpha^2} x_0 + \vec{V}, \quad k = 0,$$

where $\vec{\alpha}\vec{V} = 0, \quad \vec{\beta}\vec{V} = \vec{\gamma}\vec{\delta}, \quad \vec{\gamma}\vec{V} = -\vec{\beta}\vec{\delta};$ (4.5.45′)

6. $\vec{M}(x_0) = \vec{\alpha}, \quad \vec{N}(x_0) = \vec{\gamma}\exp(-bx_0), \quad \vec{L}(x_0) = \vec{\beta}\exp(bx_0),$

$$\vec{B}(x_0) = \vec{\alpha}\frac{2a}{\alpha^2} x_0 + \vec{V}, \quad k = 0,$$

where $\vec{\alpha}\vec{V} = 0, \quad \vec{\beta}\vec{V} = \vec{\beta}\vec{\delta}, \quad \vec{\gamma}\vec{V} = -\vec{\gamma}\vec{\delta};$

7. $\vec{M}(x_0) = \vec{e}_1, \quad \vec{N}(x_0) = \vec{e}_2, \quad \vec{L}(x_0) = \vec{e}_3,$

$\vec{B}(x_0) = 0, \quad k = 0,$

where $\vec{e}_a$ are orthonormal constant vectors, $a = 1, 2, 3$;

8. $\vec{M}(x_0) = \vec{e}_1(x_0)^{-1/2}, \quad \vec{N}(x_0) = \vec{e}_2(x_0)^{-1/2}, \quad \vec{L}(x_0) = \vec{e}_3(x_0)^{-1/2},$

$\vec{B}(x_0) = 0, \quad k = (1-\gamma)^{-1}.$

Substitution of (4.5.45), (4.5.45') into (4.5.44) gives rise to the following reduced PDEs (the notation is used: $\varphi = (\varphi^0, \vec{\varphi})$, $\phi_s = (\phi_s^0, \vec{\varphi}_s)$):

1. $(2\varphi^1 - \omega_1)\varphi_1 + (2\varphi^2 - \omega_2 + 2\omega_3)\varphi_2 - (2\varphi^3 + \omega_3 + 2\omega_2)\varphi_3 +$ (4.5.46)

$$+\phi_1 = 0,$$

where $\phi_1^0 = 2\left(\operatorname{div}\vec{\varphi} - \frac{1}{\gamma-1}\right)\varphi^0$, $\vec{\phi} = -\vec{\varphi} + 2\vec{\psi}_1 + 2\lambda(\varphi^0)^{\gamma-2}\vec{\nabla}\varphi^0$, $\vec{\psi}_1 = (0, -\varphi_3, \varphi_2)$;

2. $(2\varphi^1 - \omega_1)\varphi_1 + (2\varphi^2 - \omega_2 + 2\omega_1)\varphi_2 +$ (4.5.47)

$$+(2\varphi^3 - \omega_3 + 2\varphi^1 + 4\omega_2)\varphi_3 + \phi_2 = 0,$$

where $\phi_2^0 = \phi_1^0 + 2\left(\varphi_3^1 + \varphi_1^3 - \varphi_1^1\right)\varphi^0$, $\vec{\phi}_2 = -\vec{\varphi} + 2\vec{\psi}_2 + 2\lambda(\varphi^0)^{\gamma-2}\vec{\nabla}\varphi^0$, and $\vec{\psi}_2 = (\varphi^2, 2\varphi^3, 0)$;

3. $(2\varphi^1 + \omega_1)\varphi_1 - (2\varphi^3 - 2a + 2\omega_2)\varphi_2 - (2\varphi^2 - 2a\omega_3)\varphi_3 + \phi_3 = 0,$ (4.5.48)

where $\phi_3^0 = 2\left(\varphi_1^1 - \varphi_3^2 + \varphi_2^3 + \frac{1}{\gamma-1}\right)\varphi^0$, $\vec{\phi}_3 = \vec{\psi}_3 - 2\lambda(\varphi^0)^{\gamma-2}\vec{\nabla}\varphi^0$, and $\vec{\psi}_3 = (\varphi^1, 2a + \varphi^2, 2a - \varphi^3)$;

4. $(2\varphi^1 + \omega_1)\varphi_1 - 2(\varphi^3 - b_3)\varphi_2 - 2(\varphi^3 - \omega_3 + b_4)\varphi_3 + \phi_4 = 0,$ (4.5.49)

where $\phi_4^0 = 2\left(\varphi_1^1 - \varphi_2^2 - \varphi_3^3 + \frac{1}{\gamma-1}\right)\varphi^0$, $\vec{\phi}_4 = \vec{\psi}_4 - 2\lambda(\varphi^0)^{\gamma-2}\vec{\nabla}\varphi^0$, and $\vec{\psi}_4 = (\varphi^1, 2\varphi^2, 2b_4)$;

5. $(\varphi^1 + bc)\varphi_1 + (\varphi^2 + \omega_3)\varphi_2 + (\varphi^3 - \omega_2)\varphi_3 + \phi_5 = 0,$ (4.5.50)

where $\phi_5^0 = \left(-\varphi_1^1 + \varphi_2^2 + \varphi_3^3\right)\varphi^0$, $\vec{\phi}_5 = \vec{\psi}_5 - 2\lambda(\varphi^0)^{\gamma-2}\vec{\nabla}\varphi^0$, and $\vec{\psi}_5 = (-2a, -b\varphi^3, b\varphi^2)$;

6. $(-\varphi^1 + bc)\varphi_1 + (\varphi^2 + b\omega_2)\varphi_2 + (\varphi^3 - b\omega_3)\varphi_3 + \phi_6 = 0,$ (4.5.51)

where $\phi_6^0 = \phi_5^0$, $\vec{\phi}_6 = \vec{\psi}_6 + \lambda(\varphi^0)^{\gamma-2}\vec{\nabla}\varphi^0$, $\vec{\psi}_6 = (-2a, -b\varphi^2, b\varphi^3)$;

7. $(1 - \vec{\alpha}\vec{\varphi})\varphi_1 + (1 - \vec{\beta}\vec{\varphi})\varphi_2 + (1 - \vec{\gamma}\vec{\varphi})\varphi_3 + \phi_7 = 0,$ (4.5.52)

where $\phi_7^0 = -(\vec{\alpha}\vec{\varphi}_1 + \vec{\beta}\vec{\varphi}_2 + \vec{\gamma}\vec{\varphi}_3)\varphi^0$, $\vec{\phi}_7 = \lambda(\varphi^0)^{\gamma-2}(\vec{\alpha}\varphi_1^0 + \vec{\beta}\varphi_2^0 + \gamma\varphi_3^0)$;

8. $(\omega_1 + \vec{\alpha}\vec{\varphi})\varphi_1 + (\omega_2 + \vec{\beta}\vec{\varphi})\varphi_2 + (\omega_3 + \vec{\gamma}\vec{\varphi})\varphi_3 + \phi_8 = 0,$ (4.5.53)

where $\phi_8^0 = \frac{1}{\gamma-1}\varphi^0 - \phi_7^0$ $\vec{\phi}_8 = \vec{\phi}_7 + \frac{1}{2}\varphi$.

Knowing a solution of Equations (4.5.46)–(4.5.53) one obtains by means of (4.5.45) a partial solutions of Equations (4.5.44).

Consider Equations (4.5.46) and suppose $\varphi_1 = 0$, that is $\varphi = \varphi(\omega_2, \omega_3)$. It follows

$$(2\varphi^2 - \omega_2 + 2\omega_3)\varphi_2 - (2\varphi^3 + \omega_3 + 2\omega_2)\varphi_3 + \phi_1 = 0. \tag{4.5.54}$$

If $\varphi = \varphi(\omega_1)$ then (4.5.46) is directly reduced to the ODE

$$(2\varphi^1 - \omega_1)\varphi_1 + \phi_1 = 0,$$

which has under $\gamma = 2$ a partial solution

$$\begin{aligned} &\varphi^0 = c_0, \qquad \varphi^1 = \frac{\omega_1}{\alpha^2}, \\ &\varphi^2 = c_1\omega_1 \sin\left(\frac{\alpha}{c}\ln c_2\omega_1\right), \\ &\varphi^3 = c_1\omega_1 \cos\left(\frac{\alpha}{c}\ln c_2\omega_1\right). \end{aligned} \tag{4.5.55}$$

So, combining (4.5.55) and (4.5.45) we get a partial solution of Equations (4.5.44) under $\gamma = 2$:

$$\begin{aligned} &\rho = c_0 x_0^{-1}, \\ &\vec{u} = M(x)x_0^{-1/2}\left[\vec{\alpha} + c_1\vec{\beta}\sin\Big(\ln c_2 M(x)\Big) + c_1\vec{\gamma}\cos\Big(\ln c_2 M(x)\Big)\right] \end{aligned} \tag{4.5.56}$$

where $M(x) = x_0^{-1/2}(\vec{\alpha}\vec{x} + bx_0)$.

Equation (4.5.47) under $\varphi = \varphi(\omega_1)$ takes the form

$$(2\varphi^1 - \omega_1)\varphi_1 + \phi_2 = 0.$$

This latter system has a partial solution

$$\begin{aligned} &\varphi^0 = c_0\omega_1^{\frac{2}{\gamma-1}} \exp(-2\omega_1), \\ &\varphi^1 = a\omega_1(\omega_1 + c_2)^2 + \frac{2\lambda}{\gamma - 1}c_0^{\gamma-1}\omega_1 \exp[-2(\gamma - 2)\omega_1] + c_1\omega_1, \\ &\varphi^2 = -2a\omega_1(\omega_1 + c_2), \\ &\varphi^3 = \omega_1, \end{aligned} \tag{4.5.57}$$

where c_0, c_1, c_2 are arbitrary constants. Using this result we find a partial solution of the system (4.5.44)

$$\begin{aligned} &\rho = x_0^{\frac{1}{\gamma-1}}[B(x)]^{\frac{2}{\gamma-1}} \exp[-2B(x)], \\ &\vec{u} = x_0^{-1/2}B(x)\Big\{\vec{\alpha}\Big[a(B(x) + c_2)^2 + \frac{2\lambda}{\gamma-1}c_0^{\gamma-1}\cdot \end{aligned} \tag{4.5.58}$$

$$\cdot \exp\{-2(\gamma-1)B(x)\} + c_1\Big] - 2a\vec{\beta}(B(x)+c_2) + \vec{\gamma}\Big\} + \vec{V},$$

where $B(x) = x_0^{-1/2}(\vec{\alpha}\vec{x} - \vec{\alpha}\vec{A}(x_0))$.

Consider Equations (4.5.49). They are successfully solved when $\varphi = \varphi(\omega_1)$. In such a case Equations (4.5.49) take the form

$$(2\varphi^1 + \omega_1)\varphi_1 + \phi_4 = 0. \qquad (4.5.59)$$

Under $\gamma = 2$, $b_4 = 0$ we found its solution

$$\varphi^0 = -\frac{1}{2\lambda}\left(\omega_1^2 + c_0\right), \qquad \varphi^1 = -\tfrac{1}{2}\omega_1, \qquad \varphi^2 = 0, \qquad \varphi^3 = F(\omega_1),$$

where c_0 is a constant, F an arbitrary differentiable function. Using this result we obtain a partial solution of Equations (4.5.44)

$$\begin{aligned} \rho &= \frac{(\vec{\alpha}\vec{x} + b_1x_0)^2}{8\lambda x_0}, \\ \vec{u} &= \frac{\vec{\alpha}}{2x_0}(\vec{\alpha}\vec{x} + b_1x_0) + \vec{\beta}F\left(\frac{\vec{\alpha}\vec{x} + b_1x_0}{\sqrt{x_0}}\right), \end{aligned} \qquad (4.5.60)$$

where $\vec{\alpha} = 1$, $\vec{\alpha}\vec{\beta} = 0$, $\vec{\beta}^2 = 0$. Note, that this solution contains arbitrary function F.

Next consider Equations (4.5.50). If function $\varphi(\omega)$ does not depend on ω_2, ω_3, we get instead of (4.5.50) the system of ODEs

$$(\varphi^1 + bc)\varphi_1 + \phi_5 = 0 \qquad (4.5.61)$$

Under $\gamma = -1$ its general solution is given by

$$\varphi^0 = \sqrt{\frac{c_0^2 - \lambda}{4a(\omega_1 + c_1)}}, \qquad \varphi^1 = \frac{c_0}{\varphi^0} - bc, \qquad (4.5.62)$$

$$\varphi^2 = c_2 \sin\left[\frac{2b}{c_0}(\omega_1 + c_1)\varphi^0 + c_3\right], \qquad (4.5.63)$$

$$\varphi^3 = c_2 \cos\left[\frac{2b}{c_0}(\omega_1 + c_1)\varphi^0 + c_3\right],$$

where c_0, c_1, c_2, c_3 are arbitrary constants. The corresponding solution of the system (4.5.44) under $\gamma = -1$ has the form

$$\rho = A(x)$$

$$\vec{u} = \vec{\alpha}\left(\frac{c_0}{A(x)} + 2ax_0 - bc\right) + c_2\vec{\beta}\sin\left[\frac{2b}{c_0}(\vec{\alpha}\vec{x} + ax_0^2 + bcx_0 + c_1)A(x) + c_3\right] +$$

$$+c_2\vec{\gamma}\cos\left[\frac{2b}{c_0}(\vec{\alpha}\vec{x}+ax_0^2+bcx_0+c_1)A(x)+c_3\right]+\vec{V}, \tag{4.5.64}$$

where

$$A(x)=\sqrt{\frac{c_0^2-\lambda}{4a(\vec{\alpha}\vec{x}+ax_0^2+bcx_0+c_1)}}.$$

It is well to note, that because solutions (4.5.56), (4.5.58), (4.5.60), (4.5.64) contain arbitrary functions, they can be used to solve corresponding boundary-value or initial-value problems.

In the same way one can construct solutions of Equations (4.5.1) when $p = f(\rho)$ with arbitrary differentiable function f. In such a case Equations (4.5.1) are invariant with respect to the Galilei group G(1,3) and one can use invariants of this group and formulae (4.5.45). By substituting these ansatze into (4.5.1) one gets the reduced equations which coincide with (4.5.50)–(4.5.53) with the only exception: instead of the term $\lambda(\varphi^0)^{\gamma-2}$ we will have $(\varphi^0)^{-1}f'(\varphi^0)$.

As an example consider reduced equations corresponding (4.5.52):

$$\begin{gathered}(1-\vec{\alpha}\vec{\varphi})\varphi_1+(1-\vec{\beta}\vec{\varphi})\varphi_2+(1-\vec{\gamma}\vec{\varphi})\varphi_3+\psi_7=0,\\ \psi_7^0=\phi_7^0, \qquad \psi_7=(\vec{\alpha}\varphi_1^0+\vec{\beta}\varphi_2^0+\vec{\gamma}\varphi_3^0)(\varphi^0)^{-1}f'(\varphi^0).\end{gathered} \tag{4.5.65}$$

Supposing $\varphi=\varphi(\omega_1)$ we obtain from (4.5.65) the system of PDEs

$$\begin{gathered}(1-\vec{\alpha}\vec{\varphi})\varphi_1^0+\varphi^0\vec{\alpha}\vec{\varphi}_1=0,\\ (1-\vec{\alpha}\vec{\varphi}_1)\vec{\varphi}_1+(\varphi^0)^{-1}f'(\varphi^0)\vec{\alpha}\varphi_1^0=0.\end{gathered} \tag{4.5.66}$$

This system has a solution

$$\begin{gathered}\varphi^0=c_0\\ \vec{\varphi}=\vec{F}(\omega_1),\end{gathered} \tag{4.5.67}$$

where $\vec{F}$ is an arbitrary differentiable vector-function, satisfying the condition $\vec{\alpha}\vec{F}=1$, c_0 is a constant. There follows a partial solution of Equations (4.5.1)

$$\begin{gathered}\rho=c_0\\ \vec{u}=\vec{F}(x_0-\vec{\alpha}\vec{x}),\end{gathered} \tag{4.5.68}$$

It is appropriate to present here formulae of generating solutions of the system (4.5.1) under $p=\lambda\rho^{5/3}$.

The projective transformations have the form

$$\begin{gathered}x_0'=\frac{x_0}{1-\theta x_0}, \qquad \vec{x}'=\frac{\vec{x}}{1-\theta x_0},\\ \rho'=\rho(1-\theta x_0)^3, \qquad \vec{u}'=\vec{u}(1-\theta x_0)+\theta\vec{x},\end{gathered} \tag{4.5.69}$$

(θ is a parameter) and the corresponding formulae of generating solutions are

$$\begin{aligned} \rho &= (1-\theta x_0)^{-3} F\left(\frac{x_0}{1-\theta x_0}, \frac{\vec{x}}{1-\theta x_0}\right), \\ \vec{u} &= (1-\theta x_0)^{-1}\left[\vec{G}\left(\frac{x_0}{1-\theta x_0}, \frac{\vec{x}}{1-\theta x_0}\right) - \theta\vec{x}\right]. \end{aligned} \tag{4.5.70}$$

For example, applying (4.5.70) to a trivial constant solution of Equations (4.5.1)

$$\rho = c_0, \qquad \vec{u} = \vec{c}$$

we obtain as a result a solution which depends on all variables

$$\begin{aligned} \rho &= c_0(1-\theta x_0)^{-3}, \\ \vec{u} &= (\vec{c} - \theta\vec{x})(1-\theta x_0)^{-1}. \end{aligned} \tag{4.5.71}$$

Let us present some more solutions of Equations (4.5.44) obtained with the help of formulae (4.5.70)

$$\begin{aligned} \rho &= c_0(1-\theta x_0)^{-3}, \\ \vec{u} &= (1-\theta x_0)^{-1}\left[\vec{F}\left(\frac{x_0 - \vec{\alpha}\vec{x}}{1-\theta x_0}\right) - \theta\vec{x}\right]. \end{aligned} \tag{4.5.72}$$

Here $\vec{F}$ are the same as in (4.5.68).

4. Now we shall study symmetry of gas dynamics equations describing isochoric process (that is, the process taking place without change in volume (density)) with constant pressure. Equations in question are

$$\begin{aligned} &\vec{u}_0 + (\vec{u}\vec{\nabla})\vec{u} = 0, \\ &\operatorname{div}\vec{u} = 0 \end{aligned} \tag{4.5.73}$$

$(\vec{u} = \vec{u}(x) = \{u^1, u^2, u^3\}, \qquad x = \{x_0, x_1, x_2, x_3\})$

or

$$\vec{u}_0 + (\vec{u}\vec{\nabla})\vec{u} = 0. \tag{4.5.74}$$

Symmetry of Equation (4.5.74) was investigated in [174] and it was established the maximal in the class of linear representations 24-parameter invariance group $G = \{IGL(4,R), C(4)\}$ (IGL(4,R) is a 20-parameter group of general inhomogeneous linear transformations in $R(4)$; $C(4)$ is a 4-parameter Abelian group of pure conformal transformations in $R(4)$), the infinitesimal transformations having the form

$$\begin{aligned} x'_\mu &= x_\mu + \epsilon\xi^\mu(x) + O(\epsilon^2), \\ u^{a\prime}(x') &= u^a(x) + \epsilon\eta^a(x,u) + O(\epsilon^2), \end{aligned} \tag{4.5.75}$$

where

$$\xi^{\mu} = (\vec{\alpha}\cdot\vec{x})x_{\mu} + \beta_{\mu\nu}x_{\nu} + \gamma_{\mu}, \tag{4.5.76}$$
$$\eta^{a} = \beta_{a0} + \beta_{ab}u^{b} + (\alpha_0 + \alpha_b u^{b})x_a - [(\alpha_0 + \alpha_b u^{b})x_0 + \beta_{00} + \beta_{ab}u^{b}]u^{a};$$

where α_a, $\beta_{\mu\nu}$, γ_μ are arbitrary constants.

Note, the most general form of IFO admitted by Equation (4.5.74) is

$$X = \xi^{\mu}(x,u)\partial_{\mu} + \eta^{a}(x,u)\partial_{u^{a}} \tag{4.5.77}$$

which includes linear representation ($\xi^{\mu}(x,u) = \xi^{\mu}(x)$, $\eta^{a}(x,u)$ are linear functions of u) as a particular case. It turns out that the Lie-maximal invariance group of Equation (4.5.74) is infinite-dimensional [91].

Lemma 4.5.1. *The maximal invariance group of Equation* (4.5.74) *is infinite-dimensional, the basis elements of corresponding Lie algebra having the form* (4.5.77) *provided*

$$\begin{aligned}
\xi^{0} &= \xi^{0}(x,u),\\
\xi^{a} &= u^{a}\xi^{0} + F^{a}(\omega,u) + x_0 G^{a}(\omega,u),\\
\eta^{a} &= G^{a}(\omega,u),
\end{aligned} \tag{4.5.78}$$

where $\omega = \{\omega^{a} = x_a - x_0 u_a\}$; ξ^{0}, F^{a}, G^{a} *are arbitrary differentiable functions.*

Proof. Applying Lie algorithm to the Equation (4.5.74) one obtains the following defining equations for coordinates of IFO (4.5.77):

$$\begin{aligned}
\eta^{a} &= (\partial_0 + \vec{u}\cdot\vec{\nabla})(\xi^{a} - u^{a}\xi^{0}),\\
(\partial_0 &+ \vec{u}\cdot\vec{\nabla})\eta^{a} = 0,
\end{aligned} \tag{4.5.78$'$}$$

the general solution of which is given by (4.5.78). The Lemma is proved.

The symmetry properties of Equations (4.5.73) are summarized in the following statement.

Theorem 4.5.2. *The maximal IA of Equations* (4.5.73) *is* AIGL(4,R)*, basis elements having the form*

$$\begin{aligned}
&\partial_{\mu} = \frac{\partial}{\partial x_{\mu}}, \qquad L_{ab} = x_b\partial_a + u^{b}\partial_{u^{a}}, \qquad a \neq b,\\
&L_{0a} = x_a\partial_0 - u^{a}u^{b}\partial_{u^{b}}, \qquad G_{a0} = x_0\partial_a + \partial_{u^{a}},\\
&\mathcal{D}_0 = x_0\partial_0 - u^{b}\partial_{u^{b}}, \qquad \mathcal{D}_a = x_a\partial_a + u^{a}\partial_{u^{a}} \qquad (\text{no sum over } a).
\end{aligned} \tag{4.5.79}$$

Proof. As a result of applying Lie condition of invariance (3), one obtains defining equations for coordinates of IFO (4.5.77) which include (4.5.78′) and equations

$$\operatorname{div}\vec{\eta} = 0, \qquad \frac{\partial \xi^\mu}{\partial u^a} = 0; \qquad \mu = \overline{0,3} \tag{4.5.80}$$

The general solution of these equation is

$$\begin{aligned} \xi^\mu &= \beta_{\mu\nu}x_\nu + \gamma_\mu; \qquad \mu = \overline{0,3}; \\ \eta^a &= \beta_{a0} + \beta_{ab}u^b - (\beta_{00} + \beta_{ab}u^b)u^a \end{aligned} \tag{4.5.81}$$

which defines AIGL(4,R) (4.5.79). The theorem is proved.

Remark 4.5.2. Algebra AIGL(4,R) (4.5.79) contains as subalgebras AP(1,3) and AG(1,3). This means that Equations (4.5.73) describe relativistic as well as nonrelativistic processes. It is interesting to note that generators of Lorentz boosts J_{0a} are realized in nonlinear representation

$$J_{0a} = x_0\partial_a + x_a\partial_0 - u^a u^b \partial_{u^b} + \partial_{u^a}. \tag{4.5.82}$$

The final transformations generated by operators (4.5.82) have the form

$$\begin{aligned} x_0' &= \gamma(x_0 + \vec{x}\cdot\vec{v}), \\ \vec{x}' &= \vec{x} + \frac{\gamma - 1}{v^2}(\vec{x}\cdot\vec{v})\vec{v} + \gamma\vec{v}x_0, \end{aligned} \tag{4.5.83}$$

$$\vec{u}' = \frac{\vec{u} + \vec{v}\left[(\gamma - 1)\dfrac{\vec{u}\cdot\vec{v}}{v^2} - \gamma\right]}{\gamma(1 - \vec{u}\cdot\vec{v})}, \tag{4.5.84}$$

where $\vec{v} = \{v_1, v_2, v_3\}$ are arbitrary constants, $v^2 = \vec{v}\cdot\vec{v}$, and $\gamma = (1 - v^2)^{-1/2}$. Though the transformations (4.5.84) are nonlinear in u, they have a clear interpretation. Formula (4.5.84) is the well-known relativistic-velocity summation.

5. Here we obtain exact solution of Equations (4.5.73), (4.5.74) by making use of their symmetry.

Consider Equation (4.5.74). Since it is invariant under G(1,3), we look for its solutions in the form

$$\vec{u}(x) = \vec{M}(x_0)\varphi^1(\omega) + \vec{N}(x_0)\varphi^2(\omega) + \vec{L}(x_0)\varphi^3(\omega) + \vec{B}(x_0) \tag{4.5.85}$$

which is a particular case of the ansatz (4.5.45). It is obvious that the reduced equations following as a result of substitution (4.5.85) into (4.5.74) coincide with (4.5.46)–(4.5.53) if to put in these latter $\lambda = 0$ and omit the last equation in every one of them.

Consider some such integrable reduced equations. Putting in (4.5.46) $\vec{\varphi} = \vec{\varphi}(\omega_1)$ we get

$$(2\varphi^1 - \omega_1)\vec{\dot{\varphi}}_1 - \vec{\varphi} + 2\vec{\dot{\psi}}_1 = 0 \tag{4.5.86}$$

The general solution of Equation (4.5.86) has the form

$$\begin{aligned}
\varphi^1 &= \frac{\omega_1 \pm \sqrt{\omega_1^2 + c_1}}{2}, \\
\varphi^2 &= c_2\Big(\omega_1 \pm \sqrt{\omega_1^2 + c_1}\Big) \sin \ln \Big[c_3\Big(\omega_1 \pm \sqrt{\omega_1^2 + c_1}\Big)\Big], \\
\varphi^3 &= c_2\Big(\omega_1 \pm \sqrt{\omega_1^2 + c_1}\Big) \cos \ln \Big[c_3\Big(\omega_1 \pm \sqrt{\omega_1^2 + c_1}\Big)\Big]
\end{aligned} \tag{4.5.87}$$

where c_1, c_2, c_3 are integration constants. Formula (4.5.87) together with (4.5.85) give a partial solution to the Equation (4.5.74)

$$\vec{u} = \frac{F(x)}{\sqrt{x_0}}\left[\frac{\vec{\alpha}}{2} + c_2\vec{\beta} \sin \ln \big(c_3 F(x)\big) + c_2\vec{\gamma} \cos \ln \big(c_3 F(x)\big)\right] + \vec{v}, \tag{4.5.88}$$

where $F(x) = x_0^{-1/2}\Big(\vec{\alpha}\vec{x} + bx_0 \pm \sqrt{(\vec{\alpha}\vec{x} + bx_0)^2 + c_1 x_0}\Big)$.

Let function $\vec{\varphi}$ of Equation (4.5.47) depends on ω_1 only. Then we have the ODE

$$(2\varphi^3 - \omega_1)\vec{\dot{\varphi}}_1 - \vec{\varphi} + 2\vec{\dot{\psi}}_2 = 0 \tag{4.5.89}$$

After integrating it we get

$$\begin{aligned}
\varphi^1 &= \tfrac{1}{2}\Big(\omega_1 \pm \sqrt{\omega_1^2 + c_3}\Big)\Big[\ln^2 c_2\Big(\omega_1 \pm \sqrt{\omega_1^2 + c_3}\Big) + c_1\Big], \\
\varphi^2 &= -a\Big(\omega_1 \pm \sqrt{\omega_1^2 + c_3}\Big) \ln c_2\Big(\omega_1 \pm \sqrt{\omega_1^2 + c_3}\Big), \\
\varphi^3 &= \frac{\omega_1 \pm \sqrt{\omega_1^2 + c_3}}{2},
\end{aligned} \tag{4.5.90}$$

where c_1, c_2, c_3 are integration constants. From (4.5.90) and (4.5.85) we find a solution of Equation (4.5.74)

$$\vec{u} = \frac{M(x)}{2\sqrt{x_0}}\Big[\vec{\alpha}a^2(\ln^2 c_2 M(x) + c_1) - 2a\vec{\beta}\ln c_2 M(x) + \vec{\gamma}\Big] + \vec{v}, \tag{4.5.91}$$

where $M(x) = \dfrac{\vec{\alpha}\vec{x} + kx_0}{\sqrt{x_0}} \pm \sqrt{\dfrac{(\vec{\alpha}\vec{x} + kx_0)^2}{x_0} + c_3}$.

Let function $\vec{\varphi}$ of Equation (4.5.48) not depend on ω_2, ω_3. Then we have the ODE

$$(2\varphi^1 + \omega_1)\vec{\dot{\varphi}} + \vec{\dot{\psi}}_3 = 0. \tag{4.5.92}$$

The general solution of (4.5.92) has the form

$$\begin{aligned}\varphi^1 &= \frac{-\omega_1 \pm \sqrt{\omega_1^2 + c_1}}{2},\\ \varphi^2 &= c_2\Big(-\omega_1 \pm \sqrt{\omega_1^2 + c_1}\Big)^{-2a_+},\\ \varphi^3 &= c_3\Big(-\omega_1 \pm \sqrt{\omega_1^2 + c_1}\Big)^{-2a_-},\end{aligned} \tag{4.5.93}$$

where c_1, c_2, c_3 are integration constants. Formulae (4.5.93), (4.5.85) give a solution to the Equation (4.5.74)

$$\begin{aligned}\vec{u} &= \frac{\vec{\alpha}}{2}F(x) + c_2\vec{\beta}\big[F(x)\big]^{-2a_+} + c_3\vec{\gamma}\big[F(x)\big]^{-2a_-} + \vec{v},\\ F(x) &= x_0^{-1}\Big(-\vec{\alpha}\vec{x} - bx_0 \pm \sqrt{(\vec{\alpha}\vec{x} + bx_0)^2 + c_1x_0}\Big)\end{aligned} \tag{4.5.94}$$

Putting $\vec{\varphi} = \vec{\varphi}(\omega_3)$ we get from (4.5.48)

$$(\varphi^3 - a_+\omega_2)\vec{\varphi}_2 - \tfrac{1}{2}\vec{\psi}_3 = 0 \tag{4.5.95}$$

which has a solution

$$\begin{aligned}\varphi^1 &= c_1(\varphi^3)^{1/2a_-},\\ \varphi^2 &= c_2(\varphi^3)^{1/2a_+},\end{aligned} \tag{4.5.96}$$

function φ^3 being a solution of functional equation

$$\omega_2(\varphi^3)^{a-} + c_3(\varphi^3)^{a+} = 1. \tag{4.5.97}$$

The corresponding solution of Equation (4.5.74) can be written as

$$\vec{u} = \frac{c_1\vec{\alpha}}{\sqrt{x_0}}(\varphi^3)^{1/2a_-} + c_2\vec{\beta}x_0^{-a_+}(\varphi^3)^{a_+/a_-} + \vec{\gamma}x_0^{-a_-}\varphi^3 + \vec{v}. \tag{4.5.98}$$

Let function $\vec{\varphi}$ of Equation (4.5.49) depend only on ω_1. Then we have the ODE

$$(2\varphi^1 + \omega_1)\vec{\varphi}_1 + \vec{\psi}_4 = 0$$

the general solution of which having the form

$$\begin{aligned}\varphi^1 &= \tfrac{1}{2}\Big(-\omega_1 \pm \sqrt{\omega_1^2 + c_1}\Big),\\ \varphi^2 &= c_2\Big(-\omega_1 \pm \sqrt{\omega_1^2 + c_1}\Big)^2,\\ \varphi^3 &= 2b_4 \ln\Big(-\omega_1 \pm \sqrt{\omega_1^2 + c_1}\Big) + c_3,\end{aligned} \tag{4.5.99}$$

where c_1, c_2, c_3 are integration constants. Formulae (4.5.99), (4.5.85) give a solution to the Equation (4.5.74)

$$\vec{u} = \vec{\alpha}\left(\frac{M(x)}{2\sqrt{x_0}} + b_1\right) + \vec{\gamma}\left(c_2\frac{M^2(x)}{x_0} - b_2\right) + \vec{\beta}\,(2b_4 \ln M(x) - b_4 \ln x_0) \quad (4.5.100)$$

where $M(x)$ is the same as in (4.5.94).

Let function $\vec{\varphi}$ of Equation (4.5.49) depends only on ω_2. Then we have the ODE

$$(\varphi^3 + b_3)\vec{\varphi}_2 - \tfrac{1}{2}\vec{\psi}_4 = 0$$

which is satisfied by

$$\begin{aligned} \varphi^1 &= c_1\sqrt{\varphi^2}, \\ \varphi^3 &= b_4 \ln \varphi^2 + c_3, \end{aligned} \quad (4.5.101)$$

provided function φ^2 is a solution of functional equation

$$b_3 \ln \varphi^2 + \varphi^2 = \omega_2 + c_2 \quad (4.5.102)$$

The corresponding solution of Equation (4.5.74) can be written as

$$\vec{u} = \vec{\alpha}\left(\frac{c_1}{x_0}\sqrt{\varphi^2} + b_1\right) + \vec{\beta}\left(b_4 \ln\frac{\varphi^2}{x_0} + c_3\right) + \vec{\gamma}\left(\frac{\varphi^2}{x_0} - b_2\right). \quad (4.5.103)$$

Let function $\vec{\varphi}$ of Equation (4.5.50) depend only on ω_1. Then we have the ODE

$$(\varphi^1 + bc)\vec{\varphi}_1 + \vec{\psi}_5 = 0$$

which has solution

$$\begin{aligned} \varphi^1 &= \pm\tfrac{1}{2}\sqrt{4a\omega_1 + c_1} - bc, \\ \varphi^2 &= c_2 \sin\left(\pm\frac{b}{2a}\sqrt{4a\omega_1 + c_1} + c_3\right), \\ \varphi^3 &= c_2 \cos\left(\pm\frac{b}{2a}\sqrt{4a\omega_1 + c_1} + c_3\right). \end{aligned}$$

Thereby we find one more solution to the Equation (4.5.74)

$$\vec{u} = \vec{\alpha}\big(M(x) - 2ax_0 - bc\big) + \vec{\beta}c_2 \sin\left(\frac{b}{2a}M(x) + c_3\right) + \quad (4.5.104)$$
$$+ \vec{\gamma}c_2 \cos\left(\frac{b}{2a}M(x) + c_3\right),$$

where $M(x) = \pm\big[4a(ax_0^2 + bcx_0 + \vec{\alpha}\vec{x}) + c_1\big]^{1/2}$.

Let function $\vec{\varphi}$ of Equation (4.5.51) depends only on ω_1. Then we have the ODE

$$(-\varphi^1 + bc)\vec{\varphi}_1 + \vec{\psi}_6 = 0$$

which is satisfied by

$$\begin{aligned} \varphi^1 &= 2\sqrt{a}\sqrt{\omega_1 + c_1} - bc, \\ \varphi^2 &= c_2 \exp\left(\frac{b}{\sqrt{a}}\sqrt{\omega_1 + c_1}\right), \\ \varphi^3 &= c_2 \exp\left(-\frac{b}{\sqrt{a}}\sqrt{\omega_1 + c_1}\right), \end{aligned} \tag{4.5.105}$$

Formulae (4.5.105) and (4.5.85) give a solution to the Equation (4.5.74)

$$\vec{u} = \vec{\alpha}\big(M(x) + 2ax_0 - bc\big) + \vec{\beta}c_2 \exp\left[-\frac{b}{\sqrt{a}}M(x) + bx_0\right] + \\ + \vec{\gamma}c_2 \exp\left[\frac{b}{\sqrt{a}}M(x) - bx_0\right] + \vec{v}, \tag{4.5.106}$$

where $M(x) = \big(ax_0^2 + bcx_0 + \vec{\alpha}\vec{x} + c_1\big)^{1/2}$.

Let function $\vec{\varphi}$ of Equation (4.5.52) depends only on ω_1. Then we have the ODE

$$(1 - \vec{\alpha}\vec{\varphi})\vec{\varphi}_1 = 0$$

the general solution of which has the form

$$\vec{\varphi} = \vec{f}(\omega_1), \tag{4.5.107}$$

where $\vec{f}$ are arbitrary differentiable funçtions, such as $\vec{\alpha}\vec{f} = 1$. It follows a solution to the Equation (4.5.74)

$$\vec{u} = \vec{f}(x_0 - \vec{\alpha}\vec{x}). \tag{4.5.108}$$

Let function $\vec{\varphi}$ of Equation (4.5.53) depends only on ω_1. Then we have the ODE

$$(\omega_1 + \vec{\alpha}\vec{\varphi})\vec{\varphi}_1 = -\tfrac{1}{2}\vec{\varphi}$$

which has a partial solution

$$\vec{\varphi} = \vec{c}F(\omega_1) \tag{4.5.109}$$

provided $F(\omega_1)$ satisfies the condition

$$F^2(\vec{c}\vec{\alpha}F + \tfrac{3}{2}\omega_1) = 1.$$

It follows a solution to the Equation (4.5.74)

$$\vec{u} = \frac{\vec{c}}{\sqrt{x_0}} F(x) \tag{4.5.110}$$

where function $F(x)$ satisfies functional equation

$$F^{-2} - \vec{\alpha}\vec{c}F = \frac{3}{2}\frac{\vec{\alpha}\vec{x}}{\sqrt{x_0}}. \tag{4.5.111}$$

Remark 4.5.3. Looking for solutions of Equation (4.5.74) by means of the ansatz

$$\vec{u} = \vec{x}\varphi(\omega) \tag{4.5.112}$$

we get the reduced equation as follows

$$\Big(\partial_0\omega + \big[(\vec{x}\vec{\nabla})\omega\big]\varphi\Big)\varphi' + \varphi^2 = 0 \tag{4.5.113}$$

To require Equation (4.5.113) to be an ODE, one has to put

$$\begin{aligned} \partial_0\omega &= A(\omega), \\ (\vec{x}\vec{\nabla})\omega &= B(\omega). \end{aligned} \tag{4.5.114}$$

For example, if

$$\omega = \vec{\alpha}\vec{x}e^{\lambda x_0} \quad \text{or} \quad \omega = \sqrt{\vec{x}^2}e^{\lambda x_0}, \tag{4.5.115}$$

Equations (4.5.113) take the form

$$\omega\varphi'(\varphi + \lambda) + \varphi^2 = 0. \tag{4.5.116}$$

This latter equation has the general solution

$$\frac{1}{\varphi}e^{\lambda/\varphi} = c_1\omega. \tag{4.5.117}$$

Let us write down some formulae of generating solutions of Equation (4.5.74). Let

$$\vec{u} = \vec{F}(x) \tag{4.5.118}$$

be a solution of (4.5.74). Then using transformations (4.5.83), (4.5.84) and formulae (17), we find new solutions of Equation (4.5.74):

$$\vec{u} = \frac{1 - \vec{v}^2}{1 - \vec{v}\vec{F}(x')}\vec{F}(x') - \vec{v} \tag{4.5.119}$$

where x' is determined in (4.5.83); analogously, using conformal transformations

$$x' = \frac{x}{\sigma}, \qquad \vec{u}' = \frac{\sigma\vec{u} + (\alpha_0 + \vec{\alpha}\vec{u})\vec{x}}{\sigma + (\alpha_0 + \vec{\alpha}\vec{u})x_0}, \qquad \sigma = 1 - \alpha_0 x_0 + \vec{\alpha}\vec{x} \tag{4.5.120}$$

we find new solutions as

$$\vec{u} = \frac{\vec{F}\left(\frac{x}{\sigma}\right) - \left[\alpha_0 + \vec{\alpha}\vec{F}\left(\frac{x}{\sigma}\right)\right]\vec{x}}{1 - \left[\alpha_0 + \vec{\alpha}\vec{F}\left(\frac{x}{\sigma}\right)\right]x_0}. \tag{4.5.121}$$

Now, without going into details, we list some solutions of the system (4.5.73) obtained in the same manner as solutions of Equation (4.5.74):

$$\vec{u} = x_0^{-1/2}\exp\left[-M(x)\right]\left[\vec{\beta}\left(c_2\cos M(x) - c_1\sin M(x)\right) + \right.$$
$$\left. + \vec{\gamma}\left(c_1\cos M(x) + c_2\sin M(x)\right)\right] - \vec{v},$$

$$M(x) = \ln\frac{\vec{\alpha}\vec{x} + bx_0}{\sqrt{x_0}};$$

$$\vec{u} = \frac{c_2}{B(x)}\left[\vec{\alpha}\left(a\ln\frac{B(x)}{\sqrt{x_0}} + c_1\right) + \vec{\beta}\right] + \vec{v},$$

$$B(x) = \vec{\alpha}\vec{x} - x_0\vec{\alpha}\vec{A}(x_0);$$

$$\vec{u} = c_2\vec{\beta}(\vec{\alpha}\vec{x} + bx_0)^{-2a} + c_3\vec{\gamma}(\vec{\alpha}\vec{x} + bx_0)^{-2a-} + \vec{v};$$

$$\vec{u} = \vec{\alpha}b_1 - \vec{\beta}\left(2b_4\ln\left(\vec{\alpha}\vec{x} + b_1x_0\right) + c_3\right) + \vec{\gamma}\left(\frac{c_2}{\vec{\alpha}\vec{x} + b_1x_0} - b_2x_0\right);$$

$$\vec{u} = \vec{\alpha}\left(c_1\exp\frac{\vec{\beta}\vec{x} + b_2x_0}{2b_3} - b_1x_0^{-1/2}\right) + \vec{\beta}\frac{b_4}{b_3}(\vec{\beta}\vec{x} + b_2x_0 + c_3) - \vec{\gamma}b_2;$$

$$\vec{u} = \vec{\alpha}c_1 + \vec{\beta}c_2\left(c_3 - \frac{2c_2}{c_1}\vec{\alpha}\vec{x}\right)^{-1/2} \pm \vec{\gamma}\left(c_3 - \frac{2c_2}{c_1}\vec{\alpha}\vec{x}\right)^{1/2} + \vec{v};$$

$$\vec{u} = \vec{f}(x_0 - \vec{\alpha}\vec{x}).$$

Note that in this case of generating solutions formula (4.5.119) also holds.

Remark 4.5.4. Let

$$\vec{u} = \vec{\nabla}v \tag{4.5.122}$$

where v is scalar real function. Substitution of (4.5.122) into Equation (4.5.74) gives rise to the Hamilton-Jacobi equation (3.6.1) with $m = 1$:

$$v_0 + \tfrac{1}{2}\left(\vec{\nabla}v\right)^2 = 0.$$

Hence, one can use solutions of the Hamilton-Jacobi equation obtained in Paragraph 3.6 to construct via (4.5.122) solutions of Equation (4.5.74). Below we list some of such solutions

$$\vec{u} = \frac{\vec{\alpha}}{2x_0}\left(\vec{\alpha}\vec{x} - bx_0 \pm \sqrt{(\vec{\alpha}\vec{x} + bx_0)^2 - 8ax_0}\right), \qquad \vec{\alpha}^2 = 1;$$

$$\vec{u} = -\vec{\alpha}\left(2ax_0 + b \pm \sqrt{(\vec{\alpha}\vec{x} + ax_0^2 + bx_0 + c}\right), \qquad \vec{\alpha}^2 = 1;$$

$$\vec{u} = (1 - \theta x_0)^{-1}\left[\vec{\alpha}F\left(\frac{\vec{\alpha}\vec{x}}{1 - \theta x_0}\right) - \theta\vec{x}\right], \qquad \vec{\alpha}^2 = 0,$$

where θ, $\vec{\alpha}$, a, b are arbitrary constants, F is an arbitrary differentiable function.

In conclusion, it will be noted that since equations considered above are translationally invariant all solutions obtained here can be multiplied by making change $x_\mu \to x_\mu + \delta_\mu$ (δ_μ are constants).

4.6. *The Galilean invariant generalization of the Lame equations. The superalgebra of symmetry of the Lame equations*

The Lame equations

$$L\vec{u} \equiv \frac{\partial^2\vec{u}}{\partial x_0^2} - \alpha\,\mathrm{grad}(\mathrm{div}\,\vec{u}) - \Delta\vec{u} = 0 \tag{4.6.1}$$

where $\vec{u} = \vec{u}(x) = \{u^1, u^2, u^3\}$ is a displacement vector, $x = (x_0, \vec{x}) \in R^4$, α is a constant, are basic equations of elastodynamics and describe propagation of waves in elastic and isotropic medium. In spite of the fact that Equations (4.6.1) are quite satisfactory from the physical point of view (at least for small-amplitude oscillations) [141] the rightfulness of their use from the group-theoretic point of view gives rise to grave doubts [63]. The point is that the Lame equations do not satisfy any principle of invariance, neither Galilean nor Lorentz. As it was shown in [39] the Lie-maximal invariance algebra of Equation (4.6.1) is $\widetilde{\mathrm{AE}}(1,3)$, basis elements having the form

$$\begin{gathered} P_0 = \frac{\partial}{\partial x_0}, \qquad P_a = \frac{\partial}{\partial x_a}, \qquad D = x_0P_0 + x_aP_a \\ J_{ab} = x_aP_b - x_bP_a + u^a\frac{\partial}{\partial u^b} - u^b\frac{\partial}{\partial u^a} \end{gathered} \tag{4.6.2}$$

In book [10] this symmetry has been used for obtaining exact solutions of Equation (4.6.1). It will be noted that in [73] it is shown (with the help of non-Lie method [57]) that Lame equations (4.6.1) are invariant both under

AC(1,3)⊃$\widetilde{\mathrm{AP}}(1,3)$ and AG(1,3); however, the basis elements of these algebras are pseudodifferential operators.

Following [104] we generalize the Lame equations (4.6.1) so that they will be invariant under the point Galilean transformations

$$\vec{x}' = \vec{x} + \vec{v}_0, \qquad x_0' = x_0; \\ \vec{u}'(x') = \vec{u}(x) + \vec{v}, \qquad \vec{v} = \text{const.} \tag{4.6.3}$$

Theorem 4.6.1. [104]. *Equation*

$$L\vec{u} + F(\vec{u}, \underset{1}{\vec{u}}, \underset{2}{\vec{u}}) = 0, \qquad \underset{1}{\vec{u}} = \left\{\frac{\partial u^a}{\partial x_b}\right\}, \qquad \underset{2}{\vec{u}} = \left\{\frac{\partial^2 u^a}{\partial x_b \partial x_c}\right\} \tag{4.6.4}$$

is invariant under the Galilean transformations (4.6.3) *iff*

$$F(\vec{u}, \underset{1}{\vec{u}}, \underset{2}{\vec{u}}) = 2(\vec{u} \cdot \vec{\nabla})\vec{u}_0 - \left[\vec{u}(\vec{u} \cdot \vec{\nabla})\vec{\nabla}\right]\vec{u} \tag{4.6.5}$$

Proof. The transformations (4.6.3) are generated by the operator

$$G = v_a G_a = x_0 \vec{v} \cdot \vec{\nabla} + v_a \partial_{u^a}. \tag{4.6.6}$$

According to (5.3.8) we find

$$\begin{aligned} \vec{\nabla}' &= \exp\{x_0 \vec{v} \cdot \vec{\nabla}\}\vec{\nabla} \exp\{-x_0 \vec{v} \cdot \vec{\nabla}\} = \vec{\nabla}, \\ \partial' &= \exp\{x_0 \vec{v} \cdot \vec{\nabla}\}\partial_0 \exp\{-x_0 \vec{v} \cdot \vec{\nabla}\} = \partial_0 - \vec{v} \cdot \vec{\nabla}, \\ (\partial_0')^2 &= \partial_0^2 - 2(\vec{v} \cdot \vec{\nabla})\partial_0 + (\vec{v} \cdot \vec{\nabla})^2 \end{aligned} \tag{4.6.7}$$

Hence

$$L(\partial')\vec{u}'(x') = L\vec{u} - 2(\vec{v} \cdot \vec{\nabla})\vec{u}_0 + (\vec{v} \cdot \vec{\nabla})^2 \vec{u} \tag{4.6.8}$$

So, to ensure the invariance of Equation (4.6.4) it is necessary and sufficient to require

$$F' = F + 2(\vec{v} \cdot \vec{\nabla})\vec{u}_0 - (\vec{v} \cdot \vec{\nabla})^2 \vec{u} \tag{4.6.9}$$

whence we obtain (4.6.5). The theorem is proved.

Remark 4.6.1. One can make sure applying Lie's method that the maximal IA of Equation (4.6.4), (4.6.5) is AG(1,3) with basis elements (4.6.2), (4.6.6).

Remark 4.6.2. There is one more realization of AG(1,3), namely when

$$G_a = x_0 \partial_a + \omega^a \omega^b \partial_{\omega^b} \tag{4.6.10}$$

These operators generate the following transformations

$$\vec{x}' = \vec{x} + \vec{v}x_0, \qquad x_0' = x_0$$
$$\vec{\omega}' = \frac{\vec{\omega}}{1 - \vec{v}\vec{\omega}}.$$

However, by virtue of the substitution

$$\vec{\omega} = \frac{\vec{c}}{1 - \vec{c}\vec{u}}, \qquad \vec{c} = \text{const}$$

these transformations and operators (4.6.10) are reduced to (4.6.3) and (4.6.6).

It will be noted that Lame equations are closely connected with the wave equations

$$\frac{\partial^2 \varphi}{\partial x_0^2} = (\alpha + 1)\Delta\varphi, \qquad \frac{\partial^2 \vec{\psi}}{\partial x_0^2} = \Delta\vec{\psi}. \tag{4.6.11}$$

It is easy to verify that

$$\vec{u} = \vec{\nabla}\varphi + \operatorname{rot}\vec{\psi} \tag{4.6.12}$$

is a solution of Lame equations (4.6.1) for any φ and $\vec{\psi}$ satisfying (4.6.11). An infinite sequence of solutions of the wave equation is given in (2.3.60), (2.3.61).

2. If we seek for symmetry operators of the Lame equation (4.6.1) among the second-order differential operators with matrix coefficients

$$Q = A^{\mu\nu}(x)\partial_\mu\partial_\nu + B^\mu(x)\partial_\mu + c(x), \tag{4.6.13}$$

then we will have:

Theorem 4.6.2. [70*] *The Lame equations* (4.6.1) *admit 62 operators of the type* (4.6.13) *from which nine operators*

$$\begin{aligned} N &= (\vec{S}\vec{J})^2 - (\vec{S}\vec{J}), \\ N_a &= [P_a, N], \qquad N_{ab} = [P_b, N_a] \end{aligned} \tag{4.6.14}$$

where $\vec{J} = \vec{x} \times \vec{P} + \vec{S}$; *matrices* S_a *are written in* (4.2.21), $\vec{P} = -i\vec{\nabla}$, *do not belong to the enveloping algebra of* $\mathrm{A\tilde{E}}(1,3)$ (4.6.2).

Theorem 4.6.3. *The Lame equations are invariant with respect to the superalgebra* SQM(3,3), *basis elements having the form*

$$\begin{aligned} Q_1 &= (\vec{S}\cdot\vec{P})^2, \qquad Q_2 = N^2, \qquad Q_3 = Q_1Q_2 \\ R_1 &= (\vec{S}\cdot\vec{P})^2, \qquad R_2 = N, \qquad R_3 = R_1R_2 \end{aligned} \tag{4.6.15}$$

and satisfying relations

$$\begin{aligned}&[Q_a, Q_b] = [Q_a, R_b] = 0,\\ &[R_a, R_b]_+ \equiv R_a R_b + R_b R_a = \delta_{ab} R_b \text{ (no sum on } b).\end{aligned} \tag{4.6.16}$$

The proof of these theorems is based on the invariance condition in the form

$$[L, Q] \equiv [L, A^{\mu\nu}\partial_\mu\partial_\nu + B^\mu\partial_\mu + C] = \left(\widetilde{A}^{\mu\nu}\partial_\mu\partial_\nu + \widetilde{B}\partial_\mu + \widetilde{C}\right)L \tag{4.6.17}$$

where $\widetilde{A}^{\mu\nu}$, $\widetilde{B}^\mu$, $\widetilde{C}$ are 3×3 matrices depending on x. The truth of formulae (4.6.16) one can verify straightforwardly.

Now we present final transformations generated by operators (4.6.15) calculated according to §5.3 and Appendix 3.5. Using the identities

$$\begin{aligned}&(\vec{S}\cdot\vec{P})\vec{u} = \operatorname{rot}\vec{u},\\ &(\vec{S}\cdot\vec{P})^2\vec{u} = \operatorname{rot}\operatorname{rot}\vec{u} = \vec{\nabla}\operatorname{div}\vec{u} - \Delta\vec{u} \stackrel{\text{def}}{=} \vec{v},\\ &(\vec{S}\cdot\vec{P})^{2n}\vec{u} = (-1)^{n+1}\Delta^{n-1}\vec{v}, \qquad n = 1, 2, \dots\\ &(\vec{S}\cdot\vec{P})^{2n+1}\vec{u} = (-1)^n\Delta^n\operatorname{rot}\vec{u}, \qquad n = 0, 1, 2, \dots\end{aligned} \tag{4.6.18}$$

one can easy obtain

$$\begin{aligned}Q_1: \quad \vec{u}'(x') &= \exp\{\theta(\vec{S}\cdot\vec{p})^2\}\vec{u} = \sum_{n=0}^{\infty}\frac{\theta^n}{n!}(\vec{S}\cdot\vec{p})^{2n}\vec{u} =\\ &= \vec{u} - \sum_{n=1}^{\infty}(-1)^n\frac{\theta^n}{n!}\Delta^{n-1}\vec{v} = \vec{u} - \Delta\left(\sum_{n=0}^{\infty}\frac{(-\theta)^n}{n!}\Delta^n - 1\right)\vec{v} =\\ &= \vec{u} - \Delta^{-1}\left(e^{-\theta\Delta} - 1\right)\vec{v} =\\ &= e^{-\theta\Delta}\vec{u} + \left(1 - e^{-\theta\Delta}\right)\Delta^{-1}\operatorname{grad}\operatorname{div}\vec{u}\end{aligned} \tag{4.6.19}$$

$$x' = x$$

$$R_1: \quad \vec{u}'(x') = \vec{u}(x) + \beta\operatorname{rot}\vec{u}$$

θ is a usual real parameter; β is a Grassmann parameter.

Using the identities

$$\begin{aligned}&(\vec{S}\cdot\vec{J})\vec{u} = \vec{u} + \vec{\nabla}(\vec{x}\cdot\vec{u}) - \vec{x}\operatorname{div}\vec{u} \stackrel{\text{def}}{=} \vec{w}\\ &(\vec{S}\cdot\vec{J})^2\vec{u} = (\vec{S}\cdot\vec{J})\vec{w} = \vec{w} + \vec{\nabla}(\vec{x}\cdot\vec{w}) - \vec{x}\operatorname{div}\vec{w}\end{aligned} \tag{4.6.20}$$

we find transformations generated by the odd operator R_2:

$$\begin{aligned}u^{a\prime}(x') = u^a(x) + \beta_1\Big[&2u^a + (\vec{x}\cdot\vec{\nabla})u^a + 3\vec{x}\cdot\vec{u}_a +\\ &+ \vec{x}(\vec{x}\cdot\vec{\nabla})\vec{u}_a - \vec{x}^2\operatorname{div}\vec{u}_a + x_a\left((\vec{x}\cdot\vec{\nabla})\operatorname{div}\vec{u} - \vec{x}\cdot\Delta\vec{u} - 2\operatorname{div}\vec{u}\right)\Big]\end{aligned}$$

β_1 is a Grassmann parameter.

4.7. *Reduction and exact solutions of nonlinear Galilei-invariant spinor equations*

Following [73*] we consider the nonlinear spinor equation of the form

$$[-i(\gamma_0+\gamma_4)\partial_t + i\gamma_a\partial_a + m(\gamma_0-\gamma_4)]\psi = F(\psi^*,\psi), \tag{4.7.1}$$

where $\psi=\psi(t,x_1,\ldots,x_3)$ is a four-component complex-valued function (column), $\partial_t=\partial/\partial t$, $\partial_a=\partial/\partial x_a$; $\gamma_0,\ldots,\gamma_3$ are Dirac matrices (2.1.2), $\gamma_4=\gamma_0\gamma_1\gamma_2\gamma_3$; F is a four-component function (column) depending on ψ^*,ψ.

It is known that Equation (4.7.1) under $F=0$ is invariant with respect to the 11-dimensional Galilei algebra AG(1,3) [144,78] with basis elements

$$\begin{aligned} &P_0=\partial_t, \quad P_a=\partial_a, \quad M=2im,\\ &G_a=t\partial_a+2imx_a+\tfrac{1}{2}(\gamma_0+\gamma_4)\gamma_a,\\ &J_{ab}=x_a\partial_b-x_b\partial_a-\tfrac{1}{2}\gamma_a\gamma_b. \end{aligned} \tag{4.7.2}$$

Operators G_a generate the three-dimensional group of Galilei transformations

$$\begin{aligned} &t=t', \quad x'_a=x_a+v_at,\\ &\psi'(x')=\exp\{-2im(v_ax_a+\tfrac{1}{2}v_av_at)-\\ &\quad -\tfrac{1}{2}(\gamma_0+\gamma_4)\gamma_av_a\}\psi(x), \quad v_a=\text{const}, \quad x=\{t,x_a\}. \end{aligned} \tag{4.7.3}$$

Let us require the nonlinear equation (4.7.1) to be invariant under Galilei transformations (4.7.3). (Non-Lie symmetry of Equation (4.7.1) under $F=0$ is studied in [138*].)

Theorem 4.7.1. *Equation* (4.7.1) *is invariant under* AG(1,3) (4.7.2) *iff*

$$F=[f_1+(\gamma_0+\gamma_4)f_2]\psi \tag{4.7.4}$$

where $f_i=f_i(\overline{\psi}\psi,\overline{\psi}(\gamma_0+\gamma_4)\psi)$, $\overline{\psi}=(\psi^*)^T\gamma_0$.

The proof can be obtained by application of standard Lie algorithm.

Remark 4.7.1. When $m=0$, Equation (4.7.1) with nonlinearity (4.7.4) is invariant under the infinite-dimensional group of transformations

$$\begin{aligned} &t'=t \qquad x'_a=x_a+h_a(t),\\ &\psi'(x')=\exp\{-\tfrac{1}{2}(\gamma_0+\gamma_4)\gamma_a\dot{h}_a(t)\}, \end{aligned} \tag{4.7.5}$$

where $h_a(t)$ are arbitrary differentiable functions of t, $\dot{h} = dh/dt$.

To construct solutions of Equations (4.7.1), (4.7.4) we will act by analogy with that done in Paragraph 2.1, but in this case we use the three-dimensional subalgebras of AG(1,3) found in [71*]. Below in Table 4.7.1 we list G(1,3)-inequivalent ansatze of codimension 1 obtained as solutions of equations

$$\begin{gathered} Q_a A(x) \equiv \left[\xi_a^0(x)\partial_t + \xi_a^b(x)\partial_b + \eta(x)\right]A = 0, \\ \left[\xi_a^0(x)\partial_t + \xi_a^b(x)\partial_b\right]\omega = 0, \\ \psi(x) = A(x)\varphi(\omega), \end{gathered} \tag{4.7.6}$$

where Q_a, $a = 1, 2, 3$ are elements of a three-dimensional subalgebra of AG(1,3).

Table 4.7.1. G(1,3)-inequivalent ansatze of codimension 1 for spinor field

N	Algebra	Invar. var. ω	Ansatze $\psi(x) =$
1.	P_0, P_1, P_2	x_3	$= \varphi(\omega)$
2.	$P_1, P_2,$ $J_{12} + \alpha P_0$	x_3	$= \exp\{\frac{t}{2\alpha}\gamma_1\gamma_2\}\varphi(\omega)$
3.	$P_1, P_2,$ $P_0 + i\alpha m$	x_3	$= \exp\{-i\alpha mt\}\varphi(\omega)$
4.	$P_0, P_3,$ J_{12}	$x_1^2 + x_2^2$	$= \exp\left\{-\frac{1}{2}\gamma_1\gamma_2 \arctan\frac{x_1}{x_2}\right\}\varphi(\omega)$
5.	$P_1, P_2,$ $J_{12} + \alpha P_0 +$ $+\beta G_3$	$\beta t^2 - 2\alpha x_3$	$= \exp\left\{\frac{2}{3}im\alpha^{-2}\beta t(\beta t^2 - 3\alpha x_3) - -\alpha^{-1}\beta t\eta_3 + \frac{t}{2\alpha}\gamma_1\gamma_2\right\}\varphi(\omega)$
6.	$P_1, P_2,$ $J_{12} + \alpha G_3$	t	$= \exp\left\{\frac{x_3}{2\alpha t}(\gamma_1\gamma_2 - 2\alpha\eta_3 - -2i\alpha m x_3\right\}\varphi(\omega)$
7.	$G_1, G_2,$ $J_{12} + \alpha G_3$	t	$= \exp\left\{-\frac{im}{t}(x_1^2 + x_2^2) - -\frac{1}{t}(\eta_1 x_1 + \eta_2 x_2)\right\} \exp\left\{-\frac{x_3}{2\alpha t}\cdot \cdot(2i\alpha m x_3 + \alpha\eta_3 - \gamma_1\gamma_2)\right\}\varphi(\omega)$
8.	$P_1, P_2,$ $J_{12} + \alpha P_3$	t	$= \exp\left\{-\frac{x_3}{2\alpha}\gamma_1\gamma_2\right\}\varphi(\omega)$
9.	$G_1, G_2,$ $J_{12} + \alpha P_3$	t	$= \exp\left\{-\frac{im}{t}(x_1^2 + x_2^2) - \frac{1}{t}(\eta_1 x_1 + +\eta_2 x_2)\right\} \exp\left\{-\frac{x_3}{2\alpha}\gamma_1\gamma_2\right\}\varphi(\omega)$

Table 4.7.1. G(1,3)-inequivalent ansatze of codimension 1 for spinor field

N	Algebra	Invar. var. ω	Ansatze $\psi(x) =$
10.	P_1, P_2, P_3	t	$= \varphi(\omega)$
11.	G_1, P_2, P_3	t	$= \exp\left\{-\frac{im}{t}x_1^2 - \frac{1}{t}x_1\eta_1\right\}\varphi(\omega)$
12.	$G_1 + \alpha P_1,$ G_2, P_3	t	$= \exp\left\{-\frac{x_2}{t}(imx_2 + \eta_2) + \frac{x_1}{\alpha - t}(imx_1 + \eta_1)\right\}\varphi(\omega)$
13.	$G_1 + \alpha P_1,$ $G_2 + \beta P_2,$ G_3	t	$= \exp\left\{-\frac{x_3}{t}(imx_3 + \eta_3) + \frac{x_1}{\alpha - t}\cdot(imx_1 + \eta_1) + \frac{x_2}{\beta - t}(imx_2 + \eta_2)\right\}\varphi(\omega)$
14.	$G_1 + \alpha P_0,$ P_2, P_3	$t^2 - 2\alpha x_1$	$= \exp\left\{\frac{2}{3}i\alpha^{-2}mt(t^2 - 3\alpha x_1) - \frac{t}{\alpha}\eta_1\right\}\varphi(\omega)$
15.	$J_{12} + i\alpha m,$ P_0, P_3	$x_1^2 + x_2^2$	$= \exp\left\{(i\alpha m - \frac{1}{2}\gamma_1\gamma_2)\arctan\frac{x_1}{x_2}\right\}\varphi(\omega)$
16.	$G_1 + \alpha P_2,$ $G_2 + \alpha P_1 +$ $+\beta P_2 +$ $+(\alpha\rho - \delta\beta)P_3,$ $-G_3 + \rho G_1 +$ $+\delta G_2 + \alpha\delta P_1$	t	$= \exp\left\{-\frac{x_1}{t}(imx_1 + \eta_1) - im(\alpha x_1 + tx_2)^2\left[t(t^2 - t\beta - \alpha^2)\right]^{-1} + \frac{x_3}{\tau t}(\alpha\eta_1 + t\eta_2)\right\}\cdot \exp\left\{imz^2\left[f(t)(t^2 - t\beta - \alpha^2)\right]^{-1} + \frac{z}{f(t)}\left[\alpha\left(\frac{1}{\tau} - \frac{\delta}{t^2}\right)\eta_1 + \left(\frac{t}{\tau} - \delta\right)\eta_2 + \eta_3\right]\right\}\varphi(\omega)$

In this table:

$$\eta_a = \tfrac{1}{2}(\gamma_0 + \gamma_4)\gamma_a, \qquad \tau = \alpha\rho + \delta\beta$$
$$f(t) = \tau[\alpha(\rho t - \alpha\delta) + \delta t^2] - t[t^2 - t\beta - \alpha^2],$$
$$z = \tau(\alpha x_1 + tx_2) + [t(t - \beta) - \alpha^2]x_3,$$

α, β, ρ, δ are arbitrary real constants.

Let us substitute ansatze from Table 4.7.1 into Equation (4.7.1), (4.7.4). After some cumbersome calculations we obtain the following system of ODEs:

$1^\circ \quad i\gamma_3\dot{\varphi} + m(\gamma_0 - \gamma_4)\varphi = R.$

$$2^\circ \quad i\gamma_3\dot{\varphi} + \left[\frac{i}{2\alpha}(\gamma_0 + \gamma_4)\gamma_3 + m(\gamma_0 - \gamma_4)\right]\varphi = R. \tag{4.7.7}$$

3° $i\gamma_3\dot{\varphi} + [\alpha m(\gamma_0 + \gamma_4) + m(\gamma_0 - \gamma_4)]\,\varphi = R.$

4° $2i\sqrt{\omega}\gamma_2\dot{\varphi} + \left[\frac{i}{2\sqrt{\omega}}\gamma_2 + m(\gamma_0 - \gamma_4)\right]\varphi = R.$

5° $-2i\gamma_3\dot{\varphi} + \left[\frac{i}{2\alpha}(\gamma_0 + \gamma_4)\gamma_3 + \frac{\beta\omega}{\alpha^2}(\gamma_0 + \gamma_4) + m(\gamma_0 - \gamma_4)\right]\varphi = R.$

6° $-i(\gamma_0 + \gamma_4)\dot{\varphi} + \left[\frac{i}{2\alpha\omega}(\gamma_0\gamma_4 - \alpha(\gamma_0 + \gamma_4)) + m(\gamma_0 - \gamma_4)\right]\varphi = R.$

7° $-i(\gamma_0 + \gamma_4)\dot{\varphi} + \left[\frac{i}{2\alpha\omega}\gamma_0\gamma_4 - \frac{i}{\omega}(\gamma_0 + \gamma_4) + m(\gamma_0 - \gamma_4)\right]\varphi = R.$

8° $-i(\gamma_0 + \gamma_4)\dot{\varphi} + \left[m(\gamma_0 - \gamma_4) - \frac{i}{2\alpha}\gamma_0\gamma_4\right]\varphi = R.$

9° $-i(\gamma_0 + \gamma_4)\dot{\varphi} + \left[m(\gamma_0 - \gamma_4) - \frac{i}{\omega}(\gamma_0 + \gamma_4) - \frac{i}{2\alpha}\gamma_0\gamma_4\right]\varphi = R.$

10° $-i(\gamma_0 + \gamma_4)\dot{\varphi} + m(\gamma_0 - \gamma_4)\varphi = R.$

11° $-i(\gamma_0 + \gamma_4)\dot{\varphi} + \left[m(\gamma_0 - \gamma_4) - \frac{i}{2\omega}(\gamma_0 + \gamma_4)\right]\varphi = R.$

12° $-i(\gamma_0 + \gamma_4)\dot{\varphi} + \left[m(\gamma_0 - \gamma_4) - \frac{i}{2}\left(\frac{1}{\omega} + \frac{1}{\omega - \alpha}\right)(\gamma_0 + \gamma_4)\right]\varphi = R.$

13° $$-i(\gamma_0 + \gamma_4)\dot{\varphi} + \Bigg[m(\gamma_0 - \gamma_4) - \frac{i}{2}\left(\frac{1}{\omega} + \frac{1}{\omega - \alpha} + \frac{1}{\omega - \beta}\right)(\gamma_0 + \gamma_4)\Bigg]\varphi = R.$$

14° $-2i\alpha\gamma_1\dot{\varphi} + \left[m(\gamma_0 - \gamma_4) + \frac{m\omega}{\alpha^2}(\gamma_0 + \gamma_4)\right]\varphi = R.$

15° $2i\sqrt{\omega}\gamma_2\dot{\varphi} + \left[\frac{i}{\sqrt{\omega}}(i\alpha m\gamma_1 + \tfrac{1}{2}\gamma_2) + m(\gamma_0 - \gamma_4) + \beta m(\gamma_0 + \gamma_4)\right]\varphi = R.$

16° $$-i(\gamma_0 + \gamma_4)\dot{\varphi} + \Big\{i(\gamma_0 + \gamma_4)\Big[[2\omega f(\omega)]^{-1}(\omega^3 + \alpha(\alpha + \rho\tau)\omega - 2\delta\alpha^2\tau) - \frac{1}{\omega}\Big] + m(\gamma_0 - \gamma_4)\Big\}\varphi = R.$$

Equations 1°–16° correspond to that of ansatze in Table 4.7.1; dot means differentiation with respect to corresponding ω;

$$R = \left[f_1(\overline{\varphi}\varphi, \overline{\varphi}(\gamma_0 + \gamma_4)\varphi) + (\gamma_0 + \gamma_4)f_2(\overline{\varphi}\varphi, \overline{\varphi}(\gamma_0 + \gamma_4)\varphi)\right]\varphi.$$

Further we consider the nonlinear equation

$$\begin{gathered}\left[-i(\gamma_0+\gamma_4)\partial_t + i\gamma_a\partial_a + m(\gamma_0-\gamma_4) - \lambda\left(\overline{\psi}(\gamma_0+\gamma_4)\psi\right)^{1/2k}\right]\psi = 0,\\ \lambda,\ k = \text{const},\end{gathered} \tag{4.7.8}$$

which is a particular case of Equation (4.7.1), (4.7.4). Note that in this case corresponding reduced equations have form (4.7.7), provided $R = \lambda\left(\overline{\varphi}(\gamma_0 + \gamma_4)\varphi\right)^{1/2k}\varphi$.

We succeeded in solving Equations 6°–13° (4.7.7). In this connection the following lemma was essentially used.

Lemma 4.7.1. *The quantities*

$$\begin{gathered}I_6 = \overline{\varphi}(\gamma_0+\gamma_4)\varphi\omega, \quad I_7 = \overline{\varphi}(\gamma_0+\gamma_4)\varphi\omega^2, \quad I_8 = \overline{\varphi}(\gamma_0+\gamma_4)\varphi,\\ I_9 = \overline{\varphi}(\gamma_0+\gamma_4)\varphi\omega^2, \quad I_{10} = \overline{\varphi}(\gamma_0+\gamma_4)\varphi, \quad I_{11} = \overline{\varphi}(\gamma_0+\gamma_4)\varphi\omega,\\ I_{12} = \overline{\varphi}(\gamma_0+\gamma_4)\varphi(\omega^2-\alpha\omega), \quad I_{13} = \overline{\varphi}(\gamma_0+\gamma_4)\varphi(\omega^3 - 2(\alpha+\beta)\omega^2 + \alpha\beta\omega)\end{gathered} \tag{4.7.9}$$

are the first integrals of the systems of ODEs 6°–13° (4.7.7), *respectively.*

Proof. Consider Equation 6° of (4.7.7) (the rest of the cases are analogous). Multiplying on γ_0 the complex conjugation of Equation 6° we obtain

$$i\dot{\overline{\varphi}}(\gamma_0+\gamma_4) + \overline{\varphi}\left[m(\gamma_0-\gamma_4) + \frac{i}{2\omega}(\gamma_0+\gamma_4) + \frac{i}{2\alpha\omega}\gamma_0\gamma_4\right] = \lambda\left(\overline{\varphi}(\gamma_0+\gamma_4)\varphi\right)^{1/2k}\varphi,$$

from which

$$\begin{gathered}\dot{\overline{\varphi}}(\gamma_0+\gamma_4)\varphi + \overline{\varphi}(\gamma_0+\gamma_4)\dot{\varphi} = -\omega^{-1}\overline{\varphi}(\gamma_0+\gamma_4)\varphi \quad \text{or}\\ \frac{d}{d\omega}\left[\overline{\varphi}(\gamma_0+\gamma_4)\varphi\right] = -\omega^{-1}\overline{\varphi}(\gamma_0+\gamma_4)\varphi.\end{gathered}$$

After integrating this equation we find

$$\overline{\varphi}(\gamma_0+\gamma_4)\varphi = c\omega^{-1}, \qquad c = \text{const}.$$

So, the lemma is proved.

Resolve relations (4.7.9) with respect to $\overline{\varphi}(\gamma_0+\gamma_4)\varphi$ and inserting the result obtained in Equations 6°–13° (4.7.7) we get the linear systems of ODEs, having general solutions of the form [73*]:

$$\varphi_N(\omega) = \tfrac{1}{2}\left[f_N(\omega)(\gamma_0+\gamma_4) + g_N(\omega)(1+\gamma_0\gamma_4)\right]\chi, \tag{4.7.10}$$

where $N = \overline{6-13}$, $\quad \chi = (\chi^0\, \chi^1\, \chi^2\, \chi^3)^T$,

$$g_6(\omega) = \frac{1}{\sqrt{\omega}} \exp\{(16i\alpha^2 m\omega)^{-1} + i\phi_1(k,\omega)\}, \tag{4.7.11}$$

$$f_6(\omega) = \frac{1}{2m}\big(\widetilde{\lambda}\omega^{-1/2k} + (2i\alpha\omega)^{-1}\big) g_6(\omega);$$

$$f_7(\omega) = \frac{1}{2m}\big(\lambda\omega^{-1/k} + (2i\alpha\omega)^{-1}\big) g_7(\omega),$$

$$g_7(\omega) = \frac{1}{\omega} \exp\{(16i\alpha^2 m\omega)^{-1} + i\phi_2(k,\omega)\};$$

$$f_8(\omega) = \frac{1}{2m}\big(\widetilde{\lambda} + \frac{i}{2\alpha}\big) g_8(\omega),$$

$$g_8(\omega) = \exp\{-(1 + 4\alpha^2 m\widetilde{\lambda}^2)(16i\alpha^2 m)^{-1}\omega\};$$

$$f_9(\omega) = \frac{1}{2m}\Big(\widetilde{\lambda}\omega^{-1/k} + \frac{i}{2\alpha}\Big) g_9(\omega),$$

$$g_9(\omega) = \omega^{-1} \exp\{-(16i\alpha^2 m)^{-1}\omega + i\phi_2(k,\omega)\};$$

$$f_{10}(\omega) = \frac{\widetilde{\lambda}}{2m} \exp\left\{\frac{i\widetilde{\lambda}^2}{4m}\omega\right\} g_{10}(\omega);$$

$$g_{10}(\omega) = \exp\left\{\frac{i\widetilde{\lambda}^2}{4m}\omega\right\};$$

$$f_{11}(\omega) = \frac{\widetilde{\lambda}}{2m}\omega^{-1/2k} g_{11}(\omega),$$

$$g_{11}(\omega) = \frac{1}{\sqrt{\omega}} \exp\{i\phi_1(k,\omega)\};$$

$$f_{12}(\omega) = \frac{\widetilde{\lambda}}{2m}(\omega^2 - \alpha\omega)^{-1/2k} g_{12}(\omega),$$

$$g_{12}(\omega) = (\omega^2 - \alpha\omega)^{-1/2} \exp\left\{\frac{i\widetilde{\lambda}^2}{4m}\int^{\omega}(z^2 - \alpha z)^{-1/k}\,dz\right\};$$

$$f_{13}(\omega) = \frac{\widetilde{\lambda}}{2m}\big[\omega(\omega-\alpha)(\omega-\beta)\big]^{-1/2k} g_{13}(\omega),$$

$$g_{13}(\omega) = \big[\omega(\omega-\alpha)(\omega-\beta)\big]^{-1/2k} \exp\left\{\frac{i\widetilde{\lambda}^2}{4m}\int^{\omega}\big[z(z-\alpha)(z-\beta)\big]^{-1/k}\,dz\right\}.$$

In formulae (4.7.11) $\widetilde{\lambda} = \lambda(\overline{\chi}(\gamma_0 + \gamma_4)\chi)^{1/2k}$,

$$\phi_n(k, \omega) = \frac{\widetilde{\lambda}^2}{4m} \begin{cases} \dfrac{k}{k-n}\omega^{(k-n)/k}, & k \neq n \\ \ln \omega, & k = n. \end{cases}$$

Returning to ansatze 6°–13° from Table 4.7.1, we can easily write with the help of (4.7.10), (4.7.11) the corresponding solutions of Equation (4.7.8).

In conclusion, it will be noted that Equation (4.7.8) under $k = \frac{3}{2}$ is additionally invariant with respect to scale and projective transformations [94,73*]:

$$t' = e^{2\theta}t, \quad x'_a = e^{\theta}x_a, \quad \psi'(x') = \exp\{\theta(-2 + \tfrac{1}{2}\gamma_0\gamma_4)\}\psi(x);$$

$$t' = \frac{t}{1-\theta t}, \quad x'_a = \frac{x_a}{1-\theta t},$$

$$\psi'(x') = (1-\theta t)^2 \exp\Big\{im\theta x_a x_a(\theta t - 1)^{-1} - \\ - \frac{1}{2t}\ln(1-\theta t)[t\gamma_0\gamma_4 + (\gamma_0 + \gamma_4)\gamma_a x_a]\Big\}\psi(x).$$

4.8. *Reduction and exact solutions of the Navier-Stokes equation*

Following [63*] below we construct the complete set of $\widetilde{G}(1,3)$-inequivalent ansatze of codimension 1 for the Navier-Stokes(NS) field which reduce the NS equations to systems of ODEs. Having solved these ODEs we obtain thereby solutions of the NS equations.

The NS equations

$$\begin{aligned} &\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} - \Delta \vec{u} + \vec{\nabla} p = 0, \\ &\operatorname{div} \vec{u} = 0, \end{aligned} \tag{4.8.1}$$

where $\vec{u} = \vec{u}(x) = \{u^1, u^2, u^3\}$ is the velocity field of a fluid, $p = p(x)$ is the pressure, $x = \{t, \vec{x}\} \in R(4)$, $\vec{\nabla} = \{\partial/\partial x_a\}$, $a = 1, 2, 3$; Δ the Laplacian, are basic equations of hydrodynamics which describe motion of an incompressible viscous fluid. The problem of finding the exact solutions of nonlinear Equations (4.8.1) is rather comlicated, and the symmetry approach turns out very effective. It is well known (see, e.g. [29]) that NS equations (4.8.1) are invariant under the extended Galilei group $\widetilde{G}(1,3)$ generated by operators

$$\begin{aligned} &\partial_t = \frac{\partial}{\partial t}, \qquad \partial_a = \frac{\partial}{\partial x_a}, \\ &J_{ab} = x_a\partial_b - x_b\partial_a + u^a\partial_{u^b} - u^b\partial_{u^a}, \\ &G_a = t\partial_a + \partial_{u^a}, \\ &D = 2t\partial_t + x_a\partial_a - u^a\partial_{u^a} - 2p\partial_p, \end{aligned} \tag{4.8.2}$$

Table 4.8.1. $\widetilde{G}(1,3)$-inequivalent ansatze of codimension 1 for the NS field

N	Algebra	Invar. var. ω	Ansatze
1.	$\partial_1, \partial_2, \partial_3$	t	$u^1 = f(\omega), \quad u^2 = g(\omega),$ $u^3 = h(\omega), \quad p = \varphi(\omega)$
2.	$\partial_t, \partial_1, \partial_2$	x_3	$u^1 = f(\omega), \quad u^2 = g(\omega),$ $u^3 = h(\omega), \quad p = \varphi(\omega)$
3.	∂_t, ∂_1 $G_1 + G_2$	x_3	$u^1 = x_2 + f(\omega), \quad u^2 = g(\omega),$ $u^3 = h(\omega), \quad p = \varphi(\omega)$
4.	∂_1, ∂_2 $\partial_t + G_3$	$t^2 - 2x_3$	$u^1 = f(\omega), \quad u^2 = g(\omega),$ $u^3 = t + h(\omega), \quad p = \varphi(\omega)$
5.	∂_1, ∂_2 $\partial_t + G_1$	x_3	$u^1 = t + f(\omega),$ $u^2 = g(\omega),$ $u^3 = h(\omega), \quad p = \varphi(\omega)$
6.	$\partial_1, \partial_2 + G_1$ $\partial_t + G_3$	$t^2 - 2x_3$	$u^1 = x_2 + f(\omega), \ u^2 = g(\omega),$ $u^3 = t + h(\omega), \quad p = \varphi(\omega)$
7.	$\partial_1 + \alpha\partial_3,$ $\partial_2, \partial_t + G_3$	$t^2 + 2\alpha x_1 - 2x_3$	$u^1 = f(\omega), \quad u^2 = g(\omega),$ $u^3 = t + h(\omega), \quad p = \varphi(\omega)$
8.	$\partial_1, \partial_t + G_3,$ $G_1 + \partial_2 + \alpha\partial_3$	$\alpha x_2 - x_3 + \frac{1}{2}t^2$	$u^1 = x_2 + f(\omega), \quad u^2 = g(\omega),$ $u^3 = t + h(\omega), \quad p = \varphi(\omega)$
9.	$\partial_t, \partial_3, J_{12}$	$(x_1^2 + x_2^2)^{1/2}$	$u^1 = x_1 f(\omega) - x_2 g(\omega),$ $u^2 = x_1 g(\omega) + x_2 f(\omega),$ $u^3 = h(\omega), \quad p = \varphi(\omega)$
10.	$\partial_t + G_3,$ ∂_3, J_{12}	$(x_1^2 + x_2^2)^{1/2}$	$u^1 = x_1 f(\omega) - x_2 g(\omega),$ $u^2 = x_1 g(\omega) + x_2 f(\omega),$ $u^3 = t + h(\omega), \quad p = \varphi(\omega)$
11.	$\partial_t, \partial_3, D$	x_1/x_2	$u^1 = \dfrac{1}{x_2} f(\omega), \quad u^2 = \dfrac{1}{x_2} g(\omega),$ $u^3 = \dfrac{1}{x_2} h(\omega), \quad p = \dfrac{1}{x_2^2}\varphi(\omega),$
12.	$\partial_t, \partial_3,$ $J_{12} + \alpha D$	$\ln(x_1^2 + x_2^2) +$ $+ 2\alpha \arctan \frac{x_1}{x_2}$	$u^1 = (x_1^2 + x_2^2)^{-1} \cdot$ $\cdot (x_1 f(\omega) - x_2 g(\omega)),$ $u^2 = (x_1^2 + x_2^2)^{-1} \cdot$ $\cdot (x_1 g(\omega) + x_2 f(\omega)),$ $u^3 = (x_1^2 + x_2^2)^{-1/2} h(\omega),$ $p = (x_1^2 + x_2^2)^{-1} \varphi(\omega)$

Table 4.8.1.

N	Algebra	Invar. var.	Ansatze
13.	∂_t, J_{12}, D	$\frac{(x_1^2+x_2^2)^{1/2}}{x_3}$	$u^1 = (x_1^2+x_2^2)^{-1}\cdot(x_1 f(\omega) - x_2 g(\omega))$, $u^2 = (x_1^2+x_2^2)^{-1}\cdot(x_1 g(\omega) + x_2 f(\omega))$, $u^3 = (x_1^2+x_2^2)^{-1/2} h(\omega)$, $p = (x_1^2+x_2^2)^{-1}\varphi(\omega)$
14.	∂_3, J_{12}, D	$\frac{(x_1^2+x_2^2)^{1/2}}{t}$	$u^1 = \frac{1}{t}(x_1 f(\omega) - x_2 g(\omega))$ $u^2 = \frac{1}{t}(x_1 g(\omega) + x_2 f(\omega))$, $u^3 = \frac{1}{\sqrt{t}} h(\omega)$, $p = \frac{1}{t}\varphi(\omega)$
15.	G_3, J_{12}, D	$\frac{(x_1^2+x_2^2)^{1/2}}{t}$	$u^1 = \frac{1}{t}(x_1 f(\omega) - x_2 g(\omega))$ $u^2 = \frac{1}{t}(x_1 g(\omega) + x_2 f(\omega))$, $u^3 = \frac{1}{\sqrt{t}} h(\omega) + \frac{x_3}{t}$, $p = \frac{1}{t}\varphi(\omega)$
16.	$\partial_1, \partial_2, D$	$\frac{x_3}{\sqrt{t}}$	$u^1 = \frac{1}{\sqrt{t}} f(\omega)$, $u^2 = \frac{1}{\sqrt{t}} g(\omega)$, $u^3 = \frac{1}{\sqrt{t}} h(\omega)$, $p = \frac{1}{t}\varphi(\omega)$
17.	$\partial_1, D, G_2 + \alpha G_1$	$\frac{x_3}{\sqrt{t}}$	$u^1 = \frac{1}{\sqrt{t}} f(\omega) + \frac{\alpha x_2}{t}$, $u^2 = \frac{1}{\sqrt{t}} g(\omega) + \frac{x_2}{t}$, $u^3 = \frac{1}{\sqrt{t}} h(\omega)$, $p = \frac{1}{t}\varphi(\omega)$
18.	G_1, G_2, D	$\frac{x_3}{\sqrt{t}}$	$u^1 = \frac{1}{\sqrt{t}} f(\omega) + \frac{x_1}{t}$, $u^2 = \frac{1}{\sqrt{t}} g(\omega) + \frac{x_2}{t}$, $u^3 = \frac{1}{\sqrt{t}} h(\omega)$, $p = \frac{1}{t}\varphi(\omega)$
19.	∂_1, G_2, D	$\frac{x_3}{\sqrt{t}}$	$u^1 = \frac{1}{\sqrt{t}} f(\omega)$, $u^2 = \frac{1}{\sqrt{t}} g(\omega) + \frac{x_2}{t}$, $u^3 = \frac{1}{\sqrt{t}} h(\omega)$, $p = \frac{1}{t}\varphi(\omega)$

where $\partial_{u^a} \equiv \dfrac{\partial}{\partial u^a}$, $\partial_p = \dfrac{\partial}{\partial p}$. Recently it was shown [161,74*] that the Lie-maximal invariance algebra of the NS equations (4.8.1) consists of the 11-dimensional $A\tilde{G}(1,3)$ (4.8.2) and the infinite-dimensional algebra A^∞ with basis elements

$$\begin{aligned} Q &= f^a\partial_a + \dot{f}^a\partial_{u^a} - x_a\ddot{f}^a\partial_p \\ R &= g\partial_p \end{aligned} \tag{4.8.3}$$

where $f^a = f^a(t)$ and $g = g(t)$ are arbitrary differentiable functions of t; *dot* means differentiation with respect to t.

To reduce Equations (4.8.1) to a system of ODEs we use ansatze of codimension 1 obtained as invariants of inequivalent 3-dimensional subalgebras of $A\tilde{G}(1,3)$ described in [68,71*]. In Table 4.8.1. we list these 3-dimensional subalgebras and give corresponding invariant variable ω; $\alpha \neq 0$ is an arbitrary constant; f, g, h, φ are differentiable functions of corresponding invariant variables ω.

Let us substitute ansatze from Table 4.8.1 into the Equations (4.8.1). As a result we obtain the following system of ODEs

$$\begin{aligned}
&1^\circ \quad \dot{f} = 0, \qquad \dot{g} = 0, \qquad \dot{h} = 0.\\
&2^\circ \quad h\dot{f} - \ddot{f} = 0, \qquad h\dot{g} - \ddot{g} = 0,\\
&\qquad h\dot{h} - \ddot{h} + \dot{\varphi} = 0, \qquad \dot{h} = 0.\\
&3^\circ \quad g + h\dot{f} - \ddot{f} = 0, \qquad h\dot{g} - \ddot{g} = 0,\\
&\qquad h\dot{h} - \ddot{h} + \dot{\varphi} = 0, \qquad \dot{h} = 0.\\
&4^\circ \quad \dot{f}h + 2\ddot{f} = 0, \qquad \dot{g}h + 2\ddot{g} = 0,\\
&\qquad 1 - 2h\dot{h} - 4\ddot{h} - 2\dot{\varphi} = 0, \qquad \dot{h} = 0.\\
&5^\circ \quad 1 + h\dot{f} - \ddot{f} = 0, \qquad g\dot{h} - \ddot{g} = 0,\\
&\qquad h\dot{h} - \ddot{h} + \dot{\varphi} = 0, \qquad \dot{h} = 0.\\
&6^\circ \quad g - 2h\dot{f} - 4\ddot{f} = 0, \qquad h\dot{g} + 2\ddot{g} = 0,\\
&\qquad 1 - 2h\dot{h} - 4\ddot{h} - 2\dot{\varphi} = 0, \qquad \dot{h} = 0.\\
&7^\circ \quad (\alpha f - h)\dot{f} - 2(\alpha^2 + 1)\ddot{f} + \alpha\dot{\varphi} = 0,\\
&\qquad (\alpha f - h)\dot{g} - 2(\alpha^2 + 1)\ddot{g} = 0,\\
&\qquad (\alpha f - h)\dot{h} - 2(\alpha^2 + 1)\ddot{h} - \dot{\varphi} + \tfrac{1}{2} = 0,\\
&\qquad \alpha\dot{f} - \dot{h} = 0.
\end{aligned} \tag{4.8.4}$$

8°
$$-\dot f(h-\alpha g)+g-(\alpha^2+1)\ddot f=0,$$
$$-\dot g(h-\alpha g)+\alpha\dot\varphi-(\alpha^2+1)\ddot g=0,$$
$$1-\dot h(h-\alpha g)-\dot\varphi-(\alpha^2+1)\ddot h=0,$$
$$\dot h-\alpha\dot g=0.$$

9°
$$f^2-g^2+\omega f\dot f+\tfrac{1}{\omega}\dot\varphi=\tfrac{3}{\omega}\dot f+\ddot f$$
$$2fg+\omega f\dot g=\tfrac{3}{\omega}\dot g+\ddot g,$$
$$\omega f\dot h=\ddot h+\tfrac{1}{\omega}\dot h,\qquad 2f+\omega\dot f=0.$$

10°
$$f^2-g^2+\omega f\dot f+\tfrac{1}{\omega}\dot\varphi=\tfrac{3}{\omega}\dot f+\ddot f$$
$$2fg+\omega f\dot g=\tfrac{3}{\omega}\dot g+\ddot g,$$
$$1+\omega f\dot h=\ddot h+\tfrac{1}{\omega}\dot h,\qquad 2f+\omega\dot f=0.$$

11°
$$f\dot f-g(f+\omega\dot f)+\dot\varphi=2f+4\omega\dot f+(\omega^2+1)\ddot f,$$
$$f\dot g-g(g+\omega\dot g)-(2\varphi+\omega\dot\varphi)=2g+4\omega\dot g+(\omega^2+1)\ddot g,$$
$$f\dot h-g(h+\omega\dot h)=2h+4\omega\dot h+(\omega^2+1)\ddot h,$$
$$\dot f-(g+\omega\dot g)=0.$$

12°
$$-\tfrac{1}{2}(f^2+g^2)+(f-\alpha g)\dot f-\varphi+\dot\varphi=2(-f-\dot f+\alpha\dot g+(\alpha^2+1)\ddot f),$$
$$-(f-\alpha g)\dot g+\alpha\dot\varphi=2\big[g+\dot g+\alpha\dot f-(\alpha^2+1)\ddot g\big],$$
$$-fh+2(f-\alpha g)\dot h=h-4\dot h+4(\alpha^2+1)\ddot h,$$
$$\dot f-\alpha\dot g=0.$$

13°
$$-f^2-g^2+\omega f\dot f-\omega^2h\dot f-2\varphi+\omega\dot\varphi=\omega(-f-\omega\ddot f)+\omega^3(2\dot f+\omega\ddot f),$$
$$f\dot g-\omega^2h\dot g=\omega(-g+\omega\ddot g)+\omega^3(2\dot g+\omega\ddot g),$$
$$f(-h+\omega\dot h)-\omega^2h\dot h-\omega^2\dot\varphi=h-\omega\dot h+\omega^2\ddot h+\omega^3(2\dot h+\omega\ddot h)$$
$$\dot f-\omega\dot h=0.$$

14°
$$f^2-g^2+2\omega f\dot f+2\dot\varphi=4(2\dot f+\omega\ddot f),$$
$$g+\omega\dot g-2f(g+\omega\dot g)=-4(2\dot g+\omega\ddot g),$$
$$-(\tfrac{1}{2}h+\omega\dot h)+2\omega f\dot h+h=4(\dot h+\omega\ddot h)$$

$$f + \omega \dot{f} = 0.$$

15° $$f^2 - g^2 + 2\omega f\dot{f} + 2\dot{\varphi} = 4(2\dot{f} + \omega\ddot{f}),$$

$$g + \omega\dot{g} - 2f(g + \omega\dot{g}) = -4(2\dot{g} + \omega\ddot{g}),$$

$$-(\tfrac{1}{2}h + \omega\dot{h}) + 2\omega f\dot{h} + h = 4(\dot{h} + \omega\ddot{h})$$

$$f + \omega\dot{f} + \tfrac{1}{2} = 0.$$

16° $$-\tfrac{1}{2}(f + \omega\dot{f}) + h\dot{f} = \ddot{f},$$

$$-\tfrac{1}{2}(g + \omega\dot{g}) + h\dot{g} = \ddot{g},$$

$$-\tfrac{1}{2}(h + \omega\dot{h}) + h\dot{h} + \dot{\varphi} = \ddot{h}, \qquad \dot{h} = 0.$$

17° $$-\tfrac{1}{2}(f + \omega\dot{f}) + h\dot{f} + \alpha g = \ddot{f},$$

$$-\tfrac{1}{2}(g + \omega\dot{g}) + h\dot{g} + g = \ddot{g},$$

$$-\tfrac{1}{2}(h + \omega\dot{h}) + h\dot{h} + \dot{\varphi} = \ddot{h}, \qquad \dot{h} + 1 = 0.$$

18° $$\tfrac{1}{2}(f - \omega\dot{f}) + h\dot{f} = \ddot{f},$$

$$\tfrac{1}{2}(g - \omega\dot{g}) + h\dot{g} = \ddot{g},$$

$$-\tfrac{1}{2}(h + \omega\dot{h}) + h\dot{h} = \ddot{h}, \qquad \dot{h} + 2 = 0.$$

19° $$-\tfrac{1}{2}(f + \omega\dot{f}) + h\dot{f} = \ddot{f},$$

$$\tfrac{1}{2}(g - \omega\dot{g}) + h\dot{g} = \ddot{g},$$

$$-\tfrac{1}{2}(h + \omega\dot{h}) + h\dot{h} + \dot{\varphi} = \ddot{h}, \qquad \dot{h} + 1 = 0.$$

Enumeration 1° – 19° in (4.8.4) corresponds to that of ansatze in Table 4.8.1; *dot* means differentiation with respect to corresponding ω.

Equations 1° – 10° (4.8.4) can easily be solved and their general solutions are as follows:

1°. $f = c_1, \qquad g = c_2, \qquad \varphi = \varphi(\omega).$

(Here and in what follows c with a subscript denotes an arbitrary constant; $\varphi = \varphi(\omega)$ means that φ is an arbitrary differentiable function of ω.)

2°. $$f = \begin{cases} \dfrac{c_1}{c_3} e^{c_3\omega} + c_2, & c_3 \neq 0 \\ c_1\omega + c_2, & c_3 = 0, \end{cases}$$

$$f = \begin{cases} \dfrac{c_4}{c_3} e^{c_3\omega} + c_5, & c_3 \neq 0 \\ c_4\omega + c_5, & c_3 = 0, \end{cases}$$

$$h = c_3, \qquad \varphi = c_6.$$

3°. $$f = \begin{cases} c_1 + c_2 e^{c_3\omega} + \dfrac{c_4}{c_3^2}\left(\omega - \dfrac{1}{c_3}\right) e^{c_3\omega} - \dfrac{c_5}{c_3}\omega, & c_3 \neq 0 \\ c_1 + c_2\omega + \frac{1}{6}c_4\omega^3 + \frac{1}{2}c_5\omega^2, & c_3 = 0, \end{cases}$$

$$g = \begin{cases} \dfrac{c_4}{c_3} e^{c_3\omega} + c_5, & c_3 \neq 0 \\ c_4\omega + c_5, & c_3 = 0, \end{cases}$$

$$h = c_3, \qquad \varphi = c_6.$$

4°. $$f = \begin{cases} \dfrac{c_1}{c_3^2} e^{-\frac{1}{2}c_3\omega} + c_2, & c_3 \neq 0 \\ c_1\omega + c_2, & c_3 = 0. \end{cases}$$

$$g = \begin{cases} \dfrac{c_4}{c_3} e^{-\frac{1}{2}c_3\omega} + c_5, & c_3 \neq 0 \\ c_4\omega + c_5, & c_3 = 0, \end{cases}$$

$$h = c_3, \qquad \varphi = \tfrac{1}{2}\omega + c_6.$$

5°. $$f = \begin{cases} -\dfrac{1}{c_3}\omega + \dfrac{c_1}{c_3^2} e^{c_3\omega} + c_2 & c_3 \neq 0 \\ \frac{1}{2}\omega^2 + c_1\omega + c_2, & c_3 = 0, \end{cases}$$

$$g = \begin{cases} \dfrac{c_4}{c_3} e^{c_3\omega} + c_5, & c_3 \neq 0 \\ c_4\omega + c_5, & c_3 = 0, \end{cases}$$

$$h = c_3, \qquad \varphi = c_6.$$

6°. $$f = \begin{cases} c_1 + c_2 e^{-\frac{1}{2}c_3\omega} + \dfrac{c_5}{2c_3}\omega - \dfrac{c_4}{c_3^2}\left(\dfrac{\omega}{2} - \dfrac{1}{c_3}\right) e^{-\frac{1}{2}c_3\omega}, & c_3 \neq 0 \\ \frac{1}{4}\left(c_1 + c_2\omega + \frac{1}{2}c_5\omega^2 + \dfrac{1}{6}c_4\omega^3\right), & c_3 = 0, \end{cases}$$

$$g = \begin{cases} \dfrac{c_4}{c_3} e^{-\frac{1}{2}c_3\omega} + c_5, & c_3 \neq 0 \\ c_4\omega + c_5, & c_3 = 0, \end{cases}$$

$$h = c_3, \qquad \varphi = \tfrac{1}{2}\omega + c_6.$$

$$7^\circ \quad f = \begin{cases} c_1 \exp\left\{\dfrac{c\omega}{2(\alpha^2+1)}\right\} + c_2 - \dfrac{\alpha\omega}{2(\alpha^2+1)c}, & c \neq 0, \\ \dfrac{\alpha\omega^2}{2[2(\alpha^2+1)]^2} + c_1\omega + c_2, & c = 0 \end{cases}$$

$$g = \begin{cases} c_3 \exp\left\{\dfrac{c\omega}{2(\alpha^2+1)}\right\} + c_4, & c \neq 0, \\ c_3\omega + c_4, & c = 0 \end{cases}$$

$$h = \alpha f - c, \qquad \varphi = \frac{\omega}{2(\alpha^2+1)} + c_5.$$

$$8^\circ \quad f = \begin{cases} \dfrac{\alpha\omega^2}{2c^2(\alpha^2+1)} + \dfrac{\omega}{c}\left(\dfrac{\alpha}{c^2} - c_4\right) + & c \neq 0, \\ \quad + \left[\dfrac{c_3}{c}\left(\omega - \dfrac{\alpha^2+1}{c}\right) + c_1\right] \exp\dfrac{c\omega}{\alpha^2+1} + c_2, & \\ \dfrac{1}{\alpha^2+1}\left(\dfrac{\alpha\omega^4}{24(\alpha^2+1)^2} + \dfrac{c_3}{6}\omega^3 + \dfrac{c_4}{2}\omega^2 + c_1\omega + c_2\right), & c = 0 \end{cases}$$

$$g = \begin{cases} \dfrac{-\alpha\omega}{c(\alpha^2+1)} + c_3 \exp\left\{\dfrac{c\omega}{\alpha^2+1}\right\} + c_4, & c \neq 0, \\ \dfrac{\alpha}{2(\alpha^2+1)}\omega^2 + c_3\omega + c_4, & c = 0, \end{cases}$$

$$h = \alpha g - c, \qquad \varphi = \frac{\omega}{\alpha^2+1} + c_6.$$

$$9^\circ \quad f = \frac{c}{\omega^2}, \qquad g = c_1\omega^c + \frac{c_2}{\omega^2}, \qquad h = c_3\omega^c + c_4,$$

$$\varphi = \begin{cases} \dfrac{c_1^2}{2(c+1)}\omega^{2(c+1)} + \dfrac{2c_1c_2}{c}\omega^c - \dfrac{c_1^2+c_2^2}{2\omega^2} + c_5, & c \neq -1, 0 \\ c_1^2 \ln\omega - \dfrac{2c_1c_2}{\omega} - \dfrac{c_2^2+1}{2\omega^2} + c_5, & c = -1 \\ \dfrac{1}{2}c_1^2\omega^2 + 2c_1c_2\ln\omega - \dfrac{c_2^2}{2\omega^2} + c_5, & c = 0. \end{cases}$$

10° f, g, and φ are the same as in the previous case 9°,

$$h = \begin{cases} \dfrac{\omega^2}{2(2-c)} + c_3\omega^c + c_4, & c \neq 2, 0 \\ \dfrac{\omega^2}{4} - c_3 \ln \omega + c_4, & c = 0 \\ \dfrac{1}{2}\omega^2 \ln \omega - \dfrac{1}{4}\omega^2 + c_3\omega^2 + c_4, & c = 2 \end{cases}$$

For Equation 11° (4.8.4) we did not find solutions. A particular solution of Equations 12° (4.8.4) is

12° $f = c, \qquad g = 0, \qquad \varphi = 2c - \frac{1}{2}c^2,$

$$h = \begin{cases} c_1 e^{\lambda_1 \omega} + c_2 e^{\lambda_2 \omega}, & \dfrac{c^2}{4} > \alpha^2(1+c), \\ e^{\lambda\omega}(c_1 + c_2\omega), & \dfrac{c^2}{4} = \alpha^2(1+c), \\ e^{\lambda\omega}(c_1 \cos\beta\omega + c_2 \sin\beta\omega), & \dfrac{c^2}{4} < \alpha^2(1+c), \end{cases}$$

$$\lambda_{1,2} = \frac{1 + \dfrac{c}{2} \pm \sqrt{\dfrac{c^2}{4} - \alpha^2(1+c)}}{2(1+\alpha^2)},$$

$$\lambda = \frac{1 + \dfrac{c}{2}}{2(1+\alpha^2)}, \qquad \beta = \frac{\sqrt{\alpha^2(1+c) - \dfrac{c^2}{4}}}{2(1+\alpha^2)}.$$

A particular solution of Equation 13° (4.8.4) is

13° $\quad f = c_1, \qquad g = c_2, \qquad h = 0, \qquad \varphi = -\frac{1}{2}(c_1^2 + c_2^2).$ (4.8.5)

Consider system 14° (4.8.4). Its last equation immediately gives

$$f = \frac{c}{\omega} \tag{4.8.6}$$

(as before, c is an arbitrary constant). Substituting (4.8.6) into the rest of Equations 14° (4.8.4) we get

$$4\frac{d^2}{d\omega^2}(\omega g) + \left(1 - \frac{2c}{\omega}\right)\frac{d}{d\omega}(\omega g) = 0 \tag{4.8.7}$$

and

$$4\omega\ddot{h} + (\omega + 4 - 2c)\dot{h} + \tfrac{1}{2}h = 0. \tag{4.8.8}$$

Equation (4.8.7) can be easily integrated and the result is

$$g(\omega) = \frac{c_1}{\omega}\int^{\omega} x^{c/2}e^{-x/4}dx + \frac{c_2}{\omega}. \tag{4.8.9}$$

In particular, when $c = 0$, the general solution of Equation (4.8.7) takes the form

$$g(\omega) = \frac{c_1}{\omega} e^{\omega/4} + \frac{c_2}{\omega} \tag{4.8.10}$$

Equation (4.8.8) is in itself an equation for a degenerate hypergeometric function and it can be rewritten in standard Whittaker form

$$4x^2 \ddot{W} - (x^2 - 4kx + 4m^2 - 1)W = 0, \tag{4.8.11}$$

(where $W = W(k, m, x)$; k, m are parameters) by the substitution

$$h(\omega) = \omega^{\frac{c-2}{4}} e^{-\omega/8} W\left(\frac{c}{4}, -\frac{c}{4}, \frac{\omega}{4}\right). \tag{4.8.12}$$

When $c = 0$, the substitution

$$h(\omega) = e^{-\tau} \widetilde{Z}_0(\tau), \qquad \tau = \frac{\omega}{8} \tag{4.8.13}$$

reduces Equation (4.8.8) to the modified Bessel equation of null order, that is

$$\tau \ddot{\widetilde{Z}}_0 + \dot{\widetilde{Z}}_0 - \tau \widetilde{Z}_0 = 0. \tag{4.8.14}$$

Summarizing results (4.8.6)–(4.8.14) we can write down the general solutions of Equations 14° (4.8.4) as

14° $f = \dfrac{c}{\omega}$ (4.8.5)

$$g = \frac{c_1}{\omega} \int^{\omega} x^{c/2} e^{-x/4} dx + \frac{c_2}{\omega},$$

$$h = \omega^{\frac{c-2}{4}} e^{-\omega/8} W\left(\frac{c}{4}, -\frac{c}{4}, \frac{\omega}{4}\right),$$

$$\varphi = -\frac{c^2}{2\omega} + \frac{1}{2} \int^{\omega} g^2(y) dy + c_3.$$

(We continue to numerate solutions of reduced NS Equations 1° – 19° (4.8.5). Number $n°$ indicates corresponding ansatz of Table 1.) When $c = 0$ we get from 14° (4.8.5) the following particular solution of Equations 14° (4.8.4)

14°° $f = 0, \qquad g = \dfrac{c_1}{\omega} e^{-\omega/4} + \dfrac{c_2}{\omega}$ (4.8.5)

$$h = e^{-\omega/8} \widetilde{Z}_0\left(\frac{\omega}{8}\right),$$

$$\varphi = -\frac{c_2^2}{2\omega} + \frac{c_1^2}{2} \int^{\omega} \frac{e^{-y/2}}{y^2} dy + c_1 c_2 \int^{\omega} \frac{e^{-y/4}}{y^2} dy + c_3.$$

where $\widetilde{Z}_0$ is a modified Bessel function satisfying Equation (4.8.14). Consider system 15° (4.8.4). Its last eqaution results in

$$f = \frac{c}{\omega} - \frac{1}{2}. \tag{4.8.15}$$

The rest of Equation 15° take the form

$$2\frac{d^2}{d\omega^2}(\omega g) + \left(1 - \frac{c}{\omega}\right)\frac{d}{d\omega}(\omega g) = 0, \tag{4.8.16}$$

$$2\dot{\varphi} = \left(\frac{c}{\omega}\right)^2 + g^2 - \frac{1}{4} \tag{4.8.17}$$

$$\omega\ddot{h} + \left(\tfrac{1}{2}\omega + 1 - \tfrac{c}{2}\right)\dot{h} - \tfrac{1}{8}h = 0. \tag{4.8.18}$$

Equations (4.8.16), (4.8.17) can be easily integratyed and the result is as follows

$$g = \frac{c_1}{\omega}\int^{\omega} x^{c/2}e^{-x/2}dx + \frac{c_2}{\omega}, \tag{4.8.19}$$

$$\varphi = \frac{1}{2}\int^{\omega} g^2(y)dy - \frac{c^2}{2\omega} - \frac{1}{8}\omega. \tag{4.8.20}$$

Equation (4.8.18) is reduced to the Whittaker equation (4.8.11) by the substitution

$$h(\omega) = \omega^{\frac{c-2}{4}}E^{-\omega/4}W\left(\frac{c-3}{4}, -\frac{c}{4}, \frac{\omega}{4}\right) \tag{4.8.21}$$

Note, when $c = 3$, function $W\left(0, -\frac{3}{4}, \frac{\omega}{2}\right)$ is reduced to the modified Bessel function $\widetilde{Z}_{-\frac{3}{4}}\left(\frac{\omega}{4}\right)$. The general relation is [130,75*]

$$W(0, m, x) = \sqrt{x}\,\widetilde{Z}_m\left(\frac{x}{2}\right). \tag{4.8.22}$$

So, we can write down the general solution of reduced NS Equations 15° (4.8.4) in the form

$$15^\circ \quad f = \frac{c}{\omega} - \frac{1}{2} \tag{4.8.5}$$

$$g = \frac{c_1}{\omega}\int^{\omega} x^{c/2}e^{-x/2}dx + \frac{c_2}{\omega},$$

$$h = \omega^{\frac{c-2}{4}}e^{-\omega/4}W\left(\frac{c-3}{4}, -\frac{c}{4}, \frac{\omega}{2}\right),$$

$$\varphi = \frac{1}{2}\int^{\omega} g^2(y)dy - \frac{c^2}{2\omega} - \frac{1}{8}\omega,$$

where W satisfies the Whittaker equation (4.8.11).

Consider system 16° (4.8.4). The last two equations of 16° (4.8.4) give rise to

$$h = c, \qquad \varphi = \frac{c\omega}{2} + c_1 \tag{4.8.23}$$

Taking into account (4.8.23) we can rewrite the remaining equations of system 16° as follows:

$$\ddot{f} + (\tfrac{1}{2}\omega - c)\dot{f} + \tfrac{1}{2}f = 0, \tag{4.8.24}$$

$$\ddot{g} + (\tfrac{1}{2}\omega - c)\dot{g} + \tfrac{1}{2}g = 0, \tag{4.8.25}$$

By substituting

$$f(\omega) = F(\tau), \qquad \tau = \tfrac{1}{2}\omega - c \tag{4.8.26}$$

into (4.8.24) we obtain equation

$$\frac{d^2F}{d\tau^2} + 2\tau\frac{dF}{d\tau} + 2F = 0. \tag{4.8.27}$$

The general solution of Equation (4.8.27) is

$$F(\tau) = e^{-\tau^2}\left(c_2 + c_3\int^{\tau} e^{y^2}dy\right). \tag{4.8.28}$$

Summarizing results (4.8.23)–(4.8.28) we write the general solutions of Equations 16° (4.8.4):

16° $$f = \exp\left\{-\left(\frac{\omega}{2} - c\right)^2\right\}\left(c_2 + c_3\int^{\frac{\omega}{2}-c} e^{y^2}dy\right), \tag{4.8.5}$$

$$g = \exp\left\{-\left(\frac{\omega}{2} - c\right)^2\right\}\left(c_4 + c_5\int^{\frac{\omega}{2}-c} e^{y^2}dy\right),$$

$$h = c, \qquad \varphi = \frac{c\omega}{2} + c_1.$$

In the same way we find solutions of reduced equations 17° – 19° (4.8.4). The solutions are as follows.

17° $\alpha = 1$:

$$f = g = \left(\tfrac{3}{2}\omega - c\right)^{-1/2}\exp\left\{-\tfrac{1}{6}\left(\tfrac{3}{2}\omega - c\right)^2\right\}\times$$
$$\times W\left(-\tfrac{5}{12}, \tfrac{1}{4}, \tfrac{1}{3}\left(\tfrac{3}{2}\omega - c\right)^2\right),$$

$$h = -\omega + c, \qquad \varphi = \tfrac{3c}{2}\omega - \omega^2 + c_1,$$

where $W(.,.,.)$ satisfies the Whittaker equation (4.8.11). The above solution 17° (4.8.5) is a particular solution of Equation 17° (4.8.4) with $\alpha = 1$. When α is an arbitrary constant, the general solution of Equation 17 (4.8.4) has the form:

17°° $$g = \left(\tfrac{3}{2}\omega - c\right)^{-1/2} \exp\left\{-\tfrac{1}{6}\left(\tfrac{3}{2}\omega - c\right)^2\right\} W\left(-\tfrac{5}{12}, \tfrac{1}{4}, \tfrac{1}{3}\left(\tfrac{3}{2}\omega - c\right)^2\right),$$

$$h = -\omega + c, \qquad \varphi = \frac{3c}{2}\omega - \omega^2 + c_1,$$

and f is determined from the ODE

$$\ddot{f} + \left(\frac{3}{2}\omega - c\right)\dot{f} + \tfrac{1}{2}f - \alpha g = 0.$$

The general solution of Equation 18° (4.8.4) is

18° $$f = \left(\tfrac{5}{2}\omega - c\right)^{-1/2} \exp\left\{-\tfrac{1}{10}\left(\tfrac{5}{2}\omega - c\right)^2\right\} W\left(-\tfrac{27}{20}, \tfrac{1}{4}, \tfrac{1}{5}\left(\tfrac{5}{2}\omega - c\right)^2\right),$$

$$g = \left(\tfrac{5}{2}\omega - c\right)^{-1/2} \exp\left\{-\tfrac{1}{10}\left(\tfrac{5}{2}\omega - c\right)^2\right\} W\left(-\tfrac{27}{20}, \tfrac{1}{4}, \tfrac{1}{5}\left(\tfrac{5}{2}\omega - c\right)^2\right),$$

$$h = -2\omega + c, \qquad \varphi = \tfrac{5}{2}c\omega - 3\omega^2 + c_1.$$

19° $$f = \left(\tfrac{3}{2}\omega - c\right)^{-1/2} \exp\left\{-\tfrac{1}{6}\left(\tfrac{3}{2}\omega - c\right)^2\right\} W\left(-\tfrac{1}{12}, \tfrac{1}{4}, \tfrac{1}{3}\left(\tfrac{3}{2}\omega - c\right)^2\right),$$

$$g = \left(\tfrac{3}{2}\omega - c\right)^{-1/2} \exp\left\{-\tfrac{1}{6}\left(\tfrac{3}{2}\omega - c\right)^2\right\} W\left(-\tfrac{5}{12}, \tfrac{1}{4}, \tfrac{1}{3}\left(\tfrac{3}{2}\omega - c\right)^2\right),$$

$$h = -\omega + c, \qquad \varphi = \tfrac{3}{2}c\omega - \omega^2 + c_1.$$

In 17°–19° (4.8.5) $W(.,.,.)$ is an arbitrary solution of the Whittaker equation (4.8.11).

Remark 4.8.1. Solutions of reduced equations 1°–19° (4.8.5) should be considered together with corresponding ansatze of the Table 4.8.1, and then one gets solutions of the NS equations (4.8.1).

The above obtained solutions of the NS equations can be used as basic ones to construct multiparameter families of solutions. For this aim one has to use formulae of generating solutions (group multiplication of solutions) (see Paragraphs 1.4, 2.3). Below we list the final symmetry transformations generated by operators (4.8.2), (4.8.3) and corresponding formulae of group multiplication of solutions for the NS equations (4.8.1).

Table 4.8.2 Final symmetry transformations and corresponding formulae of group multiplication of solutions (GMS) for the NS equation (4.8.1)

N	Op.	Transformations $x \to x'$	$u(x) \to u'(x')$	Formulae of GMS
1	∂_t	$\vec{x}' = \vec{x}$, $t' = t + \delta_0$	$\vec{u}'(x') = \vec{u}(x)$	$\vec{u}_{II} = \vec{u}_I(x')$
2-4	∂_a	$\vec{x}' = \vec{x} + \vec{\delta}$	$\vec{u}'(x') = \vec{u}(x)$	$\vec{u}_{II} = \vec{u}_I(x')$
5-7	J_{ab}	$t' = t, \vec{x}' = \vec{x}\cos\alpha + (\vec{x}\times\vec{\alpha})\cdot \cdot\frac{\sin\alpha}{\alpha} + \vec{\alpha}(\vec{\alpha}\cdot\vec{x})\cdot \cdot\frac{1-\cos\alpha}{\alpha^2}$	$\vec{u}'(x') = \vec{u}\cos\alpha + +(\vec{u}\times\vec{\alpha})\frac{\sin\alpha}{\alpha} + +\vec{\alpha}(\vec{\alpha}\cdot\vec{u})\frac{1-\cos\alpha}{\alpha}$	$u^a_{II}(x) = \Big(\delta_{ab}\cos\alpha + +\epsilon_{abc}\alpha_c\frac{\sin\alpha}{\alpha} + +\alpha_a\alpha_b\frac{1-\cos\alpha}{\alpha^2}\Big)u^b_I(x')$
8-10	G_a	$\vec{x}' = \vec{x} + \vec{\theta}t$, $t' = t$	$\vec{u}' = \vec{u}(x) + \vec{\theta}$	$\vec{u}_{II}(x) = \vec{u}_I(x') - \vec{\theta}$
11	D	$\vec{x}' = e^{\beta}\vec{x}$, $t' = e^{2\beta}t$	$\vec{u}'(x') = e^{-\beta}$, $p'(x') = e^{-2\beta}p(x)$	$\vec{u}_{II}(x) = e^{\beta}\vec{u}_I(x')$, $p_{II}(x) = e^{2\beta}p_I(x')$
12	Q	$\vec{x}' = \vec{x} + \epsilon\vec{f}(t)$, $t' = t$	$\vec{u}'(x') = \vec{u}(x) + \epsilon\dot{\vec{f}}(t)$, $p'(x') = p(x) - \epsilon\vec{x}\cdot\ddot{\vec{f}}$	$\vec{u}_{II}(x) = \vec{u}_I(x') - \epsilon\dot{\vec{f}}(t)$, $p_{II}(x) = p_I(x') + \epsilon\vec{x}\cdot\ddot{\vec{f}}(t)$
13	R	$\vec{x}' = \vec{x}$, $t' = t$	$\vec{u}'(x') = \vec{u}(x)$, $p'(x') = p(x) + æg(t)$	$\vec{u}_{II}(x) = \vec{u}_I(x')$, $p_{II}(x) = p_I(x') - æg(t)$

Note: In 1−11 $p'(x') = p(x)$ and therefore $p_{II}(x) = p_I(x')$. In this table δ_0, δ_a, α_a, θ_a, θ_a, β, ϵ, æ are arbitrary constants, $\alpha = (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)^{1/2}$; $\vec{f}$ and g are arbitrary differentiable functions of t. Formulae of GMS stated above allow us to contruct new solutions $\vec{u}_{II}(x)$ of the NS equations (4.8.1) starting from one $\vec{u}_I(x)$.

Remark 4.8.2. It will be noted that operator Q given in (4.8.3) generates transformations (N 12 in Table 4.8.2) which can be considered as an invariant transition to a frame of reference which is moved arbitrarily: $\vec{x}_{ref} = \epsilon\vec{f}(t)$.

Let us give some examples of application of formulae of GMS. Having applied formulae 5−7 of Table 4.8.2 to solution 16° (4.8.5) we get the following multiparameter families of solutions of the NS equations (4.8.1)

$$\vec{u}(x) = \frac{1}{\sqrt{t}}\left\{e^{-\tau^2}\left[\vec{a}\Big(\alpha_1 + \alpha_2\int^{\tau} e^{s^2}ds\Big) + \vec{b}\Big(\alpha_3 + \alpha_4\int^{\tau} e^{s^2}ds\Big)\right] + \vec{c}\right\},$$

$$p(x) = \frac{1}{t}\left(\frac{\vec{c}\cdot\vec{x}}{2\sqrt{t}} + \alpha_5\right), \qquad \tau = \frac{\vec{c}\cdot\vec{x}}{2\sqrt{t}} - 1, \tag{4.8.29}$$

where $\alpha_1, \ldots, \alpha_5$ are arbitrary constants, $\vec{a}$, $\vec{b}$, $\vec{c}$, are arbitrary orthonormal constant vectors

$$\vec{a}^2 = \vec{b}^2 = \vec{c}^2 = 1, \qquad \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 0. \tag{4.8.30}$$

Further application of formulae of GMS N 8–10 to (4.8.29) gives rise to the following solution of Equations (4.8.1)

$$\begin{aligned} &\vec{u}(x) = \frac{1}{\sqrt{t}}\left\{e^{-y^2}\left[\vec{a}\left(\alpha_1 + \alpha_2\int^{y} e^{s^2}ds\right) + \vec{b}\left(\alpha_3 + \alpha_4\int^{y} e^{s^2}ds\right)\right] + \vec{c}\right\} - \vec{\theta}, \\ &p(x) = \frac{1}{t}(y + \alpha_5), \qquad y = \frac{\vec{c}\cdot(\vec{x} + \vec{\theta}t)}{2\sqrt{t}} - 1, \end{aligned} \tag{4.8.31}$$

where $\vec{\theta}$ are arbitrary constants, the remaining designations are the same as in (4.8.29).

The procedure of generating solutions by means of symmetry transformations can be continued until one gets ungenerative families of solutions (see Paragraphs 1.4). Without doubt, the reader can make it, by analogy with the above examples, for any solution $1° - 19°$ (4.8.5) of the NS equations.

Ansatze collected in Table 4.8.1, of course, do not exhaust all possible ansatze which reduce the NS equations. Let us consider several examples of ansatze which cannot be obtained in the framework of local Lie approach.

The ansatz

$$\vec{u} = \vec{\nabla}\varphi \tag{4.8.32}$$

where $\varphi = \varphi(x)$ is a scalar function, reduces NS equations (4.8.1) to the system of Hamilton-Jacobi and Laplace equations

$$\begin{aligned} \varphi_t + \left(\vec{\nabla}\varphi\right)^2 + p = 0, \\ \Delta\varphi = 0. \end{aligned} \tag{4.8.33}$$

This is an example of nonlocal component reduction.

Ansatz

$$\vec{u} = \vec{a}\varphi\left(t, \vec{b}\cdot\vec{x}, \vec{c}\cdot\vec{x}\right), \tag{4.8.34}$$

where $\vec{a}$, $\vec{b}$, $\vec{c}$ are constant vectors satisfying (4.8.30), reduces (4.8.1) to the two-dimensional heat equation

$$\begin{aligned} &\varphi_t - \Delta_2\varphi = 0. \qquad \Delta_2 \equiv \frac{\partial^2}{\partial\omega_1^2} + \frac{\partial^2}{\partial\omega_2^2}, \\ &\omega_1 = \vec{b}\cdot\vec{x}, \qquad \omega_2 = \vec{c}\cdot\vec{x}. \end{aligned} \tag{4.8.35}$$

Ansatz

$$\vec{u} = \vec{x}\varphi(x), \qquad p = p(x) \tag{4.8.36}$$

reduces Equations (4.8.1) to the system of PDEs

$$\begin{aligned}\vec{x}(\varphi_t + \Delta\varphi) + \vec{\nabla}(\varphi + p) = 0,\\ \varphi + (\vec{x}\cdot\vec{\nabla})\varphi = 0.\end{aligned} \tag{4.8.37}$$

New ansatze and solutions of the NS equations can be constructed within the framework of conditional symmetry considered in Paragraph 5.7.

Let us make some concluding remarks. It will be noted that the question "What spin is carried by the NS field?" has a rather strange answer [66]: The NS field carries not only spin 1 but all possible integer spins $s = 0, 1, 2, \ldots$. This is due to the fact that the space of solutions of the NS equations can be decomposed into infinite direct sum of subspaces invariant under operators $S_{ab} = u^a\partial_{u^b} - u^b\partial_{u^a}$ from algebra AO(3), and these subspaces are not invariant under operators G_a from (4.8.2) because of the unboundedness of operators ∂_{u^a}. The following linearized NS equations are often used in hydrodynamics:

$$\begin{aligned}\vec{u}_t - \Delta\vec{u} = 0\\ \operatorname{div}\vec{u} = 0.\end{aligned} \tag{4.8.38}$$

The maximal invariance algebra of Equations (4.8.38) is the 9-dimensional Lie algebra with basis elements

$$\begin{aligned}&\partial_t, \qquad \partial_a, \qquad J_{ab} = x_a\partial_b - x_b\partial_a + u^a\partial_{u^b} - u^b\partial_{u^a},\\ &D = 2t\partial_t + x_a\partial_a, \qquad I = u^a\partial_{u^a}.\end{aligned} \tag{4.8.39}$$

It is to be pointed out that Equations (4.8.38) are not Galilei invariant and therefore, generally speaking, they fail to adequately describe real hydrodynamic processes.

The ansatze which reduce NS equations (4.8.1) to linear systems of heat equations are considered in [72*]. The following is an example of such:

$$\begin{aligned}\vec{u} = \vec{a}f(t,\omega) + \vec{b}g(t,\omega) + \vec{c}h(t,\omega)\\ p = p(t,\omega) \qquad \omega = \vec{c}\cdot\vec{x},\end{aligned} \tag{4.8.40}$$

where $\vec{a}$, $\vec{b}$, $\vec{c}$ are constant vectors satisfying (4.8.30)

Ansatz (4.8.40) reduces Equations (4.8.1) to the system of PDEs

$$\begin{aligned}&f_t - f_{\omega\omega} + hf_\omega = 0,\\ &g_t - g_{\omega\omega} + hg_\omega = 0,\\ &h_t + p_\omega = 0, \qquad h_\omega = 0.\end{aligned} \tag{4.8.41}$$

The last two equations from (4.8.41) are easily integrated and after that the system (4.8.41) takes the form

$$\begin{aligned}&f_t - f_{\omega\omega} + h(t)f_\omega = 0,\\ &g_t - g_{\omega\omega} + h(t)g_\omega = 0, \quad p = -\dot{h}(t)\omega + \tilde{p}(t).\end{aligned} \tag{4.8.42}$$

Further, the change of variables

$$t \to \tau = t, \quad \omega \to y = \omega - \int^{t} h(z)dz \tag{4.8.43}$$

reduces Equations (4.8.42) to the two disconnected heat equations

$$\begin{aligned} f_\tau - f_{yy} &= 0, \\ g_\tau - g_{yy} &= 0. \end{aligned} \tag{4.8.44}$$

Chapter 5

Some Special Questions

In the present chapter we consider some problems tightly connected with group-algebraic investigations such as: finding nonlocal transformations to linearize a given nonlinear PDE, symmetry analysis of the three-body problem, calculating final transformations generated by non-Lie symmetry operators, and studying symmetry of integrodifferential equations. Here we introduce the concept of conditional invariance, and study non-Lie symmetry of quasi-relativistic generalization of the Schrödinger equation, Galilean invariance of Maxwell's equations, solutions of the Schrödinger equation invariant under the non-Lie Lorentz algebra. Finally, in the concluding topic we introduce the concept of approximate symmetry.

5.1. *On nonlocal linearization of nonlinear equations*

It is well known that information about Lie (local or point) symmetry of a given nonlinear PDE sometimes allows us to find a substitution by means of which the equation is linearized. For example, if equation

$$\Box u = F\left(x_0, \ldots, x_3, \frac{\partial u}{\partial x_0}, \ldots, \frac{\partial u}{\partial x_3}\right) \qquad (5.1.1)$$

is invariant under the conformal group C(1,3) then there exists a reversible point change of variables $u \to w = \varphi(u)$ which transforms Equation (5.1.1) to the free wave equation $\Box w = 0$ (see Paragraph 1.2). Class of nonlinear equations which can be linearized essentially extends if to make use of nonlocal transformations, that is, the transformations containing not only dependent and independent variables but also derivatives $\underset{1}{u}, \underset{2}{u}, \ldots$. Examples of such transformations were known long ago. In mechanics and hydrodynamics nonpoint transformations of Legendre, Euler, Laplace, Backlund, and hodographs have been widely used for a long time. A classical example of nonlocal transformations is the so-called Hopf-Cole transformation [40, 121]

$$u(x,t) = -2\mu \frac{v_x}{v}, \qquad v = v(t,x), \tag{5.1.2}$$

(μ is a constant) which transforms Burgers' equation

$$u_t + uu_x - \mu u_{xx} = 0 \tag{5.1.3}$$

into heat equation

$$v_t = \mu v_{xx}. \tag{5.1.4}$$

Below we show how to obtain the Hopf-Cole transformation from the group-theoretical point of view [81*]. Another interesting approach is developed in [145*].

It is not difficult to calculate the maximal IA of the Burgers' equation (5.1.3). It is five-parameter algebra, basis elements having the form

$$\begin{aligned} X: \quad & X_1 = \partial_x, \qquad X_2 = \partial_t \\ & X_3 = x\partial_x + 2t\partial_t - u\partial_u, \\ & X_4 = t\partial_x + \partial_u, \\ & X_5 = tx\partial_x + t^2\partial_t + (x - tu)\partial_u. \end{aligned} \tag{5.1.5}$$

It means that Burgers' equation is invariant with respect to space X_1 and time X_2 translations; scale changes X_3; Galilean transformations X_4; and projective transformations generated by X_5. There is a remarkable similarity between the symmetry algebra of Burgers' equation and that for the heat equation. The latter has the form

$$\begin{aligned} Y: \quad & Y_1 = \partial_x, \qquad Y_2 = \partial_t \\ & Y_3 = x\partial_x + 2t\partial_t, \\ & Y_4 = t\partial_x - \tfrac{1}{2\mu} xv\partial_v, \\ & Y_5 = tx\partial_x + t^2\partial_t - \tfrac{1}{4\mu}(x^2 + 2\mu t)v\partial_v. \\ & Y_6 = v\partial_v \end{aligned} \tag{5.1.6}$$

and in addition, the heat equation, being linear, admits an infinite-dimensional group generated by operators

$$Y_\infty = h(t,x)\partial_v \tag{5.1.7}$$

with arbitrary solution $h(t,x)$ of the equation.

Although the IAs (5.1.5) and (5.1.6) are such similar, nevertheless, the difference between them means that there is not a point change of variables which reduces Burgers' equation to the heat equation.

Suppose

$$u = \varphi(v, v_x) \tag{5.1.8}$$

and require the system of PDEs

$$\begin{aligned} &u_t + uu_x - \mu u_{xx} = 0, \\ &v_t - \mu v_{xx} = 0, \\ &u = \varphi(v, v_x) \end{aligned} \tag{5.1.9}$$

to be invariant with respect to the algebra $Z = \{X, Y\}$, i.e.,

$$\begin{aligned} Z:\quad & Z_1 = \partial_x, \qquad Z_2 = \partial_t \\ & Z_3 = x\partial_x + 2t\partial_t - u\partial_u, \\ & Z_4 = t\partial_x + \partial_u - \frac{1}{2\mu}xv\partial_v, \\ & Z_5 = tx\partial_x + t^2\partial_t + (x - tu)\partial_u - \frac{1}{4\mu}(x^2 + 2\mu t)v\partial_v. \\ & Z_6 = v\partial_v, \qquad Z_\infty = h(t,x)\partial_v. \end{aligned} \tag{5.1.10}$$

Applying the second prolongation of operators (5.1.10) to the system (5.1.9), we obtain from the condition of invariance the defining equations for function $\varphi = \varphi(v, v_x)$. So, using formula (1.1.7) we find the first prolongation of the operator Z_4

$$\underset{1}{Z}_4 = Z_4 - \frac{1}{2\mu}(v + xv_x)\partial_{v_x}.$$

Then we get

$$\underset{1}{Z}_4\big(u - \varphi(v, v_x)\big) = \left\{1 + \frac{1}{2\mu}\left[xv\varphi_v + (v + xv_x)\varphi_{v_x}\right]\right\} = 0$$

Equating coefficients with variable x and then those without x we obtain defining equations

$$\begin{aligned} v\varphi_v + v_x\varphi_{v_x} &= 0, \\ 1 + \frac{1}{2\mu}v\varphi_{v_x} &= 0. \end{aligned} \tag{5.1.11}$$

The general solution of the system (5.1.11) is

$$\varphi(v, v_x) = -2\mu\frac{v_x}{v}. \tag{5.1.12}$$

It can be easily checked that operators Z_5, Z_6, and Z_∞ leave the system (5.1.9) with function φ (5.1.12) invariant. From (5.1.8), (5.1.12) follows the Hopf-Cole transformation (5.1.2).

Formula (5.1.2) can be used to obtain a nonlinear superposition principle for the Burgers' equation. Let $u_k(x)$ be a solution of Equation (5.1.3). Inserting it into (5.1.2) and integrating, we get

$$v_k(x,t) = \exp\left\{-\frac{1}{2\mu}\int^x u_k(\tau,t)d\tau\right\} \tag{5.1.13}$$

Clearly v_k (5.1.13) satisfies the heat equation (5.1.4). Since the heat equation is linear, it possesses a linear superposition principle. It means that the function

$$v(x,t) = \sum_{k>1} v_k(x,t)$$

will be a solution of the equation as soon as every $v_k(x,t)$ is its solution. So, using (5.1.13) and (5.1.2), we find thereby the superposition principle for Burgers' equation

$$u(x,t) = -2\mu\frac{\partial}{\partial x}\ln\left[\sum_{k>1}\exp\left\{-\frac{1}{2\mu}\int^x u_k(\tau,t)d\tau\right\}\right]. \tag{5.1.14}$$

Now consider the Liouville equation

$$u_{xt} + \lambda e^u = 0. \tag{5.1.15}$$

It admits operators

$$X = f(t)\partial_t + g(x)\partial_x - (f_t + g_x)\partial_u \tag{5.1.16}$$

with arbitrary differentiable functions $f = f(t)$ and $g = g(x)$. Analogous symmetry properties have the free wave equation

$$v_{xt} = 0. \tag{5.1.17}$$

It admits operators $Y = \{Y_1, Y_2\}$:

$$Y_1 = f(t)\partial_t + g(x)\partial_x, \qquad Y_2 = v\partial_v. \tag{5.1.18}$$

The similarity between symmetry operators (5.1.16) and (5.1.18) says that a transformation may have to exist which connects Equations (5.1.15) and (5.1.17). To find the constraint equation

$$u = \varphi(v, v_x, v_t)$$

explicitly we require it to be invariant under the algebra $X \oplus Y = \{X, Y_2\}$. Therefore from the condition of invariance

$$\underset{1}{X}(u - \varphi(v, v_x, v_t)) \Big|_{\substack{u=\varphi \\ v_{xt}=0}} = \left[-(f_t + g_x) - v_t\varphi_{v_t} - v_x\varphi_{v_x}\right] \Big|_{\substack{u=\varphi \\ v_{xt}=0}} = 0,$$

$$\underset{1}{Y_2}(u - \varphi) \Big|_{u=\varphi} = v\varphi_v + v_t\varphi_{v_t} + v_x\varphi_{v_x} = 0$$

we find, setting $v = f + g$,

$$u = \varphi(v, v_x, v_t) = \ln c + \ln v_x + \ln v_t - 2\ln v = \ln \frac{cv_x v_t}{v^2} \tag{5.1.19}$$

where c is an arbitrary constant.

Substitution of (5.1.19) into (5.1.15) reduces the latter to (5.1.6) when $c = -2/\lambda$.

Using this fact, it is not difficult to write down the general solution of the Liouville equation (5.1.15)

$$u(x,t) = \ln\left(-\frac{2}{\lambda}\frac{f_t g_x}{(f+g)^2}\right)$$

(Compare with (1.6.2)).

Following [105], consider more general form of nonlocal transformations, namely the transformations

T:
$$\begin{aligned} x_\nu \to x'_\nu &= f_\nu\left(x, u, \underset{1}{u}, \dots, \underset{\ell}{u}\right), \\ u^s \to u'^s &= \varphi^s\left(x, u, \underset{1}{u}, \dots, \underset{k}{u}\right), \end{aligned} \tag{5.1.20}$$

where

$$u = \{u^s\}, \quad s = \overline{1,m} \qquad \underset{1}{u} = \frac{\partial u^s}{\partial x_\nu}, \qquad \underset{2}{u} = \frac{\partial^2 u^s}{\partial x_\nu \partial x_\nu}.$$

From (5.1.20) we get

$$\frac{\partial u'^s}{\partial x'_\nu} = \frac{\partial \varphi^s}{\partial x_\mu}\frac{\partial x_\mu}{\partial x'_\nu} + \frac{\partial \varphi^s}{\partial u^k}\frac{\partial u^k}{\partial x_\mu}\frac{\partial x_\mu}{\partial x'_\nu} + \frac{\partial \varphi^s}{\partial u^k_\mu}\frac{\partial u^k_\mu}{\partial x_\sigma}\frac{\partial x_\sigma}{\partial x'_\nu} + \cdots \tag{5.1.21}$$

With the help of the operator of total differentiation

$$D_\nu = \frac{\partial}{\partial x_\nu} + u^k_{\nu\mu}\frac{\partial}{\partial u^k_\mu} + \cdots \tag{5.1.22}$$

The equality (5.1.21) can be rewritten as follows:

$$D_\nu \varphi^s = \frac{\partial u'^s}{\partial x'_\mu} D_\nu x'_\mu. \tag{5.1.23}$$

Analogous expression hold for the second derivatives

$$D_\sigma D_\nu \varphi^s = \frac{\partial^2 u'^s}{\partial x'_\mu \partial x'_\rho}(D_\sigma x'_\mu)(D_\nu x'_\rho) + \left(\frac{\partial u'^s}{\partial x_\mu}\right) D_\sigma D_\nu x'_\mu. \tag{5.1.24}$$

Formulae (5.1.23), (5.1.24) are relations for determining the primed derivatives.

A PDE ($L_1 = 0$) is said to be *reducible* to a PDE ($L_2 = 0$) by means of transformation T if on sets of solutions of these equations and their differential consequences denoted as $[L_1] = 0$ and $[L_2] = 0$, the following relations hold

$$TL_1 \Big|_{\substack{[L_1]=0 \\ [L_2]=0}} = 0. \tag{5.1.25}$$

To illustrate what has been said, consider several examples.

Theorem 5.1.1. [105]. *The Monge-Ampere equation*

$$u_{xx}u_{tt} - u^2_{xt} = 0 \tag{5.1.26}$$

is reduced to the equation

$$v_\eta = 0, \qquad v = \phi(\xi) \tag{5.1.27}$$

by means of nonlocal change of variables

$$\xi = u_t, \qquad \eta = x, \qquad v = u_x.$$

Proof. Using formulae (5.1.23), (5.1.24) we find

$$D_x\xi = u_{xt}, \qquad D_x\eta = 1, \qquad D_x v = u_{xx},$$

$$D_t\xi = u_{tt}, \qquad D_t\eta = 0, \qquad D_t v = u_{xt},$$

$$v_\eta = -(u_{tt})^{-1}\left(u^2_{tx} - u_{xx}u_{tt}\right), \qquad u_{tt} \neq 0,$$

whence follows (5.1.27).

It will be noted that by force of Equation (5.1.27) any solution of the equation $u_x = \phi(u_t)$ also satisfies Equation (5.1.26).

One can make such in much the same way that the two-dimensional ELBI equation in the Euclidean space

$$\left(1 + u_t^2\right) u_{xx} = 2u_x u_t u_{xt} - \left(1 + u_x^2\right) u_{tt} \tag{5.1.28}$$

is reduced by means of the Monge transformation

$$u = i \int \left(1 + v_\xi^2\right)^{1/2} d\xi + i \int \left(1 + v_\eta^2\right)^{1/2} d\eta + c_1$$

$$x = \xi + \eta + c_2, \qquad t = v + c_3$$

(c_1, c_2, c_3 are arbitrary constants) to the equation

$$v_{\xi\eta} = 0, \qquad v = \phi(\xi) + \psi(\eta) \tag{5.1.29}$$

In [105, 106] is contained many examples of the linearization of nonlinear PDEs by means of nonlocal transformations.

It will be noted that nonlocal transformations can be used to construct formulae of generating solutions of a given nonlinear PDE. Let us give an example of such formulae. New solution $u^{II}(x)$ of the Korteweg-de Vries equation (KdV)

$$u_t + 6uu_x + u_{xxx} = 0 \tag{5.1.30}$$

can be derived from a known one $u^I(x)$ according to the formula [118*]

$$u^{II}(x) = -u^I(x) - 2v^2(x) = u^I(x) - 2v_x, \tag{5.1.31}$$

if there is a function $v(x)$ which satisfies the equations

$$\begin{aligned} v_x &= v^2 + u^I(x), \\ v_t &- 6v^2 v_x + v_{xxx} = 0. \end{aligned} \tag{5.1.32}$$

This statement can be easily proved. Note that Equations (5.1.32) are compatible iff $u^I(x)$ satisfies the KdV equation (5.1.30).

Let us demonstrate the usefulness of formula (5.1.31) by considering several simple examples. Using the simplest solution of the KdV equation

$$u^{(1)} = \lambda, \qquad \lambda = \text{const}$$

we get, according to (5.1.31), the new solution

$$u^{(2)} = \lambda - 2v_x^{(1)}.$$

To find $u^{(2)}$ explicitly one has to solve the Riccati equation

$$v_x^{(1)} = \left(v_x^{(1)}\right)^2 + \lambda. \tag{5.1.33}$$

There are three cases: $\lambda = 0$, $\lambda = -1$, $\lambda = 1$. Consider the first one. The general solution of Equation (5.1.33) under $\lambda = 0$ has the form

$$v^{(1)} = \frac{-1}{x + c(t)} \tag{5.1.34}$$

where $c(t)$ is an arbitrary function of t, which should be defined from the second equation of (5.1.32). From this one we find that $\dot{c} = 0$ and due to the invariance of the equation with respect to time translations we can put, without loss of generality, $c = 0$. So, starting from a trivial solution $u^{(1)} = 0$ of the KdV equation (5.1.30), we find by means of (5.1.31) the new solution $u^{(2)} = -2/x$. Let us repeat the above procedure once more, that is, make up

$$u^{(3)} = u^{(2)} - 2v_x^{(2)}, \tag{5.1.35}$$

where function $v^{(2)}$ satisfies, according to (5.1.32), the system

$$v_x^{(2)} = \left(v^{(2)}\right)^2 + u^{(2)}, \tag{5.1.36}$$

$$v_t^{(2)} - 6\left(v^{(2)}\right)^2 v_x^{(2)} + v_{xxx}^{(2)} = 0. \tag{5.1.37}$$

The solution to the Riccati equation (5.1.36) is

$$v^{(2)} = \frac{c(t) - 2x^3}{x(c(t) + x^3)}. \tag{5.1.38}$$

After substitution of (5.1.38) into (5.1.37) we find $\dot{c} = 12$, and therefore

$$u^{(3)} = \frac{6(24tx - x^4)}{(12t + x^3)^2}. \tag{5.1.39}$$

Continuing this process we will have on the nth step the formulae

$$u^{(n+1)} = u^{(n)} - 2v_x^{(n)}, \tag{5.1.40}$$

$$v_x^{(n)} = \left(v^{(n)}\right)^2 + u^{(n)}, \tag{5.1.41}$$

$$v_t^{(n)} - 6\left(v^{(n)}\right)^2 v_x^{(n)} + v_{xxx}^{(n)} = 0. \tag{5.1.42}$$

The main difficulty in application of formulae (5.1.40)–(5.1.42) for generating new solutions of the KdV equation (5.1.30) consists in solving the Riccati equation (5.1.41). However, starting from $u^{(1)} = \lambda = \text{const}$ we succeeded in obtaining the general solution of Equation (5.1.41) for any $u^{(n)}$ (found from (5.1.40)) as

$$v^{(n)} = -v^{(n-1)} - w^{(n+1)}, \qquad (5.1.43)$$

where

$$w^{(n+1)} = \left(\ln \int \exp\left\{2\sum_{m=0}^{n}(-1)^{n-m}\int w^{(m)}dx\right\}dx\right)_x \qquad (5.1.44)$$

$$n = 0, 1, 2, \ldots, \qquad w^{(0)} = 0.$$

From (5.1.42) one can define more exactly the dependence of $w^{(n)}$ on t and then, from (5.1.40), it follows

$$\lambda \to \lambda + 2\big(\ln \varphi^{(1)}\big)_{xx} \to 1 + 2\big(\ln \varphi^{(2)}\big)_{xx} \to \ldots$$

where $(\ln \varphi)_x = w$, or

$$u^{(2n)} = \lambda + 2\sum_{m=0}^{n} w_x^{(2m)}, \quad u^{(2m+1)} = \lambda + 2\sum_{m=0}^{n} w_x^{(2m+1)}. \qquad (5.1.45)$$

So, using formulae (5.1.44), (5.1.45), we have the following chain of solutions

$$0 \to -\frac{2}{x^2} \to \frac{6(24tx - x^4)}{(12t + x^3)^2} \to \ldots$$

Analogously, one obtains two chains of solutions of the KdV equation (5.1.30):

$$1 \to 1 - \frac{2}{\cos^2(x - 2t)} \to$$
$$1 - \frac{16\big[(x + 6t)\sin(2x - 4t) + \cos(2x - 4t) + 1\big]}{\big[2(x + 6t) + \sin(2x - 4t)\big]^2} \to \ldots$$

$$-1 \to -1 + \frac{2}{\operatorname{ch}^2(x + 2t)} \to$$
$$-1 + \frac{16\big[(x - 6t)\operatorname{sh}(2x + 4t) - \operatorname{ch}(2x + 4t) - 1\big]}{\big[2(x - 6t) + \operatorname{sh}(2x + 4t)\big]^2} \to \ldots$$

It will be noted that Lie's symmetry of Equation (5.1.30) results in the following formulae of generating solutions:

translations — $u^{(2)}(t,x) = u^{(1)}(t + a_0, x + a_1)$; (5.1.46)

Galilei transformations — $u^{(2)}(t,x) = u^{(1)}(t, x + \theta t) - \frac{1}{6}\theta$; (5.1.47) scale transformations — $u^{(2)}(t,x) = \beta^2 u^{(1)}(\beta^3 t, \beta x),$ (5.1.48)

where a_0, a_1, θ, β are arbitrary constants.

Formula (5.1.31) together with those of (5.1.46)–(5.1.48) allow us to construct wide classes of exact solutions (soliton-like and non-soliton) of the KdV equation (5.1.30). Particularly, the solution

$$u = -1 + \frac{2}{\operatorname{ch}^2(2t+x)}$$

after applying formulae (5.1.46)–(5.1.48), provided $\theta = -6$, $a_0 = 0$, takes the form of the well-known soliton solution

$$u = \frac{2\beta^2}{\operatorname{ch}^2\beta(x - 4\beta^2 t + a_1)}.$$

It is obvious that formula (5.1.31) is not unique for the KdV equation. There is, for example, another one

$$u^{(2)} = u^{(1)} - 2\big(v_x^{(1)} + k^{(1)}\big),$$

$$v_x^{(1)} = \big(v^{(1)}\big)^2 + k^{(1)} + u^{(1)},$$

$$v_t^{(1)} = -6\big[\big(v^{(1)}\big)^2 + k^{(1)}\big]u_x^{(1)} + v_{xxx}^{(1)},$$

where $k^{(1)}$ is an arbitray constant.

In conclusion we note that multidimensional generalization of the KdV equation (5.1.30) is suggested in [63]:

$$u_t + F(u)\left(\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_k}\right)^{1/2} + G(u)\Delta\left(\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_k}\right)^{1/2} = 0,$$

$u = u(t, x_k)$, $k = 1,2,3$; F, G are arbitrary smooth functions of u.

5.2. *Symmetry, integrals of motion, and some partial solutions of the three-body problem*

Here we consider from group-theoretic point of view the classical three-body problem in four-dimensional spacetime.

All of the ten first integrals of the three-body Newtonian problem (that is, when particles are interacting according to Newtonian laws) were known to Euler. Jacobi was the first to show that if particles are interacting with force inversely proportional to the third power of interbody distance, then equations of motions admit one more new the first integral, eleventh, the so-called generalized Jacobi integral. It is, of course, independent of those ten. Further, Poincare, Bruns, and Painleve proved a set of fundamental theorems on integrability of the classical three-body problem. They showed that any integral of motion which represented itself as an arbitrary function of coordinates and an algebraic function of velocities of the particles is just a combination of classical integrals. Yu. D. Sokolov [193] for the first time after Jacobi found a new first integral of the one-dimensional three-body problem in the case when particles have equal mass and they interact with force proportional to the third power of interbody distance. This new integral was generalized on flat and three-dimensional space in [110, 48]. Recently Golubev and Grebennikov [113] developed new fruitful methods of qualitative analysis of equations of motion of the three-body problem.

1. To study the symmetry properties of ODEs

$$\ddot{x}_k \equiv \frac{d^2x_k}{dt^2} = F_k(t, x, \dot{x}), \qquad k = \overline{1, N}. \tag{5.2.1}$$

we following [64] rewrite system (5.2.1) in equivalent form

$$\frac{dt}{1} = \frac{dx_1}{\dot{x}_1} = \cdots = \frac{dx_N}{\dot{x}_N} = \frac{d\dot{x}_1}{F_1} = \cdots = \frac{d\dot{x}_N}{F_N}. \tag{5.2.2}$$

In turn, the system (5.2.2) is equivalent to one linear PDE

$$L\psi \equiv \left(\frac{\partial}{\partial t} + \dot{x}_k\frac{\partial}{\partial x_k} + F_k\frac{\partial}{\partial \dot{x}_k}\right)\psi = 0 \tag{5.2.3}$$

for the scalar function ψ depending on t, x_k, $\dot{x}_k$. Now one can apply to Equation (5.2.3) the standard algorithm of Lie according to which the symmetry operators are looked for in the form

$$X = \xi^0\partial_t + \xi^k\partial_{x_k} + \widetilde{\xi}^k\partial_{\dot{x}_k} + \eta\partial_\psi, \tag{5.2.4}$$

where ξ^0, ξ^k, $\widetilde{\xi}^k$, η are functions on $t, x_k, \dot{x}_k, \psi$.

The invariance condition can be taken in commutator form

$$[L, X] = BL \tag{5.2.5}$$

with an arbitrary function B depending on $t, x_k, \dot{x}_k, \psi$.

Substitution of (5.2.4) into (5.2.5) results in

$$(L\xi^0)\frac{\partial}{\partial t} + (L\xi^k)\frac{\partial}{\partial x_k} + (L\widetilde{\xi}^k)\frac{\partial}{\partial \dot{x}_k} + (L\eta)\frac{\partial}{\partial \psi} - \widetilde{\xi}^k\frac{\partial}{\partial x_k} - \tag{5.2.6}$$

$$-\left[\left(\xi^0\frac{\partial}{\partial t}+\xi^s\frac{\partial}{\partial x_s}+\widetilde{\xi}^s\frac{\partial}{\partial \dot{x}_s}\right)F_k\right]\frac{\partial}{\partial \dot{x}_k}=B\left(\frac{\partial}{\partial t}+\dot{x}_k\frac{\partial}{\partial x_k}+F_k\frac{\partial}{\partial \dot{x}_k}\right)$$

After equating in (5.2.6) coefficients of the various derivatives, we get

$$\begin{aligned}
&L\xi^0=B, \qquad L\xi^k-\widetilde{\xi}^k=B\dot{x}^k,\\
&L\widetilde{\xi}^k-\left(\xi^0\frac{\partial}{\partial t}+\xi^s\frac{\partial}{\partial x_s}+\widetilde{\xi}^s\frac{\partial}{\partial \dot{x}_s}\right)F_k=BF_k,\\
&L\eta=0.
\end{aligned} \tag{5.2.7}$$

Excluding from (5.2.7) the function B, we obtain the defining equations

$$L\left(L\xi^k-\dot{x}_kL\xi^0\right)-F_kL\xi^0=\xi^0\frac{\partial F_k}{\partial t}+\xi^s\frac{\partial F_k}{\partial x_s}+\left(L\xi^s-\dot{x}_s(L\xi^0)\right) \tag{5.2.8}$$

From (5.2.7), it also follows that coordinates $\widetilde{\xi}^k$ are completely determined by means of ξ^0 and ξ^k:

$$\widetilde{\xi}^k=L\xi^k-\dot{x}_k\xi^0. \tag{5.2.9}$$

Now we apply the algorithm stated above to study the symmetry properties of equations of motion of the three-body problem.

$$\ddot{x}_a=\frac{\partial U}{\partial x_a}, \qquad \ddot{y}_a=\frac{\partial U}{\partial y_a}, \qquad \ddot{z}_a=\frac{\partial U}{\partial z_a}, \qquad a=\overline{1,3} \tag{5.2.10}$$

with potential

$$U=F\left(r_{xy}^2\right)+F\left(r_{yz}^2\right)+F\left(r_{zx}^2\right) \tag{5.2.11}$$

where x_a, y_a, z_a are coordinated of the first, second, and third particle; $F(r^2)$ is an arbitrary smooth function;

$$r_{xy}^2=\left(\vec{x}-\vec{y}\right)^2, \qquad r_{yz}^2=\left(\vec{y}-\vec{z}\right)^2, \qquad r_{zx}^2=\left(\vec{z}-\vec{x}\right)^2,$$

Theorem 5.2.1. [155] *The system of ODEs* (5.2.10) *with arbitrary smooth function* $F(r^2)$ *is invariant under the Galilei group* G(1,3), *basis elements of corresponding* AG(1,3) *having the form*

$$P_0=\partial_t, \qquad P_a=\frac{\partial}{\partial x_a}+\frac{\partial}{\partial y_a}+\frac{\partial}{\partial z_a}, \tag{5.2.12}$$

$$G_a=tP_a, \qquad J_{ab}=x_a\frac{\partial}{\partial x_b}-x_b\frac{\partial}{\partial x_a}+y_a\frac{\partial}{\partial y_b}-y_b\frac{\partial}{\partial y_a}+z_a\frac{\partial}{\partial z_b}-z_b\frac{\partial}{\partial z_a}$$

The proof is not difficult to obtain by means of algorithm described above.

Theorem 5.2.1. determines minimal IA of Equations (5.2.10). It is more or less obvious that depending on special forms of the potential U, the symmetry of the equations may extend. The following theorem describes the all possible cases of the symmetry extension.

Theorem 5.2.2. [155] *The invariance algebra* (IA) *of Equations* (5.2.10), AG(1,3) *extends if and only if the potential* (5.2.11) *has one of the following five forms determined by function* $F(r^2)$*:*

$$F(r^2) = \lambda_1 r^4 + \lambda_2 r^2, \tag{5.2.13}$$

then IA $= \{\mathrm{AG}(1,3), S\}$,

$$S = (y_i - z_i)\frac{\partial}{\partial x_i} + (z_i - x_i)\frac{\partial}{\partial y_i} + (x_i - y_i)\frac{\partial}{\partial z_i}; \tag{5.2.14}$$

$$F(r^2) = \lambda_3 r^2 \tag{5.2.15}$$

then IA $= \{P_0,\ P_a,\ G_a,\ Y_k^{ab}\}$, $a, b = \overline{1,3}$; $k = \overline{1,6}$,

$$\begin{aligned}
Y_1^{ab} &= v_{a1}\frac{\partial}{\partial v_{b1}} + v_{a2}\frac{\partial}{\partial v_{b2}} + v_{a3}\frac{\partial}{\partial v_{b3}},\\
Y_2^{ab} &= v_{a1}\frac{\partial}{\partial v_{b1}} + v_{a3}\frac{\partial}{\partial v_{b2}} + v_{a2}\frac{\partial}{\partial v_{b3}},\\
Y_3^{ab} &= v_{a2}\frac{\partial}{\partial v_{b1}} + v_{a3}\frac{\partial}{\partial v_{b2}} + v_{a1}\frac{\partial}{\partial v_{b3}},\\
Y_4^{ab} &= v_{a2}\frac{\partial}{\partial v_{b1}} + v_{a1}\frac{\partial}{\partial v_{b2}} + v_{a3}\frac{\partial}{\partial v_{b3}},\\
Y_5^{ab} &= v_{a3}\frac{\partial}{\partial v_{b1}} + v_{a1}\frac{\partial}{\partial v_{b2}} + v_{a2}\frac{\partial}{\partial v_{b3}},\\
Y_6^{ab} &= v_{a3}\frac{\partial}{\partial v_{b1}} + v_{a2}\frac{\partial}{\partial v_{b2}} + v_{a1}\frac{\partial}{\partial v_{b3}},\\
&\big(v_{a1} \equiv x_a, \qquad v_{a2} \equiv y_a, \qquad v_{a3} \equiv z_a,\big);
\end{aligned} \tag{5.2.16}$$

$$F(r^2) = \lambda_4 r^{-2}, \tag{5.2.17}$$

then IA $= \{\mathrm{AG(1,3)},\ D_1, \Pi\}$,

$$\begin{aligned}
D_1 &= 2t\partial_t + x_a\frac{\partial}{\partial x_a} + y_a\frac{\partial}{\partial y_a} + z_a\frac{\partial}{\partial z_a}\\
\Pi &= tD_1 - t^2\partial_t;
\end{aligned} \tag{5.2.18}$$

$$F(r^2) = \lambda_5 (r^2)^\beta, \qquad \beta \neq 0 \tag{5.2.19}$$

The IA $= \{\text{AG(1,3)}, D_2\}$,

$$D_2 = (1-\beta)t\partial_t + x_a\frac{\partial}{\partial x_a} + y_a\frac{\partial}{\partial y_a} + z_a\frac{\partial}{\partial z_a} \tag{5.2.20}$$

and finally, if

$$F(r^2) = \lambda_6 \ln(r^2) \tag{5.2.21}$$

then IA $= \{\text{AG}(1,3), D_3\}$,

$$D_3 = t\partial_t + x_a\frac{\partial}{\partial x_a} + y_a\frac{\partial}{\partial y_a} + z_a\frac{\partial}{\partial z_a} \tag{5.2.22}$$

Proof. It is convenient to write IFO (5.2.4) in the form

$$X = \xi\partial t + \eta^{1a}\partial x_a + \eta^{2a}\partial y_a + \eta^{3a}\partial z_a, \tag{5.2.23}$$

where ξ, η^{1a}, η^{2a}, η^{3a} depend on $t, x_1, \ldots, z_3$, provided (x_a, y_a, z_a) denotes coordinates of the ath body, $a = 1, 2, 3$.

In this case, the defining equations (5.2.8) take the form

$$\begin{aligned}
&U_{x_c}\eta^{1a}_{x_c} + U_{y_c}\eta^{1a}_{y_c} + U_{z_c}\eta^{1a}_{z_c} - 2U_{x_a}\xi_t = \left(\eta^{1a} - \eta^{1b}\right)\dot{F}_{ab} + \\
&\qquad + \left(\eta^{1a} - \eta^{1c}\right)\dot{F}_{ac} + (x_a - x_b)\ddot{F}_{ab}X(r^2_{ab}) + (x_a - x_b)\ddot{F}_{ac}X(r^2_{ac}), \\
&U_{x_c}\eta^{2a}_{x_c} + U_{y_c}\eta^{2a}_{y_c} + U_{z_c}\eta^{2a}_{z_c} - 2U_{y_a}\xi_t = \left(\eta^{2a} - \eta^{2b}\right)\dot{F}_{ab} + \\
&\qquad + \left(\eta^{2a} - \eta^{2c}\right)\dot{F}_{ac} + +(y_a - y_b)\ddot{F}_{ab}X(r^2_{ab}) + (y_a - y_c)\ddot{F}_{ac}X(r^2_{ac}), \\
&U_{x_c}\eta^{3a}_{x_c} + U_{y_c}\eta^{3a}_{y_c} + U_{z_c}\eta^{3a}_{z_c} - 2U_{z_a}\xi_t = \left(\eta^{13} - \eta^{3b}\right)\dot{F}_{ab} + \\
&\qquad = + \left(\eta^{3a} - \eta^{3c}\right)\dot{F}_{ac} + (z_a - z_b)\ddot{F}_{ab}X(r^2_{ab}) + (z_a - z_c)\ddot{F}_{ac}X(r^2_{ac}),
\end{aligned} \tag{5.2.24}$$

$$\begin{aligned}
&\xi = b_0t^2 + \gamma t + d, \\
&\eta^{1a} = b_0x_at + a^{1ab}x_b + a^{2ab}y_b + a^{3ab}z_b + a^{a0}t + d^{1a}, \\
&\eta^{2a} = b_0y_at + c^{a1b}x_b + c^{a2b}y_b + c^{a3b}z_b + c^{a0}t + d^{2a}, \\
&\eta^{3a} = b_0z_at + e^{a1b}x_b + e^{a2b}y_b + e^{a3b}z_b + e^{a0}t + d^{3a},
\end{aligned} \tag{5.2.25}$$

where

$$\dot{F}_{ab} = \frac{df(r^2_{ab})}{d(r^2_{ab})}, \qquad a \neq b \neq c, \qquad a, b, c = 1, 2, 3;$$

b_0, γ, d, a^{1ab}, a^{2ab}, a^{3ab}, a^{a0}, d^{1a}, d^{2a}, d^{3a},

$$c^{a1b},\ c^{a2b},\ c^{a3b},\ c^{a0},\ e^{1a},\ e^{a2b},\ e^{a3b},\ e^{a0}$$

are arbitrary real constants (group parameters);

$$X(r_{ab}^2) = 2(x_a - x_b)(\eta^{1a} - \eta^{1b}) + \\ + 2(y_a - y_b)(\eta^{2a} - \eta^{2b}) + 2(z_a - z_b)(\eta^{3a} - \eta^{3b}). \tag{5.2.26}$$

Further analysis of the defining equations (5.2.24), (5.2.25) results in five different cases, namely those listed in the theorem. By this remark, we end the proof.

As a consequence of Theorem 5.2.2, we note that Equations (5.2.10) with the potential

$$U = \lambda_4 \left(\frac{1}{r_{xy}^2} + \frac{1}{r_{yz}^2} + \frac{1}{r_{zx}^2} \right) \tag{5.2.27}$$

are invariant with respect to the Schrödinger group Sch(1,3). One can make sure that operators of invariance algebra stated in (5.2.12) and (5.2.18) satisfy commutation rules of ASch(1,3) (compare with (4.2.2)):

$$[P_a, P_b] = [P_0, P_a] = [G_a, G_b] = [P_0, J_{ab}] = \\ = [P_a, G_b] = [J_{ab}, D] = [J_{ab}, \Pi] = 0$$

$$[P_a, J_{bc}] = \delta_{ab} P_c - \delta_{ac} P_b,$$

$$[G_a, J_{bc}] = \delta_{ab} G_c - \delta_{ac} G_b,$$

$$[P_0, G_a] = P_a, \qquad [P_0, D] = 2P_0, \qquad [P_0, \Pi] = D,$$

$$[P_a, D] = P_a, \qquad [P_a, \Pi] = 2G_a, \qquad [D, \Pi] = 2\Pi.$$

Let us write down the final transformations generated by (5.2.14). In the new notations they have the form:

$$x'_a = \left(x_a - \frac{x_1 + x_2 + x_3}{3} \right) \cos\sqrt{3}\theta + \frac{1}{2\sqrt{3}} \epsilon_{abc}(x_b - x_c) \sin\sqrt{3}\theta + \frac{x_1 + x_2 + x_3}{3},$$

$$y'_a = \left(y_a - \frac{y_1 + y_2 + y_3}{3} \right) \cos\sqrt{3}\theta + \frac{1}{2\sqrt{3}} \epsilon_{abc}(y_b - y_c) \sin\sqrt{3}\theta + \frac{y_1 + y_2 + y_3}{3},$$

$$z'_a = \left(z_a - \frac{z_1 + z_2 + z_3}{3} \right) \cos\sqrt{3}\theta + + \frac{1}{2\sqrt{3}} \epsilon_{abc}(z_b - z_c) \sin\sqrt{3}\theta + \frac{z_1 + z_2 + z_3}{3},$$

where θ is an arbitrary real parameter. In this notation operator S takes the form

$$S = \tfrac{1}{2} \epsilon_{abc} \left[(x_a - x_b) \frac{\partial}{\partial x_c} + (y_a - y_b) \frac{\partial}{\partial y_c} + (z_a - z_b) \frac{\partial}{\partial z_c} \right].$$

2. Now consider integrals of motion of the three-body problem. It is well known that invariance of the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\left(\dot{\vec{x}}^2 + \dot{\vec{y}}^2 + \dot{\vec{z}}^2\right) - U \qquad (5.2.30)$$

of the three-body problem (5.2.10) with respect to 10-parameter Galilei group G(1,3) means that there are 10 quantities which are constants of motion of the system (also known as *first integrals*):

$$\begin{aligned} H &= \tfrac{1}{2}\left(\dot{\vec{x}}^2 + \dot{\vec{y}}^2 + \dot{\vec{z}}^2\right) + U, \\ \mathcal{P}_a &= \dot{x}_a + \dot{y}_a + \dot{z}_a, \\ \mathcal{Y}_a &= x_a + y_a + z_a - t(\dot{x}_a + \dot{y}_a + \dot{z}_a), \\ j_{ab} &= \dot{x}_a x_b - \dot{x}_b x_a + \dot{y}_a y_b - \dot{y}_b y_a + \dot{z}_a z_b - \dot{z}_b z_a \end{aligned} \qquad (5.2.31)$$

Additional first integrals come into being when function $F(r^2)$ defining the potential U has the form (5.2.13) and (5.2.17). In the first case (that is when $F(r^2)$ is given by (5.2.13)), the Lagrangian (5.2.30) takes the form

$$\begin{aligned} \mathcal{L}_4 = \tfrac{1}{2}\left(\dot{\vec{x}}^2 + \dot{\vec{y}}^2 + \dot{\vec{z}}^2\right) - \\ - \lambda_1\left[(r_{xy}^2)^2 + (r_{yz}^2)^2 + (r_{zx}^2)^2\right] - \lambda_2\left(r_{xy}^2 + r_{yz}^2 + r_{zx}^2\right). \end{aligned} \qquad (5.2.32)$$

The equations of motion following from the Lagrangian (5.2.32) possess an additional symmetry operator (5.2.14). This means that there exists an additional conserved quantity. To construct it, one can use the Noether theorem (the condition of the theorem is fulfilled: $S\mathcal{L}_4 = 0$). It yields the 11th first integral

$$\overline{S} = (y_a - z_a)\dot{x}_a + (z_a - x_a)\dot{y}_a + (x_a - y_a)\dot{z}_a,$$

or in new notations (5.2.33)

$$\overline{S} = \tfrac{1}{2}\epsilon_{abc}\left[(x_a - x_b)\dot{x}_c + (y_a - y_b)\dot{y}_c + (z_a - z_b)\dot{z}_c\right].$$

This integral was obtained for the first time by Yu. D. Sokolov in [193] for one-dimensional space and then it was extended on flat and three-dimensional space in [48, 110].

In the second case (that is when the function $F(r^2)$ is given by (5.2.17)), the Lagrangian has the form

$$\mathcal{L}_2 = \tfrac{1}{2}(\dot{\vec{x}}^2 + \dot{\vec{y}}^2 + \dot{\vec{z}}^2) - \lambda_4\left(\frac{1}{r_{xy}^2} + \frac{1}{r_{yz}^2} + \frac{1}{r_{zx}^2}\right). \qquad (5.2.34)$$

According to the Theorem (5.2.2) the equations of motion following from the Lagrangian (5.2.34) have two additional symmetry operators D_1 and Π written in (5.2.18). But since $\Pi\mathcal{L}_2 \neq 0$, we cannot make use of the Noether theorem to construct the corresponding first integral. Nevertheless, the first integral can be constructed in the following way, which holds for any equation of the form (5.2.3). For any symmetry operator Q of a given equation (5.2.3) and for the arbitrary solution ψ of this equation the function

$$\psi_1 = Q\psi \tag{5.2.35}$$

will also have a solution. An analogous statement holds for the conserved quantities. It means, in particular, that the expressions

$$\pi_1 = \Pi H_2 = \left(x_a\dot{x}_a + y_a\dot{y}_a + z_a\dot{z}_a\right) - 2H_2 t, \tag{5.2.36}$$

$$\pi_2 = \Pi\pi_1 = \vec{x}^2 + \vec{y}^2 + \vec{z}^2 - 2H_2t^2 - 2\pi_1 t, \tag{5.2.37}$$

where

$$H_2 = \tfrac{1}{2}\left(\dot{\vec{x}}^2 + \dot{\vec{y}}^2 + \dot{\vec{z}}^2\right) + \lambda_4\left(\frac{1}{r_{xy}^2} + \frac{1}{r_{yz}^2} + \frac{1}{r_{zx}^2}\right),$$

give new constants of motion. Integral π_2 is known as the Jacobi integral. Since

$$\Pi\mathcal{P}_a = \mathcal{G}_a, \qquad \Pi\mathcal{G}_a = \Pi j_{ab} = \Pi\pi_2 = 0$$

we do not obtain any new integrals by this procedure.

When the potential U is determined by (5.2.19) and (5.2.21), the equations of motion (5.2.10) are additionally invariant under operators D_2 (5.2.20) and D_3 (5.2.22) respectively. One finds that

$$D_2\mathcal{L}_\beta \neq 0, \qquad D_3\mathcal{L}_0 \neq 0,$$

where $\mathcal{L}_\beta$ and $\mathcal{L}_0$ are corresponding Lagrangians. It means that the Noether theorem is impossible to apply here. On the other hand we find

$$D_2\mathcal{P}_a = \beta\mathcal{P}_a, \qquad D_2\mathcal{G}_a = \mathcal{G}_a, \qquad D_2H_\beta = 2\beta H_\beta,$$

$$D_2 j_{ab} = (1+\beta)j_{ab}, \quad D_2\pi_1 = 0, \qquad D_2\pi_2 = 2\pi_2 \quad (\beta = -1);$$

$$D_3\mathcal{P}_a = 0, \qquad D_3\mathcal{G}_a = \mathcal{G}_a,$$

$$D_3 j_{ab} = j_{ab}, \quad D_3H_0 = 2\lambda_6;$$

where

$$H_\beta = \tfrac{1}{2}\left(\dot{\vec{x}}^2 + \dot{\vec{y}}^2 + \dot{\vec{z}}^2\right) + \lambda_5\left[(r_{xy}^2)^\beta + (r_{yz}^2)^\beta + (r_{zx}^2)^\beta\right],$$

$$H_0 = \tfrac{1}{2}\left(\dot{\vec{x}}^2 + \dot{\vec{y}}^2 + \dot{\vec{z}}^2\right) + \lambda_6\left[\ln(r_{xy}^2) + \ln(r_{yz}^2) + \ln(r_{zx}^2)\right].$$

It means that operators (5.2.20) and (5.2.22) do not give new integrals of motion.

Equations (5.2.10) with the potential U defined by (5.2.15), as follows from the theorem (5.2.2), are invariant with respect to the 61-parameter Lie algebra (5.2.16). This case is studied in full detail in [125, 126] and therefore we do not consider it here.

3. In this point we describe some exact solutions of the three-body problem.

The problem of integration of equations of motion of the three-body system was studied by many workers since Newtonian times. Until now only three cases of full integrability of this problem are known, namely the potential of interaction is defined by (5.2.13) and (5.2.17). Some partial solutions of Equations (5.2.10) are obtained for other potentials.

The equations of motion (5.2.10) represent themselves the system of PDEs for 18 unknown functions. The fact that this system possesses 10 first integrals (5.2.31) allows us to reduce the number of unknown functions to 6. When equations of motion possess additional first integrals (they are written in (5.2.33), (5.2.36), (5.2.37)), the number of unknowns is reduced to 4. The reduction is achieved by means of appropriate change of variables.

Consider the three-body system with the potential of interaction given by (5.2.11), (5.2.13). As it was already said in this case, equations of motion have an additional first integral (5.2.33). The one-dimensional case of this problem was fully integrated by Yu. S. Sokolov [194]. The change of variables found in [194] was generalized in the case of flat (two-dimensional) space which resulted in some partial solutions of the corresponding equations of motion. Some exact solutions in three-dimensional space were obtained in [204].

Below we construct some partial solutions of equations of motion (5.2.10) with the potential (5.2.11) defined by (5.2.13).

For the sake of convenience we combine the center-of-mass system with the reference system, that is, we put

$$x_a + y_a + z_a = 0. \tag{5.2.38}$$

Further, we make the following change of variables

$$\begin{aligned}
3x_k &= S_1 \cos\left(\alpha_1 + \frac{2\pi(k-1)}{3}\right) + S_2 \cos\left(\alpha_2 - \frac{2\pi(k-1)}{3}\right) + \\
&\quad + S_3 \cos\left(\frac{\alpha_1 - \alpha_2}{2} + \frac{2\pi(k-1)}{3}\right) + S_4 \sin\left(\frac{\alpha_1 - \alpha_2}{2} + \frac{2\pi(k-1)}{3}\right), \\
3y_k &= S_1 \cos\left(\alpha_1 + \frac{2\pi k}{3}\right) + S_2 \cos\left(\alpha_2 - \frac{2\pi(k+1)}{3}\right) + \qquad (5.2.39) \\
&\quad + S_3 \cos\left(\frac{\alpha_1 - \alpha_2}{2} + \frac{2\pi(k-1)}{3}\right) + S_4 \sin\left(\frac{\alpha_1 - \alpha_2}{2} + \frac{2\pi(k-1)}{3}\right),
\end{aligned}$$

$$3z_k = S_1 \cos\left(\alpha_1 + \frac{2\pi(k+1)}{3}\right) + S_2 \cos\left(\alpha_2 - \frac{2\pi k}{3}\right) +$$
$$+ S_3 \cos\left(\frac{\alpha_1 - \alpha_2}{2} + \frac{2\pi(k-1)}{3}\right) + S_4 \sin\left(\frac{\alpha_1 - \alpha_2}{2} + \frac{2\pi(k-1)}{3}\right),$$
$$k = \overline{1,3}$$

In the two-dimensional case an analogous change of variables was used in [48].

In new variables expressions of kinetic and potential energy take the following form

$$T = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \tfrac{1}{4}\Big[\dot{S}_1^2 + \dot{S}_2^2 + \dot{S}_3^2 + \dot{S}_4^2 + \tag{5.2.40}$$
$$+ (S_1\dot{\alpha}_1)^2 + (S_2\dot{\alpha}_2)^2 + (S_3^2 + S_4^2)\left(\frac{\dot{\alpha}_1 - \dot{\alpha}_2}{2}\right)^2 + (\dot{S}_3 S_4 - S_3\dot{S}_4)(\dot{\alpha}_1 - \dot{\alpha}_2)\Big],$$

$$U = \lambda_1\left(r_{12}^4 + r_{23}^4 + r_{31}^4\right) + \lambda_2\left(r_{12}^2 + r_{23}^2 + r_{31}^2\right) =$$
$$= \tfrac{3}{2}\lambda_2\left(S_1^2 + S_2^2 + S_3^2 + S_4^2\right) + \tfrac{3}{4}\lambda_1\Big[\left(S_1^2 + S_2^2 + S_3^2 + S_4^2\right)^2 + \tag{5.2.41}$$
$$+ \tfrac{1}{2}\left(S_4^2 - S_3^2 - 2S_1S_2\right)^2 + 2\left(S_3S_4\right)^2\Big].$$

Equations of motion in new variables are written as follows

$$\tfrac{1}{2}\ddot{S}_1 - \tfrac{1}{2}S_1\dot{\alpha}_1^2 = -\frac{\partial U}{\partial S_1},$$
$$\tfrac{1}{2}\ddot{S}_2 - \tfrac{1}{2}S_2\dot{\alpha}_2^2 = -\frac{\partial U}{\partial S_2}, \tag{5.2.42}$$
$$\frac{d}{dt}\Big[\tfrac{1}{2}\dot{S}_3 + \tfrac{1}{8}S_4(\dot{\alpha}_1 - \dot{\alpha}_2)\Big] - \tfrac{1}{2}S_3\left(\frac{\dot{\alpha}_1 - \dot{\alpha}_2}{2}\right)^2 + \tfrac{1}{8}\dot{S}_4(\dot{\alpha}_1 - \dot{\alpha}_2) = -\frac{\partial U}{\partial S_3},$$
$$\frac{d}{dt}\Big[\tfrac{1}{2}\dot{S}_4 + \tfrac{1}{8}S_3(\dot{\alpha}_1 - \dot{\alpha}_2)\Big] - \tfrac{1}{2}S_4\left(\frac{\dot{\alpha}_1 - \dot{\alpha}_2}{2}\right)^2 + \tfrac{1}{8}\dot{S}_3(\dot{\alpha}_2 - \dot{\alpha}_1) = -\frac{\partial U}{\partial S_4},$$
$$\frac{d}{dt}\Big[\dot{S}_1^2\dot{\alpha}_1 - \tfrac{1}{8}(S_3^2 + S_4^2)(\dot{\alpha}_1 - \dot{\alpha}_2) + \tfrac{1}{2}(\dot{S}_3S_4 - S_3\dot{S}_4)\Big] = 0,$$
$$\frac{d}{dt}\Big[\dot{S}_2^2\dot{\alpha}_2 + \tfrac{1}{8}(S_3^2 + S_4^2)(\dot{\alpha}_2 - \dot{\alpha}_1) + \tfrac{1}{2}(S_3\dot{S}_4 - \dot{S}_3S_4)\Big] = 0.$$

Integrals of motion $\mathcal{P}_a$ and $\mathcal{G}_a$ in new variables become zero.

Let S_1, S_2, S_3, $S_4, \dot{\alpha}_1$, $\dot{\alpha}_2$ be constant, then the system (5.2.42) is reduced to the system of algebraic equations

$$S_1\dot\alpha_1^2 = \lambda_1\left[3(S_1^2+S_2^2+S_3^2+S_4^2)S_1 + \tfrac{3}{4}(S_1S_2+S_3^2-S_4^2)S_2\right] + 3\lambda_2 S_1,$$

$$S_2\dot\alpha_2^2 = \lambda_1\left[3(S_1^2+S_2^2+S_3^2+S_4^2)S_2 + \tfrac{3}{4}(S_1S_2+S_3^2-S_4^2)S_1\right] + 3\lambda_2 S_2,$$

$$S_3(\dot\alpha_1-\dot\alpha_2)^2 = 8\lambda_1\left[3(S_1^2+S_2^2+S_3^2+S_4^2)S_3+\tfrac{3}{2}(S_1S_2+S_3^2-S_4^2)S_3+3S_3S_4^2\right]+ \\ +24\lambda_2 S_3,$$

$$S_4(\dot\alpha_1-\dot\alpha_2) = 8\lambda_1\left[3(S_1^2+S_2^2+S_3^2+S_4^2)S_4-\tfrac{3}{2}(S_1S_2+S_3^2-S_4^2)S_4+3S_2^2S_4\right]+, \\ +24\lambda_2 S_4,$$

$$S_1^2\dot\alpha_1 + \tfrac{1}{4}(S_3^2+S_4^2)(\dot\alpha_2-\dot\alpha_2) = c_1, \qquad (5.2.43)$$

$$S_1^2\dot\alpha_1 + \tfrac{1}{4}(S_3^2+S_4^2)(\dot\alpha_2-\dot\alpha_2) = c_1,$$

where c_1 and c_2 are constants of integration.

Further, we put $S_1 = S_2$, $S_3 = S_4$ (and therefore $\dot\alpha_1 = \dot\alpha_2$, $c_1 = c_2$). After this suggestion, the system (5.2.43) takes the form

$$\begin{aligned} &\dot\alpha_1^2 = \lambda_1\left[6(S_1^2+S_2^2)+\tfrac{3}{4}S_1^2\right]+3\lambda_2 \\ &\lambda_1\left(\tfrac{5}{2}+3S_3^2\right)+\lambda_2, \qquad \dot\alpha S_1^2 = c_1 \end{aligned} \qquad (5.2.44)$$

The system (5.2.44) has a solution when $\lambda_1\lambda_2 < 0$. Having expressed S_3^2 and $\dot\alpha_1$ from the second and third equation of the system (5.2.44) via S_1^2, we get cubic equation with respect to S_1^2

$$\begin{aligned} &\dot\alpha_1 = \frac{c_1}{S_1^2}, \qquad S_3^2 = \frac{\lambda_2}{3\lambda_1} - \tfrac{5}{6}S_1^2, \\ &\tfrac{7}{4}\lambda_1(S_1^2)^3 + \lambda_2 S_1^2 - c_1^2 = 0. \end{aligned} \qquad (5.2.45)$$

Using Cardano formula, it is not difficult to obtain the solution of Equation (5.2.45)

$$S_1 = \left\{\frac{4}{7\lambda_1}\left[\frac{c_1^2}{2} - \left(\frac{\lambda_2}{3}\right)^3\left(\frac{4}{7\lambda_1}\right)^2\right] + \frac{4}{7}\left|\frac{c_1}{\lambda_1}\right|\left[\frac{c_1}{4} - \left(\frac{\lambda_2}{3}\right)^3\left(\frac{4}{7\lambda_1}\right)^2\right]^{1/2}\right\}^{1/3} + \\ + \left\{\frac{4}{7\lambda_1}\left[\frac{c_1}{2} - \left(\frac{\lambda_2}{3}\right)^3\left(\frac{4}{7\lambda_1}\right)^2\right] - \frac{4}{7}\left|\frac{c_1}{\lambda_1}\right|\left[\frac{c_1^2}{4} - \left(\frac{\lambda_2}{3}\right)^3\left(\frac{4}{7\lambda_1}\right)^2\right]^{1/2}\right\}^{1/3} - \\ - \frac{4}{21}\frac{\lambda_2}{\lambda_1},$$

$$\frac{c_1^2}{4} \geq \left(\frac{\lambda_2}{3}\right)^3 \left(\frac{4}{7\lambda_1}\right)^2 \tag{5.2.46}$$

From (5.2.45), we find

$$\begin{aligned} \alpha_1 &= \frac{c_1}{S_1^2}t + c_3, \qquad \alpha_2 = \frac{c_1}{S_1^2}t + c_4, \\ S_3^2 &= S_4^2 = \frac{\lambda_2}{3\lambda_1} - \frac{5}{6}S_1^2, \end{aligned} \tag{5.2.47}$$

where c_1, c_3, c_4 are constants of integration.

Formulae (5.2.46), (5.2.47), and (5.2.39) give a solution of equations of motion (5.2.10) of the three-body system with the potential (5.2.11), (5.2.13).

5.3 *Non-Lie symmetry and nonlocal transformations*

The infinitesimal Lie method is far from providing the possibility of finding all symmetries which a system of differential equations possesses. A familiar example of a "non-Lie" symmetry is the invariance of the Schrödinger equation for the hydrogen atom under the group 0(4) first discovered by Fock [53].

Let us consider an arbitrary linear system of PDEs

$$L(x, \partial)\psi(x) = 0, \tag{5.3.1}$$

where $L(x,\partial)$ is a linear operator, ψ is a multi-component function with components $\{\psi^1, \psi^2, \ldots, \psi^m\}$, $x \in R^n$, $\partial = \left\{\frac{\partial}{\partial x_\mu}\right\}$ $\mu = \overline{0, n-1}$.

In Lie's approach, the infinitesimal operators of the invariance algebra are sought in the form of first-order differential operators (2). Taking into account that the manifold of solutions of linear system (5.3.1) is also linear, we can simplify the general form of the symmetry operators (2) as follows

$$X = \xi^\mu(x)\frac{\partial}{\partial x^\mu} - (\eta(x)\psi)^k \frac{\partial}{\partial \psi^k} \Leftrightarrow Q = \xi^\mu(x)\frac{\partial}{\partial x^\mu} + \eta(x), \tag{5.3.2}$$

where $\eta(x)$ are matrices of dimension $m \times m$. Here we used different notations for the same operator to emphasize that operator Q acts in the linear m-dimensional space $\{\psi(x)\}$ while the operator X acts in linear $(m+n)$-dimensional space $\{x, \psi\}$.

By means of operator Q (5.3.2), the invariance condition of the system (5.3.1) can be written in the form (9), (10):

$$[L, Q]\psi(x)\Big|_{L\psi=0} = 0. \tag{5.3.3}$$

It is obvious that this condition does not impose any restrictions on the order of generators Q. So the formulation of the problem of investigating symmetry properties of the system (5.3.1) can be considerable generalized by extending the class of the desired operators Q. For example, it is possible to seek IA in a class of second-order differential operators or even of integrodifferential operators. It will be emphasized that the condition of invariance (5.3.3) does not guarantee for the set of invariance operators Q of second and higher orders to form a Lie algebra as it was for the first-order differential operators in Lie approach, and what is more operators Q can form, for example, superalgebra or possess other specific algebraic properties. If we desire operators Q to form a Lie algebra, we must also require the following relations to hold:

$$[Q_A, Q_B] = C_{ABC} Q_C, \tag{5.3.4}$$

where C_{ABC} are structure constants. A set of operators $\{Q_A\}$ satisfying condition (5.3.) and (5.3.4) is called an invariance algebra (IA) of the system (5.3.1). This non-Lie algebraic approach was suggested in 1971 in [57] and since then new IA of the Dirac equations. Maxwell's equations and many equations of quantum mechanics have been found in just this manner. A largely complete review of these results the reader will find in [82,87*].

Below we consider how to find final transformations generated by non-Lie symmetry operators. It is clear that these transformations cannot be local; naturally they are nonlocal.

An arbitrary differential operator of any order can be written as follows:

$$Q \equiv Q(x, \partial) = \xi^\mu(x)\partial_\mu + \eta(x, \partial), \tag{5.3.5}$$

where $\xi^\mu(x)$ are scalar functions, $\eta(x, \partial)$ is the matrix-differential part of Q does not contain terms like $\xi^\mu(x)\partial_\mu$.

Theorem 5.3.1. [99] *If operator Q (5.3.5) satisfies condition of invariance (5.3.3) on the manifold of solutions of the system (5.3.1), then the system (5.3.1) is invariant under one-parameter group of transformations.*

$$\begin{aligned} x_\mu \to x'_\mu &= \exp\{\theta\xi\partial\} x_\mu \exp\{-\theta\xi\partial\}, \\ \psi(x) \to \psi(x') &= \exp\{\theta\xi\partial\} \exp\{-\theta Q\}\psi(x), \end{aligned} \tag{5.3.6}$$

where θ is an arbitrary real constant (group parameter), generated by this operator Q.

Proof. As a result of transformations $x_\mu \to x'_\mu$ from (5.3.6), any differential operator constructed from x_μ and ∂_μ, and of course, the operator of the system (5.3.1), $L(x, \partial)$ transforms according to the law

$$L(x, \partial) \to L(x', \partial') = \exp\{\theta\xi\partial\} L(x, \partial) \exp\{-\theta\xi\partial\}.$$

then using the expression for $\psi(x)$ from (5.3.6) and taking into account (5.3.3) we find

$$L(x',\partial')\psi'(x') = \exp\{\theta\xi\partial\}L(x,\partial)\exp\{-\theta\xi\partial\}\cdot$$
$$\cdot\exp\{\theta\xi\partial)\exp\{-\theta Q\}\psi(x) = \exp\{\theta\xi\partial)L(x,\partial)\exp\{-\theta Q\}\psi(x) =$$
$$= \exp\{\theta\xi\partial\}L(x,\partial)\left(1 - \theta Q + \frac{(\theta Q)^2}{2!} \mp \cdots\right)\psi(x) = 0.$$

It proves the first part of the theorem, that is, the system (5.3.1) is indeed invariant with respect to transformations (5.3.6). And now we convince ourselves that transformations (5.3.6) form a one-parameter group. We find

$$x''_\mu = (\exp\{\beta\xi\partial\})'\, x'_\mu\, (\exp\{-\beta\xi\partial\})' =$$
$$= \exp\{\theta\xi\partial\}\exp\{\beta\xi\partial\}\exp\{-\theta\xi\partial\}\exp\{\theta\xi\partial\}x_\mu\cdot$$
$$\cdot\exp\{-\theta\xi\partial\}\exp\{\theta\xi\partial\}\exp\{-\beta\xi\partial\}\exp\{-\theta\xi\partial\} =$$
$$= \exp\{(\theta+\beta)\xi\partial\}x_\mu\exp\{-(\theta+\beta)\xi\partial\},$$

$$\psi''(x'') = (\exp\{\beta\xi\partial\}\exp\{-\beta Q\})'\,\psi'(x') =$$
$$= \exp\{\theta\xi\partial\}\exp\{\beta\xi\partial\}\exp\{-\beta Q\}\exp\{-\theta\xi\partial\}\exp\{\theta\xi\partial\}\exp\{-\theta Q\}\psi(x) =$$
$$= \exp\{(\theta+\beta)\xi\partial\}\exp\{-(\theta+\beta)\xi\partial\}\psi(x),$$

which completes the proof.

Remark 5.3.1. If operator Q satisfying invariance condition (5.3.2) belongs to the class of Lie-type operators (5.3.2), then expressions (5.3.6) give a formal solution to the Lie equations.

$$\begin{aligned} \frac{dx'_\mu}{d\theta} &= \xi^\mu(x'), \qquad x'_\mu\Big|_{\theta=0} = x_\mu, \\ \frac{d\psi'(x')}{d\theta} &= -\eta(x')\psi'(x'), \qquad \psi'(x')\Big|_{\theta=0} \equiv \psi(x). \end{aligned} \tag{5.3.7}$$

Indeed, by substituting (5.3.6) into (5.3.7), we obtain identities

$$x'_\mu\Big|_{\theta=0} \equiv x_\mu, \qquad \psi'(x')\Big|_{\theta=0} \equiv \psi(x),$$

$$\frac{dx'_\mu}{d\theta} = \exp\{\theta\xi\partial\}[\xi\partial, x_\mu]\exp\{-\theta\xi\partial\} = \exp\{\theta\xi\partial\}\xi^\mu(x)\exp\{-\theta\xi\partial\} \equiv \xi^\mu(x'),$$

$$\frac{d\psi'(x')}{d\theta} = \exp\{\theta\xi\partial\}(\xi\partial - Q)\exp\{-\theta Q\}\psi(x) = -\exp\{\theta\xi\partial\}\eta(x)\exp\{-\theta\xi\partial\}\cdot$$

$$\cdot \exp\{\theta\xi\partial\} \exp\{-\theta Q\}\psi(x) \equiv -\eta(x')\psi(x').$$

If operator Q does not belong to the class of Lie's operators (5.3.2), it generates nonlocal transformations. As a simple example of non-Lie operator, let us consider the operator

$$Q = \frac{\partial^2}{\partial x^2}.$$

According to (5.3.6), we have

$$x' = x,$$

$$\psi'(x') = \exp\left\{\theta \frac{\partial^2}{\partial x^2}\right\} \psi(x) = \frac{1}{\sqrt{4\pi\theta}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x-y)^2}{4\theta}\right\} \psi(y) dy.$$

This result follows by noting that the $F(\theta, x) \equiv \psi'(x')$ satisfies the Cauchy problem

$$\frac{\partial F}{\partial \theta} = \frac{\partial^2 F}{\partial x^2}, \qquad F(\theta = 0, x) = \psi(x),$$

and, therefore, we may use the well-known expression for the fundamental solution of the Cauchy problem for the heat equation

$$F(\theta, x) \equiv \psi'(x') = \frac{1}{\sqrt{4\pi\theta}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x-y)^2}{4\theta}\right\} \psi(y) dy$$

To obtain functional form of final transformations (5.3.6), it is necessary to do additional calculations connected with expansion of operational exponents. So, we further consider this question.

Let $\mathcal{A}$ be an algebra with the following properties: for any elements $\widehat{x}$, $\widehat{y}$ from $\mathcal{A}$ their commutator $[\widehat{x}, \widehat{y}] = \widehat{x}\widehat{y} - \widehat{y}\widehat{x}$ belongs to $\mathcal{A}$, and the sum of a infinite convergent series with elements belonging to $\mathcal{A}$ also belongs to $\mathcal{A}$. In what follows we will use hat as the notation of elements of algebra $\mathcal{A}$.

Theorem 5.3.2. *For any $\widehat{x}$, $\widehat{y} \in \mathcal{A}$ the following expansion holds (the Campbell-Baker-Hausdorff (CBH) formula)*

$$\exp\{-\theta\widehat{x}\}\widehat{w}\exp\{\theta\widehat{x}\} = \sum_{n=0}^{\infty} \frac{\theta^n}{n!}\{\widehat{w}, \widehat{x}^n\}, \tag{5.3.8}$$

where

$$\begin{aligned} &\{\widehat{w}, \widehat{x}^0\} \equiv \widehat{w}, \\ &\{\widehat{w}, \widehat{x}\} \equiv [\widehat{w}, \widehat{x}] \equiv \widehat{w}\widehat{x} - \widehat{x}\widehat{w}, \\ &\{\widehat{w}, \widehat{x}^k\} \equiv [\{\widehat{w}, \widehat{x}^{k-1}\}, \widehat{x}], \qquad k = 1, 2, \ldots; \end{aligned} \tag{5.3.9}$$

θ is an arbitrary real constant.

Proof. Consider the function of θ

$$F(\theta) = \exp\{-\theta\widehat{x}\}\widehat{w}\exp\{\theta\widehat{x}\} \tag{5.3.10}$$

Let us expand it in Maclaurin's series. But at first we find

$$\begin{aligned}\frac{\partial F}{\partial\theta} &= \exp\{-\theta\widehat{x}\}[\widehat{w},\widehat{x}]\exp\{\theta\widehat{x}\}\\ \frac{\partial^2 F}{\partial\theta^2} &= \exp\{-\theta\widehat{x}\}\left(-\widehat{x}[\widehat{w},\widehat{x}] + [\widehat{w},\widehat{x}]\widehat{x}\right)\exp\{\theta\widehat{x}\} =\\ &= \exp\{-\theta\widehat{x}\}[\widehat{w},\widehat{x}^2]\exp\{\theta\widehat{x}\}\end{aligned}$$

. . . .

whence follows

$$\begin{aligned}F(\theta) = \sum_{n=0}^{\infty}\frac{\theta^n}{n!}\left(\frac{\partial^n F}{\partial\theta^n}\right)\Bigg|_{\theta=0} &= \widehat{w} + \frac{\theta}{1!}\{\widehat{w},\widehat{x}\} +\\ + \frac{\theta^2}{2!}\{\widehat{w},\widehat{x}^2\} + \ldots &= \sum_{n=0}^{\infty}\frac{\theta^n}{n!}\{\widehat{w},\widehat{x}^n\}.\end{aligned}$$

The theorem is proved.

Theorem 5.3.3. *For any $\widehat{x}$, $\widehat{y} \in \mathcal{A}$, the following identity holds (the formula of Hausdorff)*

$$e^{\widehat{x}}e^{\widehat{y}} = e^{\widehat{z}}, \qquad \widehat{z} \in \mathcal{A}, \tag{5.3.11}$$

where

$$\begin{aligned}&\widehat{z} = \sum_{n=0}^{\infty}\widehat{z}_n,\\ &\widehat{z}_0 = \widehat{y}\\ &\widehat{z}_1 = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}B_n\{\widehat{x},\widehat{y}^n\} = \widehat{x} + \tfrac{1}{2}[\widehat{x},\widehat{y}] + \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\{\widehat{x},\widehat{y}^{2k}\},\\ &\widehat{z}_2 = \tfrac{1}{2}(\widehat{z}_1\partial_{\widehat{y}})\widehat{z}_1,\ldots, \quad \widehat{z}_k = \frac{1}{k}(\widehat{z}_1\,\partial_{\widehat{y}})\widehat{z}_{k-1}, \quad k = 1,2,\ldots,\end{aligned} \tag{5.3.12}$$

where B_n are the Bernouilli numbers:

$$\begin{aligned}&B_0 = 1, \qquad B_1 = -\tfrac{1}{2}, \qquad B_2 = \tfrac{1}{6},\\ &B_3 = B_5 = \ldots = B_{2k+1} = \ldots = 0,\end{aligned} \tag{5.3.13}$$

$B_4 = -\frac{1}{30}, \qquad B_6 = \frac{1}{42}, \ldots;$

$\widehat{z}_1\partial_{\widehat{y}}$ *is the so-called Hausdorff operator which operates in the following manner: for example,*

$$(\widehat{z}_1\partial_{\widehat{y}})F\widehat{y}^2G\widehat{y} = F\widehat{z}_1\widehat{y}G\widehat{y} + F\widehat{y}\widehat{z}_1G\widehat{y} + F\widehat{y}^2G\widehat{z}_1, \tag{5.3.14}$$

where F *and* G *are operator-valued functions independent of* $\widehat{y}$.

Before proceeding to the proof of this theorem, we will first prove three lemmas.

Lemma 5.3.1. *For smooth operator-valued function* $F(\widehat{x})$, *the analog of Taylor expansion holds true*

$$F(\widehat{x}+\widehat{u}) = F(\widehat{x}) + \left(\widehat{u}\partial_{\widehat{x}}\right)F(x) + \tfrac{1}{2}\left(\widehat{u}\partial_{\widehat{x}}\right)^2 F(\widehat{x}) + \cdots \tag{5.3.15}$$

Proof. Since in both parts of equality (5.3.15), function $F(\widehat{x})$ comes linearly, it is sufficient to prove the lemma for the function $\widehat{x}^n$. For $n = 2$ we have

$$(\widehat{x}+\widehat{u})^2 \equiv (\widehat{x}+\widehat{u})(\widehat{x}+\widehat{u}) = \widehat{x}^2 + \widehat{x}\widehat{u} + \widehat{u}\widehat{x} + \widehat{u}^2.$$

From the other hand, if $F(\widehat{x}) = \widehat{x}^2$, we find the same result from (5.3.15)

$$(\widehat{x}+\widehat{u})^2 = \widehat{x}^2 + \left(\widehat{u}\partial_{\widehat{x}}\right)\widehat{x}^2 + \frac{1}{2!}\left(\widehat{u}\partial_{\widehat{x}}\right)^2\widehat{x}^2 = \widehat{x}^2 + \widehat{u}\widehat{x} + \widehat{x}\widehat{u} + \widehat{u}^2.$$

To complete the proof, one has to apply further the method of mathematical induction.

Remark 5.3.2. Analogous expansion can be written in two-dimensional case

$$\begin{aligned} F(\widehat{x}+\widehat{u}, \widehat{y}+\widehat{v}) = F(\widehat{x},\widehat{y}) + \left(\widehat{u}\partial_{\widehat{x}} + \widehat{v}\partial_{\widehat{y}}\right)F(\widehat{x},\widehat{y}) + \\ + \frac{1}{2!}\left(\widehat{u}\partial_{\widehat{x}} + \widehat{v}\partial_{\widehat{y}}\right)^2 F(\widehat{x},\widehat{y}) + \cdots. \end{aligned} \tag{5.3.16}$$

Lemma 5.3.2. *In algebra* $\mathcal{A}$, *the following identities are valid*

$$\left(\widehat{u}\partial_{\widehat{x}}\right)\widehat{e^{x}} = \widehat{e^{x}}\varphi(\widehat{u},\widehat{x}), \tag{5.3.17}$$

$$\left(\widehat{u}\partial_{\widehat{x}}\right)\widehat{e^{x}} = \psi(\widehat{u},\widehat{x})\widehat{e^{x}}, \tag{5.3.18}$$

where

$$\varphi(\widehat{u},\widehat{x}) = \widehat{u} + \frac{1}{2!}\{\widehat{u},\widehat{x}\} + \frac{1}{3!}\{\widehat{u},\widehat{x}^2\} + \ldots = \sum_{n=0}^{\infty} \frac{\{\widehat{u},\widehat{x}^n\}}{(n+1)!} \stackrel{\text{def}}{=} \widehat{p}, \qquad (5.3.19)$$

$$\psi(\widehat{u},\widehat{x}) = \varphi(\widehat{u},-\widehat{x}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!}\{\widehat{u},\widehat{x}^n\} \stackrel{\text{def}}{=} \widehat{q}. \qquad (5.3.20)$$

Proof. It follows as a trivial consequence of (5.3.15) that

$$\left(\widehat{u}\partial_{\widehat{x}}\right) e^{\widehat{x}} = \widehat{u} + \frac{1}{2!}(\widehat{u}\widehat{x} + \widehat{x}\widehat{u}) + \frac{1}{3!}(\widehat{u}\widehat{x}^2 + \widehat{x}\widehat{u}\widehat{x} + \widehat{x}^2\widehat{u}) + \cdots.$$

Setting in this expansion

$$\widehat{u} = [\widehat{w},\widehat{x}] \qquad (5.3.21)$$

we find

$$\left([\widehat{w},\widehat{x}]\partial_{\widehat{x}}\right) e^{\widehat{x}} = [\widehat{w},\widehat{x}] + \frac{1}{2!}[\widehat{w},\widehat{x}^2] + \frac{1}{3!}[\widehat{w},\widehat{x}^3] + \cdots = [\widehat{w},e^{\widehat{x}}] \qquad (5.3.22)$$

Further, we use CBH formula (5.3.8) under $\theta = 1$

$$[\widehat{w},e^{\widehat{x}}] \equiv \widehat{w}e^{\widehat{x}} - e^{\widehat{x}}\widehat{w} \equiv e^{\widehat{x}}\left(e^{-\widehat{x}}\widehat{w}e^{\widehat{x}} - \widehat{w}\right) = e^{\widehat{x}}\left(\{\widehat{w},\widehat{x}\} + \frac{1}{2!}\{\widehat{w},\widehat{x}^2\} + \cdots\right)$$

It yields together with (5.3.21), (5.3.22) formulae (5.3.17), (5.3.19):

$$\left(\widehat{u}\partial_{\widehat{x}}\right) e^{\widehat{x}} = e^{\widehat{x}}\left(\widehat{u} + \frac{1}{2!}\{\widehat{u},\widehat{x}\} + \frac{1}{3!}\{\widehat{u},\widehat{x}^2\} + \cdots\right) = e^{\widehat{x}}\varphi(\widehat{u},\widehat{x}).$$

By noting that

$$[\widehat{w},e^{\widehat{x}}] \equiv \widehat{w}e^{\widehat{x}} - e^{\widehat{x}}\widehat{w} \equiv \left(\widehat{w} - e^{-\widehat{x}}\widehat{w}e^{\widehat{x}}\right)e^{\widehat{x}} =$$

$$= \left(\{\widehat{w},\widehat{x}\} - \frac{1}{2!}\{\widehat{w},\widehat{x}^2\} + \frac{1}{3!}\{\widehat{u},\widehat{x}^2\} \mp \cdots\right)e^{\widehat{x}}$$

one can easily show the validity of (5.3.18), (5.3.20).

Lemma 5.3.3. *The series* (5.3.19) *is reversible, the reversion having the form*

$$\widehat{u} = \sum_{n=0}^{\infty} \frac{B_n}{n!}\{\widehat{p},\widehat{x}^n\}, \qquad (5.3.23)$$

where B_n are Bernoulli numbers (5.3.13).

Proof. Starting from (5.3.19), we can successively write

$$\widehat{p} = \widehat{u} + \frac{1}{2!}\{\widehat{u}, \widehat{x}\} + \frac{1}{3!}\{\widehat{u}, \widehat{x}^2\} + \cdots + \frac{1}{(n+1)!}\{\widehat{u}, \widehat{x}^n\} + \cdots$$

$$\{\widehat{p}, \widehat{x}\} = \{\widehat{u}, \widehat{x}\} + \frac{1}{2!}\{\widehat{u}, \widehat{x}^2\} + \cdots + \frac{1}{n!}\{\widehat{u}, \widehat{x}^n\} + \cdots \tag{5.3.24}$$

$$\{\widehat{p}, \widehat{x}^2\} = \{\widehat{u}, \widehat{x}^2\} + \frac{1}{2!}\{\widehat{u}, \widehat{x}^3\} + \cdots + \frac{1}{(n-1)!}\{\widehat{u}, \widehat{x}^n\} + \cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\{\widehat{p}, \widehat{x}^{n-1}\} = \{\widehat{u}, \widehat{x}^{n-1}\} + \frac{1}{2!}\{\widehat{u}, \widehat{x}^n\} + \cdots$$

$$\{\widehat{p}, \widehat{x}^n\} = \{\widehat{u}, \widehat{x}^n\} + \cdots$$

Now we make the sum $\sum_{n=0}^{\infty} b_n\{\widehat{p}, \widehat{x}^n\}$ and require the coefficients $b_n\,(n = 0, 1, \ldots; b_0 = 1)$ to satisfy the relations

$$\frac{1}{(n+1)!} + \frac{b_1}{n!} + \cdots + \frac{b_{n-1}}{2!} + b_n = 0. \tag{5.3.25}$$

All terms in the right-hand side of our sum vanish except the addend u. Thus we get

$$\widehat{u} = \sum_{n=0}^{\infty} b_n\{\widehat{p}, \widehat{x}^n\}. \tag{5.3.26}$$

Equations (5.3.25) should be considered as recurrent relations for determination of coefficients $b_1, b_2, \ldots$. These relations can be written as a single identity

$$\left(\sum_{n=0}^{\infty} \frac{t^n}{(n+1)!}\right)\left(\sum_{n=0}^{\infty} b_n t^n\right) = 1.$$

But as it is well known

$$\sum_{n=0}^{\infty} \frac{t^n}{(n+1)!} = \frac{e^t - 1}{t}.$$

Therefore, numbers b_n are just the coefficients of the expansion of the function $t/(e^t - 1)$ in a power series. So we have

$$\sum_{n=0}^{\infty} b_n t^n = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n,$$

(B_n are Bernouilli numbers (5.3.13)) whence follows

$$b_n = \frac{B_n}{n!}. \tag{5.3.27}$$

From (5.3.26) and (5.3.27) one obtains the formula (5.3.23).

Remark 5.3.3. Similarly one can reverse the series (5.3.20). But since $\widehat{q} = \varphi(\widehat{u}, -\widehat{x})$ we get from (5.3.23)

$$\widehat{u} = \sum_{n=0}^{\infty} \frac{(-1)^n B_n}{n!} \{\widehat{q}, \widehat{x}^n\}. \tag{5.3.28}$$

Now we are ready to prove Theorem 5.3.3. Let $\widehat{x}$ is changed to $\widehat{x} + \theta\widehat{u}$ and $\widehat{y}$ is changed to $\widehat{y} - \theta\widehat{v}$, where $\widehat{u}$ and $\widehat{v}$ are arbitrary, in such a way that

$$\widehat{z}(\widehat{x} + \theta\widehat{u}, \widehat{y} - \theta\widehat{v}) = \widehat{z}(\widehat{x}, \widehat{y}) \tag{5.3.29}$$

Then according to Lemma 5.3.1 using (5.3.16) we get identity

$$\left(\widehat{u}\partial_{\widehat{x}}\right) \widehat{z} = \left(\widehat{v}\partial_{\widehat{y}}\right) \widehat{z}. \tag{5.3.30}$$

The condition (5.3.29) means that

$$e^{\widehat{x}+\theta\widehat{u}} e^{\widehat{y}-\theta\widehat{v}} = \left(e^{\widehat{x}} + \theta \left(\widehat{u}\partial_{\widehat{x}}\right) e^{\widehat{x}} + \cdots\right) \left(e^{\widehat{y}} - \theta \left(\widehat{v}\partial_{\widehat{y}}\right) e^{\widehat{y}} + \cdots\right) = e^{\widehat{x}} e^{\widehat{y}}$$

And together with Lemmas 5.3.1 and 5.3.2, it yields

$$\varphi(\widehat{u}, \widehat{x}) = \psi(\widehat{v}, \widehat{y}) \tag{5.3.31}$$

Since $\widehat{u}$ is arbitrary, we may choose it equal to $\widehat{x}$, from which we may find

$$\left(\widehat{x}\partial_{\widehat{x}}\right) \widehat{z} = \left(\widehat{v}\partial_{\widehat{y}}\right) \widehat{z}. \tag{5.3.32}$$

and

$$\varphi(\widehat{x}, \widehat{x}) \equiv \widehat{x} = \psi(\widehat{v}, \widehat{y}). \tag{5.3.33}$$

Having used formulae (5.3.20), (5.3.28) we obtain

$$\widehat{v} = \sum_{n=0}^{\infty} \frac{(-1)^n B_n}{n!} \{\widehat{x}, \widehat{y}^n\} \tag{5.3.34}$$

Let us expand $\widehat{z}$ as a series

$$\widehat{z} = \widehat{z}_0 + \widehat{z}_1 + \widehat{z}_2 + \cdots, \tag{5.3.35}$$

where $\widehat{z}_k$ is a polynomial containing k factors of $\widehat{x}$. Since $\widehat{v}$ contains $\widehat{x}$ only to the first power (see (5.3.34)), we can rewrite equality (5.3.32) as

$$\widehat{z}_1 + 2\widehat{z}_2 + 3\widehat{z}_3 + \cdots = \left(\widehat{v}\partial_{\widehat{y}}\right) (\widehat{z}_0 + \widehat{z}_1 + \widehat{z}_2 + \cdots)$$

and after equating coefficients of the terms with equal power of $\hat{x}$, we get identities

$$\hat{z}_1 = \left(\hat{v}\partial_{\hat{y}}\right)\hat{z}_0, \qquad 2\hat{z}_2 = \left(\hat{v}\partial_{\hat{y}}\right)\hat{z}_1, \qquad 3\hat{z}_3 = \left(\hat{v}\partial_{\hat{y}}\right)\hat{z}_2, \cdots.$$

To find $\hat{z}_0$ we set in (5.3.35) and in (5.3.11) $\hat{x} = O$. It immediately yields

$$\hat{z}\Big|_{\hat{x}=0} = \hat{y} = \hat{z}_0.$$

Thus Theorem 5.3.3 is proved.

Remark 5.3.4. In the same way, using the substitution $\hat{v} = \hat{y}$, we find

$$\left(\hat{y}\partial_{\hat{y}}\right)\hat{z} = \left(\hat{u}\partial_{\hat{x}}\right)\hat{z}, \qquad \psi(\hat{y},\hat{y}) \equiv \hat{y} = \varphi(\hat{u},\hat{x}). \tag{5.3.36}$$

Using (5.3.19), (5.3.23) we obtain another equivalent expression for $\hat{z}$:

$$\hat{z} = \hat{x} + \hat{r}_1 + \hat{r}_2 + \cdots, \tag{5.3.37}$$

$$\hat{r}_1 = \sum_{n=0}^{\infty} \frac{B_n}{n!}\{\hat{y},\hat{x}^n\} = \hat{y} - \tfrac{1}{2}[\hat{y},\hat{x}] + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\{\hat{y},\hat{x}^{2k}\},$$

$$\hat{r}_2 = \tfrac{1}{2}(\hat{r}_1\partial_{\hat{x}})\hat{r}_1, \qquad \hat{r}_3 = \tfrac{1}{3}(\hat{r}_1\partial_{\hat{x}})\hat{r}_2, \ldots.$$

Consequence 5.3.1. When operators $\hat{x}$ and $\hat{y}$ satisfy commutation relations

$$[\hat{x},[\hat{x},\hat{y}]] = [\hat{y},[\hat{x},\hat{y}]] = 0, \tag{5.3.38}$$

formulae (5.3.11)–(5.3.13), (5.3.37) result in the well-known identity

$$e^{\hat{x}}e^{\hat{y}} = e^{\hat{x}+\hat{y}+\frac{1}{2}[\hat{x},\hat{y}]}, \tag{5.3.39}$$

which is widely used in quantum mechanics and in the theory of coherent states [151,164].

Remark 5.3.5. The problem of finding an explicit expression for $\hat{z} = \ln\left(e^{\hat{x}}e^{\hat{y}}\right)$ was first attacked by Campbell in 1898 and was soon thereafter followed by investigations of Baker and of Hausdorff. The author last named found an expression for $\hat{z}$ in terms of repeated commutators of $\hat{x}$ and $\hat{y}$ [117]. Hausdorff results were expounded in [38,146,189,206,208] and here we used these works. More recently Dynkin [45] found explicit form of coefficients of the commutator series for $\hat{z}$ but this result was quite unwieldy and therefore is not used in applications, while the Hausdorff formulae (5.3.11), (5.3.12), (5.3.37) give, as we see it, an effective way of finding $\hat{z}$.

Below we consider an example of using the Hausdorff formula to find nonlocal transformations of invariance of the Dirac equation.

$$(i\gamma_\nu \partial^\nu - m)\psi(x) = 0 \tag{5.3.40}$$

where m is a constant (mass of particle), γ_ν are Dirac matrices written in (2.1.2).

As it is well known the Dirac equation (5.3.40) is invariant with respect to the Poincare group P(1,3) and this group is the maximal one in the sense of Lie (here we do not consider transformations which mix up ψ and $\overline{\psi}$). But the Poincare group does not exhaust all symmetry properties of the Dirac equation. It was shown in [64] by means of non-Lie method the duality of the spacetime symmetry of the Dirac equation. This duality is stipulated by the existence of two IA: the known one AP(1,3) with basis elements (2.1.11)), and the algebra AF(1,3) with basis elements

$$\begin{aligned}
&P_0 = i\partial_0, \qquad \mathcal{P}_0 = H = \gamma_0\gamma_a P_a + \gamma_0 m, \qquad P_a = -i\partial_a,\\
&J_{ab} = x_a P_b - x_a P_a + S_{ab} \equiv \epsilon_{abc} J_c,\\
&R_a = x_0 P_a - \tfrac{1}{2}(\mathcal{P}_0 x_a + x_a \mathcal{P}_0),
\end{aligned} \tag{5.3.41}$$

which satisfy commutation rules

$$\begin{aligned}
&[\mathcal{P}_0, P_0] = [\mathcal{P}_0, P_a] = [P_a, P_b] = [P_0, P_a] = [\mathcal{P}_0, R_a] = 0,\\
&[P_0, R_a] = iP_a, \qquad [P_a, R_b] = i\delta_{ab}\mathcal{P}_0,\\
&[J_a, J_b] = i\epsilon_{abc}J_c, \qquad [J_a, R_b] = -i\epsilon_{abc}R_c,\\
&[R_a, R_b] = -i\epsilon_{abc}J_c.
\end{aligned} \tag{5.3.42}$$

As operators $\mathcal{P}_0$ and R_a are of non-Lie type, to find their group action, we shall use the formulae (5.3.6).

Theorem 5.3.4 [99, 189]. *The Dirac equation* (5.3.40) *is invariant under Galilean transformations*

$$\begin{aligned}
&x_0' = x_0, \qquad x_1' = x_1 + \theta x_0, \qquad x_2' = x_2, \quad x_3' = x_3,\\
&\psi'(x') = \exp\left\{ i\theta\left[(1 - \tfrac{1}{2}\theta \,\mathrm{cth}\, \tfrac{1}{2}\theta) x_0 P_1 + \tfrac{1}{2}(Hx_1 + x_1 H + \theta x_0 H) \right] \right\}\psi(x).
\end{aligned} \tag{5.3.43}$$

generated by the operator R_1 *from* AF(1, 3) (5.3.41).

Proof. At first we convince ourselves that operator

$$R_1 = x_0 P_1 - \tfrac{1}{2}(Hx_1 + x_1 H) = -i\big(x_0\partial_1 - \tfrac{1}{2}i(Hx_1 + x_1 H)\big) \tag{5.3.44}$$

satisfies the invariance condition (5.3.3), and to do it we rewrite Equation (5.3.40) into the equivalent form

$$L\psi \equiv (P_0 - H)\psi = 0, \qquad H = \gamma_0\gamma_a P_a + \gamma_0 m. \tag{5.3.45}$$

It is easy to find

$$[L, R_1] = iP_1 + \tfrac{1}{2}[H^2, x_1]$$

and using the identities

$$H^2 \equiv (\gamma_0\gamma_a P_a + \gamma_0 m)^2 = P_a^2 + m^2, \qquad [H^2, x_1] = -2iP_1$$

we get

$$[L, R_1] = 0.$$

So, the invariance condition (5.3.3) is satisfied and one can use formulae (5.3.6) for finding the group action of the operator (5.3.44). According to (5.3.6) we have

$$x'_\mu = e^{\theta x_0 \partial_1} x_\mu e^{-\theta x_0 \partial_1}, \tag{5.3.46}$$

$$\psi'(x') = \exp\{\theta x_0\partial_1\} \exp\left\{-\theta[x_0\partial_1 - \tfrac{1}{2}i(Hx_1 + x_1 H)]\right\} \psi(x). \tag{5.3.47}$$

To obtain the functional form of transformations (5.3.46), (5.3.47) we apply formulae (5.3.8), (5.3.11)–(5.3.13).

From (5.3.46) and (5.3.8) we find

$$x'_\mu = x_\mu - \theta[x_\mu, x_0\partial_1] = x_\mu + \theta x_0 \delta_{\mu 1}. \tag{5.3.48}$$

To use expansion (5.3.11) let us introduce notations

$$\widehat{x} = \theta x_0 \partial_1, \qquad \widehat{y} = -\widehat{x} + \tfrac{1}{2} i\theta (Hx_1 + x_1 H) \tag{5.3.49}$$

Then we successively calculate

$$\{\widehat{x}, \widehat{y}^0\} \equiv \widehat{x},$$

$$\{\widehat{x}, \widehat{y}\} \equiv [\widehat{x}, \widehat{y}] = i\theta^2 x_0 H,$$

$$\{\widehat{x}, \widehat{y}^2\} = [\{\widehat{x}, \widehat{y}\}, \widehat{y}] = \theta^2 \widehat{x},$$

....

$$\{\widehat{x}, \widehat{y}^{2n}\} = \theta^{2n}\widehat{x}, \qquad n = 1, 2, \ldots,$$

whence follows that the series for $\widehat{z}_1$ from (5.3.12) takes the form

$$\begin{aligned} \widehat{z}_1 &= \widehat{x} + \frac{1}{2}[\widehat{x}, \widehat{y}] + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\{\widehat{x}, \widehat{y}^{2k}\} = \\ &= \frac{1}{2} i\theta^2 x_0 H + \left(\frac{\theta}{2} \operatorname{cth} \frac{\theta}{2}\right)\widehat{x}, \qquad |\theta| < 2\pi. \end{aligned} \tag{5.3.50}$$

Here we used the known expansion

$$\frac{\theta}{2}\,\mathrm{cth}\,\frac{\theta}{2} = \sum_{k=0}^{\infty} \frac{B_{2k}}{(2k)!}\theta^{2k}, \qquad |\theta| < 2\pi. \tag{5.3.51}$$

To find the second term in the expansion of (5.3.12) we note that

$$(\widehat{z}_1\,\partial_{\widehat{y}})\{\widehat{x},\widehat{y}\} = \{\widehat{x},\widehat{z}_1\} = 0,$$

$$(\widehat{z}_1\,\partial_{\widehat{y}})\{\widehat{x},\widehat{y}^2\} = [[\widehat{x},\widehat{z}_1]\widehat{y}] + [[\widehat{x},\widehat{y}]\widehat{z}_1] = 0,$$

$$(\widehat{z}_1\,\partial_{\widehat{y}})\{\widehat{x},\widehat{y}^3\} = \theta(\widehat{z}_1,\partial_{\widehat{y}})\{\widehat{x},\widehat{y}\} = 0,$$

. . . .

and therefore

$$\widehat{z}_2 = \frac{1}{2}\left(\widehat{z}_1\partial_{\widehat{y}}\right)\widehat{z}_1 = \frac{1}{2}\widehat{z}_1\partial_{\widehat{y}}\left(\widehat{x} + \frac{1}{2}[\widehat{x},\widehat{y}] + \sum_{k=0}^{\infty}\frac{B_{2k}}{(2k)!}\{\widehat{x},\widehat{y}^{2k}\}\right) = 0$$

and hence $\widehat{z}_3 = \widehat{z}_4 = \cdots = 0$. Finally we obtain the following expression for $\widehat{z}$:

$$\widehat{z} = \widehat{y} + \widehat{z}_1 = \widehat{y} + \frac{i}{2}\theta^2 x_0 H + \left(\frac{\theta}{2}\,\mathrm{cth}\,\frac{\theta}{2}\right)\widehat{x}, \qquad |\theta| < 2\pi,$$

or in the terms of the original notations (5.3.49)

$$\widehat{z} = \left(\frac{\theta}{2}\,\mathrm{cth}\,\frac{\theta}{2} - 1\right)\theta x_0\partial_1 + \frac{1}{2}i\theta(Hx_1 + x_1 H) + \frac{1}{2}i\theta^2 x_0 H.$$

So the theorem is proved.

Remark 5.3.6. One can make sure in the validity of Theorem 5.3.4 straightforwardly. Indeed, under the Galilei transformations (5.3.48) we have

$$P_0' = i\frac{\partial}{\partial x_0'} = P_0 + \theta P_1, \qquad P_a' = -i\frac{\partial}{\partial x_a'} = P_a. \tag{5.3.52}$$

For $\psi'(x')$ (5.3.43) we find

$$\psi'(x') = \psi(x) + \tfrac{1}{2}i\theta(Hx_1 + x_1 H)\psi(x) + \cdots \tag{5.3.53}$$

After substitution of (5.3.52), (5.3.53) into the primed Equation (5.3.40)

$$(P_0' - H')\psi'(x') = 0$$

and retaining terms linear in θ, we successively obtain

$$(P_0 + \theta P_1)\left(\psi + \tfrac{1}{2}i\theta(Hx_1 + x_1H)\psi\right) - H\psi - \tfrac{1}{2}i\theta H(Hx_1 + x_1H)\psi = 0,$$

$$(P_0 - H)\psi + \theta\left[\tfrac{1}{2}i(Hx_1 + x_1H)P_0\psi + P_1\psi - \tfrac{1}{2}i(H^2x_1 + Hx_1H)\psi\right] = 0,$$

$$(P_0 - H)\psi + \theta\left[\tfrac{1}{2}i(Hx_1P_0 + x_1HP_0 - H^2x_1 - Hx_1H) + P_1\psi\right] = 0.$$

Since on the manifold of solutions of the Dirac equation (5.3.40), we have $P_0\psi = H\psi$ then the latter equality we can write as

$$\left[\tfrac{1}{2}i(x_1H^2 - H^2x_1) + P_1\right]\psi \equiv 0,$$

which proves our statement.

Remark 5.3.7. As follows from the proof of Theorem 5.3.4 the formula (5.3.43) for $\psi'(x')$ is valid when $|\theta| < 2\pi$. This restriction is stipulated by the range of convergence of the series (5.3.51). But for real values of the parameter, the series (5.3.51) is convergent everywhere in R. This means that there is no restriction on the value of the parameter which can be treated as the velocity of an inertial reference frame, while in the case of the Lorentz transformation a restriction exists: the speed of light.

Remark 5.3.8. Theorem 5.3.4 can be easily generalized on equations of arbitrary spin. These equations should have the form

$$P_0\psi = H\psi \tag{5.3.54}$$

and the Hamiltonian H must satisfy the condition

$$[H^2, x_a] = -2iP_a. \tag{5.3.55}$$

(A wide class of such equations for particles of arbitrary spin is described in [76,82,87*].)

Formulae (5.3.43) are written in the form suitable for any equation of the type (5.3.54), (5.3.55).

In conclusion, we note that some other applications of CBH and Hausdorff formulae are given in Sections 5.4, 5.5, 5.8–5.10 and in Appendix 3.

5.4 *Lie-Backlund symmetry of the Dirac equation*

Following [75, 82, 211] we shall study the symmetry properties of the Dirac equation (5.3.40) in the class of the first-order matrix-differential operators. The investigation will be done in two different approaches: non-Lie and Lie-Backlund.

Theorem 5.4.1 [74,82]. *The Dirac equation* (5.3.40) *is invariant under an eight-dimensional Lie algebra defined over the field of real numbers. The basis elements of this algebra have the form*

$$\begin{aligned}\widehat{\Sigma}_{\mu\nu} &= \tfrac{1}{2}im[\gamma_\mu,\gamma_\nu] + i(1-i\gamma_4)(\gamma_\mu P_\nu - \gamma_\nu P_\mu),\\ \widehat{\Sigma}_0 &= I, \qquad \widehat{\Sigma}_1 = \gamma_4 m - i(1-i\gamma_4)\gamma_\nu P^\nu,\end{aligned} \tag{5.4.1}$$

where $\gamma_4 = \gamma_0\gamma_1\gamma_2\gamma_3$ *and* I *is a unit matrix. In the case* $m \neq 0$ *the algebra* (5.4.1) *is isomorphic to the Lie algebra of the group* U(2)⊗U(2), *while for* $m = O$ *the operators* (5.4.1) *form an abelian algebra.*

Proof. The validity of the theorem can most simply be seen by direct verification. One has

$$\begin{aligned}&\left[\widehat{\Sigma}_{\mu\nu}, L\right] = i(\gamma_4 P_\nu - \gamma_\nu P_\mu)L, \qquad (L \equiv \gamma_\nu P^\nu - m),\\ &\left[\widehat{\Sigma}_1, L\right] = -i2\gamma_4\gamma_\nu P^\nu L, \qquad \left[\widehat{\Sigma}_0, L\right] = 0,\\ &\left[\widehat{\Sigma}_{\mu\nu}, \widehat{\Sigma}_{\lambda\sigma}\right] = 2im^2\left(g_{\mu\sigma}\widehat{\Sigma}_{\nu\lambda} + g_{\nu\lambda}\widehat{\Sigma}_{\mu\sigma} - g_{\mu\lambda}\widehat{\Sigma}_{\nu\sigma} - g_{\nu\sigma}\widehat{\Sigma}_{\mu\lambda}\right),\\ &\left[\widehat{\Sigma}_1, \widehat{\Sigma}_0\right] = \left[\widehat{\Sigma}_1, \widehat{\Sigma}_{\mu\nu}\right] = \left[\widehat{\Sigma}_0, \widehat{\Sigma}_{\mu\nu}\right] = 0,\end{aligned} \tag{5.4.2}$$

whence it is evident that operators (5.4.1) satisfy the invariance condition (10).

According to (5.4.2), in the case $m = 0$ the operators (5.4.1) commute. If $m \neq 0$ then, setting

$$\Sigma_{\mu\nu} = \frac{1}{m}\widehat{\Sigma}_{\mu\nu}, \qquad \Sigma_0 = \widehat{\Sigma}_0, \qquad \Sigma_1 = \frac{1}{m}\widehat{\Sigma}_1 \tag{5.4.3}$$

from (5.4.2) we find that operators $\Sigma_{\mu\nu}$, Σ_0, Σ_1 form the algebra

$$\begin{aligned}&[\Sigma_{\mu\nu}, \Sigma_{\lambda\sigma}] = 2i\left(g_{\mu\sigma}\Sigma_{\nu\lambda} + g_{\nu\lambda}\Sigma_{\mu\sigma} - g_{\mu\lambda}\Sigma_{\nu\sigma} - g_{\nu\sigma}\Sigma_{\mu\lambda}\right),\\ &[\Sigma_1, \Sigma_0] = [\Sigma_1, \Sigma_{\mu\nu}] = [\Sigma_0, \Sigma_{\mu\nu}] = 0.\end{aligned} \tag{5.4.4}$$

The algebra (5.4.4) is isomorphic to the Lie algebra of the group U(2)⊗U(2). This isomorphism can be established by the following relations:

$$\Sigma_{ab} \leftrightarrow i\epsilon_{abc}Q_c, \qquad \Sigma_{0a} \leftrightarrow iQ_{3+a}, \qquad \Sigma_0 \leftrightarrow iQ_7, \quad \Sigma_1 \leftrightarrow iQ_8;$$

$$[Q_a, Q_b] = -[Q_{3+a}, Q_{3+b}] = -\epsilon_{abc}Q_c$$

$$[Q_{3+a}, Q_b] = \epsilon_{abc}Q_{3+c} \qquad [Q_7, Q_8] = [Q_8, Q_A] = 0 \qquad A = \overline{1,8}.$$

Over the field of real numbers all elements of the algebra (5.4.3) are linearly independent. In order to see this, it suffices to subject the operators (5.4.3) to the transformation

$$\Sigma_{\mu\nu} \to \Sigma'_{\mu\nu} = V\Sigma_{\mu\nu}V^{-1} = \tfrac{1}{2}i[\gamma_\mu, \gamma_\nu],$$
$$\Sigma_0 \to \Sigma'_0 = V\Sigma_0 V^{-1} = 1 \qquad \Sigma_1 \to \Sigma'_1 = V\Sigma_1 V^{-1} = \gamma_4,$$

where

$$V = \exp\left\{-\frac{1}{2m}(1 - i\gamma_4)\gamma_\nu P^\nu\right\} \equiv 1 - \frac{1}{2m}(1 - i\gamma_4)\gamma_\nu P^\nu.$$

The theorem is proved.

Since operators (5.4.1) do not belong to the class of first-order differential operators of the form (7), they correspond to nonlocal symmetry of the Dirac equation (5.3.40). In order to find the group action of the operators (5.4.1) we shall use formulae (5.3.6). Since $\xi^\mu(x) = 0$ for all operators (5.4.1) it means that $x'_\mu = x_\mu$. For operator $\theta_a\Sigma_{0a}$ (θ_a are arbitrary constants) one successively finds

$$\begin{aligned}\psi'(x') &= \exp\{-i\Sigma_{0a}\theta_a\}\psi(x) = \\ &= \left[\sum_{k=0}^{\infty}\frac{1}{(2k)!}\left(i\Sigma_{0a}\theta_a\right)^{2k} + \sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\left(-i\Sigma_{0a}\theta_a\right)^{2k+1}\right]\psi(x) = \qquad (5.4.5)\\ &= \left[\sum_{k=0}^{\infty}\frac{\theta^{2k}}{(2k)!}+\frac{1}{\theta}\sum_{k=0}^{\infty}\frac{\theta^{2k+1}}{(2k+1)!}\Big(\gamma_0\gamma_a+\frac{1}{m}(1-i\gamma_4)(\gamma_0P_a-\gamma_aP_0)\Big)\theta_a\right]\psi(x) = \\ &= (\operatorname{ch}\theta + \frac{\operatorname{sh}\theta}{\theta}\gamma_0\gamma_a\theta_a)\psi(x) - \frac{i}{\theta m}\operatorname{sh}\theta(1 - i\gamma_4)\left(\gamma_0\frac{\partial\psi}{\partial x_a} + \gamma_a\frac{\partial\psi}{\partial x_0}\right)\theta_a\psi(x),\end{aligned}$$

where $\theta \equiv \left(\theta_1^2 + \theta_2^2 + \theta_3^2\right)^{1/2}$. Here we have used the identity

$$\left(i\Sigma_{0a}\theta_a\right)^2 = \theta^2.$$

Similarly one obtains

$$\begin{aligned}\psi'(x') &= \exp\left\{-\tfrac{1}{2}i\epsilon_{abc}\Sigma_{ab}\beta_c\right\}\psi(x) = \qquad (5.4.6)\\ &= \left[\cos\beta - \frac{1}{2\beta}\sin\beta\epsilon_{abc}\gamma_a\gamma_b\beta_c + \frac{1}{m\beta}\sin\beta(1 - i\gamma_4)\epsilon_{abc}\gamma_aP_b\beta_c\right]\psi(x),\end{aligned}$$

$$\psi'(x') = \exp\{-\Sigma_1\alpha\}\psi(x) = \left[\operatorname{ch}\alpha + i\gamma_4\operatorname{sh}\alpha + \frac{1}{m}\operatorname{sh}\alpha(1 - i\gamma_4)\gamma_\nu P^\nu\right]\psi(x),$$

where β_a, α are arbitrary constants, $\beta = (\beta_1^2 + \beta_2^2 + \beta_3^2)^{1/2}$.

The principal difference of the transformations (5.4.5), (5.4.6) from Lorentz transformations for the Dirac spinor $\psi(x)$ is that the function $\psi'(x')$ depends not only on $\psi(x)$ (and parameters of the transformation) but also on the derivatives $\partial\psi/\partial x_\mu$.

2. We will show that operators (5.4.1) can be obtained within the framework of Lie-Backlund approach.

As is known [161], the Lie-Backlund symmetry operators are looked for in the form

$$X = \eta^\sigma\left(x, \psi^\alpha, \psi^\alpha_{\mu_1}, \psi^\alpha_{\mu_1\mu_2}, \ldots\right)\frac{\partial}{\partial\psi^\sigma} + \\ + \sum_{\substack{0\le\mu_k\le 3,\\ k\ge 1}} (D_{\mu_1}D_{\mu_2}\ldots D_{\mu_k})\eta^\sigma\frac{\partial}{\partial\psi^\sigma_{\mu_1\mu_2\ldots\mu_k}}, \tag{5.4.7}$$

where

$$D_\mu = \frac{\partial}{\partial x^\mu} + \psi^\sigma_\mu\frac{\partial}{\partial\psi^\sigma} + \cdots + \psi^\sigma_{\mu\mu_1\cdots\mu_k}\frac{\partial}{\partial\psi^\sigma_{\mu_1\ldots\mu_k}} + \cdots,$$

$$\psi^\sigma_\mu \equiv \frac{\partial\psi^\sigma}{\partial x^\mu}, \qquad \psi^\sigma_{\mu\nu} \equiv \frac{\partial^2\psi^\sigma}{\partial x^\mu\partial x^\nu}, \ldots .$$

The problem of finding Lie-Backlund symmetry is the problem of finding functions $\eta(x, \psi^\alpha, \psi^\alpha_\mu, \ldots)$. The functions η^σ are found from the condition of invariance.

$$X[L]\Big|_{[L]=0} = 0, \tag{5.4.8}$$

where X is given in (5.4.7); $[L] = 0$ denotes the manifold of solutions of the equation and all its differential consequences. It is rather difficult (if not impossible) to solve the problem in such general formulation without any restrictions on the coordinates η^σ of IFO (5.4.7). The needed calculations are enormous even in the simplest cases. Besides that one has to study additionally the algebraic properties of the found operators, which is rather difficult by itself.

Below we apply the Lie-Backlund method to the study of symmetry properties of the Dirac equation (5.3.40). Following [211] we rewrite it as eight-component system of PDEs

$$i\gamma_\mu\frac{\partial\psi}{\partial x_\mu} - m\psi = 0, \\ i\frac{\partial\overline{\psi}}{\partial x_\mu}\gamma_\mu + m\overline{\psi} = 0, \tag{5.4.9}$$

or in equivalent form

$$i\Gamma_\mu\partial^\mu\phi + m\phi = 0, \\ \Gamma_4\Gamma_5\phi + \Gamma_2\phi^* = 0, \tag{5.4.10}$$

where $\phi = \phi(x)$ is an eight-component spinor, $\Gamma_\mu = \begin{pmatrix}\gamma_\mu & 0\\ 0 & \gamma_\mu\end{pmatrix}$,

$$\Gamma_4 = \begin{pmatrix}0 & \gamma_4\\ \gamma_\mu & 0\end{pmatrix}, \quad \Gamma_5 = i\begin{pmatrix}0 & \gamma_4\\ -\gamma_\mu & 0\end{pmatrix}, \quad \Gamma_6 = \begin{pmatrix}\gamma_4 & 0\\ 0 & -\gamma_4\end{pmatrix} \tag{5.4.11}$$

We look for the Lie-Backlund symmetry operators (5.4.7) setting $\eta = A_\mu \phi_\mu + F(\phi)$, A_μ are 8-column constant matrices.

Theorem 5.4.2 [211]. *The basis of Lie-Backlund symmetry operators* (5.4.7) *under* $\eta = A_\mu\phi_\mu + F(\phi)$ *of the Dirac equation* (5.4.10) *is given by the following operators*

$$
\begin{aligned}
&I = \phi, \qquad \widehat{Q}_a = Q_a\phi\\
&P_\mu^{(0)} = \phi_\mu, \qquad P_\mu^{(a)} = Q_a P_\mu^{(0)};\\
&\Sigma_{\mu\nu}^{(0)} = i(I - i\widetilde{\Gamma}_4)(\Gamma_\mu\phi_\nu - \Gamma_\nu\phi_\mu) - 2imG_{\mu\nu}\phi,\\
&\Sigma_{\mu\nu}^{(a)} = Q_a\Sigma_{\mu\nu}^{(0)};\\
&\Sigma_\mu^{(0)} = \widetilde{\Gamma}_4\phi_\mu + im\Gamma_\mu\Gamma_4\phi, \qquad \Sigma_\mu^{(a)} = Q_a\Sigma_\mu^{(0)},
\end{aligned}
\tag{5.4.12}
$$

where

$$
Q_1 = \Gamma_4\Gamma_5, \qquad Q_2 = i\Gamma_5\Gamma_6, \qquad Q_3 = i\Gamma_4\Gamma_6, \qquad \widetilde{\Gamma}_4 = \Gamma_0\Gamma_1\Gamma_2\Gamma_3,
$$
$$
G_{\mu\nu} = \tfrac{1}{4}(\Gamma_\mu\Gamma_\nu - \Gamma_\nu\Gamma_\mu).
$$

Proof. Since the calculations are involved in the present case, we shall state them in a sketchy manner, for the sake of brevity. From the condition of invariance (5.4.8) with $\eta^\sigma = (A_\mu\phi_\mu)^\sigma + F^\sigma(\phi)$ we get the defining equations

$$
\begin{aligned}
&(\Gamma_\mu B\phi_\mu + \Gamma_\mu A_\nu\phi_{\mu\nu} - im\eta)\Big|_{[i\Gamma_\mu\phi_\mu + m\phi = 0]} = 0,\\
&\Gamma_4\Gamma_5\eta + \Gamma_2\eta^* = 0,
\end{aligned}
\tag{5.4.13}
$$

where $B = \left(B_\beta^\alpha \equiv \partial\eta^\alpha/\partial\phi^\beta;\ \alpha,\beta = \overline{0,7}\right)$; $[i\Gamma_\mu\phi_\mu + m\phi = 0]$ is the totality of all differential consequences of Equation (5.4.10). Taking into account differential consequences we obtain from (5.4.13)

$$
\begin{aligned}
&\Gamma_0 B(-\Gamma_0\Gamma_a\phi_a + im\Gamma_0\phi) + \Gamma_a B\phi_a + \Gamma_0 A_0(\phi_{aa} - m^2\phi) +\\
&\quad + (\Gamma_0 A_a + \Gamma_a A_0)(-\Gamma_0\Gamma_b\phi_{ba} + im\Gamma_0\phi_a) + \Gamma_0 A_b\phi_{ab} - im\eta = 0,\\
&\Gamma_4\Gamma_5\eta + \Gamma_2\eta^* = 0,
\end{aligned}
\tag{5.4.14}
$$

Decomposing the first equation of (5.4.14) into independent variables ϕ_{ab}, one finds the defining equations for matrices A_μ:

$$
-\Gamma_0 A_a\Gamma_0\Gamma_b - \Gamma_a A_0\Gamma_0\Gamma_b + \Gamma_a A_b + \Gamma_b\Gamma_a - (\Gamma_0 A_b + \Gamma_b A_0)\Gamma_0\Gamma_a = 0, \qquad a \neq b
$$
$$
\Gamma_0 A_0 - (\Gamma_0 A_a + \Gamma_a A_0) + \Gamma_a A_a = 0, \text{ (no sum over } a),
$$

whence follows

$$A_\mu = \begin{pmatrix} A_\mu^{(1)} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & A_\mu^{(2)} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ A_\mu^{(3)} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & A_\mu^{(4)} \end{pmatrix},$$

where

$$A_\mu^{(k)} = a^{\mu(k)} I + b_\alpha^{\mu(k)} \gamma^\alpha + C_{\alpha\beta}^{\mu(k)} \sigma^{\alpha\beta} + d_\alpha^{\mu(k)} \gamma_4 \gamma^\alpha + e^{\mu(k)} \gamma_4,$$

$$b_0^{0(k)} = -b_a^{a(k)}, \qquad b_\nu^{\mu(k)} = -b_\mu^{\nu(k)}, \qquad \mu \neq \nu$$

$$d_0^{0(k)} = -d_a^{a(k)}, \qquad b_\nu^{\mu(k)} = -d_\mu^{\nu(k)}, \qquad \mu \neq \nu$$

$$c_{ab}^{a(k)} = -c_{0b}^{0(k)}, \qquad c_{0b}^{a(k)} = -c_{ab}^{0(k)}, \qquad c_{bc}^{a(k)} = -c_{ac}^{b(k)},$$

$$\sigma^{\alpha\beta} = \tfrac{1}{4}(\gamma^\alpha \gamma^\beta - \gamma^\beta \gamma^\alpha).$$

Here there is no sum over a. It leads to (5.4.12).

By substituting this result into (5.4.13) it is cumbersome but not difficult to convince ourselves that the operators (5.4.12) exhaust the set of linearly independent solutions of the defining Equations (5.4.13). So, the theorem is proved.

In much the same way one can prove another statement.

Theorem 5.4.3 [211]. *The basis of Lie-Backlund symmetry operators* (5.4.7) *under* $\eta = A_\mu \phi_\mu + F(\phi)$ *of the Dirac equation* (5.4.10) *with* $m = 0$ *is given by the following operators*

$$I = \phi, \qquad \widehat{Q}_a = Q_a \phi \tag{5.4.12'}$$

$$P_\mu^{(0)} = \phi_\mu, \qquad P_\mu^{(a)} = Q_a \phi_\mu;$$

$$\Sigma_{\mu\nu}^{(0)} = i(I - \widetilde{\Gamma}_4)(\Gamma_\mu \phi_\nu - \Gamma_\nu \phi_\mu), \qquad \Sigma_{\mu\nu}^{(a)} = Q_a \Sigma_{\mu\nu}^{(0)}$$

$$\Sigma_\mu^{(0)} = \widetilde{\Gamma}_4 \phi_\mu, \qquad \Sigma_\mu^{(a)} = Q_a \Sigma_\mu^{(0)},$$

$$\Sigma^{(0)} = \widetilde{\Gamma}_4 \phi, \qquad \Sigma^{(a)} = Q_a \Sigma^{(0)}.$$

Remark 5.4.1. It is easy to show that the sets of operators (5.4.12), (5.4.12′) contain operators (5.4.1) and (5.4.1) with $m = 0$, respectively.

3. Following [81] we will show that any relativistic wave equation for a particle with non-zero mass and arbitrary spin is additionally invariant under the Lie algebra of the group $GL(2s+1, C)$.

Let us write an arbitrary linear (differential or integrodifferential) equation

$$L\psi = 0 \tag{5.4.15}$$

where L is a linear operator defined on a vector space $\mathcal{H}$,

$$\psi = \psi(x) \in \mathcal{H}.$$

Definition 5.4.1. Equation (5.4.15) is Poincare invariant and describes a particle of mass m and spin s if it has the 10 symmetry operators P_μ, $J_{\mu\nu}$, $\mu, \nu = \overline{0,3}$, which form the basis of the Lie algebra of the Poincare group, and any solution ψ satisfies the conditions

$$P_\mu P^\mu \psi = m^2 \psi, \qquad W_\mu W^\mu \psi = -m^2 s(s+1)\psi, \tag{5.4.16}$$

where W_μ is the Lubansky-Pauli vector

$$W_\mu = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma. \tag{5.4.17}$$

We consider only such equations as (5.4.15) which satisfy the given definition and so may be interpreted as equations for relativistic particle of spin s and mass m. The symmetry operators P_μ, $J_{\mu\nu}$ of such an equation satisfy the commutation relations of AP(1,3).

$$\begin{aligned} &[P_\mu, P_\nu] = 0, \qquad [P_\mu, J_{\nu\sigma}] = i(g_{\mu\nu}P_\sigma - g_{\mu\sigma}P_\nu), \\ &[J_{\mu\nu}, J_{\lambda\sigma}] = i(g_{\mu\sigma}J_{\nu\lambda} + g_{\nu\lambda}J_{\mu\sigma} - g_{\mu\lambda}J_{\nu\sigma} - g_{\nu\sigma}J_{\mu\lambda}). \end{aligned} \tag{5.4.18}$$

The eigenvalues of the corresponding Casimir operators $P_\mu P^\mu$ and $W_\mu W^\mu$ are fixed and given by the relations (5.4.16). It is to be pointed out that we do not make any supposition on the explicit form of the operators P_μ and $J_{\mu\nu}$, they can be as differential operators of the first order as nonlocal (integrodifferential) ones.

Theorem 5.4.4 [81]. *Any Poincare invariant equation for a particle of mass m and spin s is invariant under the algebra* AGL$(2s+l, C)$.

Proof. Let P_μ, $J_{\mu\nu}$ be the symmetry operators of Equation (5.4.15). Then according to the definition and conditions (5.4.16) the following combinations

$$Q^\pm_{\mu\nu} = \frac{1}{m^2}\left[\epsilon_{\mu\nu\rho\sigma} W^\rho P^\sigma \pm (P_\mu W_\nu - P_\nu W_\mu)\right] \tag{5.4.19}$$

are also the symmetry operators of this equation.

Using (5.4.18) and the relations

$$[W_\mu, P_\nu] = 0, \qquad [W_\mu, W_\nu] = i\epsilon_{\mu\nu\rho\sigma} P^\rho W^\sigma \tag{5.4.20}$$

one can make sure that the operators (5.4.19) satisfy the conditions

$$[Q^{\pm}_{\mu\nu}, Q^{\pm}_{\lambda\sigma}] = i\big(g_{\mu\sigma}Q^{\pm}_{\nu\lambda} + g_{\nu\lambda}Q^{\pm}_{\mu\sigma} - \\ - g_{\mu\lambda}Q^{\pm}_{\nu\sigma} - g_{\nu\sigma}Q^{\pm}_{\mu\lambda}\big)m^{-4}(P_\rho P^\rho)^2 \tag{5.4.21}$$

and

$$\begin{aligned} c_1 &= \tfrac{1}{4}Q^{\pm}_{\mu\nu}Q^{\pm\mu\nu} = -m^{-4}W_\lambda W^\lambda P_\sigma P^\sigma, \\ c_2 &= \tfrac{1}{4}\epsilon_{\mu\nu\rho\sigma}Q^{\pm\mu\nu}Q^{\pm\rho\sigma} = \mp i m^{-4}W_\lambda W^\lambda P_\sigma P^\sigma, \end{aligned} \tag{5.4.22}$$

It follows from (5.4.16) and (5.4.21) that on the set of solutions of Equation (5.4.15) operators (5.4.19) satisfy the commutation relations

$$[Q^{\pm}_{\mu\nu}, Q^{\pm}_{\lambda\sigma}] = i\big(g_{\mu\sigma}Q^{\pm}_{\nu\lambda} + g_{\nu\lambda}Q^{\pm}_{\mu\sigma} - g_{\mu\lambda}Q^{\pm}_{\nu\sigma} - g_{\nu\sigma}Q^{\pm}_{\mu\lambda}\big)\psi \tag{5.4.23}$$

which characterize the Lie algebra of the group SL(2,C). From (5.4.16) and 5.4.22) one obtains the eigenvalues of corresponding Casimir operators

$$c_1\psi = \tfrac{1}{2}(\ell_0^2 + \ell_1^2 - 1)\psi, \qquad c_2\psi = -i\ell_0\ell_1\psi, \tag{5.4.24}$$

where $\ell_0 = s$, $\ell_1 = \pm(s+1)$.

So we demonstrated that any Poincare-invariant equation of non-zero mass and spin s is additionally invariant under the algebra ASL(2,C), the basis elements of which belong to the enveloping algebra of AP(1,3) and are given exactly by the relations (5.4.19). According to (5.4.24) operators (5.4.19) realize the representation $D(\ell_0, \ell_1) = D(s, \pm(s+l))$ of AGL(2,C). Now we see that this invariance algebra may be extended to $2(2s+l)$-dimensional Lie algebra isomorphic to the algebra AGL$(2s+l, C)$. Exactly the basis elements of AGL$(2s+l, C)$ have the following form on the set of solutions of Equation (5.4.15):

$$\begin{aligned} \lambda_{n+k\,n} &= a_{kn}(Q^+_{23} - Q^+_{02})P^s_n, \\ \lambda_{n\,n+k} &= a_{kn}P^s_n(Q^+_{23} + Q^+_{02}), \\ \widetilde{\lambda}_{m\,n} &= Q_1\lambda_{m\,n}, \end{aligned} \tag{5.4.25}$$

where

$$P^s_n = \prod_{n' \neq n} \frac{Q_{12} - s - 1 + n'}{n' - n}, \qquad Q_1 = \frac{\epsilon_{abc}}{2s(s+1)}Q^+_{0a}Q^+_{bc},$$

$$m, n = 1, 2, \ldots, 2s+1; \qquad k = 0, 1, \ldots, 2s - n$$

and a_{kn} are coefficients determined by the recurrent relations

$$\begin{aligned} &a_{0n} = 1, \qquad a_{1n} = [n(2s+1-n)]^{-1/2}, \\ &a_{\lambda n} = a_{\lambda-1\,n}a_{\lambda-1\,n+\lambda-1}, \qquad \lambda = 2, 3, \ldots, 2s - n. \end{aligned}$$

Actually, the polynomials of the symmetry operators $Q^{\pm}_{\mu\nu}$ given by the relations (5.4.25) are the symmetry operators of the equation (5.4.15). Operators (5.4.25) form the basis of the algebra AG($2s+l,C$) inasmuch as they satisfy the commutation relations of AGL($2s+l,C$)

$$[\lambda_{ab},\lambda_{cd}]=-[\widetilde{\lambda}_{ab},\lambda_{cd}]=\delta_{bc}\lambda_{ad}-\delta_{ad}\lambda_{bc},$$
$$[\lambda_{ab},\widetilde{\lambda}_{cd}]=\delta_{bc}\widetilde{\lambda}_{ad}-\delta_{ad}\widetilde{\lambda}_{bc},\quad a,b,c,d=1,2,\ldots,2s+1. \tag{5.4.26}$$

The relations (5.4.26) are correct on the manifold of solutions of Equation (5.4.15). The validity of the above formulae can be verified by direct calculation using the equivalent matrix representation for the basis elements of ASL(2,C)

$$Q^{+}_{ab}=\epsilon_{abc}S_c,\qquad Q^{+}_{0a}=-iS_a$$

where S_a are matrices realizing the representation $D(s)$ of ASO(3) in the Gelfand-Zetlin basis. Thus the theorem is proved.

So, if Equation (5.4.15) is Poincare invariant and describes a particle of spin s and mass $m>0$, it is invariant also under AGL($2s+l,C$) basis elements of which belong to the enveloping algebra of AP(1,3). The operators (5.4.25) together with the Poincare generators P_μ and $J_{\mu\nu}$ form the basis of the $[10+2(2s+l)]$-dimensional Lie algebra isomorphic to the algebra AP(1,3)$\oplus$AGL($2s+l,C$). The last statement can easily be verified by moving to the new basis

$$P_\mu\to P_\mu,\qquad J_{\mu\nu}\to J_{\mu\nu}-Q_{\mu\nu},\qquad \lambda_{mn}\to\lambda_{mn},\qquad \widetilde{\lambda}_{mn}\to\widetilde{\lambda}_{mn},$$

where

$$Q_{12}=\sum_n(s-n+1)\lambda_{nn},\qquad Q_{03}=\sum_n(s-n+1)\widetilde{\lambda}_{nn}$$

$$Q_{23}=\sum_n\frac{1}{2a_{1n}}(\lambda_{n\,n+1}+\lambda_{n+1\,n}),\qquad Q_{31}=-i[Q_{12},Q_{23}],$$

$$Q_{02}=i[Q_{23},Q_{03}],\qquad Q_{01}=-i[Q_{31},Q_{03}].$$

The theorem proved has a constructive character insofar as it gives the explicit form of the basis elements of additional invariance algebra via the Poincare generators. Starting, for example, from the Poincare generators for the Dirac equation (2.1.11) we obtain by means of the formula (5.4.19) the additional symmetry operators

$$Q^{\pm}_{\mu\nu}=\frac{i}{4}[\gamma_\mu,\gamma_\nu]+\frac{i}{2m}(\gamma_\mu P_\nu-\gamma_\nu P_\mu)(1\pm i\gamma_4), \tag{5.4.27}$$

which one easily recognizes as operators from (5.4.1).

In conclusion, let us present the result on complete symmetry of the Dirac equation (5.3.40) in the class of matrix differential operators of the first order (that is operators of the form $Q = A^{\mu}(x)\partial_{\mu} + B(x)$, where A_{μ} and B are 4-column matrices. Note, in Lie approach A_{μ} are obligatory scalar functions). Let us denote this class as M_1.

It turns out that operators (2.1.11) and (5.4.1) do not exhaust all symmetry operators of the Dirac equation (5.3.40) in the class M_1.

Theorem 5.4.5 [81]. *The Dirac equation* (5.3.40) *has only 26 linearly independent symmetry operators* $Q \in M_1$, *These operators include the Poincare generators* (2.1.11), *identity operator and 15 operators given below*

$$\begin{aligned} \eta_{\mu} &= \frac{i}{4}\gamma_4(P_{\mu} - m\gamma_{\mu}), \\ \omega_{\mu\nu} &= mS_{\mu\nu} + \frac{i}{2}(\gamma_{\mu}P_{\nu} - \gamma_{\nu}P_{\mu}), \\ A_{\mu} &= \omega_{\mu\nu}x^{\nu} + x^{\nu}\omega_{\mu\nu} - i\gamma_{\mu}, \\ B &= i\gamma_4(D - m\gamma_{\nu}x^{\nu}), \end{aligned} \tag{5.4.28}$$

where

$$D = x^{\nu}P_{\nu} + \tfrac{3}{2}i, \qquad S_{\mu\nu} = \frac{i}{4}[\gamma_{\mu}, \gamma_{\nu}], \qquad \mu, \nu = \overline{0, 3}. \tag{5.4.29}$$

The proof consists in constructing the general solution of the defining equations following from the invariance condition

$$[L, Q] = f_Q L, \tag{5.4.30}$$

where $L = \gamma P - m$, Q, and f_Q are unknown operators belonging to M_1.

It will be noted that operators (5.4.28) do not form the basis of a Lie algebra since commutators $[\omega_{\mu\nu}, \omega_{\lambda\sigma}]$ do not belong to the class M_1.

5.5 *Symmetry of integrodifferential equations*

Here we consider (following [88*]) a method of studying local symmetry properties of nonlinear systems of integrodifferential equations (IDEs) of the form

$$L\psi + \lambda_1 F(\psi) + \lambda_2 \int\limits_{R^n} K\big(x, y, \psi(x), \psi(y)\big)\, dy = 0, \tag{5.5.1}$$

where L is a linear differential operator: $x \in R^n$; ψ, F, K are columns with m components; λ_1, λ_2 are arbitrary constants. In what follows we shall suppose that the integral of (5.5.1) exists.

Before we proceed to formulate the principal result we note that the standard Lie algorithm is inapplicable to the IDE (5.5.1). The symmetry properties of Equation (5.5.1) can be studied by means of method of differential forms [116, 84], but in this case one faces a problem of unwieldy calculations which become really enormous when the order of the differential operator L and the number of components of ψ increase. This circumstance essentially restricts the applicability of the method of differential forms.

We shall investigate symmetry properties of IDE (5.5.1) in the class of first-order differential operators of the form (7), (8). As a matter of convenience we write here formulae (7), (8) once again

$$\begin{aligned} &Q = \xi^{\mu}(x)\frac{\partial}{\partial x_{\mu}} + \eta(x) \Leftrightarrow \\ &X = \xi^{\mu}(x)\frac{\partial}{\partial x_{\mu}} - \big(\eta(x)\psi\big)^{k}\frac{\partial}{\partial \psi^{k}}, \\ &\mu = \overline{0, n-1}, \qquad k = \overline{1, m}. \end{aligned} \tag{5.5.2}$$

Theorem 5.5.1. *The maximal invariance algebra of the system of IDE* (5.5.1) *in the class of first-order differential operators* (5.5.2) *is determined by the following defining equations*

1° $[L, Q] = \lambda(x)L$, *($\lambda(x)$ is an m-component matrix)*

$$2^\circ \quad \left(\eta(x) + \lambda(x) - (\eta(x)\psi)^{k}\frac{\partial}{\partial\psi^{k}}\right)F(\psi) = 0,$$

$$\begin{aligned} 3^\circ \quad &\left\{\frac{\partial \xi^{\mu}(y)}{\partial y_{\mu}} + \eta(x) + \lambda(x) + \xi^{\mu}(x)\frac{\partial}{\partial x^{\mu}} + \xi^{\mu}(y)\frac{\partial}{\partial y^{\mu}} - \right. \\ &\left. -\big(\eta(y)\psi(y)\big)^{k}\frac{\partial}{\partial\psi^{k}(y)} - \big(\eta(x)\psi(x)\big)^{k}\frac{\partial}{\partial\psi^{k}(x)}\right\}K\big(x, y, \psi(x), \psi(y)\big) = 0. \end{aligned} \tag{5.5.3}$$

Proof. Using formulae (5.3.6) one finds transformations generated by operator (5.5.2):

$$\begin{aligned} &x'_{\mu} = e^{\theta\xi\partial}x_{\mu}e^{-\theta\xi\partial} = x_{\mu} + \theta\xi^{\mu}(x) + \cdots \\ &\psi'(x') = e^{\theta\xi\partial}e^{-\theta Q}\psi(x) = \psi(x) - \theta\eta(x)\psi(x) + \cdots \\ &L(x', \partial') = e^{\theta\xi\partial}L(x, \partial)e^{-\theta\xi\partial} = L(x, \partial) + \theta[\xi\partial, L] + \cdots \end{aligned} \tag{5.5.4}$$

Further, according to the fundamental theorem of integral calculus, under arbitrary coordinate transformation $y_{\mu} \rightarrow y'_{\mu}$ an element of volume $dy =$

$dy_0 \ldots dy_{n-1}$ transforms as

$$dy \to dy' = \det\left(\frac{\partial y'}{\partial y}\right) dy, \tag{5.5.5}$$

or in infinitesimal form, when $y'_\nu = y_\nu + \theta\xi^\nu(y) + \cdots$,

$$dy' = \left(1 + \theta\frac{\partial\xi^\nu(y)}{\partial y_\nu} + \cdots\right) dy \tag{5.5.6}$$

Now we substitute (5.5.4), (5.5.6) into the primed Equation (5.5.1):

$$L(x', \partial')\psi'(x') + \lambda_1 F(\psi') + \lambda_2 \int_{R^n} K\big(x', y', \psi'(x'), \psi'(y')\big)dy' =$$

$$= L(x, \partial)\psi(x) + \lambda_1 F(\psi) + \lambda_2 \int_{R^n} K\big(x, y, \psi(x), \psi(y)\big)dy +$$

$$+ \theta\Big\{ \big([Q(x,\partial), L(x.\partial)] - \eta(x)L(x,\partial)\big)\psi(x) - \lambda_1\big(\eta(x)\psi(x)\big)^k\frac{\partial}{\partial\psi^k}F(\psi) +$$

$$+\lambda_2 \int_{R^n}\Big[\frac{\partial\xi^\nu(y)}{\partial y_\nu} + \xi^\mu(x)\frac{\partial}{\partial x_\mu} + \xi^\mu(y)\frac{\partial}{\partial y_\mu} - \big(\eta(x)\psi(x)\big)^k\frac{\partial}{\partial\psi^k} -$$

$$-\big(\eta(y)\psi(y)\big)^k\frac{\partial}{\partial\psi^k(y)}\Big]K\big(x, y, \psi(x), \psi(y)\big)\Big\}dy + O(\theta^2).$$

The requirement of invariance means that the above expression should be equal to zero on the set of solutions of Equation (5.5.1). It immediately gives the condition

$$\begin{aligned} &\big([Q, L] - \eta(x)L\big)\psi(x) - \lambda_1\big(\eta(x)\psi(x)\big)^k\frac{\partial}{\partial\psi^k}F(\psi) + \\ &\quad +\lambda_2 \int_{R^n}\Big(\frac{\partial\xi^\nu(y)}{\partial y_\nu} + \xi^\nu(x)\frac{\partial}{\partial x_\nu} + \xi^\nu(y)\frac{\partial}{\partial y_\nu} - \\ &\quad -\big(\eta(x)\psi(x)\big)^k\frac{\partial}{\partial\psi^k(x)} - \big(\eta(y)\psi(y)\big)^k\frac{\partial}{\partial\psi^k(y)}\Big)K\big(x, y, \psi(x), \psi(y)\big) = 0. \end{aligned} \tag{5.5.7}$$

The condition (5.5.7) together with Equation (5.5.1) lead to Equations (5.5.3). The general solutions of Equations (5.5.3) determines the maximal IA of IDE (5.5.1) in the class of operators (5.5.2). The theorem is proved.

Now we consider several examples of IDE of the type (5.5.1).

Theorem 5.5.2. [88*] *The equation*

$$\Box u + \lambda_1 u^3 + \lambda_2 \int\limits_{R^4} \frac{u^{3-\alpha}(x)u^{4-\alpha}(y)}{|x-y|^{2\alpha}} G\big(|x-y|^2 u(x)u(y)\big)dy = 0, \tag{5.5.8}$$

where $|x-y| = [(x_\nu - y_\nu)(x^\nu - y^\nu)]^{1/2}$; $\nu = \overline{0,3}$; *G is an arbitrary integrable function; $u = u(x)$ is a real scalar function; λ_1, λ_2, α are arbitrary constants, is invariant under the conformal group C(1,3) and has the most general form in the class of conformally invariant IDEs of the type*

$$\Box u + \lambda_1 F(u) + \lambda_2 \int\limits_{R^4} K\big(x, y, u(x), u(y)\big)dy = 0. \tag{5.5.9}$$

Proof. Let us use Theorem 5.5.1 and apply it to Equation (5.5.9). It is appropriate to recall that the free wave equation $\Box u = 0$ is invariant under AC(1,3) (basis elements are given in (1.1.2), (1.3.2)), provided

$$\begin{aligned} &[\Box, P_\mu] = [\Box, J_{\mu\nu}] = 0, \\ &[\Box, D] = 2\Box, \qquad [\Box, K_\mu] = 4x_\mu\Box. \end{aligned} \tag{5.5.10}$$

Using (5.5.10) we can write the defining equations 2°, 3° from (5.5.3):

$$\begin{aligned} &\left(\frac{\partial}{\partial x_\mu} + \frac{\partial}{\partial y_\mu}\right) K\big(x, y, u(x), u(y)\big) = 0, \\ &\left(x_\mu \frac{\partial}{\partial x_\nu} - x_\nu \frac{\partial}{\partial x_\mu} + y\frac{\partial}{\partial y_\nu} - y_\nu \frac{\partial}{\partial y_\mu}\right) K = 0 \\ &\left(7 + x_\mu \frac{\partial}{\partial x_\mu} + y_\mu \frac{\partial}{\partial y_\mu} - u(x)\frac{\partial}{\partial u(x)} - u(y)\frac{\partial}{\partial u(y)}\right) K = 0, \\ &3F - u\frac{\partial F}{\partial u} = 0. \end{aligned} \tag{5.5.11}$$

The general solution of the system (5.5.11) has the form

$$F = F(u) = \lambda_1 u^3,$$

$$K = K\big(x, y, u(x), u(y)\big) = \lambda_2 \frac{u^{3-\alpha}(x)u^{4-\alpha}(y)}{|x-y|^{2\alpha}} G\big(|x-y|^2 u(x)u(y)\big)$$

Thus the theorem is proved.

Remark 5.5.1. One can directly verify the conformal invariance of Equation (5.5.8) by means of final conformal transformations (see formulae (2.3.2), 2.3.33))

$$x'_\mu = \frac{x_\mu - c_\mu x^2}{\sigma(x,c)}, \qquad u'(x') = \sigma(x,c)u(x), \qquad \sigma(x,c) = 1 - 2cx + c^2x^2,$$

$$|x' - y'| = \frac{|x-y|}{\sqrt{\sigma(x,c)\sigma(y,c)}}, \qquad dy' = \frac{dy}{\sigma^4(y,c)}. \tag{5.5.12}$$

In much the same way one can prove the validity of the following statements.

Theorem 5.5.3. *The nonlinear IDE for Dirac spinor field*

$$\Big\{ i\gamma\partial + \lambda_1(\overline{\psi}\psi)^{1/3} + \lambda_2 \frac{(\overline{\psi}\gamma_\nu\psi)\gamma^\nu}{\left[(\overline{\psi}\gamma_\mu\psi)(\overline{\psi}\gamma^\mu\psi)\right]^{1/3}} + \tag{5.5.13}$$
$$+ \lambda_3 \int\limits_{R^4} \frac{[\overline{\psi}(y)\psi(y)]^\alpha}{|x-y|^{2(4-3\alpha)}} [\overline{\psi}(x)\psi(x)]^{\alpha-1} \cdot$$
$$\cdot G\Big(|x-y|^6 (\overline{\psi}(x)\psi(x)) \, (\overline{\psi}(y)\psi(y))\Big) dy \Big\} \psi(x) = 0,$$

where λ_1, λ_2, λ_3, α are arbitrary constants; G is an arbitrary integrable function (other notations see in Paragraphs 2.1, 2.3), is invariant the conformal group C(1,3).

Theorem 5.5.4. *The nonlinear IDE for complex scalar field*

$$\left(i\frac{\partial}{\partial x_0} + \frac{\Delta}{2m} \right) \psi + \lambda_1(\psi^*\psi)^{2/3}\psi + \tag{5.5.14}$$
$$+ \lambda_2 \int\limits_{R^4} \exp\left\{ i\frac{|\vec{x}-\vec{y}|^2}{2(x_0-y_0)} \right\} \psi(y)(x_0-y_0)^{-7/2} \cdot$$
$$\cdot \phi \left(\frac{\psi^*(x)\psi(x)\psi^*(y)\psi(y)}{(x_0-y_0)^3}, \exp\left\{ -i\frac{|\vec{x}-\vec{y}|}{(x_0-y_0)} \frac{\psi^*(y)\psi(x)}{\psi^*(x)\psi(y)} \right\} \right) d^4y = 0.$$

where ψ is an arbitrary integrable function, is invariant under the Schrödinger group Sch(1,3) *and generalizes the Schrödinger equation.*

5.6. *On exact and approximate solutions of the multidimensional Van der Pol equation*

In [63] the multidimensional generalization of Van der Pol equation was suggested. Here we following [191] construct some exact and approximate solutions of this multidimensional Van der Pol equation.

So, the equation in question is

$$\Box u + \lambda_1 u + \lambda_2(1 - \lambda_3 u^2)\frac{\partial u}{\partial x_0} = 0 \tag{5.6.1}$$

where $u = u(x)$ is a real scalar function, $x \in R^4$; $(\lambda_1, \lambda_2, \lambda_3) = \text{const}$. By means of Lie's method one can show that the Lie-maximal IA of Equation (5.6.1) is the 7-dimensional Lie algebra AE(3)⊕P_0 with basis elements

$$P_0 = \partial_0, \qquad P_a = \partial_a, \qquad J_{ab} = x_a P_b - x_b P_a. \tag{5.6.2}$$

Using the algorithm stated in Paragraph 1.4 and symmetry operators (5.6.2) one can construct an ansatze to solve Equation (5.6.1). Below we consider some of such ansatze.

Let

$$u(x) = \varphi(\omega), \qquad \omega = \beta_0 x_0 + \vec{\beta} \cdot \vec{x}, \qquad \beta_0^2 - \vec{\beta}^2 \equiv \beta^2 \neq 0. \tag{5.6.3}$$

The substitution of (5.6.3) into (5.6.1) results in the original Van der Pol equation for function $\varphi(\omega)$:

$$\beta^2 \frac{d^2\varphi}{d\omega^2} + \lambda_1 \varphi + \lambda_2 \beta_0 (1 - \lambda_3 \varphi^2) \frac{d\varphi}{d\omega} = 0. \tag{5.6.4}$$

Unfortunately, exact solutions of this equation are not known yet. Therefore, we apply the Krilov-Bogolubov-Mitropolski method [32] to obtain an approximate solution of Equation (5.6.4). So, in a first approximation we find

$$\varphi(\omega) = a_0 e^{\epsilon\omega/2} \left[1 + a_0^2 (e^{\epsilon\omega} - 1)/4\right]^{-1/2} \cos\left(\sqrt{\frac{\lambda_1}{\beta^2}}\, \omega + \theta\right)$$

where a_0, θ are arbitrary constants; $\epsilon = -\dfrac{\lambda_2 \beta_0}{\beta^2}$. Inserting this result into (5.6.3) we obtain an approximate solution of Equation (5.6.1):

$$u(x) = \frac{a_0 \exp\left\{\frac{1}{2}\epsilon\beta x\right\}}{\left[1 + \frac{1}{4} a_0^2 (\exp\{\epsilon\beta x\} - 1)\right]^{-1/2}} \cos\left(\sqrt{\frac{\lambda_1}{\beta^2}}\, \beta x + \theta\right) \tag{5.6.5}$$

Setting in (5.6.4) $\lambda_1 = 0$ we get the ODE

$$\beta^2 \frac{d^2\varphi}{d\omega^2} + \lambda_2 \beta_0 (1 - \lambda_3 \varphi^2) \frac{d\varphi}{d\omega} = 0$$

which can be fully integrated. Its general solution has the form

$$\varphi(\omega) = \pm\sqrt{\frac{3}{\lambda_3}} \left[1 - \left(\frac{\lambda_2 \lambda_3 \beta_0}{3\beta^2}\right)^2 \exp\left\{\frac{2\lambda_2 \beta_0}{\beta^2}\omega + c_2\right\}\right]^{-1/2}, \quad (c_2 = \text{const}).$$

Correspondingly, we can write a partial solution to Equation (5.6.1) under $\lambda_1 = 0$:

$$u(x) = \pm\sqrt{\frac{3}{\lambda_3}}\left[1 - \left(\frac{\lambda_2\lambda_3\beta_0}{3\beta^2}\right)^2 \exp\left\{\frac{2\lambda_2\beta_0}{\beta^2}\beta^\nu x_\nu + c_2\right\}\right]^{-1/2}, \qquad (5.6.6)$$

Consider the ansatz

$$\begin{aligned} u(x) &= \varphi(\omega), \qquad \omega = \tfrac{1}{2}(\vec{\alpha}\cdot\vec{x})^2 - \theta\vec{\beta}\cdot\vec{x}, \\ \vec{\alpha}^2 &= \theta^2, \qquad \vec{\beta}^2 = -\ell^2 \neq 0, \end{aligned} \qquad (5.6.7)$$

which reduces Equation (5.6.1) to the ODE

$$\ell^2\theta^2\frac{d^2\varphi}{d\omega^2} + \lambda_1\varphi = 0 \qquad (5.6.8)$$

The general solution of Equation (5.6.8) has the form

$$\varphi(\omega) = c_1\cos\sqrt{\frac{\lambda_1}{\ell^2\theta^2}}\,\omega + c_2\sin\sqrt{\frac{\lambda_1}{\ell^2\theta^2}}\,\omega, \qquad \lambda_1 > 0;$$

$$\varphi(\omega) = c_1\exp\left\{\sqrt{\frac{-\lambda_1}{\ell^2\theta^2}}\,\omega\right\} + c_2\exp\left\{-\sqrt{\frac{-\lambda_1}{\ell^2\theta^2}}\,\omega\right\}, \qquad \lambda_1 < 0.$$

Inserting this result into (5.6.7) we obtain solutions of Equation (5.6.1):

$$\begin{aligned} u(x) &= c_1\cos\left(\sqrt{\frac{\lambda_1}{\ell^2\theta^2}}\left[\tfrac{1}{2}(\vec{\alpha}\cdot\vec{x})^2 - \theta\vec{\beta}\cdot\vec{x}\right]\right) + \\ &\quad + c_2\sin\left(\sqrt{\frac{\lambda_1}{\ell^2\theta^2}}\left[\tfrac{1}{2}(\vec{\alpha}\cdot\vec{x})^2 - \theta\vec{\beta}\cdot\vec{x}\right]\right), \qquad \lambda_1 > 0; \\ u(x) &= c_1\exp\left\{\sqrt{\frac{-\lambda_1}{\ell^2\theta^2}}\left[\tfrac{1}{2}(\vec{\alpha}\cdot\vec{x})^2 - \theta\vec{\beta}\cdot\vec{x}\right]\right\} + \\ &\quad + c_2\exp\left\{-\sqrt{\frac{-\lambda_1}{\ell^2\theta^2}}\left[\tfrac{1}{2}(\vec{\alpha}\cdot\vec{x})^2 - \theta\vec{\beta}\cdot\vec{x}\right]\right\}, \qquad \lambda_1 < 0. \end{aligned} \qquad (5.6.9)$$

Consider another ansatz

$$u(x) = \varphi(\omega), \qquad \omega = x_0 + \theta\ln\vec{\alpha}\cdot\vec{x}, \qquad \vec{\alpha}^2 = 0. \qquad (5.6.10)$$

Which reduces Equation (5.6.1) to the original Van der Pol equation

$$\frac{d^2\varphi}{d\omega^2} + \lambda_1\varphi + \lambda_2(1 - \lambda_3\varphi^2)\frac{d\varphi}{d\omega} = 0. \qquad (5.6.11)$$

By analogy with Equation (5.6.4) we construct an approximate solution of Equation (5.6.11):

$$\varphi(\omega) = \frac{a_0 e^{\epsilon\omega/2}}{\left[1 + \frac{1}{4}a_0^2\left(e^{\epsilon\omega} - 1\right)\right]^{1/2}} \cos\left(\sqrt{\lambda_1}\,\omega + \theta\right) \tag{5.6.12}$$

Integrating (5.6.11) under $\lambda_1 = 0$ we find a family of solutions of Equations (5.6.1) under $\lambda_1 = 0$:

$$u(x) = \pm\sqrt{\frac{3}{\lambda_3}}\left[1 - \left(\frac{\lambda_2\lambda_3}{3}\right)^2 \exp\left\{2\lambda_2(x_0 + \theta \ln \vec{\alpha}\cdot\vec{x}) + c_2\right\}\right]^{-1/2}. \tag{5.6.13}$$

In conclusion, let us note that exact solutions of Equations (5.6.1) can be used for obtaining new approximate solutions of the equation. Indeed, let $u(x) = \varphi(x)$ be an exact solution of Equation (5.6.1), then we shall look for an approximate solution in the form

$$u(x) = \varphi(x) + \epsilon g(x) \tag{5.6.14}$$

where ϵ is a small parameter. The substitution of (5.6.14) into (5.6.1) and taking only first order in ϵ gives a linear PDE for function $g = g(x)$:

$$\Box g + \lambda_1 g + \lambda_2\left[(1 - \lambda_3\varphi^2)\frac{\partial g}{\partial t} - 2\lambda_3\varphi\frac{\partial\varphi}{\partial t}g\right] = 0$$

which should be considered as defining an equation for function $g(x)$.

5.7 *Conditional symmetry of PDEs*

Generalizing results of works [94, 67, 108, 7*, Appendix 4 in [82]] here we introduce the concept of *conditional invariance* (briefly: c-invariance) of PDEs or, in other words, invariance of PDEs on submanifolds of their solutions. The usefulness and efficiency of this conception is illustrated on a set of nonlinear equations from heat-conduction, Born-Infeld theory, gas dynamics, nonlinear acoustics, and some others. With the help of c-invariance, we construct new classes of exact solutions of these equations, which cannot be obtained based on the Lie approach.

1. The conditional invariance. Consider a system of PDEs of sth order

$$L(x, u, \underset{1}{u}, \ldots, \underset{s}{u}) = 0, \tag{5.7.1}$$

where $u = u(x)$, $x \in R^n$, $\underset{s}{u}$ is a totality of s-order derivatives. Equation (5.7.1) admits (in sense of Lie) an algebra $A = \{X\}$ (the general form of operators X

is written in (2)). It means that condition of invariance (3) holds true for any operator $X \in A$. Here we rewrite (3) as follows

$$\underset{s}{X}L = RL, \tag{5.7.2}$$

where $R = R(x, u, \theta, \partial)$ are some differential operators, depending on x and u.

Definition 5.7.1. A system of PDEs (5.7.1) we shall call c-invariant if there is an additional condition (equation), compatible with (5.7.1), which enlarges the symmetry of the system (5.7.1).

A clear example of c-symmetry gives us Maxwell's equations. The maximal IA of the system

$$\frac{\partial \vec{E}}{\partial t} = \operatorname{rot} \vec{H}, \qquad \frac{\partial \vec{H}}{\partial t} = -\operatorname{rot} \vec{E}$$

is 10-dimensional [82, p.121]:

$$A_{10} = \langle P_0 = \partial_t, \quad P_a = \partial_a, \quad J_{ab} = x_a\partial_b - x_b\partial_a + E_a\partial_{E_b} - E_b\partial_{E_a} +$$
$$+ H_a\partial_{H_b} - H_b\partial_{H_a}, \quad D = t\partial_t + x_a\partial_a, \quad F = E_a\partial_{H_a} - H_a\partial_{E_a},$$
$$I = E_a\partial_{E_a} + H_a\partial_{H_a}\rangle$$

and we see that it is not Lorentz invariant. But, by supplementing the above system to the Maxwell's equations by the well-known additional conditions

$$\operatorname{div} \vec{E} = 0, \qquad \operatorname{div} \vec{H} = 0$$

we thereby essentially enlarge its symmetry: the IA of Maxwell's equation is the 16-dimensioinal Lie algebra AC(1,3)⊗F.

If A is the maximal IA of Equation (5.7.1), then for any operator Y not belonging to A, the condition of invariance (5.7.2) does not hold. Instead of (5.7.2), we will have

$$\underset{s}{Y}L = R_0L + R_1L_1, \qquad R_1 \neq 0, \tag{5.7.3}$$

where R_0, R_1, L_1 are some differential expressions.

Definition 5.7.2. System of PDEs (5.7.1) we shall call c-invariant under an operator Y which does not belong to IA of (5.5.1) if it is invariant under this operator only together with an additional condition

$$L_1(x, u, \underset{1}{u}, \ldots, \underset{s}{u}) = 0, \tag{5.7.4}$$

or, in other words, instead of (5.7.2), now we have (5.7.3) and

$$\underset{s}{Y}L_1 = R_2L + R_3L_1. \tag{5.7.5}$$

The additional condition (5.7.4) select from the whole set of solutions of Equation (5.7.1) a subset and this subject may well be true to have the symmetry wider than that of the whole set. It is most desirable to know how to select such subsets of solutions.

It is obvious that the concept of c-invariance makes sense only if the couple of equations (5.7.1) and (5.7.4) is consistent. Relation (5.7.5) represents the necessary condition of compatibility for Equations (5.7.1), (5.7.4). So, to find c-invariant solutions one has to solve compiling systems (5.7.1), (5.7.4) and the equation

$$Yu = 0 \tag{5.7.6}$$

and for which purpose to find the very operator Y. In general case, it is rather complicated to solve this problem. However, if additional condition (5.7.4) coincides with (5.7.6), then one succeeds in obtaining a constructive algorithm for finding c-invariant operators.

Definition 5.7.3. System of PDEs (5.7.1) we shall call $c(Y)$-invariant (or Y-conditional invariant) if

$$\underset{s}{Y}L = R_0 L + R_1(Yu). \tag{5.7.7}$$

As in the previous case (see Definition 5.7.2) relation (5.7.7) represents a necessary condition of compatibility of system (5.7.1), (5.7.6). It will be noted that even in Lie's case, when $R_1 = 0$, fulfillment of relation (5.7.7) does not guarantee compatibility of system (5.7.1), (5.7.6). For example, any linear inhomogeneous PDE of first order $Qu = \text{const} \neq 0$ is, clearly, invariant under the operator Q, though it has no Q-invariant solutions because it is inconsistent with the condition $Qu = 0$ which defines such solutions.

Definition 5.7.3 can be generalized on differential operators Y of second or higher order. To do it one has to use the condition of invariance (5.7.7) in the Lie-Backlund form.

It is clear that ansatze constructed within the framework of conditional symmetry include Lie's in particular, as well as others which we shall call *non-Lie ansatze.*

2. Conditional invariance of nonlinear heat equation under the Galilei algebra. As is well known (see §3.3) the linear heat equation

$$u_0 + \frac{1}{2m}\Delta u = 0$$

admits AG$(1, n)$, provided

$$G_a = x_0 \partial_a + m x_a u \partial_u, \tag{5.7.8}$$

and there is no one nonlinear heat equation of the form

$$u_0 + \vec{\nabla}[f(u)\vec{\nabla}u] = 0, \qquad f(u) \neq \text{const}, \tag{5.7.9}$$

which would be invariant under Galilean transformations. But it only means that the whole set of solutions of Equations (5.7.9) is not invariant under Galilean transformations [66], though there may be a subset of solutions which will be Galilean invariant. An affirmative answer on this question gives the following theorem.

Theorem 5.7.1. [39*] *The nonlinear heat equation* (5.7.9) *is invariant under Galilean operators*

$$G_a = x_0\partial_a + M(u)x_a\partial_u \tag{5.7.10}$$

if

$$u_0 + \frac{(\vec{\nabla}u)^2}{2M(u)} = 0, \tag{5.7.11}$$

where

$$M(u) = \frac{u}{2f(u)}. \tag{5.7.12}$$

Proof. Having acted on (5.7.9) by $\underset{2}{G}a$ ($\underset{2}{G}a$ is constructed by virtue of formulae (1.1.7)), we find

$$\underset{2}{G_a}\{u_0 + \vec{\nabla}[f(u)\vec{\nabla}u]\} = [M'u_0 + (Mf)''(\vec{\nabla}u)^2 + \tag{5.7.13}$$

$$+(fM)'\Delta u]x_a + [2(Mf)' - 1]u_a$$

whence follows

$$2(Mf)' - 1 = 0,$$

which gives (5.7.12). Substituting (5.7.12) into (5.7.13), we get

$$\underset{2}{G_a}\{u_0 + \vec{\nabla}[(f(u)\vec{\nabla}u]\} = \frac{x_a}{2f(u)}\{u_0 + \vec{\nabla}[f(u)\vec{\nabla}u)]\} - \\ - \frac{x_a f'(u)}{2f(u)}\left[u_0 + \frac{(\vec{\nabla}u)^2}{2M(u)}\right] \tag{5.7.14}$$

whence follows (5.7.11).

To complete the proof, one can make sure that

$$\underset{1}{G_a}\left[u_0 + \frac{(\vec{\nabla}u)^2}{2M(u)}\right] = M'(u)x_a\left[u_0 + \frac{(\vec{\nabla}u)^2}{2M(u)}\right]. \tag{5.7.15}$$

Theorem 5.7.2. *The nonlinear heat equation* (5.7.9) *is* $c(G_a)$*-invariant under Galilean operators* (5.7.10) *if*

$$\begin{aligned} f(u) &= \frac{1}{2m}u^k, \qquad M(u) = \frac{2m}{kn+2}u^{1-k}; \\ f(u) &= e^u, \qquad M(u) = 1, \end{aligned} \tag{5.7.16}$$

where m, k *are arbitrary constants,* $kn + 2 \neq 0$, $m \neq 0$ *).*

Proof. Conditions (5.7.6) in this case have the form

$$G_a u = x_0 u_a - M(u)x_a = 0 \tag{5.7.17}$$

whence follows

$$\begin{aligned} u_a &= x_0^{-1}[G_a u + M(u) \cdot x_a] \\ \Delta u &= x_0^{-1}\partial_a G_a u + x_0^{-2}M(u)(nx_0 + M(u)\vec{x}^2). \end{aligned} \tag{5.7.18}$$

Substituting (5.7.18) into (5.7.13), we have

$$\begin{aligned} \underset{2}{\mathcal{G}_a}[u_0 + \vec{\nabla}[f(u)\vec{\nabla}u]] &= M'x_a\{u_0 + \vec{\nabla}[f(u)\vec{\nabla}u]\} + x_0[2(Mf)' - 1]\delta_{ab}\}G_b u + \\ &+ x_0^{-2}x_a\{[(Mf)'' - M'f'](G_b u + 2Mx_b) + Mf'x_0\partial_b + \\ &+ x_0^{-2}[2(Mf)' + nMf' - 1]x_0 + M(Mf)''\vec{x}^2\}, \end{aligned} \tag{5.7.19}$$

whence follows (5.7.16). Since $[G_a, G_b] = 0$, the theorem is proved.

It will be noted, that if $f(u) = \frac{1}{m}u^k$ ($m \neq 0$, $k \neq -\frac{2}{n}$), then Equation (5.7.9) is c-invariant under operators G_a (5.7.10), provided $M(u)$ has two different forms, (5.7.12) and (5.7.16).

So, to describe nonlinear heat condition, one may use the completed Equations (5.7.9), (5.7.11), that is the system

$$\begin{aligned} u_0 + \vec{\nabla}[f(u)\vec{\nabla}u] &= 0, \\ u_0 + \frac{(\vec{\nabla}u)^2}{2M(u)} &= 0, \end{aligned} \tag{5.7.20}$$

where $M(u) = \dfrac{u}{2f(u)}$.

Next we study the maximal IA of the system (5.7.20).

Theorem 5.7.3. *The maximal IA of coupled Equations* (5.7.20) *is* $\mathrm{A}\widetilde{\mathrm{G}}(2, n)$, *basis element having the form*

$$\partial_0, \ \partial_w, \ \partial_a, \qquad J_{ab} = x_a\partial_b - x_b\partial_a,$$

$$G_{1a} = x_0\partial_a + mx_a\partial_w, \qquad G_{2a} = w\partial_a + mx_a\partial_0, \tag{5.7.21}$$
$$\mathcal{D}_1 = 2x_0\partial_0 + x_a\partial_a, \qquad \mathcal{D}_2 = 2w\partial_w + x_a\partial_a,$$

where $a, b = \overline{1,n}$, $m = \text{const} \neq 0$,

$$w = 2m\int \frac{f(u)}{u}du. \tag{5.7.22}$$

Proof. After changing variables according to (5.7.22), Equations (5.7.20) take the form

$$\Delta w = 0, \qquad w_0 + \frac{(\vec{\nabla} w)^2}{2m} = 0. \tag{5.7.23}$$

Then applying to (5.7.23) the Lie algorithm, we find that maximal IA of these coupled equations is AG(2, n) with basis elements (5.7.21).

Now we find some exact solutions of Equations (5.7.20). From equation

$$\theta_a G_a u = 0 \tag{5.7.24}$$

where θ_a are arbitrary constants, G_a are given in (5.7.10), (5.7.12) we find the ansatz:

$$\frac{1}{2m}w = \int \frac{f(u)}{u}du = \varphi(\vec{\omega}) + \frac{\vec{x}^2}{4x_0}, \tag{5.7.25}$$

where $\varphi(\vec{\omega})$ is an unknown function of variables $\vec{\omega} \in R^n$. In three-dimensional space $\vec{\omega}$ are as follows

$$\omega_1 = x_0, \qquad \omega_2 = \alpha\vec{x}, \qquad \omega_3 = \beta\vec{x}, \tag{5.7.26}$$

where $\vec{\alpha}$, $\vec{\beta}$ are constant vectors ($\vec{\alpha}^2 = \vec{\beta}^2 = 1$, $\vec{\alpha}\cdot\vec{\beta} = 0$).

Substituting ansatz (5.7.25) into (5.7.23) we get

$$\begin{cases} \varphi_{22} + \varphi_{33} + \dfrac{n}{2\omega_1} = 0, \\ \varphi_2^2 + \varphi_3^2 + \dfrac{1}{\omega_1}\omega_a\varphi_a = 0. \end{cases} \tag{5.7.27}$$

A particular solution of Equations (5.7.27) has the form

$$\varphi = -\frac{n}{4}\frac{\omega_2^2}{\omega_1}, \qquad (n = 1). \tag{5.7.28}$$

Then the formula

$$\int \frac{f(u)}{u}du = \frac{\vec{x}^2 - n(\vec{\alpha}\vec{x})^2}{4x_0}, \qquad (n = 1) \tag{5.7.29}$$

gives a Galilean-invariant solution of Equation (5.7.9)

Consider the nonlinear equation

$$u_0 + \frac{1}{m}\vec{\nabla}(u^k\vec{\nabla}u) = 0. \qquad (5.7.30)$$

From (5.7.17), (5.7.10), (5.7.16), we find the ansatz

$$u^k = \varphi(x_0) + \frac{km}{kn+2}\frac{\vec{x}^2}{2x_0} \qquad (5.7.31)$$

which reduces (5.7.30) to the following ODE

$$\varphi' + \frac{kn}{kn+2}\frac{1}{x_0}\varphi = 0 \qquad (5.7.32)$$

The general solution of (5.7.32) can easily be obtained and therefore via (5.7.31) we find a solution of (5.7.30)

$$u^k = \lambda x_0^{-kn/(kn+2)} + \frac{km}{kn+2}\frac{\vec{x}^2}{2x_0}, \qquad k \neq -\frac{2}{n}, \quad \lambda = \text{const} \qquad (5.7.33)$$

3. Conditional invariance of the Born-Infeld equation under conformal algebra. In §1.9 we established that the maximal IA of the n-dimensional Born-Infeld equation

$$(1 - u_\nu u^\nu)\Box u + u^\mu u^\nu u_{\mu\nu} = 0 \qquad (5.7.34)$$

is $\widetilde{\text{AP}}(1, n+1)$.

Theorem 5.7.4. *Born-Infeld equation* (5.7.34) *is invariant under* AC$(1, n+1)$ *if*

$$1 - u_\nu u^\nu = 0$$

(Basis elements of AC$(1, n+1)$ *are written in* (1.2.2)*).*

Remark 5.7.1. The Born-Infeld equation (5.7.34) is a differential consequence of the eikonal equation.

$$(1 - u_\nu u^\nu)\Box u + u^\mu u^\nu u_{\mu\nu} = (\Box u - \tfrac{1}{2}u^\mu\partial_\mu)(1 - u_\nu u^\nu).$$

Except for conformally invariant subset of solutions, Equation (5.7.34) possesses a subset of solutions which is invariant under an infinite-dimensional algebra.

Theorem 5.7.5. *Equation* (5.7.34) *is invariant under the infinite-dimensional algebra* AP$(1, n)$ *with basis elements written in* (1.2.19) *if*

$$u_\nu u^\nu = 0$$

Theorem 5.7.4, 5.7.5 can be proved analogously to Theorem 5.7.1.

Using invariants of AC$(1, n+1)$

$$\omega_1 = \frac{x_A x^A}{\alpha_A x^A}, \qquad \omega_2 = \frac{\beta_A x^A}{\alpha_A x^A}, \qquad (A = \overline{0, n+1},\ x_{n+1} \equiv u)$$

and invariants of $\mathrm{A}\widetilde{\mathrm{P}}_\infty(1,n)$

$$\omega_1 = d_\mu(u) x^\mu, \qquad \omega_2 = u, \quad d_\mu d^\mu = 0$$

where α_A, β_A are arbitrary constants, $d_\mu(u)$ are arbitrary differential functions, we obtain the following ansatze

$$\frac{x_A x^A}{\alpha_A x^A} = \varphi\left(\frac{\beta_A x^A}{\alpha_A x^A}\right), \tag{5.7.35}$$

$$d_\mu(u) x^\mu = \varphi(\omega) \tag{5.7.36}$$

However, these ansatze are not invariant under IA of the Equation (5.7.34), nevertheless they reduce it. Substitution ansatz (5.7.35) into (5.7.34) we get the ODE

$$(\alpha_A \alpha^A \omega_2^2 - 2\alpha_A \beta^A \omega_2 + \beta_A \beta^A)\varphi'^2 - 2(\alpha_A \alpha^A \omega_2 - \alpha_A \beta^A)\varphi\varphi' + \alpha_A \alpha^A \varphi^2 = 0.$$

Ansatz (5.7.36) satisfies (5.7.34) under arbitrary φ.

When

$$\alpha_A \alpha^A = \alpha_A \beta^A = \beta_A \beta^A = 0,$$

formula (5.7.35) gives a solution of the Born-Infeld equation (5.7.34) with arbitrary differentiable function φ; otherwise it is a linear function of w_2:

$$\varphi = \lambda_1 \omega_2 + \lambda_2, \qquad \lambda_1, \lambda_2 = \mathrm{const}.$$

4. *Extension of symmetry of acoustic nonlinear equations.*

In many cases equations of nonlinear acoustics have the form

$$u_{00} = C(x, u, \underset{1}{u})\Delta u \tag{5.7.37}$$

where $u = u(x)$, $x(x_0, \vec{x}) \in \mathrm{R}^{1+n}$, $C(x, u, \underset{1}{u})$ is a smooth function depending on x, u, $\underset{1}{u}$. One can make sure that Equation (5.7.37) is not Galilean invariant, but, as we shall show it, it has Galilean invariant subset of solutions.

Theorem 5.7.6. *Equation (5.7.37) is conditionally G_a-invariant under G_a from (5.7.10) if*

$$M(u) = m = \text{const}, \quad C(x, u, \underset{1}{u}) = F(\vec{w}, \vec{\nabla} w_2) + \frac{\vec{x}^2}{n x_0^2}, \tag{5.7.38}$$

where F is a arbitrary differentiable function, $w \in R^3$,

$$w_1 = x_0, \qquad w_2 = u - \frac{m\vec{x}^2}{2x_0}, \qquad w_3 = u_0 + \frac{(\vec{\nabla} u)^2}{2m}. \tag{5.7.39}$$

Ansatz following as a solution of equations $G_a u = 0$

$$u = \varphi(x_0) + \frac{m}{2}\frac{\vec{x}^2}{x_0} \tag{5.7.40}$$

reduces (5.7.37), (5.7.38) to the ODE

$$x_0 \varphi'' = nmF(x_0, \varphi, \varphi', \vec{O}) \tag{5.7.41}$$

When $n = 1$, $C(x, u, \underset{1}{u}) = u$ Equation (5.7.37) takes the form

$$u_{00} = u u_{11}. \tag{5.7.42}$$

As is shown in [160] Equation (5.7.42) is reduced to the ODE by the ansatz obtained with the help of operator

$$Q = \partial_0 + 2x_0 \partial_1 + 8x_0 \partial_u \tag{5.7.43}$$

which does not belong to IA of this equation.

Using condition (5.7.7) one can obtain more operators of c-invariance of Equation (5.7.42).

Theorem 5.7.7. *Equation (5.7.42) is c(Y)-invariant, provided*

$$Y = A(x)\partial_0 + B(x)\partial_1 + [\alpha(x)u + \beta(x)]\partial_u \tag{5.7.44}$$

and functions $A(x)$, $B(x)$, $\alpha(x)$, $\beta(x)$ satisfy the following equations:

a). $A \neq 0. \quad B \neq 0$

$$\alpha = 2(B_1 - A_0 - \frac{B}{A}A_1), \quad \beta = 2\frac{B}{A}B_0, \quad \alpha_{11} = \frac{\alpha}{A}A_{11} + 2\left(\frac{\alpha}{A}\right)_1 A_1,$$

$$\alpha_{00} + 2\frac{\alpha}{A}\alpha_0 - \left[\frac{\alpha}{A}A_{00} + 2\left(\frac{\alpha}{A}\right)_1 B_0\right] = \beta_{11} - \left[\frac{\beta}{A}A_{11} + 2\left(\frac{\beta}{A}\right)_1 A_1\right],$$

$$\beta_{00} + 2\frac{\beta}{A}\alpha_0 - \left[\frac{\beta}{A}A_{00} + 2\left(\frac{\alpha}{A}\right)_1 B_0\right] = 0, \tag{5.7.45}$$

$$B_{11} - 2\alpha_1 - \left[\frac{B}{A}A_{11} + 2\left(\frac{B}{A}\right)_1 A_1\right] + 2\frac{\alpha}{A}A_1 = 0,$$

$$B_{00} + 2\frac{B}{A}\alpha_0 - \left[\frac{B}{A}A_{00} + 2\left(\frac{B}{A}\right)_1 B_0\right] + 2\frac{\alpha}{A}B_0 = 0.$$

b). $A = 0$, $B \neq 0$ *(without loss of generality one can put* $B = 1$*)*

$$\alpha_0 = 0, \qquad \alpha_{11} + 3\alpha\alpha_1 + \alpha^3 = 0, \tag{5.7.46}$$

$$\beta_{11} + \alpha\beta_1 + (3\alpha_1 + 2\alpha^2)\beta = 0, \qquad \beta_{00} - \beta\beta_1 - \alpha\beta^2 = 0. \tag{5.7.47}$$

c). $A = 1$, $B = 0$

$$\begin{gathered} \alpha_1 = 0, \qquad \alpha_{00} + \alpha\alpha_0 - \alpha^3 = \beta_{11}, \\ \beta(\beta_0 + \alpha\beta) = 0, \qquad \beta_{00} + \alpha_0\beta - \alpha^2\beta = 0. \end{gathered} \tag{5.7.48}$$

The proof of this theorem is not complicated but is rather cumbersome and for which reason we shall omit them.

Partial solutions of Equations (5.7.45)–(5.7.48) give explicit forms of operators (5.7.44) and we list them in Table 5.7.1., where we used notations: a_i, b_j $(i, j = \overline{1, 10})$ are arbitrary constants, $\omega = \frac{1}{2}a_4x_0^2 + a_5x_0x_1$, $\wp$ is a Weierstrass elliptic function, which satisfies the equation

$$\wp'' = \wp^2 \tag{5.7.49}$$

$f(x_0)$ is a solution of the Lame equation

$$f'' = \wp f \tag{5.7.50}$$

and $F(x_0)$ satisfies the equation

$$F''(x_0) = F^2(x_0)\left(\int F^{-2}(x_0)dx_0 + b_9\right). \tag{5.7.51}$$

Having solved some reduced ODEs from Table 5.7.1, we succeed in finding solutions of Equation (5.7.4):

$$\begin{gathered} u = P_1(x_0)Q_1(x_1), \qquad u = x_0^{-2}[3x_1^2 + Q_1(x_1)] + x_0^3R_1(x_1), \\ u = \tfrac{1}{2}\wp(x_0)x_1^2 + f(x_0)x_1 + f^2(x_0), \\ u = \left[P_2(x_0) + Q_2(x_1)\int F^{-2}(x_0)dx_0\right]F(x_0), \end{gathered} \tag{5.7.52}$$

Table 5.7.1. Operators Y of $c(Y)$-invariance of Equation (5.7.42), ansatz and reduced ODEs.

N	Operator Y	Ansatz $u =$	Reduced ODE
1.	$\partial_1 + a_1\partial_u$	$= \varphi(x_0) + a_1 x_1$	$\varphi'' = 0$
2.	$\partial_0 + (a_2 x_1 + a_3)\partial_u$	$= \varphi(x_1) + x_0(a_2 x_1 + a_3)$	$\varphi'' = 0$
3.	$\partial_0 + (a_4 x_0 + a_5)\cdot$ $\cdot(\partial_1 + 2a_4\partial_u)$	$= \varphi(\omega) + 2a_4 x_1$	$(\varphi - 2a_4\omega -$ $-a_5)\varphi'' = a_4\varphi'$
4.	$\partial_1 + [\wp(x_0)x_1 +$ $+f(x_0)]\partial_u$	$= \wp(x_0)\frac{x_1^2}{2} + f(x_0)x_1 + \varphi(x_0)$	$\varphi'' = \wp\varphi$
5.	$x_0\partial_0 + (u + a_7 x_1 +$ $+a_8)\partial_u$	$= x_0\varphi(x_1) - (a_7 x_1 + a_8)$	$\varphi'' = 0$
6.	$x_0\partial_0 + [x_0^3(a_9 x_1 +$ $+a_{10}) - 2u]\partial_u$	$= x_0^{-2}\varphi(x_1) + \frac{x_0^3}{5}(a_9 x_1 + a_{10})$	$\varphi'' = 6$
7.	$x_1\partial_1 + (u + b_1 x_0 +$ $+b_2)\partial_u$	$= x_1\varphi(x_0) - (b_1 x_0 + b_2)$	$\varphi'' = 0$
8.	$x_1\partial_1 + [u + \wp(x_0)\frac{x_1^2}{2} -$ $-f(x_0)]\partial_u$	$= \wp(x_0)\frac{x_1^2}{2} + \varphi(x_0)x_1 + f(x_0)$	$\varphi'' = \wp\varphi$
9.	$x_0^3\partial_0 + (3x_0^2 u - 15x_1^2 +$ $+b_3 x_1 + b_4)\partial_u$	$= x_0^3\varphi(x_1) + 3x_0^{-2}x_1^2 -$ $-\frac{1}{5}(b_3 x_1 - b_4)x_0^{-2}$	$\varphi'' = 0$
10.	$x_0^2 x_1\partial_1 + (x_0^2 u + 3x_1^2 +$ $+b_5 x_0^5 + b_6)\partial_u$	$= x_1\varphi(x_0) + 3x_0^{-2}x_1^2 -$ $-b_5 x_0^3 - b_6 x_0^{-2}$	$x_0^2\varphi'' = 6\varphi$
11.	$\wp(x_0)\partial_0 + \wp'(x_0)u\partial_u$	$= \wp(x_0)\varphi(x_1)$	$\varphi'' = 1$
12.	$F(x_0)\partial_0 + [F'(x_0)u +$ $+\frac{1}{2}x_1^2 + b_7 x_1 + b_8]\partial_u$	$= F(x_0)\varphi(x_1) + (\frac{1}{2}x_1^2 + b_7 x_1 +$ $+ b_8)F(x_0)\int F^{-2}(x_0)dx_0$	$\varphi'' = b_9$

$$u = \varphi(\omega) + a_4 x_0(a_4 x_0 + 2a_5), \tag{5.7.53}$$

where $P_k(x)$, $Q_k(x)$, $R_k(x)$ are arbitrary polynomials of order $k = 1, 2$, and the other notations are the same as in Table 5.7.1.

Note, when $a_4 = 2$, $a_5 = 0$, operator Y_3 coincides with (5.7.43). The corresponding solution (5.7.53) is also obtained in [160].

The above results on two-dimensional Equation (5.7.42) can be generalized on the multi-dimensional case, that is, for the equation

$$u_{00} = u\Delta u. \tag{5.7.54}$$

We present these results in Table 5.7.2.

Table 5.7.2. (In this table $\vec{\alpha}$ is a constant vector, $\vec{\alpha}^2 = 1$, $\omega = \frac{1}{2}c_2x_0^2 + c_3x_0\vec{\alpha}\vec{x}$.)

N	Operator Y	Ansatz $u =$	Reduced ODE
1.	$\vec{\nabla} + \vec{\alpha}\partial_u$	$= \varphi(x_0) + \vec{\alpha}\vec{x}$	$\varphi'' = 0$
2.	$\partial_0 + (\vec{\alpha}\vec{x} + c_1)\partial_u$	$= \varphi(\vec{x}) + x_0(\vec{\alpha}\vec{x} + c_1)$	$\Delta\varphi = 0$
3.	$\vec{\alpha}\partial_0 + (c_2x_0 + c_3)\vec{\nabla} + {}+2c_2(c_2x_0 + c_3)\vec{\alpha}\partial_u$	$= \varphi(\omega) + 2c_2\vec{\alpha}\vec{x}$	$(\varphi - 2c_2\omega - {}-c_3^2)\varphi'' = c_2\varphi'$
4.	$\vec{\nabla} + [\wp(x_0)\vec{x} + f(x_0)\vec{\alpha}]\partial_u$	$= \frac{1}{2n}\wp(x_0)\vec{x}^2 + {}+\frac{1}{n}f(x_0)\vec{\alpha}\vec{x} + \frac{1}{n}\varphi(x_0)$	$\varphi'' = \wp\varphi$
5.	$x_0\partial_0 + (u + \vec{\alpha}\vec{x} + c_1)\partial_u$	$= x_0\varphi(\vec{x}) - \vec{\alpha}\vec{x} - c_1$	$\Delta\varphi = 0$
6.	$x_0\partial_0 + [x_0^3(\vec{\alpha}\vec{x} + c_1) - 2u]\partial_u$	$= x_0^{-2}\varphi(\vec{x}) + \frac{x_0^3}{5}(\vec{\alpha}\vec{x} + c_1)$	$\Delta\varphi = 6$
7.	$(\vec{\alpha}\vec{x})\vec{\nabla} + \vec{\alpha}(u + c_2x_0 + c_3)\partial_u$	$= \vec{\alpha}\vec{x}\varphi(x_0) - (c_2x_0 + c_3)$	$\varphi'' = 0$
8.	$(\vec{\alpha}\vec{x})\vec{\nabla} + \vec{\alpha}[u + {}+\wp(x_0)\frac{(\vec{\alpha}\vec{x})^2}{2} - f(x_0)]\partial_u$	$= \frac{1}{2}\wp(x_0)(\vec{\alpha}\vec{x})^2 + {}+\varphi(x_0)\vec{\alpha}\vec{x} + f(x_0)$	$\varphi'' = \wp\varphi$
9.	$x_0^3\partial_0 + (3x_0^3u - 15\vec{x}^2 + {}+\vec{\alpha}\vec{x} + c_1)\partial_u$	$= x_0^3\varphi(\vec{x}) + \frac{3}{\eta}x_0^{-2}\vec{x}^2 - {}-\frac{1}{5}x_0^{-2}(\vec{\alpha}\vec{x} + c_1)$	$\Delta\varphi = 0$
10.	$x_0^2(\vec{\alpha}\vec{x})\vec{\nabla} + \vec{\alpha}[x_0^2u + 3(\vec{\alpha}\vec{x})^2 + {}+c_4x_0^5 + c_3]\partial_u$	$= \vec{\alpha}\vec{x}\varphi(x_0) + 3x_0^{-2}(\vec{\alpha}\vec{x})^2 - {}-c_4x_0^3 - c_3x_0^{-2}$	$x_0^2\varphi'' = 6\varphi$
11.	$\wp(x_0)\partial_0 + \wp'(x_0)u\partial_u$	$= \wp(x_0)\varphi(\vec{x})$	$\Delta\varphi = 1$
12.	$F(x_0)\partial_0 + [F'(x_0)u + {}+\frac{1}{2}\vec{x}^2 + \vec{\alpha}\vec{x} + c_1]\partial_u$	$= F(x_0)\varphi(\vec{x}) + (\frac{1}{2}\vec{x}^2 + \vec{\alpha}\vec{x} + {}+c_1)F(x_0)\int F^{-2}(x_0)dx_0$	$\Delta\varphi = b_9$

5. *Extension of symmetry of equations* $\Box u + \lambda uu_0 = 0$, *Hamilton-Jacobi, gas dynamics, and Navier-Stokes*

Theorem 5.7.8. *Equation*

$$\Box u + \lambda uu_0 = 0 \tag{5.7.55}$$

is $c(Y)$*-invariant with*

$$Y = x_0^2\partial_0 + \left(x_0u - \frac{4}{\lambda}\right)\partial_u \tag{5.7.56}$$

Theorem 5.7.9. *The Hamilton-Jacobi equation*

$$u_0 + \frac{(\vec{\nabla}u)^2}{2m} = 0 \tag{5.7.57}$$

is $c(Y)$-invariant with

$$Q_a = r\partial_a + x_a\left(x_0 + \sqrt{x_0^2 - 2mr}\right)\partial_u, \qquad \left(r = \sqrt{\vec{x}^2}\right) \tag{5.7.58}$$

Theorem 5.7.10. *Equations of gas dynamics*

$$\begin{cases} \vec{u}_0 + (\vec{u}\,\vec{\nabla})\vec{u} = -\dfrac{1}{\rho}\vec{\nabla}\left(\dfrac{\lambda}{2}\rho^2\right), \\ \rho_0 + \operatorname{div}(\rho\vec{u}) = 0 \end{cases} \tag{5.7.59}$$

are $c(Y)$-invariant with

$$Y_a = x_0^{n+1}\partial_a + x_0^n\partial_{u^a} + \alpha_a\partial_\rho, \qquad (\alpha_a = \text{const}). \tag{5.7.60}$$

Theorem 5.7.11. *The Navier-Stokes equations*

$$\vec{u}_0 + (\vec{u}\,\vec{\nabla})\vec{u} + \lambda\Delta\vec{u} = 0, \tag{5.7.61}$$

when $n = 1$, is $c(Y)$-invariant with

$$Y = x_1^2\partial_1 - (x_1 u^1 - 4\lambda)\partial_{u^1}. \tag{5.7.62}$$

Ansatz obtained by virtue of operator (5.7.56) has the form

$$u = x_0\varphi(\vec{x}) + 2/\lambda x_0$$

and it reduces Equation (5.7.55) to the nonlinear Laplace equation

$$\Delta\varphi = \lambda\varphi^2. \tag{5.7.63}$$

Operators (5.7.58) give the ansatz

$$u = \varphi(x_0) + x_0 r - \frac{(x_0^2 - 2mr)^{2/3}}{3m} \tag{5.7.64}$$

which reduces the Hamilton-Jacobi equation to the ODE

$$m\varphi' + x_0^2 = 0.$$

This latter can be easy solved, and as a result we find a solution of Equation (5.7.57)

$$u = x_0 r - \frac{(x_0^2 - 2mr)^{3/2} + x_0^3}{3m}, \qquad \left(r \le \frac{x_0^2}{2m}\right). \tag{5.7.65}$$

Operators (5.7.60) give the ansatz

$$\vec{u} = \vec{\varphi}(x_0) + \frac{\vec{x}}{x_0}, \qquad \rho = \varphi^0(x_0) + x_0^{-n-1}\vec{\alpha}\vec{x} \tag{5.7.66}$$

which reduces the gas-dynamics equations to the system of ODEs

$$\begin{cases} x_0\vec{\varphi}_0 + \vec{\varphi} + \lambda x_0^{-n}\vec{\alpha} = 0, \\ x_0\varphi_0^0 + n\varphi^0 + x_0^{-n}\vec{\alpha}\vec{\varphi} = 0. \end{cases} \tag{5.7.67}$$

This latter ODE can be easily integrated, and as a result we find a solution of Equations (5.7.59)

$$\begin{cases} \vec{u} = (\vec{x} + \vec{c})x_0^{-1} + \dfrac{\lambda}{n-1}x_0^{-n}\vec{\alpha}, \\ \rho = c_0 x_0^{-n} + \vec{\alpha}(\vec{x} + \vec{c})x_0^{-n-1} + \dfrac{\lambda\vec{\alpha}^2 x_0^{-2n}}{n(n-1)}. \end{cases} \tag{5.7.68}$$

We do not succeed in generalizing operator (5.7.62) for the multi-dimensional case, but the ansatz

$$u^1 = x_1\varphi(x_0) + \frac{2\lambda}{x_1} \tag{5.7.69}$$

is easily generalized as

$$\vec{u} = \vec{x}\left[\varphi(x_0) + \frac{2\lambda(2-n)}{\vec{x}^2}\right]. \tag{5.7.70}$$

Ansatz (5.7.70) reduces the multi-dimensional Navier-Stokes equation (5.7.61) to the ODE

$$\varphi' + \varphi^2 = 0$$

(it will be note, that in this case reduction is made on independent and dependent variables) whence follows a solution of Equations (5.7.61)

$$\vec{u} = \vec{x}\left[\frac{1}{x_0} + \frac{2\lambda(2-n)}{\vec{x}^2}\right]. \tag{5.7.71}$$

Remark 5.7.2. Since some reduced equations have wider symmetry than the initial equation, we can use this wider symmetry to construct nontrivial formulae of GS. In particular, the nonlinear Laplace equation (5.7.63), under $n = 6$, is conformally invariant, and if it has a solution $u_I(x)$, then

$$u_{II}(x) = x_0\sigma^2 u_I\left(\frac{\vec{x} + \vec{\theta}\vec{x}^2}{\sigma}\right) + \frac{2}{\lambda x_0}, \qquad (\sigma = 1 + 2\vec{\theta}\vec{x} + \vec{\theta}^2\vec{x}^2) \tag{5.7.72}$$

will be also its solution.

6. *Conditional symmetry and exact solutions of the Boussinesq equation*

It is known that the maximal invariance algebra of the Boussinesq equation

$$u_{00} = \tfrac{1}{2}\Delta u^2 + \Delta^2 u = 0, \quad u = u(x), \quad x = (x_0, \vec{x}) \in \mathrm{R}^{1+n} \tag{5.7.73}$$

is the extended Euclid algebra $\mathrm{A\widetilde{E}(1,n)}$ with basis elements

$$\partial_0 \equiv \frac{\partial}{\partial x_0}, \quad \partial_a \equiv \frac{\partial}{\partial x_a}, \quad J_{ab} = x_a\partial_b - x_b\partial_a, \quad D = 2x_0\partial_0 + x_a\partial_a - 2u\partial_u. \tag{5.7.74}$$

All inequivalent ansatze which reduce the two-dimensional ($n = 1$) equation (5.7.73) to the ODE constructed with the help of operators (5.7.74) are as follows

$$u = \varphi(\omega), \quad \omega = \alpha_0 x_0 + \alpha_1 x_1, \quad \alpha_0, \alpha_1 = \text{const};$$
$$u = \frac{1}{x_0}\varphi(\omega), \qquad \omega = \frac{x_1}{\sqrt{x_0}}. \tag{5.7.75}$$

Olver and Rosenau [160] considered ansatz

$$u = \varphi(\omega) - 4\mu^2 x_0^2, \quad \omega = x_1 + \mu x_0^2, \quad \mu = \text{const}, \tag{5.7.76}$$

which reduces the two-dimensional equation (5.7.73) to the ODE

$$\dddot{\varphi} + \varphi\dot{\varphi} + 2\mu\varphi = 8\mu^2\omega + c_1. \tag{5.7.77}$$

Ansatz (5.7.76) is invariant under the operator

$$Q = \partial_0 - 2\lambda x_0\partial_1 - 8\lambda^2 x_0\partial_u, \quad \lambda = -2\mu, \tag{5.7.78}$$

which does not belong to the invariance algebra (5.7.74), and therefore it is a non-Lie ansatz. Below (following [83*]) we systematically describe in terms of conditional symmetry non-Lie ansatze of type (5.7.76) which reduce the two-dimensional Boussinesq equation (5.7.73) to ODEs. Similar results were independently obtained by Levi and Winternitz [85*].

It will be noted that Clarkson and Kruskal [84*] have described ansatze

$$u(x) = f(x)\varphi(\omega) + g(x) \tag{5.7.79}$$

which reduce the two-dimensional Boussinesq equation to ODEs using direct substitution. The main difference of our approach from that of theirs consists crucially in using the conditional invariance of the equation which allows us to obtain not only the ansatze found in [84*] but other ones which cannot be obtained within the framework of the direct method [84*].

So, consider the two-dimensional Boussinesq equation

$$u_{00} + uu_{11} + u_1^2 + u_{1111} = 0. \qquad (5.7.80)$$

Theorem 5.7.12. [83*] *Equation* (5.7.80) *is Q-conditionally invariant under the operator*

$$Q = A(x)\partial_0 + B(x)\partial_1 + [\alpha(x)u + \beta(x)]\partial_u \qquad (5.7.81)$$

if functions $A(x)$, $B(x)$, $\alpha(x)$, $\beta(x)$ *satisfy the following equations:*

1) $A \neq 0$ (without loss of generality one can put $A = 1$)

$$\begin{aligned}
&\alpha = -2B_1, \quad \alpha_1 = B_{11} = 0, \quad \beta = -2B(B_0 + 2BB_1),\\
&\beta_1 = \tfrac{1}{2}B_{00} + (\alpha B)_0 + B_1(B_0 - 2BB_1 + \alpha B),\\
&\beta_{11} = -(\partial_0 + 4B_1)(\alpha_0 + \alpha^2), \qquad (5.7.82)\\
&\beta_{00} - 2B_0\beta_1 + 4B_1(\beta_0 - \beta_1 B + \alpha\beta) + 2\alpha_0\beta = 0;
\end{aligned}$$

2) $A = 0$, $B = 1$,

$$\begin{aligned}
&\alpha_0 = 0, \quad \alpha_{11} + 5\alpha\alpha_1 + 2\alpha^3 = 0,\\
&\beta_{11} + 3\alpha\beta_1 + 4\alpha^2\beta + 5\alpha_1\beta + 5\alpha_{11}(\alpha^2 - \alpha_1) + 5\alpha\alpha_1(\alpha_1 + 2\alpha^2) = 0,\\
&\beta_{1111} + 4\alpha_{111}\beta + 6\alpha_{11}(\beta_1 + \alpha\beta) + 4\alpha_1\big[(\alpha^2 + \alpha_1)\beta + \qquad (5.7.83)\\
&\qquad + (\beta_1 + \alpha\beta)_1 + \beta_{00} + 3\beta\beta_1 + 2\alpha\beta^2\big] = 0.
\end{aligned}$$

The proof is obtained by direct calculation according to formula (5.7.7).

In the first case there exists the general solutions of Equations (5.7.82) and it results in the operator

$$\begin{aligned}
Q = \partial_0 + (ax_1 + b)\partial_1 - 2[au + a(a' + 2a^2)x_1^2 + \\
+ (a'b + ab' + 4a^2b)x_1 + b(b' + 2ab)]\partial_u,
\end{aligned} \qquad (5.7.84)$$

where $a = a(x_0)$, $b = b(x_0)$ satisfy the ODEs

$$a'' + 2aa' - 4a^3 = 0, \quad b'' + 2ab' - 4a^2b = 0 \qquad (5.7.85)$$

Depending on explicit form of a, b there are several inequivalent operators

$$Q_1 = \partial_0 + x_0\partial_1 - 2x_0\partial_u, \quad (a = 0,\ b = x_0); \qquad (5.7.86)$$

$$Q_2 = x_0\partial_0 - (x_1 + 6x_0^5)\partial_1 + 2[u + 3(x_1^2x_0^{-2} - 24x_0^8 + 2x_1x_0^3)]\partial_u,$$

$$\left(a = -\frac{1}{x_0},\ b = 6x_0^5\right);$$

$$Q_3 = 2x_0\partial_0 + (x_1 - 3x_0^2)\partial_1 - 2(u - 3x_1 + 9x_0^2)\partial_u, \quad \left(a = \frac{1}{2x_0},\ b = -\frac{3}{2}x_0\right);$$

$$Q_4 = 2W\partial_0 + W'\left[x_1\partial_1 - (2u + Wx_1^2)\partial_u\right], \quad \left(a = \frac{W'}{2W},\ b = 0\right);$$

$$Q_5 = 2W\partial_0 + W'(x_1 + \rho)\partial_1 - \left[2W'u + WW'(x_1 + \rho)^2 + x_1 + \rho\right]\partial_u,$$

$$\left(a = \tfrac{1}{2}\frac{W'}{W},\ b = a\rho,\ \rho = \int \frac{W}{(W')^2}dx_0\right),$$

where $W = W(x_0)$ is the Weierstrass elliptic function satisfying the ODE $W'' = W^2$ or $(W')^2 = \frac{2}{3}(W^3 + \text{const})$.

In the second case we succeeded in obtaing several partial solutions of Equations (5.7.83) which results in the following operators

$$Q_6 = x_0^2\partial_1 + (x_0^5 - 2x_1)\partial_u, \quad \alpha = 0,\ \beta = x_0^3 - 2x_1x_0^{-2};$$

$$Q_7 = \partial_1 + (\Lambda - \tfrac{1}{3}Wx_1)\partial_u, \quad \alpha = 0,\ \beta = \Lambda + \tfrac{1}{3}Wx_1;$$

$$Q_8 = x_1\partial_1 + 2u\partial_u, \quad \alpha = \frac{2}{x_1},\ \beta = 0; \tag{5.7.87}$$

$$Q_9 = x_1^3\partial_1 + 2(x_1^2u + 24)\partial_u, \quad \alpha = \frac{2}{x_1},\ \beta = \frac{48}{x_1^3},$$

where $\Lambda = \Lambda(x_0)$ is the Lame function satisfying equation $\Lambda'' = W\Lambda$. Using operators (5.7.86), (5.7.87) we find ansatze:

$$1^\circ.\quad u = \varphi(\omega) - 4x_0^2, \quad \omega = x_1 + x_0^2; \tag{5.7.88}$$

$$2^\circ.\quad u = x_0^2\varphi(\omega) - \left(\frac{x_1}{x_0} + 6x_0^4\right)^2, \quad \omega = x_0(x_1 + x_0^5);$$

$$3^\circ.\quad u = x_0^{-1}\varphi(\omega) + 2\left(x_1 - x_0^2\right), \quad \omega = \frac{x_1 + x_0^2}{\sqrt{x_0}};$$

$$4^\circ.\quad u = \frac{1}{W}\varphi(\omega) - \frac{1}{6}Wx_1^2, \quad \omega = \frac{x_1}{\sqrt{W}};$$

$$5^\circ.\quad u = \frac{1}{W}\varphi(\omega) - \frac{1}{4}\left(\frac{W'}{W}\right)^2(x_1 + \rho)^2, \quad \omega = \frac{x_1}{\sqrt{W}} - \frac{1}{2}\int \frac{W'}{W^{3/2}}\rho\, dx_0;$$

$$6^\circ.\quad u = \varphi(\omega) - x_0^{-2}x_1^2 + x_0^3x_1, \quad \omega = x_0;$$

$$7^\circ.\quad u = \varphi(\omega) - \tfrac{1}{6}x_1^2W + \Lambda x_1, \quad \omega = x_0;$$

8°. $u = x_1^2\varphi(\omega), \quad \omega = x_0;$

9°. $u = x_1^2\varphi(\omega) - 12x_1^{-2}, \quad \omega = x_0.$

After substitution of ansatze (5.7.88) into (5.7.80) we get, respectively, the following reduced ODEs:

$$1°.\quad \varphi''' + \varphi\varphi' + 2\varphi = 8\omega + c_1; \tag{5.7.89}$$

$$2°.\quad \varphi''' + \varphi\varphi' + 30\varphi = 1800\omega + c_2;$$

$$3°.\quad \varphi'''' + (\varphi + \tfrac{1}{4}\omega^2)\varphi'' + (\varphi')^2 + \tfrac{7}{4}\omega\varphi' + 2\varphi = 0;$$

$$4°.\quad \varphi'''' + \varphi\varphi'' + (\varphi')^2 + \frac{\lambda}{6}(\omega^2\varphi'' + 7\omega\varphi' + 8\varphi) = 0;$$

$$5°.\quad \varphi'''' + \varphi\varphi'' + (\varphi')^2 + \frac{\lambda}{2}(\omega\varphi' + 2\varphi - \lambda\omega) = 0;$$

$$6°.\quad \varphi'' - \frac{2}{\omega^2}\varphi + \omega^6 = 0;$$

$$7°.\quad \varphi'' - \tfrac{1}{3}W\varphi + \Lambda^2 = 0;$$

$$8°, 9°.\quad \varphi'' + 6\varphi^2 = 0,$$

where c_1, c_2 are arbitrary constants.

Having solved the reduced Equations (5.7.89) and using once more formulae (5.7.88) we obtain solutions of Equation (5.7.80):

$$\begin{gathered} u = -\tfrac{1}{6}x_1^2 W(x_0), \quad u = -12x_1^{-2}, \\ u = -\tfrac{1}{6}x_1^2 W(x_0) - 12x_1^{-2}, \quad u = 2(x_1 - x_0^2) \\ u = 2(x_1 - x_0^2) - \frac{12}{(x_1 + x_0^2)^2}, \quad u = \frac{x_1^2}{x_0^2} - 6c_3^2x_0^8 + 18c_3x_0^3x_1 + \frac{c_4}{x_0} + c_5x_0^2. \end{gathered} \tag{5.7.90}$$

Now let us give some results on studying conditional symmetry of the multidimensional Boussinesq equation.

Theorem 5.7.13. [83*] *Equation* (5.7.73) *under* $n = 6$ *is invariant with respect to the conformal algebra* AC(6) *with basis elements*

$$\begin{gathered} \partial_a = \frac{\partial}{\partial x_a}, \quad J_{ab} = x_a\partial_b - x_b\partial_a, \quad D = x_a\partial_a - 4u\partial_u, \\ K_a = 2x_aD - x^2\partial_a, \quad a, b = 1, 2, \ldots, 6, \quad x^2 \equiv x_ax_a, \end{gathered} \tag{5.7.91}$$

if

$$\Delta u + \tfrac{1}{2}u^2 = 0. \tag{5.7.92}$$

An ansatz obtained by means of operator K_a has the form

$$u = \frac{1}{x^2}\varphi(\omega_1, \omega_2), \quad \omega_1 = x_0, \quad \omega_2 = \frac{bx - b^2x^2}{x^4}, \tag{5.7.93}$$

where b_a are arbitrary constants. The corresponding reduced equations are

$$\varphi_{11} = 0, \quad 2\omega_2\varphi_{22} + 5\varphi_2 = \frac{1}{4b^2}\varphi^2 \tag{5.7.94}$$

Partial solution of Equation (5.7.94) is $\varphi = -4b^2\omega_2^{-1}$ and it results in the following solution of the Boussinesq equation (5.7.73) under $n = 6$:

$$u = \frac{4}{x^2 - (\alpha x)^2}, \qquad \alpha_a = \text{const}, \quad \alpha^2 = 1.$$

7. In this point we give a brief review of conditional symmetry carried out by us and our collaborators up to 1991.

1°. Nonlinear wave equation

$$\Box u = \frac{n}{u} \tag{5.7.95}$$

is conformally invariant under the condition (see Appendix 6)

$$1 - u_\nu u^\nu = 0. \tag{5.7.96}$$

2°. Nonlinear wave equation [100*]

$$\Box u = \lambda_1 u^k \tag{5.7.97}$$

is conformally invariant under the condition

$$u_\nu u^\nu = \lambda_2 u^{k+1}. \tag{5.7.98}$$

The Liouville equation

$$\Box u = \lambda_1 e^u \tag{5.7.99}$$

is conformally invariant under the condition

$$u_\nu u^\nu = \lambda_2 e^u. \tag{5.7.100}$$

The Monge-Ampere equation

$$\det(u_{\mu\nu}) = 0 \tag{5.7.101}$$

is conformally invariant under the condition (5.7.96).

3°. Equations of the form

$$u_{00} - \vec{\nabla}\left[f(u)\vec{\nabla}u\right] = F(x, u, \underset{1}{u}) \qquad (5.7.102)$$

were considered in [108*] and it has been established that the following equations

$$\begin{aligned} &u_{00} - \partial_1(u^{-2}u_1) = -u^{-1}, \\ &u_{00} - \partial_1(\operatorname{ch}^{-2}u\, u_1) = -(u_0 + \operatorname{th} u), \\ &u_{00} - \partial_1(\cos^{-2} u\, u_1) = -(u_0 + \tan u), \\ &u_{00} - \vec{\nabla}(e^{2u}\vec{\nabla}u) = 0 \end{aligned} \qquad (5.7.103)$$

are invariant under the Poincare algebra if the conditions

$$\begin{aligned} &u_0^2 - u^{-2}u_1^2 = 1, \\ &u_0^2 - \operatorname{ch}^{-2}u\, u_1^2 = 1, \\ &u_0^2 - \cos^{-2} u\, u_1^2 = 1, \\ &u_0^2 - e^{2u}(\vec{\nabla}u)^2 = 1 \end{aligned} \qquad (5.7.104)$$

hold true, respectively.

Equations

$$\begin{aligned} &u_{00} - \partial_1(u^{-2}u_1) = -u^{-2}u_1 \operatorname{cotan} x_1, \\ &u_{00} - \partial_1(\operatorname{ch}^{-2}u_1) = 2\operatorname{sh}^{-1}2u, \\ &u_{00} - \partial_1(\cos^{-2} u_1) = 2\sin 2u, \\ &u_{00} - \vec{\nabla}(e^{2u}\vec{\nabla}u) = -n(u_0 \operatorname{th} x_0 + 1) \end{aligned} \qquad (5.7.105)$$

are conformally invariant under the conditions (5.7.104), respectively.

4°. Equation of nonlinear acoustic

$$u_{01} - \partial_1(f(u)u_1) - u_{22} - u_{33} = 0 \qquad (5.7.106)$$

which coincides, under $f(u) = u$, with the Khokhlov-Zabolotskaya equation

$$u_{01} - \partial_1(uu_1) - u_{22} - u_{33} = 0 \qquad (5.7.107)$$

was considered in [58*,106*]. In particular, there it is established that Equation (5.7.107) under the condition

$$u_0u_1 - uu_1^2 - u_2^2 - u_3^2 = 0 \qquad (5.7.108)$$

is invariant under infinite-dimensional algebra of special form.

It will be noted that Equations (5.7.103) under corresponding conditions (5.7.104) are locally equivalent to the system

$$\Box w = 0, \quad w_\nu w^\nu = 1, \tag{5.7.109}$$

and Equations (5.7.105) under corresponding conditions (5.7.104) are locally equivalent to the system

$$\Box w = \frac{n}{w}, \quad w_\nu w^\nu = 1. \tag{5.7.110}$$

5°. Q-conditional symmetry of the nonlinear wave equation

$$u_{00} - \partial_1(f(u)u_1) = g(u) \tag{5.7.111}$$

under

$f(u) = <\lambda e^u,\ \lambda u^k,\ \lambda u,\ \lambda/\sqrt{u},\ \lambda u^{-2/3},\ e^u + \lambda^2,\ \lambda u^{-4/5},\ \lambda u^4,\ \lambda u^{-4/3},\ \lambda u^{-4}>$,

$g(u) = <0,\ \lambda u,\ \lambda u^{-1/3},\ \lambda u^{-(2k+1)},\ \lambda e^{2u}>$

was studies in [94*, 105*].

6°. In [192] it is established that the d'Alembert equation

$$\Box u = 0$$

is invariant under the infinite-dimensional algebra $\widetilde{\mathrm{P}}^\infty(1, n)$ if the condition

$$u_\nu u^\nu = 0$$

holds true.

7°. In [56*, 90*, 129*] was treated the Q-conditional invariance of the nonlinear heat equation

$$u_0 + u_{11} = F(u), \tag{5.7.112}$$

which especially distinguished the following cases

$$\begin{aligned} u_0 + u_{11} &= \lambda u^3, \\ u_0 + u_{11} &= \lambda(u^3 - u), \\ u_0 + u_{11} &= \lambda(u^3 - u), \\ u_0 + u_{11} &= \lambda(u^3 - 3u + 2) \end{aligned} \tag{5.7.113}$$

8°. In [91*,104*] we considered the nonlinear heat equation

$$u_0 + \vec{\nabla}(f(u)\vec{\nabla}u) = F(u). \tag{5.7.114}$$

Generators of Q-conditional symmetry for the equations

$$\begin{aligned} &u_0 + \partial_1(e^u u_1) = 0, \\ &u_0 + \partial_1(u^{-1/2}u_1) = 0, \end{aligned} \tag{5.7.115}$$

were found, and it was established that the equation

$$u_0 + \vec{\nabla}(e^u\vec{\nabla}u) + \lambda e^{-u} = 0 \tag{5.7.116}$$

under the condition

$$u_0 + \frac{n}{2}e^u(\vec{\nabla}u)^2 + \frac{\lambda}{2}e^{-u} = 0 \tag{5.7.117}$$

is conformally invariant.

9°. Conditional symmetry of the Kadomtsev-Petviashvili equation

$$u_{01} + (u^2)_{11} + u_{22} + u_{1111} = 0 \tag{5.7.118}$$

is studied in [93*].

10°. In [92*] is considered the generalized Korteweg-de Vries equation

$$u_0 + F(u)u_1^k + u_{111} = 0. \tag{5.7.119}$$

It is shown that Equation (5.7.119) is Q-conditionally invariant with respect to the Galilei algebra under the conditions

$$\begin{aligned} &F(u) = \lambda_1 u^{(2-k)/2} + \lambda_2 u^{(1-k)/2}, \\ &F(u) = u^{1-k}(\lambda_1 \ln u + \lambda_2), \\ &F(u) = (1-u^2)^{(1-k)/2}(\lambda_1 \arcsin u + \lambda_2), \\ &F(u) = (1+u^2)^{(1-k)/2}(\lambda_1 \operatorname{arcsh} u + \lambda_2). \end{aligned} \tag{5.7.120}$$

11°. In [99*,101*] it is established that the Boussinesq equation

$$u_0 + \Delta u^2 = 0 \tag{5.7.121}$$

is Q-conditionally invariant under the Galilei algebra.

12°. Q-conditional symmetry of the Born-Infeld equation (5.7.35) is studied in [102*].

13°. In [107*] it is established that the nonlinear polywave equation

$$\Box^m u = \lambda u^{1-2m} \tag{5.7.122}$$

is invariant under the conformal algebra provided

$$\Box u = \frac{n}{u}, \quad u_\nu u^\nu = 1.$$

14°. In [95*,96*] Q-conditional symmetry of the gas dynamics equations is studied:

$$\begin{aligned} u_0^1 - u_1^2 &= 0, \\ u_0^2 - u_1^3 &= 0, \\ u_0^3 - F(u^1, u^3)u_1^2 &= 0 \end{aligned} \tag{5.7.123}$$

under

$$F(u^1, u^3) = < \lambda u^1, \lambda(u^1)^k, \lambda(u^1)^{-2}, \lambda(u^1)^{-1}, \lambda(u^1)^{-3/2}, \lambda(u^1)^2, \lambda(u^3)^{-1} > .$$

15°. In [31*, 137*] it is shown that the nonlinear Dirac equation

$$(i\gamma\partial - \lambda\overline{\psi}\psi)\psi = 0 \tag{5.7.124}$$

under the condition $\overline{\psi}\psi = 1$ is invariant with respect to the operators

$$Q_1 = i\partial_0 - \lambda\gamma_0, \quad Q_2 = i\partial_3 + \lambda\gamma_3. \tag{5.7.125}$$

16°. In [57*,103*,109*] it is studied the conditional symmetry of the nonlinear Schrödinger equation

$$\left(P_0 - \frac{\vec{p}^2}{2m}\right)\psi = F(|\psi|)\psi. \tag{5.7.126}$$

It is shown that Equation (5.7.126) with the nonlinearity

$$F(|\psi|) = \lambda_1|\psi|^{-k} + \lambda_2|\psi|^k \tag{5.7.127}$$

is invariant under the operator

$$Q = x_a P_a - \frac{2}{k}I + \ln\frac{\psi}{\psi^*}\left(\psi\partial_\psi - \psi^*\partial_{\psi}\right), \tag{5.7.128}$$

provided

$$\lambda_1\Delta|\psi| + \lambda_2|\psi|^{k+1} = 0. \tag{5.7.129}$$

5.8 *Nonlocal symmetry of quasirelativistic wave equation*

Consider a PDE of fourth order (on spatial variables) which, following [60,75], we shall call a *quasirelativistic* wave equation

$$P_0\psi = H(P)\psi, \qquad H(P) \equiv a_0\frac{P^2}{2m_0} + a_4\frac{P^4}{8}, \tag{5.8.1}$$

where $\psi = \psi(x)$ is a complex scalar function: $x = \{x_0 \equiv t,\ x_a\} \in \mathrm{R}^4$; a_0, a_4, m_0 are arbitrary real constants;

$$P_0 = i\frac{\partial}{\partial t}, \qquad P_a = -i\frac{\partial}{\partial x_a}, \qquad P^2 = P_aP_a, \qquad a = 1, 2, 3. \tag{5.8.2}$$

Under $a_0 = a_4 = 0$ Equation (5.8.1) coincides with the well-known Schrödinger equation, and under $a_0 = m_0$, $a_4 = -m_0^3$ the Hamiltonian of (5.8.1) represents by itself the first three terms of Taylor expansion of the relativistic Hamiltonian

$$\mathcal{H}(P) = \left(m_0^2 + P^2\right)^{1/2} \tag{5.8.3}$$

By means of Lie's method one can make sure that the maximal IA of Equation (5.8.1) is the 8-parameter Lie algebra A_8=AE(1,3), with basis elements

$$P_0,\ P_a,\ J_{ab} = x_aP_b - x_bP_a, \quad I.$$

This result means that Equation (5.8.1) is invariant neither under Lorentz transformations nor under Galilean ones. But, of course, it does not mean that symmetry properties of Equation (5.8.1) are exhausted by the first-order differential operators of AE(1,3). Equation (5.8.1) possesses a wide nonlocal symmetry.

Theorem 5.8.1 [60,75]. *Equation* (5.8.1) *is invariant under the 20-dimensional Lie algebra* $\mathrm{A}_{20} \supset \mathrm{A}_8$ *with basis elements*

$$P_0,\ P_a,\ J_{ab},\ I,$$
$$V_a = i[H, x_a] = \left(\frac{1}{m_0} + \frac{a_4}{2}P^2\right)P_a,$$
$$G_a = tV_a - x_a, \tag{5.8.4}$$
$$R_{ab} = -a_4(P_aP_b + \tfrac{1}{2}\delta_{ab}P^2).$$

Proof. One can directly verify that operators (5.8.4) satisfy invariance condition (10). Operators P_0, P_a, V_a, R_{ab}, and I commute and together with operators J_{ab}, G_a satisfy the following commutation relations

$$
\begin{aligned}
&[J_{ab}, J_{cd}] = i(\delta_{ac}J_{bd} + \delta_{bd}J_{ac} - \delta_{ad}J_{bc} - \delta_{bc}J_{ad}),\\
&[J_{ab}, G_c] = i(\delta_{ac}G_b - \delta_{bc}G_a), \qquad [J_{ad}, P_0] = 0,\\
&[P_a, G_b] = i\delta_{ab}I, \qquad [G_a, G_b] = 0,\\
&[P_0, G_a] = iV_a, \qquad [J_{ab}, P_c] = i(\delta_{ac}P_b - \delta_{bc}P_a),\\
&[V_a, G_b] = i(R_{ab} - \delta_{ab}a_2 I),\\
&[J_{ab}, R_{cd}] = i(\delta_{ac}R_{bd} + \delta_{bd}R_{ac} - \delta_{bc}R_{ad} - \delta_{ad}R_{bc}),\\
&[J_{ab}, V_c] = i(\delta_{ac}V_b - \delta_{bc}V_a),\\
&[G_a, R_{bc}] = ia_4(\delta_{ab}P_c + \delta_{bc}P_a + \delta_{ac}P_b),
\end{aligned}
\tag{5.8.5}
$$

and thereby form a Lie algebra. It will be noted that operators V_a, G_a, R_{ab} belong to the class of third-order differential operators and hence they generate nonlocal transformations.

Using formulae (5.3.6) we calculate final transformations generated by the operator

$$
Q = imv_a G_a = t\left(1 + \frac{m_0}{2}a_4 P^2\right) v_a \partial_a - im\vec{x}\cdot\vec{v}, \tag{5.8.6}
$$

where v_a are arbitrary real parameters. So, we have

$$
\begin{aligned}
t' &= e^{tv_a\partial_a} t e^{-tv_a\partial_a} = t,\\
x'_b &= e^{tv_a\partial_a} x_b e^{-tv_a\partial_a} = x_b + v_b t,
\end{aligned}
\tag{5.8.7}
$$

and

$$
\psi'(x') = e^{tv_a\partial_a} e^{-Q}\psi(x) \tag{5.8.8}
$$

Since

$$
\begin{aligned}
-[tv_a\partial_a, Q] &= im_0 v^2 t,\\
[tv_a\partial_a, im_0 v^2 t] &= [Q, im_0 v^2 t] = 0,
\end{aligned}
$$

one can use formula (5.3.39). Applying it to (5.8.8) one gets

$$
\begin{aligned}
\psi'(x') &= \exp\left\{tv_a\partial_a - Q + \frac{i}{2}m_0 v^2 t\right\}\psi(x) =\\
&= \exp\left\{im\left[x_a v_a + \frac{v^2}{2}t - \frac{1}{2}a_4 t P^2 v_a P_a\right]\right\}\psi(x).
\end{aligned}
\tag{5.8.9}
$$

So, we can conclude that Equation (5.8.1) is invariant under Galilean transformations (5.8.7), provided ψ-function is transformed according to (5.8.9). Unlike Galilean transformations given in the Table 4.1.2, the transformation (5.8.9) is nonlocal. It becomes local when $a_4 = 0$, and in this case it coincides with the standard Galilean transformation of the Schrödinger wave function.

Now we will show that Equation (5.8.1) describes uniform motion of a particle with rest mass m_0 and takes into account dependence of the full mass of the particle on its velocity. But before doing this we shall demonstrate that the relativistic integrodifferential wave equation $P_0\psi = \mathcal{H}(P)\psi$ with the Hamiltonian given in (5.8.3) describes the well-known effect of the growing mass of a particle by increasing its velocity according to the law

$$m = \frac{m_0}{\sqrt{1-v^2}}. \tag{5.8.10}$$

With the help of formulae (A.3.5), (A.3.6) one easily finds

$$\begin{aligned} \dot{x}_a \equiv V_a = i[\mathcal{H}(P), x_a] = i\left[\sqrt{P^2+m_0^2}, x_a\right] = \frac{P_a}{\sqrt{P^2+m_0^2}}; \\ \ddot{x}_a = \dot{V}_a = i[\mathcal{H}(P), V_a] = 0, \end{aligned} \tag{5.8.11}$$

where dot means differentiation with respect to t. Let us recall that in (5.8.11) quantities x_a, $\dot{x}_a \equiv V_a$, $\dot{V}_a$, P_a are operators of coordinates, velocities, accelerations, and of impulse, respectively. The transition to corresponding classical quantities is made by means of the change

$$x_a \to x_a, \qquad P_a = -i\frac{\partial}{\partial x_a} \to p_a \tag{5.8.12}$$

So, one finds from (5.8.11),(5.8.12) the expression for particle velocity

$$v_a = \frac{p_a}{\sqrt{p^2+m_0^2}} \tag{5.8.13}$$

On the other hand, according to the definition of velocity, we have

$$v_a = \frac{p_a}{m} \tag{5.8.14}$$

where m is the full mass of the particle. Equating these two expressions (5.8.13) and (5.8.14) and then solving the obtained equation with respect to m, we get the formula (5.8.10).

In the same spirit we shall act in the case of Equation (5.8.4), we can write according to (5.8.12) the corresponding classical expression

$$v_a = \frac{p_a}{m_0} + \frac{a_4}{2}p^2 p_a. \tag{5.8.15}$$

Substituting (5.8.14) into (5.8.15) we get a cubic equation with respect to m:

$$\frac{m}{m_0} + \frac{a_4}{2}m^3 v^2 - 1 = 0. \tag{5.8.16}$$

Let $a_4 < 0$. Solution of Equation (5.8.16) can be looked for in the form

$$m = \frac{m_0}{c} \sin \alpha \tag{5.8.17}$$

where α, c are arbitrary real constants. Substitution of (5.8.17) into (5.8.16) followed by multiplication on $3c$, yields

$$3 \sin \alpha - 4 \sin^3 \alpha \left(\frac{3}{8} m_0^3 |a_4| \frac{v^2}{c^2} \right) = 3c. \tag{5.8.18}$$

Let us define c so that

$$\frac{3}{8} m_0^3 |a_4| \frac{v^2}{c^2} = 1,$$

that is

$$c = \left(\frac{3}{8} m_0^3 |a_4| \right)^{1/2} v \stackrel{\text{def}}{=} \frac{1}{3} w, \qquad w = \left(\frac{3}{2} \right)^{3/2} v \sqrt{m_0^3 |a_4|}. \tag{5.8.19}$$

Now we can rewrite Equation (5.8.18) as

$$3c = 3 \sin \alpha - 4 \sin^3 \alpha \equiv \sin 3\alpha$$

whence follows

$$\alpha = \frac{1}{3} \arcsin 3c \equiv \frac{1}{3} \arcsin w \equiv \frac{1}{3} \arctan \frac{w}{\sqrt{1 - w^2}}.$$

Finally, we get the formula

$$m = m_0 \frac{3}{w} \sin \left(\frac{1}{3} \arctan \frac{w}{\sqrt{1 - w^2}} \right). \tag{5.8.20}$$

Starting from (5.8.20) we can conclude that in mechanics based on Equation (5.8.1) with $a_4 < 0$, as well as in relativistic mechanics, the limiting speed exists. But unlike relativistic mechanics, where the mass of a particle infinitely increases when velocity v tends to its limit $v \longrightarrow 1$ (see formula (5.8.10)), in the considered case we have according to (5.8.20)

$$m \underset{w \to 1}{\longrightarrow} \frac{3}{2} m_0. \tag{5.8.21}$$

Now consider the case of Equation (5.8.1) with $a_4 > 0$. Solution of Equation (5.8.16) we look for in the form

$$m = \frac{m_0}{c} \operatorname{sh} \alpha.$$

As a result we obtain

$$m = m_0 \frac{3}{w} \operatorname{sh} \left(\frac{1}{3} \ln \left(w + \sqrt{1+w^2} \right) \right), \tag{5.8.22}$$

where w is defined in (5.8.19). So, in mechanics based on the Equation (5.8.1) with $a_4 > 0$ the limiting speed does not exist, and when velocity v aims at infinity the mass of particle tends to zero.

In conclusion, it will be noted that the same analysis can be made for an equation of the type (5.8.1) with the Hamiltonian

$$H(P) = \sum_{n=0}^{N} a_{2n} P^{2n}, \qquad N < \infty.$$

5.9 *Are Maxwell's equations invariant under Galilean transformations?*

More than eighty years ago Lorentz, Poincare, and Einstein gave negative answer on this question we are about to discuss. At present it is common knowledge that Maxwell equations (ME)

$$\begin{aligned} \dot{\vec{E}} &\equiv \frac{\partial \vec{E}}{\partial t} = \operatorname{rot} \vec{H}, \qquad \operatorname{div} \vec{E} = 0, \\ \dot{\vec{H}} &\equiv \frac{\partial \vec{H}}{\partial t} = -\operatorname{rot} \vec{E}, \qquad \operatorname{div} \vec{H} = 0 \end{aligned} \tag{5.9.1}$$

are invariant under the Lorentz transformations and are not invariant under the Galilei ones

$$x'_a = x_a + v_a t, \qquad t' = t; \qquad a = 1, 2, 3, \tag{5.9.2}$$

where v_a are arbitrary constants (the speed of inertial frame of reference). The negative answer on the discussed question is connected with the implicit suggestion that the relationship between $\vec{E}$, $\vec{H}$ and $\vec{E}'$, $\vec{H}'$ is local, that is, the field transformation $\vec{E} \to \vec{E}'$, $\vec{H} \to \vec{H}'$ depends only on unprimed fields (and, of course, on parameters v_a) and do not depend on their derivatives. But if we assume that field transformations may be nonlocal, we will have new possibilities, and our question will have a positive answer [111*].

Theorem 5.9.1. *Maxwell equations* (5.9.1) *are invariant under Galilei transformations* (5.9.2), *provided the electric and magnetic fields are transformed as*

$$\begin{aligned} \vec{E}' &= \vec{E} - \vec{v} \times \vec{H} - (\vec{v} \cdot \vec{x}) \operatorname{rot} \vec{H} + O(v^2), \\ \vec{H}' &= \vec{H} + \vec{v} \times \vec{E} + (\vec{v} \cdot \vec{x}) \operatorname{rot} \vec{E} + O(v^2). \end{aligned} \tag{5.9.3}$$

Proof. We offer a straightforward proof of this theorem. As follows from (5.9.2)

$$\vec{\nabla}' = \vec{\nabla}, \qquad \partial_{t'} = \partial_t - \vec{v}\cdot\vec{\nabla} \tag{5.9.4}$$

Substituting (5.9.3), (5.9.4) into primed ME we get, omitting terms quadratic in v_a:

$$\begin{aligned}(\operatorname{div}\vec{E})' &= \operatorname{div}\vec{E}' = \operatorname{div}\vec{E} - \operatorname{div}(\vec{v}\times\vec{H}) - \operatorname{div}[(\vec{v}\cdot\vec{x})\operatorname{rot}\vec{H}]\\ &= \operatorname{div}\vec{E} + \vec{v}\cdot\operatorname{rot}\vec{H} - \vec{v}\cdot\operatorname{rot}\vec{H} = \operatorname{div}\vec{E} = 0,\end{aligned}$$

where we used the identities

$$\begin{aligned}\operatorname{div}(\vec{v}\times\vec{H}) &= -\vec{v}\cdot\operatorname{rot}\vec{H},\\ \operatorname{div}[(\vec{v}\cdot\vec{x})\operatorname{rot}\vec{H}] &= \vec{v}\cdot\operatorname{rot}\vec{H}.\end{aligned}$$

Further

$$\begin{aligned}\partial_{t'}\vec{E}' - (\operatorname{rot}\vec{H})' = (\partial_t - \vec{v}\cdot\vec{\nabla})\vec{E}' - \operatorname{rot}\vec{H}' = \dot{\vec{E}} - \vec{v}\times\dot{\vec{H}} - (\vec{v}\cdot\vec{x})\operatorname{rot}\dot{\vec{H}} -\\ - (\vec{v}\cdot\vec{\nabla})\vec{E} - \operatorname{rot}\vec{H} - \operatorname{rot}(\vec{v}\times\vec{E}) - \operatorname{rot}[(\vec{v}\cdot\vec{x})\operatorname{rot}\vec{E}],\end{aligned}$$

(dot means differentiation with respect to t). Taking into account the identities

$$\begin{aligned}\operatorname{rot}(\vec{v}\times\vec{E}) &= -\vec{v}\cdot\vec{E} + \vec{v}\operatorname{div}\vec{E},\\ \operatorname{rot}[(\vec{v}\cdot\vec{x})\operatorname{rot}\vec{E}] &= \vec{v}\cdot\vec{x}\operatorname{rot}\operatorname{rot}\vec{E} + \vec{v}\times\operatorname{rot}\vec{E}\end{aligned}$$

and using (5.9.1) we get

$$\begin{aligned}\partial_{t'}\vec{E}' - (\operatorname{rot}\vec{H})' &= \dot{\vec{E}} - \vec{v}\times\dot{\vec{H}} - (\vec{v}\cdot\vec{x})\operatorname{rot}\dot{\vec{H}} - (\vec{v}\cdot\vec{\nabla})\vec{E} - \operatorname{rot}\vec{H} +\\ &+ (\vec{v}\cdot\vec{\nabla})\vec{E} - \vec{v}\operatorname{div}\vec{E} - (\vec{v}\cdot\vec{x})\operatorname{rot}\operatorname{rot}\vec{E} - \vec{v}\times\operatorname{rot}\vec{E} =\\ &= \dot{\vec{E}} - \operatorname{rot}\vec{H} - \vec{v}\times(\dot{\vec{H}} + \operatorname{rot}\vec{E}) - (\vec{v}\cdot\vec{x})\operatorname{rot}(\dot{\vec{H}} + \operatorname{rot}\vec{E}) - \vec{v}\operatorname{div}\vec{E} = 0.\end{aligned}$$

In the same way one can test the invariance of the rest of the equations of the system (5.9.1) with respect to Galilei transformations (5.9.2), (5.9.3). Thus the theorem is proved.

Comparing transformations (5.9.2), (5.9.3) with the infinitesimal Lorentz transformations

$$\vec{x}' = \vec{x} + \vec{v}t + O(v^2), \qquad t' = t + \vec{v}\cdot\vec{x} + O(v^2); \tag{5.9.5}$$

$$\begin{aligned}\vec{E}' &= \vec{E} - \vec{v}\times\vec{H} + O(v^2),\\ \vec{H}' &= \vec{H} + \vec{v}\times\vec{E} + O(v^2)\end{aligned} \tag{5.9.6}$$

one immediately sees that the simplicity of geometrical transformations (5.9.2) results in complexity (nonlocality) of transformations (5.9.3). To clarify the meaning of the result obtained, let us write down the system (5.9.1) in another equivalent form

$$i\frac{\partial \psi}{\partial t} = \mathcal{H}\psi, \qquad \mathcal{H} = i\widehat{\sigma}_2(\vec{S}\cdot\vec{\nabla}),$$
$$\operatorname{div}\vec{E} = 0, \qquad \operatorname{div}\vec{H} = 0, \tag{5.9.7}$$

where

$$\psi = \psi(x) = \text{column}(E_1\, E_2\, E_3\, H_1\, H_2\, H_3), \qquad \widehat{\sigma}_2 = i\begin{pmatrix} \widehat{0} & -I_3 \\ I_3 & \widehat{0}\end{pmatrix} \tag{5.9.8}$$

I_3, $\widehat{0}$ are 3×3 unit and zero matrices, respectively;

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.9.9}$$

It is not difficult to show that system (5.9.7) is invariant under the following two Poincare algebras

$$P^I_\mu = P_\mu = \frac{\partial}{\partial x^\mu}, \qquad \mu = \overline{0,3}, \qquad x^0 \equiv t,$$
$$J^I_{ab} = x_a P_b - x_b P_a + i\epsilon_{abc}\begin{pmatrix} S_c & 0 \\ 0 & S_c \end{pmatrix}, \tag{5.9.10}$$
$$J^I_{0a} = x_0 P_a - x_a P_0 + \widehat{\sigma}_2 S_a;$$

and

$$P^{II}_0 = -i\mathcal{H}, \qquad P^{II}_a = P^I_a, \qquad J^{II}_{ab} = J^I_{ab},$$
$$J^{II}_{0a} = tP_a - \frac{i}{2}(\mathcal{H}x_a + x_a\mathcal{H}) + \tfrac{1}{2}\widehat{\sigma}_2 S_a. \tag{5.9.11}$$

Operators J^I_{0a} generate the well-known Lorentz transformations, while operators J^{II}_{0a} generate another (nonlocal) transformation. Since J^{II}_{0a} are not differential operators of the first order (in the Lie sense), we cannot use Lie's equations to find the corresponding group action. In this case we shall use formulae (5.3.6), according to which we find

$$t' = e^{t(\vec{v}\cdot\vec{\nabla})} t e^{-t(\vec{v}\cdot\vec{\nabla})} = t$$
$$\vec{x}' = e^{t(\vec{v}\cdot\vec{\nabla})} \vec{x} e^{-t(\vec{v}\cdot\vec{\nabla})} = \vec{x} + \vec{v}t; \tag{5.9.12}$$

$$\psi'(x') = \exp\{t(\vec{v}\cdot\vec{\nabla})\}\exp\{-t(\vec{v}\cdot\vec{\nabla}) + \widehat{\sigma}_2(\vec{S}\cdot\vec{v} + \vec{x}\cdot\vec{v}\vec{S}\cdot\vec{\nabla})\}\psi(x), \tag{5.9.13}$$

where function $\psi(x)$ is defined in (5.9.8).

Infinitesimal transformations (5.9.3) result from (5.9.13) in first order of v_a. Final geometrical transformations (5.9.12) coincide with the Galilei ones (5.9.2). As we see, representations of AP(1,3) given by (5.9.10), (5.9.11) are inequivalent because they result in different group actions. But on the manifold of solutions of ME (5.9.7) we have

$$J^{I}_{0a}\psi = J^{II}_{0a}\psi, \qquad (J^{I}_{0a})^2\psi = (J^{II}_{0a})^2\psi, \ldots \tag{5.9.14}$$

where ψ is an arbitrary solution of ME, and therefore they are equivalent in a sense. The idea of duality of spacetime symmetry of relativistic equations was first put forward in [59] (see also §5.3).

An important application of transformations (5.9.12), (5.9.13) consists in the possibility of correctly introducing the concept of approximate Galilei invariance with absolute time for relativistic equations [111*]. It is obvious that we can interpret transformations (5.9.2), (5.9.3) as an approximate Galilei invariance of ME. Formula (5.9.13) allows us to calculate Galilei transformations for the electromagnetic field in any approximation in v_a. So, for example, the second approximation has the form

$$\begin{aligned}
\vec{E}' &= \vec{E} - \vec{v}\times\vec{H} - \left(\vec{v}\cdot\vec{x} + \tfrac{1}{2}\vec{v}^2 t\right)\ \mathrm{rot}\,\vec{H} + \\
&+\tfrac{1}{2}\left[\vec{v}^2\vec{E} - \vec{v}(\vec{v}\cdot\vec{E}) + (\vec{x}\cdot\vec{v})\big((\vec{v}\cdot\vec{\nabla})\vec{E} - 2\vec{v}\times\ \mathrm{rot}\,\vec{E}\big) + (\vec{x}\cdot\vec{v})^2\Delta\vec{E}\right] + O(v^3), \\
\vec{H}' &= \vec{H} + \vec{v}\times\vec{E} + \left(\vec{v}\cdot\vec{x} + \tfrac{1}{2}\vec{v}^2 t\right)\ \mathrm{rot}\,\vec{E} + \\
&+\tfrac{1}{2}\left[\vec{v}^2\vec{H} - \vec{v}(\vec{v}\cdot\vec{H}) + (\vec{x}\cdot\vec{v})\big((\vec{v}\cdot\vec{\nabla})\vec{H} - 2\vec{v}\times\ \mathrm{rot}\,\vec{H}\big) + (\vec{x}\cdot\vec{v})^2\Delta\vec{H}\right] + O(v^3).
\end{aligned} \tag{5.9.15}$$

The Lorentz transformations in second approximation look like

$$\begin{aligned}
\vec{x}' &= \vec{x} + \vec{v}t + \tfrac{1}{2}\vec{v}(\vec{x}\cdot\vec{v}) + O(v^3), \\
t' &= t + \vec{x}\cdot\vec{v} + \tfrac{1}{2}\vec{v}^2 t + O(v^3), \\
\vec{E}' &= \vec{E} - \vec{v}\times\vec{H} + \tfrac{1}{2}\left[\vec{v}^2\vec{E} - \vec{v}(\vec{v}\cdot\vec{E})\right] + O(v^3), \\
\vec{H}' &= \vec{H} + \vec{v}\times\vec{E} + \tfrac{1}{2}\left[\vec{v}^2\vec{H} - \vec{v}(\vec{v}\cdot\vec{H})\right] + O(v^3).
\end{aligned} \tag{5.9.16}$$

The main difference between transformations (5.9.2), (5.9.15) and (5.9.16) (as well as between (5.9.2), (5.9.3) and (5.9.5), (5.9.6)) is that the time t in Galilei transformations (5.9.2) is unchanged in contrast with Lorentz transformations (5.9.5), (5.9.16). This advantage of Galilei transformations is due to the nonlocal law of transformations of $\vec{E}$ and $\vec{H}$ (compare (5.9.3), (5.9.6) and (5.9.15), (5.9.16)).

5.10 *Nonlocal spacetime symmetry of the Klein-Gordon equation*

Consider the Klein-Gordon wave equation

$$(\Box + m^2)\varphi = 0 \tag{5.10.1}$$

and write it down in equivalent form [59]

$$\begin{aligned} &i\frac{\partial \phi}{\partial t} = H\phi, \\ &H = \frac{1}{2æ}\left[(E^2 + æ^2)\sigma_1 + (E^2 - æ^2)i\sigma_2\right], \\ &E^2 = -\Delta + m^2, \end{aligned} \tag{5.10.2}$$

where ϕ is the two-component function

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \qquad \phi_1 = \frac{i}{æ}\frac{\partial \varphi}{\partial t}, \qquad \phi_2 = \varphi \tag{5.10.3}$$

$æ \neq 0$ is an arbitrary constant, σ_1 and σ_2 are 2×2 Pauli matrices (2.1.3). According to Remark 5.3.8 and Theorem 5.3.4, Equation (5.10.2) is invariant under Galilei transformation (5.3.43) (with H given in (5.10.2)).

It will be noted that, in contrast with the Dirac equation, operators

$$J_{0a}^{II} = x_0 P_a - \tfrac{1}{2}(Hx_a + x_a H) \tag{5.10.4}$$

are nonlocal even on the set of solutions of Equation (5.10.2), where they take the form [111*]

$$J_{0a}^{I} = t\partial_a + x_a\partial_t + \frac{i}{2æ}(\sigma_1 + i\sigma_2)\partial_a \tag{5.10.5}$$

Operators (5.10.5) generate transformation (A.3.3) and

$$\phi'(x') = \tfrac{1}{2}\left[(\sigma_0 + \sigma_3)\operatorname{ch}\theta + \sigma_0 - \sigma_3 - \frac{i}{æ}(\sigma_1 + i\sigma_2)\frac{\vec{\theta}\cdot\vec{\nabla}}{\theta}\operatorname{sh}\theta\right]\phi(x), \tag{5.10.6}$$

where $\vec{\theta} = \{\theta_1, \theta_2, \theta_3\}$, $\theta = (\theta_1^2 + \theta_2^2 + \theta_3^2)^{1/2}$, σ_0 is a unit 2×2 matrix. It will be stressed that Equation (5.10.2) is not Lorentz invariant in the Lie sense. As we see, Lorentz transformations of function $\phi(x)$ (5.10.6) are nonlocal and therefore non-Lie.

From (5.10.6) one can derive, according to the general relation (17), the formula of generating new solutions. This formula has the form [111*]

$$\phi_{II}(x) = \tfrac{1}{2}\left[(1 + \sigma_3)\operatorname{sech}\theta + 1 - \sigma_3 + \frac{i}{æ}(\sigma_1 + i\sigma_2)\frac{\vec{\theta}\cdot\vec{\nabla}}{\theta}\operatorname{th}\theta\right]\phi_I(x'), \tag{5.10.7}$$

where x' are given in (A.3.3).

5.11 *On solutions of the Schrödinger equation, invariant under the Lorentz algebra*

The Lie-maximal invariance algebra of the Schrödinger Equation (3.3.24) is given in (3.3.25). The algebra ASch(1,3) does not contain the Lorentz algebra ASO(1,3). But if to look for symmetry operators among non-Lie operators we will have new possibilities. So, in [115*] it is established that the Schrödinger equation (3.3.24) is invariant under the Lorentz algebra with basis elements belonging to pseudodifferential operators. Indeed, one can make sure that the operators

$$J_{0a} = \frac{1}{2m}(PG_a + G_aP), \qquad J_{ab} = x_aP_b - x_bP_a, \tag{5.11.1}$$

where $P = \sqrt{P_aP_a}$ ($P_a = -i\partial_a$, $G_a = tP_a - mx_a$), satisfy invariance condition

$$[\overset{\vee}{S}, J_{\mu\nu}]u = 0; \qquad \overset{\vee}{S} \equiv P_0 - \frac{\vec{P}^2}{2m}, \qquad \mu,\nu = \overline{0,3} \tag{5.11.2}$$

and form the Lorentz algebra ASO(1,3) with commutation relations written in (A.2.2). The action of pseudodifferential operators J_{0a} from (5.11.1) is defined by means of integral Fourier transformation in (1.3.17).

Following [89*], we look for solutions of the Schrödinger equation (3.3.24), invariant under the operators (5.11.1). The corresponding ansatz is defined from the condition

$$J_{0a}u = 0, \qquad J_{ab}u = 0. \tag{5.11.3}$$

If we go over in (5.11.3) to Fourier transform, we get

$$\begin{gathered}\left[\left(\frac{t}{m} - \frac{i}{2k^2}\right)k_a - i\frac{\partial}{\partial k_a}\right]\widetilde{u}(t,\vec{k}) = 0,\\ \left(k_a\frac{\partial}{\partial k_b} - k_b\frac{\partial}{\partial k_a}\right)\widetilde{u}(t,\vec{k}) = 0.\end{gathered} \tag{5.11.4}$$

The general solution of Equations (5.11.4) has the form

$$\widetilde{u}(t,\vec{k}) = (\vec{k}^2)^{-1/4}\exp\left\{-it\frac{\vec{k}^2}{2m}\right\}\varphi(t). \tag{5.11.5}$$

Substitution of (5.11.5) into the Fourier transform of the Schrödinger equation results in

$$\frac{\partial\varphi}{\partial t} = 0. \tag{5.11.6}$$

So, we obtain that the solution of the Schrödinger equation which is invariant under ASO(1,3) (5.11.1)

$$u(t,\vec{x}) = c\int\limits_{R^3} e^{i\vec{k}\cdot\vec{x}}(\vec{k}^2)^{-1/4}\exp\left\{-it\frac{\vec{k}^2}{2m}\right\}d^3k, \tag{5.11.7}$$

where c is an arbitrary constant. To simplify the expression (5.11.7) let us go over to spherical coordinates, choosing the polar axis along the vector $\vec{x}$. In this case expression (5.11.7) can be rewritten as

$$u(t,\vec{x}) = c\int\limits_0^\infty k^{3/2}dk\int\limits_0^\pi \sin\theta d\theta\int\limits_0^{2\pi} d\varphi\exp\left\{ikx\cos\theta - it\frac{\vec{k}^2}{2m}\right\} =$$

$$= \frac{4\pi c}{x}\int\limits_0^\infty k^{1/2}\exp\left\{-it\frac{\vec{k}^2}{2m}\right\}\sin kx\, dk, \tag{5.11.8}$$

where $x = \sqrt{\vec{x}^2}$, $k = \sqrt{\vec{k}^2}$. After calculation of the last integral, we finally find [89*]

$$u(t,\vec{x}) = \frac{c\sqrt{x}}{t^{3/2}}e^{i\omega}\left(J_{-1/4}(\omega) + iJ_{3/4}(\omega)\right), \tag{5.11.9}$$

where $\omega = mx^2/4t$; $J_{-1/4}$, $J_{3/4}$ are Bessel functions. One can make sure that (5.11.9) satisfies Schrödinger equation (3.3.24) and it is solution invariant under the Lorentz algebra (5.11.1).

In conclusion, it will be noted that the substitution $t \to -it$ transforms the Schrödinger equation (3.3.24) into the heat equation

$$\frac{\partial u}{\partial t} - \frac{1}{2m}\Delta u = 0. \tag{5.11.10}$$

If we do the same transformation with expression (5.11.9), we obtain the following (real) solution of the heat equation (5.11.10):

$$u = \frac{c\sqrt{x}}{t^{3/2}}e^{-\omega}\left(I_{-1/4}(\omega) - I_{3/4}(\omega)\right), \qquad \omega = \frac{mx^2}{4t}, \tag{5.11.11}$$

where c is an arbitrary real constant, $I_{-1/4}$, $I_{3/4}$ are modified Bessel functions.

It will be appropriate to give here the solution of the Schrödinger equation (3.3.24) invariant under 6-dimensional Galilei algebra

$$G_a = tP_a - mx_a, \qquad J_{ab} = x_aP_b - x_bP_a. \tag{5.11.12}$$

The corresponding ansatz is (compare with (5.11.5)):

$$u(t,\vec{x}) = \exp\left\{i\frac{m\vec{x}^2}{2t}\right\}\varphi(t). \tag{5.11.13}$$

Substitution of (5.11.13) into (3.3.24) results in the ODE

$$t\dot{\varphi} + \tfrac{3}{2}\varphi = 0,$$

the general solution of which is $\varphi = \text{const} \cdot t^{-3/2}$. So, the solution of the Schrödinger equation invariant under the Galilei algebra (5.11.12) is

$$u(t, \vec{x}) = ct^{-3/2} \exp\left\{i\frac{m\vec{x}^2}{2t}\right\}, \tag{5.11.14}$$

and one can compare it with the solution (5.11.9). It will be stressed that solution (5.11.9) is not Lorentz invariant in the standard meaning, because basis elements of the Lorentz algebra (5.11.1), under which (5.11.9) is invariant, are non-Lie (nonlocal).

5.12 *On approximate symmetry and approximate solutions of the nonlinear wave equation with a small parameter*

Following [62*], we introduce the concept of approximate symmetry and describe all nonlinearities $F(u)$ with which the nonlinear wave equation

$$\Box u + \lambda u^3 + \epsilon F(u) = 0 \tag{5.12.1}$$

($\Box = \partial_\mu \partial^\mu$, $\mu = \overline{0,3}$; λ is an arbitrary constant, $\epsilon \ll 1$ is a small parameter, $u = u(x)$, $x \in \mathrm{R}(1,3)$) is approximately scale and conformally invariant.

By means of Lie's method one can make sure that when $F(u) \neq 0$ and $F(u) \neq u^3$, Equation (5.12.1) is invariant under the Poincare group P(1,3) only, because the term $\epsilon F(u)$ breaks down the scale and conformal symmetry of the equation $\Box u + \lambda u^3 = 0$.

Let us represent an arbitrary solution, analytic in ϵ, of Equation (5.12.1) in the form

$$u = w + \epsilon v, \tag{5.12.2}$$

where w and v are some smooth functions of x. After sustitution of (5.12.2) into (5.12.1) and equating to zero the ccoefficients of the zero and first power of ϵ, we get the following system of PDEs

$$\begin{aligned} \Box w + \lambda w^3 &= 0 \\ \Box v + 3\lambda w^2 v + F(w) &= 0. \end{aligned} \tag{5.12.3}$$

Definition 5.12.1. We shall call the approximate symmetry of Equation (5.12.1) the *exact* symmetry of the system (5.12.3).

Theorem 5.12.1. [62*] *Equation* (5.12.1) *is approximately scale invariant (in the sense of the definition above) if and only if*

$$F(u) = \begin{cases} \dfrac{2\lambda b}{k+1}u^3 + \dfrac{3\lambda c}{k}u^2 + au^{2-k}, & k \neq 0, -1; \\ 2\lambda b u^3 + 3\lambda c u^2 \ln u + au^2, & k = 0; \\ 2\lambda b u^3 \ln u - 3\lambda c u^2 + au^3, & k = -1 \end{cases} \tag{5.12.4}$$

(k, a, b, c are arbitrary constants), with the generator of scale transformations having the form

$$D = x\partial - w\partial_w + (kv + bw + c)\partial_v \tag{5.12.5}$$

Proof. Using Lie's algorithm we find from the condition of invariance that the generator of scale transformations should have the form

$$D = x\partial - w\partial_w + \eta^2(v, w)\partial_v,$$

provided that from the invariance of the second equation of system (5.12.3),

$$\begin{aligned} &\eta^2_{vv} = \eta^2_{ww} = \eta^2_{wv} = 0 \Rightarrow \eta^2 = kv + bw + c, \\ &2\lambda b w^3 + 3\lambda c w^2 + (2-k)F - w\frac{dF}{dw} = 0. \end{aligned} \tag{5.12.6}$$

The general solution of Equation (5.12.6) is given in (5.12.4). Thus the theorem is proved.

In particular, as follows fron the above equation, the equation

$$\Box u + \lambda u^3 + \epsilon u = 0 \tag{5.12.7}$$

is approximately scale invariant and the corresponding generator has the form $D = x\partial - w\partial_w + v\partial_v$. This statement holds true even if $\lambda = 0$.

Theorem 5.12.2. [62*] *Equation* (5.12.1) *is approximately conformally invariant if and only if*

$$F(u) = -3\lambda\beta u^2 + au^3, \tag{5.12.8}$$

with the generator of conformal transformations having the form

$$\mathcal{K} = 2cx\left[x\partial - w\partial_w - (v-\beta)\partial_v\right] - x^2 c\partial \tag{5.12.9}$$

where β, a, c_μ are arbitrary constants. The proof of Theorem 5.12.2 is performed in the same spirit as that of the Theorem 5.12.1.

It will be noted that Equation (5.12.1) with $F(u)$ given in (5.12.8) is reduced within to ϵ to the equation $\Box u + \lambda' u^3 = 0$, where $\lambda' = \lambda + \epsilon a$ be means of the transformation $u \to u + \epsilon\beta$. It is clear that the exact solutions of the associated system (5.12.3) produce the approximate solutions of the Equation (5.12.1). To find exact solutions of the system (5.12.3) one may use its symmetry under $\tilde{P}(1,3)$ and C(1,3) described above, applying the algorithm we have used throughout the book. Below, we consider another possiblilty. Suppose that in (5.12.2)

$$v = f(w), \tag{5.12.10}$$

where f is an arbitrary differentiable function. In this case the system (5.12.3) takes the form

$$\Box w + \lambda w^3 = 0 \tag{5.12.11}$$

$$\begin{gathered} (w_\mu w^\mu)\ddot{f} + (\Box w)\dot{f} + 3\lambda w^2 f + F(w) = 0, \\ w_\mu = \frac{\partial w}{\partial x^\mu}, \qquad \dot{f} = \frac{df}{dw} \end{gathered} \tag{5.12.12}$$

From the condition of splitting of Equation (5.12.12) one has to put

$$w_\mu w^\mu = A(w), \tag{5.12.13}$$

where A is some function of w. Equation (5.12.13) is compatible with (5.12.11) if $A(w) = \lambda w^4$, that is

$$w_\mu w^\mu = \lambda w^4 \tag{5.12.14}$$

(see Appendix 6).

Taking account of (5.12.11) and (5.12.14) we rewrite (5.12.12) as

$$\lambda(w^2 \ddot{f} - w\dot{f} + 3f) + w^{-2} F(w) = 0. \tag{5.12.15}$$

So, if we find function $f(w)$ as solution of Equation (5.12.15), we thereby obtain by means of expressions (5.12.2), (5.12.10) approximate solutions of Equation (5.12.1). It will be noted that a subset of such solutions of Equation (5.12.11) and (5.12.14) is conformally invariant since the corresponding approximate system (5.12.11) and (5.12.15) is conformally invariant. Solutions of Equation (5.12.15) for functions $F(w)$ given in (5.12.4) have the form

$$f(w) = \begin{cases} -\dfrac{a}{\lambda[k(k+2)+3]w^k} - \dfrac{b}{k+1}w - \dfrac{c}{k}, & k \neq -1, 0 \\ -c\ln w - bw - \dfrac{1}{3}\left(2c + \dfrac{a}{\lambda}\right), & k = 0 \\ -w\left(\dfrac{a}{2\lambda} + b\ln w\right) + c, & k = -1 \end{cases} \tag{5.12.16}$$

The solution of the system (5.12.11) and (5.12.14) is the function

$$w = \pm\left[\lambda(x_\nu + a_\nu)(x^\nu + a^\nu)\right]^{-1/2} \tag{5.12.17}$$

where a_ν are arbitrary constants.

When $\lambda = 0$, the nontrivial condition of splitting of Equation (5.12.12) compatible with the Equation $\Box w = 0$ is

$$w_\mu w^\mu = 1 \tag{5.12.18}$$

So, in this case we find approximate solutions of equation (5.12.1) be means of expressions (5.12.2) and (5.12.10), where function is determined from the equation

$$\ddot{f} + F(w) = 0 \tag{5.12.19}$$

and w, in turn, is determined from the system

$$\Box w = 0, \qquad w_\mu w^\mu = 1. \tag{5.12.20}$$

The system (5.12.20) is invariant under the extended Poincare group $\widetilde{\mathrm{P}}(1,4)$ and has solution

$$w = \alpha_\nu x^\nu + a, \qquad \alpha_\nu a^\nu = 1 \tag{5.12.21}$$

where α_ν, a are arbitrary constants.

In particular, equation

$$\Box u + \epsilon u = 0 \tag{5.12.22}$$

is approximately invariant under the group $\widetilde{\mathrm{P}}(1,4)$ on the subset of solutions

$$u = w - \epsilon\left(\frac{1}{6}w^3 + a_1 w + a_2\right), \tag{5.12.23}$$

where w is given in (5.12.21) and a_1, a_2 are arbitrary constants.

In conclusion, let us note some generalizations of the concept of approximate symmetry studied above. First of all, one can consider higher orders of approximation of u in ϵ, i.e., $u = w + \epsilon v^{(1)} + \epsilon^2 v^{(2)} + \cdots$, and can study the symmetry of the corresponding approximate system of PDEs for functions w, $v^{(1)}$, $v^{(2)}$, and so on. Secondly, one can expand in ϵ-series not only dependent variables, but also independent ones, e.g., $x_0 = t + \epsilon z^{(1)} + \epsilon^2 z^{(2)} + \cdots$, and can construct in this way the corresponding approximate system and then study its symmetry. Another approach to the study of approximate symmetry is to use some special approximates, say the two-point Padé approximants

$$u = \sum_{k=0}^{m} \epsilon^k f_k \left(\sum_{j=0}^{n} \epsilon^j g_j\right)^{-1}, \tag{5.12.24}$$

where functions f_k, g_j are determined from the condition: when $\epsilon \to 0$ expression (5.12.24) coincides with the expansion

$$u = v^{(0)} + \epsilon v^{(1)} + \epsilon^2 v^{(2)} + \cdots \qquad \epsilon \ll 1,$$

and when $\epsilon \to \infty$, (5.12.24) coincides with the expansion

$$u = w^{(0)} + \epsilon^{-1} w^{(1)} + \epsilon^{-2} w^{(2)} + \cdots \qquad \epsilon \gg 1.$$

Appendix 1

Jacobi elliptic functions [3,4,24]

Jacobi elliptic functions $E(z, k)$ (z is the variable argument, k is the parameter) are solutions of the nonlinear ODE

$$\ddot{E} + aE + bE^3 = 0, \qquad (\dot{E})^2 + aE^2 + \tfrac{1}{2}bE^4 = c \tag{A.1.1}$$

where dot means differentiation with respect to z; $a = a(k)$. $b = b(k)$, $c = c(k)$ are constants which depend on the paramenter k. For the first time elliptic functions were introduced as inverse functions of an elliptic integral. This problem of inversion was solved in 1827 by Jacobi and Abel independently.

There are twelve elliptic functions. The three basic elliptic functions are determined as follows

$$\begin{aligned} \operatorname{sn}(z, k) &= \sin\varphi \\ \operatorname{cn}(z, k) &= \cos\varphi \\ \operatorname{dn}(z, k) &= \frac{d\varphi}{dz} = (1 - k^2 \sin^2\varphi)^{1/2}, \end{aligned} \tag{A.1.2}$$

where φ is implicitly defined by the elliptic integral of the first kind

$$z = \int_0^{\varphi} \frac{d\tau}{\sqrt{1 - k^2 \sin^2 \tau}}. \tag{A.1.3}$$

The rest of the nine elliptic functions are reciprocals of these three functions, and the quotients of any two of them.

Below in Table A.1.1 we present, following [4], the constants a, b, c for the twelve Jacobi elliptic functions.

Table A.1.1. Jacobi elliptic functions and parameters of the Equation (A.1.1).

N	$E = E(z, k)$	a	b	c
1	sn	$1 + k^2$	$-2k^2$	1
2	cn	$1 - 2k^2$	$2k^2$	$1 - k^2$
3	dn	$-(2 - k^2)$	2	$-(1 - k^2)$
4	$\mathrm{ns} \equiv 1/\mathrm{sn}$	$1 + k^2$	-2	k^2
5	$\mathrm{nc} \equiv 1/\mathrm{cn}$	$1 - 2k^2$	$-2(1 - k^2)$	$-k^2$
6	$\mathrm{nd} \equiv 1/\mathrm{dn}$	$-(2 - k^2)$	$2(1 - k^2)$	-1
7	$\mathrm{sc} \equiv \mathrm{sn}/\mathrm{cn}$	$-(2 - k^2)$	$-2(1 - k^2)$	1
8	$\mathrm{sd} \equiv \mathrm{sn}/\mathrm{dn}$	$1 - 2k^2$	$2k^2(1 - k^2)$	1
9	$\mathrm{cs} \equiv \mathrm{cn}/\mathrm{sn}$	$-(2 - k^2)$	-2	$1 - k^2$
10	$\mathrm{cd} \equiv \mathrm{cn}/\mathrm{dn}$	$1 + k^2$	$-2k^2$	1
11	$\mathrm{ds} \equiv \mathrm{dn}/\mathrm{sn}$	$1 - 2k^2$	-2	$-k^2(1 - k^2)$
12	$\mathrm{dc} \equiv \mathrm{dn}/\mathrm{cn}$	$1 + k^2$	-2	k^2

From the above definition of the basic elliptic functions it follows immediately that

$$\mathrm{sn}\,(0, k) = 0, \qquad \mathrm{cn}\,(0, k) = \mathrm{dn}\,(0, k) = 1, \tag{A.1.4}$$

and

$$\begin{aligned} \dot{\mathrm{sn}} &= \mathrm{cn}\ \mathrm{dn}, \qquad \dot{\mathrm{cn}} = -\,\mathrm{sn}\ \mathrm{dn}, \\ \dot{\mathrm{dn}} &= -k^2\,\mathrm{sn}\ \mathrm{cn}; \\ \mathrm{sn}^2 &= 1 - \mathrm{cn}^2 = \frac{1 - \mathrm{dn}^2}{k^2}, \\ \mathrm{cn}^2 &= 1 + \frac{\mathrm{dn}^2 - 1}{k^2}, \\ \mathrm{dn}^2 &= 1 - k^2\,\mathrm{sn}^2 = 1 - k^2 + k^2\,\mathrm{cn}^2 \end{aligned} \tag{A.1.5}$$

Jacobi elliptic functions are doubly periodic functions of the complex argument z. Specifically,

$$\begin{aligned} &\mathrm{sn}\,(z + 4N_1\mathcal{K} + i2N_2\mathcal{K}') = \mathrm{sn}\,z, \\ &\mathrm{cn}\,\big(z + 4N_1\mathcal{K} + 2N_2(\mathcal{K} + i\mathcal{K}')\big) = \mathrm{cn}\,z, \\ &\mathrm{dn}\,(z + 2N_1\mathcal{K} + i4N_2\mathcal{K}') = \mathrm{dn}\,z, \end{aligned} \tag{A.1.6}$$

where N_1 and N_2 are any integers and

$$\mathcal{K} = \mathcal{K}(k) = \int_0^{\pi/2} \frac{d\tau}{\sqrt{1 - k^2 \sin^2 \tau}},$$
$$\mathcal{K}' = \mathcal{K}'(k) = \mathcal{K}(k'), \qquad k' = \sqrt{1 - k^2} \tag{A.1.7}$$

are complete elliptic integrals. Under $k = 0$ and $k = 1$ one of the periods becomes infinite and the elliptic functions degenerate:

$k = 0$, then $\mathcal{K}' = \infty$, $\mathcal{K} = \pi/2$,

and $\operatorname{sn} z = \sin z, \quad \operatorname{cn} z = \cos z, \quad \operatorname{dn} z = 1;$

$k = 1$, then $\mathcal{K} = \infty$, $\mathcal{K}' = \pi/2$, (A.1.8)

and $\operatorname{sn} z = \operatorname{th} z, \quad \operatorname{cn} z = \operatorname{dn} z = 1/\operatorname{ch} z.$

Restricting z to real values we see that $\operatorname{sn} z$, $\operatorname{cn} z$, and $\operatorname{dn} z$ have periods $4\mathcal{K}$, $4\mathcal{K}$, $2\mathcal{K}$, respectively. The shortest period corresponds to $k = 0$, when $\mathcal{K}(0) = \pi/2$. For $k > 0$, $\mathcal{K}(k) > \pi/2$. All Jacobi elliptic functions are real for real z and for $0 \leq k^2 \leq 1$.

The basic functions $\operatorname{sn} z$, $\operatorname{cn} z$, and $\operatorname{dn} z$ are finite everywhere on the real-z axis and have the following zeros on this axis

$$\operatorname{sn}(z, k) = 0 \text{ at } z = 2N\mathcal{K},$$
$$\operatorname{cn}(z, k) = 0 \text{ at } z = (2N + 1)\mathcal{K},$$
$$\operatorname{dn}(z, k) \neq 0 \text{ for } k < 1,$$

where N is any integer. Only for parameter $k = 1$ does $\operatorname{dn} z$ have zeros on the real axis, namely, at $z = \pm\infty$.

The Jacobi identities

$$\begin{aligned} \operatorname{sn}(iz, k) &= i\operatorname{sc}(z, k'), \qquad \left(k' = \sqrt{1 - k^2}\right) \\ \operatorname{cn}(iz, k) &= \operatorname{nc}(z, k'), \\ \operatorname{dn}(iz, k) &= \operatorname{dc}(z, k') \end{aligned} \tag{A.1.9}$$

enable one to change from real to imaginary arguments or conversely.

As an example of the use of elliptic functions we describe, following [4], elliptic solutions of the nonlinear wave equation

$$\Box u + \lambda u^3 = 0, \tag{A.1.10}$$

where $u = u(x)$ is a scalar function, $x \in \mathrm{R}(1, 3)$.

Let $f(x)$ be a given explicit solution of Equation (A.1.10) with λ equal to $-a\lambda/b$, that is

$$\Box f - \frac{a\lambda}{b} f^3 = 0. \tag{A.1.11}$$

Then the ansatz

$$u(x) = f(x)E(\omega, k). \tag{A.1.12}$$

where $E(\omega, k)$ is an elliptic function which satisfies ODE (A.1.1), will be a solution of Equation (A.1.10) provided

$$\begin{gathered} 2\frac{\partial f}{\partial x^\nu}\frac{\partial \omega}{\partial x_\nu} + f\Box\omega = 0, \\ \frac{\partial \omega}{\partial x_\nu}\frac{\partial \omega}{\partial x^\nu} - \frac{\lambda}{b} f^2 = 0. \end{gathered} \tag{A.1.13}$$

The system (A.1.13) can be considered as defining equations for the argument $\omega = \omega(x)$ of an elliptic function $E(\omega, k)$. If a solution of this system exists then an elliptic generalization of $f(x)$ exists. Solving the conditions (A.1.13) for $\omega(x)$ is the essential step in the construction.

As far as there are 12 Jacobi elliptic functions, one might expect that 12 different elliptic generalizations of $f(x)$ exist. Actually, five of these are redundant. The remaining ones are [4]:

$$\begin{aligned} u &= f\,\mathrm{sn}\left(\frac{\omega'}{\sqrt{-1-k^2}}, k\right) = if\mathrm{sc}\left(\frac{\omega'}{\sqrt{2-k'^2}}, k'\right), \\ u &= f\,\mathrm{cn}\left(\frac{\omega'}{\sqrt{-1+2k^2}}, k\right) = f\mathrm{nc}\left(\frac{\omega}{\sqrt{-1+2k'^2}}, k'\right), \\ u &= f\,\mathrm{dn}\left(\frac{\omega'}{\sqrt{2-k^2}}, k\right) = f\mathrm{dc}\left(\frac{\omega'}{\sqrt{-1-k'^2}}, k'\right), \\ u &= f\mathrm{ns}\left(\frac{\omega'}{\sqrt{-1-k^2}}, k\right) = if\,\mathrm{cs}\left(\frac{\omega'}{\sqrt{2-k'^2}}, k'\right), \\ u &= f\,\mathrm{nd}\left(\frac{\omega'}{\sqrt{2-k^2}}, k\right) = f\mathrm{cd}\left(\frac{\omega'}{\sqrt{-1-k'^2}}, k'\right), \\ u &= f\mathrm{sd}\left(\frac{\omega'}{\sqrt{-1+2k^2}}, k\right) = if\mathrm{sd}\left(\frac{\omega'}{\sqrt{-1+2k'^2}}, k'\right), \\ u &= f\mathrm{ds}\left(\frac{\omega'}{\sqrt{-1+2k^2}}, k\right) = if\mathrm{ds}\left(\frac{\omega'}{\sqrt{-1+2k'^2}}, k'\right), \end{aligned} \tag{A.1.14}$$

where $\omega' = \sqrt{-a(k)}\omega$, and Jacobi imaginary identities (A.1.9) have been used to obtain the second form of the solution with parameter $k' = \sqrt{1-k^2}$.

Appendix 2

$A\widetilde{P}(1,3)$-nonequivalent one-dimensional subalgebras of the extended Poincare algebra $A\widetilde{P}(1,3)$

Here we would like to demonstrate as example of finding one-dimensional subalgebras of a given algebra. As a given algebra we take the extended Poincare algebra $A\widetilde{P}(1,3)$ whose one-dimensional subalgebras have been used in Paragraph 2.2 in constructing $\widetilde{P}(1,3)$-nonequivalent ansatze for spinor field.

In what follows, an essential role will be played by the CBH formula, which we write once again for the sake of convenience

$$\exp\{-\theta Q_2\}Q_1 \exp\{\theta Q_2\} = \sum_{n=0}^{\infty} \frac{\theta^n}{n!}\{Q_1, Q_2^n\}, \tag{A.2.1}$$

where $\{Q_1, Q_2^0\} = Q_1$, $\{Q_1, Q_2^k\} = [\{Q_1, Q_2^{k-1}\}, Q_2]$,

with $k = 1, 2, \ldots$, and Q_1, Q_2 are some operators.

Lemma A.2.1. [109,11*] *By the transformation*

$$Q \to Q' = VQV^{-1}$$

where $V = \exp\{\theta^{\mu\nu} J_{\mu\nu}\}$, *the operator*

$$Q = C_{\mu\nu} J_{\mu\nu} = A_k M_k + B_k N_k$$

where $M_k = -\frac{1}{2}\epsilon_{kij}J_{ij}$ *and* $N_k = J_{0k}$ *(*A_k*,* B_k *are arbitrary constants) can be reduced to one of the following forms*

1° $Q' = \alpha J_{01} + \beta J_{23}$, if $(\vec{A}\cdot\vec{B})^2 + (\vec{A}^2 - \vec{B}^2)^2 \neq 0$,

2° $Q' = \alpha(J_{01} + J_{12})$, if $\vec{A}\cdot\vec{B} = \vec{A}^2 - \vec{B}^2 = 0$.

Proof. Let us introduce a new operator

$$J_a = \frac{i}{2}(M_a + iN_a), \qquad K_a = \frac{i}{2}(M_a - iN_a).$$

Using commutation relations of AP(1,3)

$$\begin{aligned} &[P_\mu, P_\nu] = 0, \qquad [P_\mu, J_{\nu\sigma}] = i(g_{\mu\nu}P_\sigma - g_{\mu\sigma}P_\nu), \\ &[J_{\mu\nu}, J_{\lambda\sigma}] = i(g_{\mu\sigma}J_{\nu\lambda} + g_{\nu\sigma}J_{\mu\sigma} - g_{\mu\lambda}J_{\nu\sigma} - g_{\nu\sigma}J_{\mu\lambda}), \end{aligned} \tag{A.2.2}$$

one can easily check the validity of relations

$$\begin{aligned} &[J_a, J_b] = i\epsilon_{abc}J_c, \\ &[K_a, K_b] = i\epsilon_{abc}K_c, \qquad [J_a, K_b] = 0. \end{aligned} \tag{A.2.3}$$

Operator Q can be rewritten as $Q = a_k J_k + b_a K_a$, provided

$$a_k = -B_k - iA_k, \qquad b_k = B_k - iA_k.$$

By means of (A.2.1) and (A.2.3) one obtains

$$Q' = V_1 Q V_1' = \sqrt{\vec{a}^2}J_1 + (\sqrt{\vec{a}^2})^* K_1 \equiv \alpha J_{01} + \beta J_{23},$$

where

$$\begin{aligned} V_1 = \exp\left\{-i\arctan\frac{a_2}{a_3}J_1\right\}\exp\left\{i\left(\arctan\frac{a_1}{\sqrt{a_2^2+a_3^2}} + \frac{\pi}{2}\right)J_2\right\}\cdot \\ \cdot\exp\left\{-i\arctan\frac{b_2}{b_3}K_1\right\}\exp\left\{i\left(\arctan\frac{b_1}{\sqrt{b_2^2+b_3^2}} + \frac{\pi}{2}\right)K_2\right\} \end{aligned} \tag{A.2.4}$$

It is evident that these formulae lose their validity in the case

$$\vec{a}^2 \equiv a_1^2 + a_2^2 + a_3^2 = 0 \Leftrightarrow \vec{A}^2 = \vec{B}^2, \qquad \vec{A}\cdot\vec{B} = 0$$

Therefore, one can apply operator (A.2.4) only in the case 1°. Let us now consider the second case 2°. It follows from (A.2.1) that

$$\exp\{\theta M_a\}A_kM_k\exp\{-\theta M_a\} = A_kM_k\cos\theta + \tag{A.2.5}$$
$$+A_aM_a(1-\cos\theta) + \epsilon_{ak\ell}A_kM_\ell\sin\theta \qquad \text{(no sum over } a\text{)},$$

$$\exp\{\theta M_a\}B_kN_k\exp\{-\theta M_a\} = B_kN_k\cos\theta + \tag{A.2.6}$$
$$+B_aN_a(1-\cos\theta) + \epsilon_{ak\ell}B_kN_\ell\sin\theta \qquad \text{(no sum over } a\text{)},$$

Using identities (A.2.5), (A.2.6) one can make sure that the following equality holds

$$Q' = V_2QV_2^{-1} = V_2(\vec{A}\cdot\vec{M} + \vec{B}\cdot\vec{N})V_2^{-1} = -\sqrt{\vec{A}^2}\,\mathrm{sign}\,A_3(J_{01}+J_{12}),$$

where

$$V_2 = \exp\left\{\arctan\frac{A_1}{A_2}M_3\right\}\exp\left\{\arctan\frac{\sqrt{A_1^2+A_2^2}}{A_3}M_1\right\}\cdot$$
$$\cdot\exp\left\{\left[\arctan\frac{B_3\sqrt{\vec{A}^2}}{B_2A_1-A_2B_1} + \pi\theta(B_1A_2-B_2A_1)\right]M_3\right\} \tag{A.2.7}$$

$$\mathrm{sign}\,x = \begin{cases}1, & x\geq 0\\ -1, & x<0\end{cases}, \qquad \theta(x) = \begin{cases}1, & x>0\\ 0, & x<0\end{cases}$$

This completes the proof.

Theorem A.2.1. [109,212,11*] *The operator*

$$Q = A_kM_k + B_kN_k + CD + C^\mu P_\mu \tag{A.2.8}$$

(A_k, B_k, C, C^μ are constants)

can be reduced by means of transformation

$$Q\to Q' = VQV^{-1}, \qquad V = \exp\{\theta^{\mu\nu}J_{\mu\nu} + \theta D + \theta^\mu P_\mu\} \tag{A.2.9}$$

to one of the following forms: *If $\vec{A}\cdot\vec{B}=0$, $\vec{A}^2=\vec{B}^2$, then*

1) $Q' = J_{01} + J_{12} + aD$,

2) $Q' = J_{01} + J_{12} + \beta P_3 - P_0$,

3) $Q' = J_{01} + J_{12} + \beta P_3$,

If $(\vec{A}\cdot\vec{B})^2 + (\vec{A}^2-\vec{B}^2)^2 \neq 0$, then

4) $Q' = J_{23} + aD$,

5) $Q' = J_{01} + bJ_{23} + aD$, (A.2.10)

6) $Q' = J_{01} + bJ_{23} + D + \beta P_0$,

7) $Q' = J_{01} + P_2$,

8) $Q' = J_{23} + \alpha_1 P_0 + \alpha_2 P_1$,

If $\vec{A} = \vec{B} = 0$, *then*

9) $Q' = D$,

10) $Q' = P_0 + P_1$,

11) $Q' = P_0$,

12) $Q' = P_1$.

Proof. Let us consider at first the case when $\vec{A}^2 + \vec{B}^2 \neq 0$. Then, as follows from Lemma A.2.1, there exists an operator $V_1(V_2)$ of the form (A.2.9) such that

(a) under $\vec{A} \cdot \vec{B} = \vec{A}^2 - \vec{B}^2 = 0$,

$$V_1 Q V_1^{-1} = \alpha(J_{01} + J_{12}) + \theta D + \theta^\mu P_\mu,$$

(b) under $(\vec{A} \cdot \vec{B})^2 + (\vec{A}^2 - \vec{B}^2)^2 \neq 0$,

$$V_2 Q V_2^{-1} = \alpha J_{01} + \beta J_{23} + \theta D + \theta^\mu P_\mu,$$

Below we shall use the identities

$$\begin{aligned} &\exp\{i\lambda^\mu P_\mu\} J_{\alpha\beta} \exp\{-i\lambda^\mu P_\mu\} = J_{\alpha\beta} + \lambda_\beta P_\alpha - \lambda_\alpha P_\beta \\ &\exp\{i\lambda^\mu P_\mu\} D \exp\{-i\lambda^\mu P_\mu\} = D - \lambda^\mu P_\mu, \\ &\exp\{i\lambda^\mu P_\mu\} P_\alpha \exp\{-i\lambda^\mu P_\mu\} = P_\alpha, \end{aligned} \tag{A.2.11}$$

which can be easily proved be means of the CBH formula (A.2.1). In the case (a) we have

$$\begin{aligned} Q' \to Q'' &= \exp\{i\lambda^\mu P_\mu\}(J_{01} + J_{12} + \theta D + \theta^\alpha P_\alpha) \exp\{-i\lambda^\mu P_\mu\} = \\ &= J_{01} + J_{12} + \theta D + \theta^\mu P_\mu + \lambda_1 P_0 - \lambda_0 P_1 - \lambda_2 P_1 - \lambda_1 P_2 - \theta \lambda^a P_\alpha. \end{aligned}$$

Under $\theta \neq 0$ one can always choose λ_α so that

$$Q'' = J_{01} + J_{12} + \theta D$$

and under $\theta = 0$ so that

$$Q'' = J_{01} + J_{12} + \alpha P_0 + \beta P_3, \qquad \alpha \leq 0$$

If in this latter operator $\alpha \neq 0$, then

$$Q''' = \exp\{-i\ln|\alpha|D\}(J_{01} + J_{12} + \alpha P_0 + \beta P_3)\exp\{i\ln|\alpha|D\} =$$
$$= J_{01} + J_{12} - \operatorname{sign}\alpha P_0 + \frac{\beta}{|\alpha|}P_3$$

If $\alpha = 0$, then

$$Q'' = J_{01} + J_{12} + \beta P_3.$$

Let us now consider the case (b). If $\alpha \neq 0$, then on dividing into α and on transforming the operator Q according to (A.2.11) we obtain

$$Q' = \exp\{i\lambda^\mu P_\mu\}(J_{01} + bJ_{23} + \theta D + \theta^\mu P_\mu)\exp\{-i\lambda^\mu P_\mu\} =$$
$$= J_{01} + \lambda_1 P_0 - \lambda_0 P_1 + bJ_{23} + b(\lambda_3 P_2 - \lambda_2 P_3) + \theta D - \theta\lambda^\mu P_\mu + \theta^\mu P_\mu.$$

Under $\theta \neq \pm 1$, $\theta^2 + b^2 \neq 0$ it is always possible to choose λ_μ so that

$$Q' = J_{01} + bJ_{23} + \theta D.$$

Under $\theta = \pm 1$, one can choose λ_μ so that

$$Q' = J_{01} + bJ_{23} \pm D + \beta P_0.$$

Under $\theta = b = 0$ there exist such λ_μ that

$$Q' = J_{01} + P_2.$$

Under $\alpha = 0$, using identities (A.2.11), one can reduce the operator Q' to one of the following forms:

$$Q'' = J_{23} + \theta D, \qquad \theta \neq 0$$
$$Q'' = J_{23} + \alpha_1 P_0 + \alpha_2 P_1, \qquad \theta = 0.$$

Remaining to be considered is the case $\vec{A} = \vec{B} = 0$, i.e., $Q = \theta D + \theta^\mu P_\mu$. Using (A.2.11) one easily obtains under $\theta \neq 0$:

$$\exp\left\{\frac{i}{\theta}\theta^\mu P_\mu\right\} Q \exp\left\{\frac{-i}{\theta}\theta^\mu P_\mu\right\} = \theta D.$$

If $\theta = 0$, then analyzing three possibilities $\theta_\mu\theta^\mu = 0$, $\theta_\mu\theta^\mu > 0$, $\theta_\mu\theta^\mu < 0$ we obtain operators 10)–12) from (A.2.10). Thus the theorem is proved.

Note A.2.1. When proving the theorem we used only commutation relations of $\widetilde{\mathrm{AP}}(1,3)$ given by (A.2.2) and

$$[P_\mu, D] = iP_\mu, \qquad [J_{\mu\nu}, D] = 0 \qquad \text{(A.2.12)}$$

and we did not use any concrete representation.

Appendix 3

Some applications of Campbell-Baker-Hausdorff operator calculus

In the present Appendix we consider some applications of operator calculus expounded in Paragraph 5.3.

1. Let us calculate finite transformations of spacetime variables for the Lorentz group. We take generators of the Lorentz algebra AO(1,3) in the form

$$J_a = (\vec{x} \times \vec{\nabla})_a, \qquad J_{0a} = x_0 \partial_a + x_a \partial_0, \qquad a = 1, 2, 3 \tag{A.3.1}$$

and consider two operators

$$M = \alpha_a J_a = \vec{\alpha} \cdot (\vec{x} \times \nabla),$$
$$N = \beta_a J_{0a} = x_0 \vec{\beta} \cdot \vec{\nabla} + \vec{\beta} \cdot \vec{x} \partial_0,$$

where α_a, β_a are arbitrary real constants (group parameters). Using formulae

(5.3.6), (5.3.8), (5.3.11) we find

$$x'_0 = \exp\{M\} x_0 \exp\{-M\} = x_0,$$
$$x'_a = \exp\{M\} x_a \exp\{-M\} = x_a +$$
$$+ \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\alpha^{2n+1}\right) \frac{(\vec{x}\times\vec{\alpha})_a}{\alpha} + \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}\alpha^{2n}\right) \frac{x_a\alpha^2 - \alpha_a\vec{\alpha}\cdot\vec{x}}{\alpha^2} =$$
$$= x_a \cos\alpha + \frac{(\vec{x}\times\vec{\alpha})_a}{\alpha}\sin\alpha + \frac{\alpha_a(\vec{\alpha}\cdot\vec{x})}{\alpha^2}(1-\cos\alpha), \tag{A.3.2}$$

$$x'_0 = \exp\{N\} x_0 \exp\{-N\} = x_0 \sum_{n=0}^{\infty} \frac{\beta^{2n}}{(2n)!} + \frac{\vec{\beta}\cdot\vec{x}}{\beta}\sum_{n=0}^{\infty}\frac{\beta^{2n+1}}{(2n+1)!} =$$
$$= x_0 \operatorname{ch}\beta + \frac{\vec{\beta}\cdot\vec{x}}{\beta}\operatorname{sh}\beta, \tag{A.3.3}$$
$$x'_a = \exp\{N\} x_a \exp\{-N\} = x_a + \frac{\beta_a\vec{\beta}\cdot\vec{x}}{\beta^2}(\operatorname{ch}\beta - 1) + \beta_a x_0 \frac{\operatorname{sh}\beta}{\beta},$$

where $\alpha = (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)^{1/2}$, $\beta = (\beta_1^2 + \beta_2^2 + \beta_3^2)^{1/2}$.

2. Here we derive useful formulae for evaluating commutator of operator-valued function with an operator.

Theorem A.3.1. *Let $\widehat{x}$, $\widehat{w}$ be some non-commuting operators belonging to the algebra $\mathcal{A}$ (defintion of $\mathcal{A}$ see in Paragraph 5.3). Let $f(\widehat{x})$ be an operator-valued differentiable function which can be represented as a power series*

$$f(\widehat{z}) = \sum_{n=0}^{\infty} a_n \widehat{x}^n \tag{A.3.4}$$

(a_n are numerical coefficients). Then in $\mathcal{A}$ the following identities hold

$$[\widehat{w}, f(\widehat{x})] = \sum_{n=1}^{\infty} \frac{1}{n!}\left(\partial_x^n f(\widehat{x})\right)\{\widehat{w}, \widehat{x}^n\}, \tag{A.3.5}$$

$$[f(\widehat{x}), \widehat{w}] = \sum_{n=1}^{\infty} \frac{1}{n!}\{\widehat{x}^n, \widehat{w}\}\left(\partial_x^n f(\widehat{x})\right), \tag{A.3.6}$$

Proof. At first we evaluate the commutator $[\widehat{w}, \widehat{x}^n]$. Using CBH formula (5.3.8) we find

$$e^{-\theta\widehat{x}}\widehat{w}e^{\theta\widehat{x}} - \widehat{w} \equiv e^{-\theta\widehat{x}}[\widehat{w}, e^{\theta\widehat{x}}] = \sum_{n=1}^{\infty}\frac{\theta^n}{n!}\{\widehat{w}, \widehat{x}^n\}.$$

Therefore

$$[\widehat{w}, e^{\theta\widehat{x}}] = e^{\theta\widehat{x}} \sum_{n=1}^{\infty} \frac{\theta^n}{n!} \{\widehat{w}, \widehat{x}^n\}.$$

$$\sum_{n=0}^{\infty} \frac{\theta^n}{n!} [\widehat{w}, \widehat{x}^n] = \left(\sum_{k=0}^{\infty} \frac{\theta^k}{k!} \widehat{x}^k \right) \left(\sum_{l=1}^{\infty} \frac{\theta^l}{l!} \{\widehat{w}, \widehat{x}^l\} \right) =$$

$$= \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} \frac{\theta^{k+l}}{k!\,l!} \widehat{x}^k \{\widehat{w}, \widehat{x}^l\}.$$

After equating in this identity terms with equal degree of θ we get

$$\frac{1}{n!} [\widehat{w}, \widehat{x}^n] = \sum_{l=1}^{n} \frac{1}{l!(n-l)!} \widehat{x}^{n-l} \{\widehat{w}, \widehat{x}^l\} \equiv \sum_{l=1}^{n} \frac{1}{n!} C_n^l \widehat{x}^{n-l} \{\widehat{w}, \widehat{x}^l\},$$

$$C_n^l \equiv \frac{n!}{l!(n-l)!}.$$

Hence

$$[\widehat{w}, \widehat{x}^n] = \sum_{l=1}^{n} C_n^l \widehat{x}^{n-l} \{\widehat{w}, \widehat{x}^l\} \qquad \text{(A.3.7)}$$

From (A.3.4) we find

$$\partial^l_{\widehat{x}} f(\widehat{x}) = \sum_{n=0}^{\infty} a_n \frac{n!}{(n-l)!} \widehat{x}^{n-l}. \qquad \text{(A.3.8)}$$

Using formulae (A.3.4), (A.3.7), (A.3.8) we obtain (A.3.5):

$$[\widehat{w}, f(\widehat{x})] = \sum_{n=0}^{\infty} a_n [\widehat{w}, \widehat{x}^n] =$$

$$= \sum_{n=0}^{\infty} a_n \sum_{l=1}^{n} C_n^l \widehat{x}^{n-l} \{\widehat{w}, \widehat{x}^l\} =$$

$$= \sum_{l=1}^{\infty} \sum_{n=l}^{\infty} a_n C_n^l \widehat{x}^{n-l} \{\widehat{w}, \widehat{x}^l\} =$$

$$= \sum_{l=1}^{\infty} \frac{1}{l!} \left(\partial^l_{\widehat{x}} f(\widehat{x}) \right) \{\widehat{w}, \widehat{x}^l\}.$$

In the same way one can derive formula (A.3.6). To do it one should start from the indentity

$$[\widehat{x}^n, \widehat{w}] = \sum_{k=1}^{n} C_n^k \{\widehat{x}^k, \widehat{w}\} \widehat{x}^{n-k}, \qquad \text{(A.3.7')}$$

which follows from (5.3.8) analogously to (A.3.7). For examples of applications of these formulae see Sec. 1.3.

3. In this point we prove the basic Lie's theorems for local groups by means of the CBH operator calculus. For the first time such a proof had been done in 1906 by Hausdorff [117]. Below we follow [38,189].

As it was shown in Paragraph 5.3, final transformations generated by infinitesimal operator of Lie type

$$X = \xi^{\mu}(x,\psi)\frac{\partial}{\partial x_{\mu}} + \eta^{k}(x,\psi)\frac{\partial}{\partial \psi^{k}}, \qquad \mu = \overline{0, n-1},\ k = \overline{1,m} \tag{A.3.9}$$

can be obtained be means of the formula

$$\begin{aligned} x_A \to x'_A &= \exp\{\theta X\}x_A \exp\{-\theta X\}, \qquad x_A = \{x_{\mu}, \psi^k\}, \\ A &= 0,1,\dots,n+m-1. \end{aligned} \tag{A.3.10}$$

Here it is convenient not to distinguish independent and dependent variables, but further we will write instead of index A the index μ bearing in mind that μ can always be extended to $A = n+m-1$.

Theorem A.3.2. *Let operators* (A.3.9) *form an r-dimensional Lie algebra* AL(r), *in which the following relations hold*

$$[X_i, X_j] = c_{ij}^{k} X_k, \qquad i,j,k = \overline{1,r} \tag{A.3.11}$$

$$c_{ij}^{k} = -c_{ji}^{k}, \qquad c_{ij}^{s}c_{ks}^{t} + c_{jk}^{s}c_{is}^{t} + c_{ki}^{s}c_{js}^{t} = 0 \qquad \textit{(Jacobi identity)} \tag{A.3.12}$$

(c_{ij}^{k} are certain constants, called the structure constants). Then there exists an r-dimensional group of point transformations $G(r)$ which corresponding to this AL(r).

Proof. Consider two infinitesimal operators

$$\widehat{x} = a^{i}X_i, \qquad \widehat{y} = b^{j}X_j \tag{A.3.13}$$

where a^i, b^j are arbitrary constants, $i,j=\overline{1,r}$. According to (A.3.10) we have

$$x'_{\mu} = e^{\widehat{y}}x_{\mu}e^{-\widehat{y}}, \qquad x''_{\mu} = e^{\widehat{x}}e^{\widehat{y}}x_{\mu}e^{-\widehat{y}}e^{-\widehat{x}},$$

which can be rewritten be means of (5.3.11) as

$$x''_{\mu} = e^{\widehat{x}}e^{\widehat{y}}x_{\mu}e^{-\widehat{y}}e^{-\widehat{x}} = e^{\widehat{z}}x_{\mu}e^{-\widehat{z}}. \tag{A.3.14}$$

The theorem will be proved if we convince ourselves that operator $\widehat{z}$ from (A.3.14) belong to the AL(r), that is it can be represented as

$$\widehat{z} = c^i X_i, \qquad c^i = \varphi^i(a, b), \tag{A.3.15}$$

where c^i are parameters depending on a and b. Note: formula (A.3.10) as well as (A.3.14) with $\widehat{z}$ given by (A.3.15) lead to local (point) transformations insofar as the expression

$$f(x') = e^{\theta\xi\partial} f(x) e^{-\theta\xi\partial} = \sum_{n=0}^{\infty} \frac{\theta^n}{n!} (\xi\partial)^n f(x)$$

is just a Maclaurin's series of an expansion of function $f(x')$.

To find $\widehat{z}$ from (A.3.14) let us use formulae (5.3.12). Taking into account (A.3.11) we successively obtain

$$\begin{aligned} [\widehat{x}, \widehat{y}] &= a^i b^j [X_i, X_j] = a^i b^j c^k_{ij} X_k = a^i \beta^k_i X_k, \\ \{\widehat{x}, \widehat{y}^2\} &\equiv [[\widehat{x}, \widehat{y}], \widehat{y}] = a^i \beta^k_i \beta^s_k X_s = a^i (\beta^2)^s_i X_s, \end{aligned} \tag{A.3.16}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\{\widehat{x}, \widehat{y}^n\} = a^i (\beta^n)^s_i X_s, \qquad n = 1, 2, \dots,$$

where $\beta = (\beta^k_i \equiv b^j c^k_{ij})$. So we can write the first term $\widehat{z}_1$ of the series (5.3.12):

$$\widehat{z}_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} B_n \{\widehat{x}, \widehat{y}^n\} = \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} B_n \beta^n \right)^s_i a^i X_s \stackrel{\text{def}}{=} f^s X_s. \tag{A.3.17}$$

where

$$f^s = a^i \mathcal{B}^s_i, \qquad \mathcal{B} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} B_n \beta^n = \frac{\beta e^\beta}{e^\beta - 1}. \tag{A.3.18}$$

As it is known from linear algebra a power series $f(\beta)$ converges if and only if all characteristic roots of the matrix β lie inside the ring of convergence of the series $f(z)$. As was already said (see formula (5.3.51)) the seriess

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} B_n z^n$$

converges if $|z|$ is less than 2π. Speaking more precisely, this series converges everywhere except points where $z = 2\pi i k$. Therefore element $\widehat{z}_1$ is well-defined everywhere except the points where $\det|\beta - 2\pi i k| = 0$, $k = 1, 2, \dots$.

To find the rest terms of the series (5.3.12) we shall use the following statement.

Lemma A.3.1. *Let*

$$\widehat{f} = f(\widehat{y}) = f^s(b)X_s, \qquad \widehat{u} = d^j X_j, \qquad d_j \text{ are constants}$$

Then the equality holds

$$\left(\widehat{u}\,\partial_{\widehat{y}}\right)\widehat{f} = d^j \frac{\partial f^s(b)}{\partial b^j} X_s. \tag{A.3.19}$$

Proof. According to the Taylor series expansion of operator-valued function $f(\widehat{y} + \theta\widehat{u})$ (see Lemma 5.3.1) we have

$$f(\widehat{y} + \theta\widehat{u}) = f(\widehat{y}) + \theta\left(\widehat{u}\partial_{\widehat{y}}\right) f(\widehat{y}) + \cdots$$

On the other hand we can write

$$f(\widehat{y} + \theta\widehat{u}) = f^s(b + \theta d)X_s = \left[f^s(b) + \theta d^j \frac{\partial f^s}{\partial b^j} + \cdots\right] X_s.$$

After equating these two expressions we obtain the identity (A.3.19). Thus the lemma is proved.

By means of (A.3.19) we find $\widehat{z}_2$, $\widehat{z}_3, \ldots$ from the expansion (5.3.12):

$$\begin{aligned} \widehat{z}_2 &= \tfrac{1}{2}\left(\widehat{z}_1\partial_{\widehat{y}}\right)\widehat{z}_1 = \tfrac{1}{2} f^j \frac{\partial f^s}{\partial b^j} X_s = -\tfrac{1}{2}[f^s, \widehat{q}], \\ \widehat{z}_3 &= \frac{1}{3}\left(\widehat{z}_1\partial_{\widehat{y}}\right)\widehat{z}_2 = \frac{1}{3!}\{f^s, \widehat{q}^2\}, \ldots, \end{aligned} \tag{A.3.20}$$

where $\widehat{q} = f^j \dfrac{\partial}{\partial b^j}$. Hence

$$\widehat{z} = \widehat{y} + \left(\widehat{e^{q} f^s e^{-q}}\right) X_s \tag{A.3.21}$$

and thereby we get

$$c^s = b^s + \widehat{e^{q} f^s e^{-q}} = b^s + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\{f^s, (f^k \partial_{b^k})^n\}, \tag{A.3.22}$$

where f^s and $\widehat{q}$ are defined in (A.3.18) and (A.3.20), respectively. In particular, in a first approximation we have

$$\mathcal{B} = 1 + \tfrac{1}{2}\beta + \cdots, \qquad \mathcal{B}_i^s = \delta_i^s + \tfrac{1}{2} b^i c_{ij}^k + \cdots$$

$$f^s = a^s + \tfrac{1}{2} c_{ij}^s a^i b^j$$

and therefore

$$c^s = a^s + b^s + \tfrac{1}{2} c_{ij}^s a^i b^j + \cdots \tag{A.3.23}$$

Analogously, in a second approximation (A.3.22) results in [189]:

$$c^s = a^s + b^s + \tfrac{1}{2} c^s_{ij} a^i b^j + \frac{1}{12} c^k_{ij} c^s_{kl} a^i b^j (b^l - a^l) + \cdots \tag{A.3.24}$$

This completes the proof of Theorem A.3.2.

Now we shall obtain the so-called Maurer-Cartan equations. Consider an operator-valued function

$$\exp\{\widehat{y} + \epsilon\widehat{u} + \delta\widehat{v}\} \tag{A.3.25}$$

where $\widehat{y} = b^j X_j$, $\widehat{u} = d^j X_j$, $\widehat{v} = h^j X_j$, $j = \overline{1,r}$; and b^j, d^j, h^j, ϵ, δ are arbitrary constants. Expanding it in the Taylor series according to (5.3.15) and using the formula (5.3.18) one successively obtains

$$\exp\{\widehat{y} + \epsilon\widehat{u} + \delta\widehat{v}\} = \exp\{\widehat{y} + \epsilon\widehat{u}\} + \delta\left(\widehat{v}\partial_{\widehat{y}}\right)\exp\{\widehat{y} + \epsilon\widehat{u}\} + \cdots =$$

$$= \exp\{\widehat{y} + \epsilon\widehat{u}\} + \delta\psi\,(\widehat{v}, \widehat{y} + \epsilon\widehat{u})\exp\{\widehat{y} + \epsilon\widehat{u}\} + \cdots =$$

$$= \left[1 + \delta\psi(\widehat{v},\widehat{y}) + \epsilon\delta\left(\widehat{u}\partial_{\widehat{y}}\right)\psi(\widehat{v},\widehat{y}) + \cdots\right]\left(\exp\{\widehat{y}\} + \epsilon\psi(\widehat{u},\widehat{y})\exp\{\widehat{y}\} + \cdots\right) =$$

$$= \left[1 + \epsilon\psi(\widehat{u},\widehat{y}) + \delta\psi\,(\widehat{v},\widehat{y}) + \epsilon\delta(\widehat{u},\partial_{\widehat{y}})\psi(\widehat{v},\widehat{y}) + \epsilon\delta\psi(\widehat{v},\widehat{y})\psi(\widehat{u},\widehat{y}) + \cdots\right]\exp\{\widehat{y}\}.$$

Let us write down a coefficient under $\epsilon\delta$. It is

$$\left[(\widehat{u}\,\partial_{\widehat{y}})\psi(\widehat{v},\widehat{y}) + \psi(\widehat{v},\widehat{y})\psi(\widehat{u},\widehat{y})\right]\exp\{\widehat{y}\}.$$

Interchanging $\widehat{u}$ and $\widehat{v}$ and equating the two expressions, we get the indentity

$$\left(\widehat{u}\,\partial_{\widehat{y}}\right)\psi(\widehat{v},\widehat{y}) - \left(\widehat{v}\,\partial_{\widehat{y}}\right)\psi(\widehat{u},\widehat{y})\psi\,(\widehat{u},\widehat{y}) = \tag{A.3.26}$$

$$= \psi(\widehat{u},\widehat{y})\psi(\widehat{v},\widehat{y}) - \psi(\widehat{v},\widehat{y})\psi(\widehat{u},\widehat{y}).$$

But according to (5.3.20) we have

$$\psi(\widehat{u},\widehat{y}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!}\{\widehat{u},\widehat{y}^n\} = \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!}\beta^n\right)^s_i d^i X_s = \tag{A.3.27}$$

$$= \psi^s_i(\beta) d^i X_s,$$

where matrix $\psi(\beta)$ (see also (A.3.18)) is

$$\psi(\beta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!}\beta^n = \mathcal{B}^{-1} \tag{A.3.28}$$

Analogous expression one obtains for $\psi(\widehat{v},\widehat{y})$:

$$\psi(\widehat{v},\widehat{y}) = h^i \psi^s_i(\beta) X_s. \tag{A.3.29}$$

Note that the series (A.3.28) is convergent for any matrix β. Using (A.3.19) we get

$$\begin{aligned}\left(\widehat{u}\,\partial_{\widehat{y}}\right)\psi(\widehat{v},\widehat{y}) &= d^i h^j \frac{\partial\psi_j^k(b)}{\partial b^i}X_k,\\ \left(\widehat{v}\,\partial_{\widehat{y}}\right)\psi(\widehat{u},\widehat{y}) &= d^i h^j \frac{\partial\psi_i^k(b)}{\partial b^j}X_k.\end{aligned} \tag{A.3.30}$$

Substituting (A.3.27)–(A.3.30) into (A.3.26) and equating coefficients under equal products of $d^i h^j$, we obtain equations

$$\frac{\partial\psi_i^k}{\partial b^j} - \frac{\partial\psi_j^k}{\partial b^i} = c_{sl}^k \psi_i^s \psi_j^l, \tag{A.3.31}$$

which are known as the Maurer-Cartan equations.

Consider operators

$$\widehat{q}_i = \mathcal{B}_i^s \frac{\partial}{\partial b^s} \tag{A.3.32}$$

acting in dual space of parameters. The third Lie's theorem will be proved if we show that operators (A.3.32) are linearly independent and form a Lie algebra with structure constants c_{ij}^k (A.3.12).

Linear independence of operators (A.3.32) follows from the fact that $\det|\mathcal{B}| \neq 0$ insofar as the inverse matrix $\mathcal{B}^{-1} = \psi(\beta)$ exists (series (A.3.28) is convergent for any β). Let us compute the commutator

$$[\widehat{q}_i, \widehat{q}_j] = \left(\mathcal{B}_i^s \frac{\partial \mathcal{B}_j^t}{\partial b^s} - \mathcal{B}_j^s \frac{\partial \mathcal{B}_i^t}{\partial b^s}\right)\frac{\partial}{\partial b^t} \stackrel{\text{def}}{=} R_{ij}^t \frac{\partial}{\partial b^t}. \tag{A.3.33}$$

Differentiating the identity

$$\psi_t^k \mathcal{B}_i^t = \delta_i^k \tag{A.3.34}$$

with respect to b^s we get

$$\psi_t^k \frac{\partial \mathcal{B}_i^t}{\partial b^s} + \mathcal{B}_i^t \frac{\partial \psi_t^k}{\partial b^s} = 0. \tag{A.3.35}$$

Multiplying (A.3.33) by ψ_i^k and using (A.3.35) we find

$$\mathcal{B}_i^s \mathcal{B}_j^t \left(\frac{\partial\psi_s^k}{\partial b^t} - \frac{\partial\psi_t^k}{\partial b^s}\right) = \psi_t^k R_{ij}^t. \tag{A.3.36}$$

By means of the Maurer-Cartan equations (A.3.31) we can rewrite (A.3.36) as follows

$$\mathcal{B}_i^s \mathcal{B}_J^t c_{lm}^k \psi_s^l \psi_t^m = c_{lm}^k \mathcal{B}_i^s \psi_s^l \mathcal{B}_j^t \psi_t^m = c_{ij}^k = \psi_t^k R_{ij}^t$$

As a result we have

$$R_{ij}^s = \mathcal{B}_k^s c_{ij}^k \tag{A.3.37}$$

Substituting (A.3.37) into (A.3.33) we get

$$[\widehat{q}_i, \widehat{q}_j] = c_{ij}^k \mathcal{B}_k^s \frac{\partial}{\partial b^s} \equiv c_{ij}^k \widehat{q}_k. \qquad \text{(A.3.38)}$$

Thus we have proved the third Lie's theorem.

4. An important extension of the Hausdorff formula results when one considers the solutions of the ODE for operator-valued function $\widehat{y}(t)$ depending on real variable t,

$$\dot{\widehat{y}} \equiv \frac{d\widehat{y}}{dt} = \widehat{q}(t)\widehat{y}, \qquad \widehat{y}(0) = I, \qquad \text{(A.3.39)}$$

where $\widehat{q}(t)$ is a given operator. Following [146] we look for a solution of (A.3.39) in the form

$$\widehat{y}(t) = \exp\{\widehat{x}(t)\} \qquad \text{(A.3.40)}$$

and the problem then becomes one of finding an expression for $\widehat{x}(t)$. It follows from (5.3.18), (5.3.20) that

$$\begin{aligned} \dot{\widehat{y}} &= \left(\dot{\widehat{x}}\,\partial_{\widehat{x}}\right) e^{\widehat{x}} = \psi(\dot{\widehat{x}}, \widehat{x}) e^{\widehat{x}} = \widehat{q} e^{\widehat{x}}, \\ \widehat{q} &= \psi\left(\dot{\widehat{x}}, \widehat{x}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \{\dot{\widehat{x}}, \widehat{x}\}, \end{aligned} \qquad \text{(A.3.41)}$$

Using Lemma 5.3.3 we can invert the series for $\widehat{q}$ (A.3.41) according to formula (5.3.28):

$$\dot{\widehat{x}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} B_n \{\widehat{q}, \widehat{x}^n\},$$

where B_n are the Bernouilli numbers (5.3.13). Finally, this equation is solved by iteration, by setting

$$\widehat{x}_0 = 0,$$

$$\widehat{x}_1(t) = \int_0^t \widehat{q}(\tau) d\tau,$$

$$\widehat{x}_2(t) = \int_0^t \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} B_n \left\{ \widehat{q}(\tau), \left(\int_0^\tau \widehat{q}(\tau_1) d\tau_1 \right)^n \right\} \right) d\tau,$$

....

$$\widehat{x}_k(t) = \int_0^t \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} B_n \left\{ \widehat{q}(\tau), x_{k-1}^n(\tau) \right\} \right) d\tau \qquad \text{(A.3.42)}$$

. . . .

and putting

$$\widehat{x}(t) = \lim_{k\to\infty} \widehat{x}_k(t). \tag{A.3.43}$$

This result was first obtained by Magnus [146], and formulae (A.3.42), (A.3.43) are the continuous analog to the Hausdorff fromula.

In particular, if

$$\left[\widehat{q}(\tau), \int_0^\tau \widehat{q}(\tau_1)d\tau_1\right] = 0, \tag{A.3.44}$$

then solution of Equation (A.3.39) has the form

$$\widehat{y}(t) = \exp\left\{\int_0^t \widehat{q}(\tau)d\tau\right\}. \tag{A.3.45}$$

Consider another example of Equation (A.3.39), namely when operator $\widehat{q}(t)$ has the form

$$\widehat{q}(t) = \sum_{j=1}^{n} a_j(t)Q_j, \qquad n < \infty \tag{A.3.46}$$

where $a_j(t)$ are scalar differentiable functions of t, and the Q_j are time-independent operators forming an n-dimensional Lie algebra AL(n), n is finite. The following statement holds.

Theorem A.3.3 [26*,27*]. *The solution of Equation* (A.3.46) *with the operator $\widehat{q}(t)$ given by* (A.3.46) *may be expressed in the form (at least in a neighborhood of $t = 0$)*

$$\widehat{y}(t) = \prod_{i=1}^{n} \exp\{g_i(t)Q_i\} \tag{A.3.47}$$

where $g_i(t)$ are scalar functions which are solutions of the following system of nonlinear ODEs:

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ \xi_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \xi_{n1} & \cdots & \cdots & \xi_{nn} \end{pmatrix} \begin{pmatrix} \dot{g}_1 \\ \dot{g}_2 \\ \vdots \\ \dot{g}_n \end{pmatrix}, \quad g_i(0) = 0, \tag{A.3.48}$$

ξ_{ij} are certain analytic functions of g. If the operators Q_i form a solvable Lie algebra, then Equations (A.3.48) *take the form*

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \xi_{11} & 0 & 0 & \dots & 0 \\ \xi_{21} & \xi_{22} & 0 & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ \xi_{n1} & \dots & \dots & \dots & \xi_{nn} \end{pmatrix} \begin{pmatrix} \dot{g}_1 \\ \dot{g}_2 \\ \vdots \\ \dot{g}_n \end{pmatrix}, \quad g_i(0) = 0, \tag{A.3.49}$$

and in this case solution (A.3.47) *exists globally.*

The proof of Theorem A.3.3 based on the CBH formula (5.3.8) and on the properties of Lie algebras, in particular of solvable Lie algebras. Let us remind some definitions.

A Lie algebra is said to be solvable if its derived series (also called a commutant series)

$$L \supset L^{(1)} \supset L^{(2)} \supset \ldots \supset L^{(k)} \supset \ldots \tag{A.3.50}$$

ends by zero ideal $L^{(k)}$, provided k is finite. A subalgebra S is called an *ideal* if the commutator $[\widehat{x}, \widehat{y}]$, of $\widehat{x} \in S$ and $\widehat{y} \in L$, is in S.

The set of those elements of algebra AL which are the result of commututation of elements $\widehat{u}$ and $\widehat{v}$, where $\widehat{u} \in N$, $\widehat{v} \in M$ (N, M are some subspaces of AL) forms a Lie product and is denoted by $[N, M]$.

The ideal $L^{(1)} = [L, L]$ is called a derived algebra or commutant. By analogy with $L^{(1)}$ we define $L^{(2)} = [L^{(1)}, L^{(1)}]$, $L^{(3)} = [L^{(2)}, L^{(2)}]$, and so on.

As an example to what has been said above we consider, following [28*], the time-dependent Fokker-Planck equation

$$\frac{\partial u}{\partial t} = \left\{ -\frac{\partial}{\partial x} \left(a(t)x + b(t)\right) + c(t)\frac{\partial^2}{\partial x^2} \right\} u, \tag{A.3.51}$$

where $u = u(x, t)$ is a scalar function. We look for solution of this equation in the form

$$u(x, t) = \widehat{y}(t)u(x, 0), \tag{A.3.52}$$

where operator $\widehat{y}(t)$ satisfies Equation (A.3.39), provided

$$\widehat{q}(t) = -a(t)\frac{\partial}{\partial x}x - b(t)\frac{\partial}{\partial x} + c(t)\frac{\partial^2}{\partial x^2}. \tag{A.3.53}$$

Comparing (A.3.53) with (A.3.46) we have

$$Q_1 = \frac{\partial}{\partial x}x \equiv x\frac{\partial}{\partial x} + 1, \qquad Q_2 = \frac{\partial}{\partial x}, \qquad Q_3 = \frac{\partial^2}{\partial x^2}. \tag{A.3.54}$$

Operators (A.43.54) form a Lie algebra with commutation relations

$$[Q_1, Q_2] = -Q_2, \qquad [Q_2, Q_3] = 0, \qquad [Q_1, Q_3] = -2Q_3,$$

whence follows that it is a solvable algebra insofar as

$$L^{(1)} = < Q_2, Q_3 >, \qquad L^{(2)} = < 0 >$$

and commutant series (A.3.50) ends under $k = 2$.

Searching for operator-valued function $\hat{y}(t)$ in the form (A.3.47)

$$\hat{y}(t) = \exp\left\{\alpha(t)\frac{\partial}{\partial x}x\right\}\exp\left\{\beta(t)\frac{\partial}{\partial x}\right\}\exp\left\{\gamma(t)\frac{\partial^2}{\partial x^2}\right\} \tag{A.3.55}$$

we find for the unknown functions $\alpha(t), \beta(t)$, $\gamma(t)$ the linear system of ODEs:

$$\begin{pmatrix} -a(t) \\ -b(t) \\ c(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\alpha(t)} & 0 \\ 0 & 0 & e^{-2\alpha(t)} \end{pmatrix} \begin{pmatrix} \dot{\alpha}(t) \\ \dot{\beta}(t) \\ \dot{\gamma}(t) \end{pmatrix}, \tag{A.3.56}$$

with $\alpha(0) = \beta(0) = \gamma(0) = 0$; whence follows

$$\begin{aligned} \alpha(t) &= -\int_0^t a(\tau)d\tau, \\ \beta(t) &= -\int_0^t b(\tau)e^{\alpha(\tau)}d\tau, \\ \gamma(t) &= -\int_0^t c(\tau)e^{2\alpha(\tau)}d\tau. \end{aligned} \tag{A.3.57}$$

Using the easily derived identities

$$\begin{aligned} &\exp\left\{f(t)\frac{\partial}{\partial x}\right\}u(x) = u\big(x + f(t)\big), \\ &\exp\left\{f(t)\frac{\partial}{\partial x}x\right\}u(x) = e^{f(t)}u\big(e^{f(t)}x\big), \\ &\exp\left\{f(t)\frac{\partial^2}{\partial x^2}\right\}u(x) = \frac{1}{\sqrt{4\pi f(t)}}\int_{-\infty}^{\infty}\exp\left\{-\frac{(x-y)^2}{4f(t)}\right\}u(y)dy \end{aligned} \tag{A.3.58}$$

we successively obtain be means of (A.3.52), (A.3.55), (A.3.57) the solution of the Fokker-Planck equation (A.3.51):

$$u(x,t) = \exp\left\{\alpha(t)\frac{\partial}{\partial x}x\right\}\exp\left\{\beta(t)\frac{\partial}{\partial x}\right\}\exp\left\{\gamma(t)\frac{\partial^2}{\partial x^2}\right\}u(x,0) =$$

$$= \exp\left\{\alpha(t)\frac{\partial}{\partial x}x\right\}\exp\left\{\beta(t)\frac{\partial}{\partial x}\right\}\frac{1}{\sqrt{4\pi\gamma(t)}}\int_{-\infty}^{\infty} dy\, u(y,0)\exp\left\{-\frac{(y-x)^2}{4\gamma(t)}\right\} =$$

$$= \exp\left\{\alpha(t)\frac{\partial}{\partial x}x\right\}\frac{1}{\sqrt{4\pi\gamma(t)}}\int_{-\infty}^{\infty} dy\, u(y,0)\exp\left\{-\frac{[y-\big(x+\beta(t)]^2}{4\gamma(t))}\right\} =$$

$$= \frac{e^{\alpha(t)}}{\sqrt{4\pi\gamma(t)}} \int_{-\infty}^{\infty} \exp\left\{-\frac{[y-(xe^{\alpha(t)}+\beta(t))]^2}{4\gamma(t)}\right\} u(y,0)dy, \tag{A.3.59}$$

where functions $\alpha(t)$, $\beta(t)$, $\gamma(t)$ are given in (A.3.57).

It will be noted that the Fokker-Planck equation (A.3.51) is reduced by means of the change of variables (see §3.8)

$$\begin{aligned} u(x,t) &= e^{\alpha(t)}w(y,\tau),\\ y &= e^{\alpha(t)}x+\beta(t),\\ \tau &= -\gamma(t) \end{aligned} \tag{A.3.60}$$

to the standard heat equation

$$w_\tau = w_{yy}. \tag{A.3.61}$$

5. The CBH formula (5.3.8) can be used for finding final transformations generated by operators forming a superalgebra. Without going into details let us present an example of such calculations.

Consider, following [199], an operator

$$X = a_j X_j, \qquad j = 1,2,3, \tag{A.3.62}$$

where X_j are basis elements of the superalgebra SQM(1,2), satisfying relations

$$\begin{aligned} &[X_1,X_2] = [X_1,X_3] = 0,\\ &[X_2,X_3]_+ \equiv X_2X_3 + X_3X_2 = X_1,\\ &[X_2,X_2]_+ = [X_3,X_3]_+ = 0. \end{aligned} \tag{A.3.63}$$

Operator X_1 is called *even* and two others are called *odd*. In the above definition of operator X (A.3.62) parameters a_j are as follows: a_1 is a real number, a_2, a_3 are Grassmann numbers. It should be kept in mind that an odd generator anticommutes with a Grassmann variable. For example,

$$X_2a_3 = -a_3X_2, \tag{A.3.64}$$

The superalgebra SQM(1,2) (A.3.63) has a faithful 2×2 matrix representation

$$X_1 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad X_2 = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \quad X_3 = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}. \tag{A.3.65}$$

We adopt the convention that matrices with Grassmann variable entries are linear combinations of the generators and their products, multiplied from the left by Grassmann parameters. The matrices multiplying the product $a_2 a_3$ are treated as even.

So, for example,

$$\left[a_1\begin{pmatrix}1&0\\0&1\end{pmatrix}+a_2\begin{pmatrix}0&1\\0&0\end{pmatrix}+a_3\begin{pmatrix}0&0\\1&0\end{pmatrix}\right]^2=$$

$$=a_1\begin{pmatrix}1&0\\0&1\end{pmatrix}a_1\begin{pmatrix}1&0\\0&1\end{pmatrix}+a_1\begin{pmatrix}1&0\\0&1\end{pmatrix}a_2\begin{pmatrix}0&1\\0&0\end{pmatrix}+$$

$$+a_1\begin{pmatrix}1&0\\0&1\end{pmatrix}a_3\begin{pmatrix}0&0\\1&0\end{pmatrix}+a_2\begin{pmatrix}0&1\\0&0\end{pmatrix}a_1\begin{pmatrix}1&0\\0&1\end{pmatrix}+$$

$$+a_2\begin{pmatrix}0&1\\0&1\end{pmatrix}a_3\begin{pmatrix}0&0\\1&0\end{pmatrix}+a_3\begin{pmatrix}0&0\\1&0\end{pmatrix}a_1\begin{pmatrix}1&0\\0&1\end{pmatrix}+$$

$$+a_3\begin{pmatrix}0&0\\1&0\end{pmatrix}a_2\begin{pmatrix}0&1\\0&0\end{pmatrix}= \tag{A.3.66}$$

$$=a_1^2\begin{pmatrix}1&0\\0&1\end{pmatrix}+2a_1a_2\begin{pmatrix}0&1\\0&0\end{pmatrix}+2a_1a_3\begin{pmatrix}0&0\\1&0\end{pmatrix}-$$

$$-a_2a_3\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix}-a_3a_2\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}=$$

$$=a_1^2\begin{pmatrix}1&0\\0&1\end{pmatrix}+2a_1a_2\begin{pmatrix}0&1\\0&0\end{pmatrix}+2a_1a_3\begin{pmatrix}0&0\\1&0\end{pmatrix}-$$

$$-a_2a_3\begin{pmatrix}1&0\\0&0\end{pmatrix}+a_2a_3\begin{pmatrix}0&0\\0&1\end{pmatrix}.$$

Let us determine a one-parameter subgroup of the supergroup as

$$\exp\{\theta X\},\qquad \theta=\text{const}. \tag{A.3.67}$$

The expression (A.3.67) in the case of X given by (A.3.62) takes the form

$$\begin{aligned}\exp\{\theta X\}\equiv\exp\{\theta a_jX_j\}&=1+\theta a_jX_j+\frac{\theta^2}{2!}\left(a_jX_j\right)^2+\dots=\\&=e^{\theta a_1}\begin{pmatrix}1-\frac{\theta^2}{2}a_2a_3&\theta a_2\\\theta a_3&1+\frac{\theta^2}{2}a_2a_3\end{pmatrix}=\\&=e^{\theta a_1}\begin{pmatrix}e^{-\frac{\theta^2}{2}a_2a_3}&\theta a_2\\\theta a_3&e^{\frac{\theta^2}{2}}a_2a_3\end{pmatrix}.\end{aligned} \tag{A.3.68}$$

For more details on superalgebras see [112, 134, 199, 144*] and the references cited therein.

Appendix 4

Differential invariants (DI) of Poincare algebras AP(1,n), A$\widetilde{\text{P}}$(1,n) and conformal algebra AC(1,n)

Following [43*] we give functional bases of second-order differential invariants (DI) of algebras AP(1,n), A$\widetilde{\text{P}}(1,n)$, and AC(1,$n$) which are realized on $(n+1+m)$-dimensional space $\mathrm{R}^{n+1}(x) \times \mathrm{R}^m(u)$, $x = (x_0, x_1, \ldots, x_n)$, $u = (u^1, \ldots, u^m) \equiv u(x)$.

1°. DI of AP(1,n) in scalar case $m = 1$:

$$S_k = S_k(u_{\mu\nu}) = u_{\mu_0\mu_1} u_{\mu_1\mu_2} \cdots u_{\mu_k\mu_0},$$

$$R_k = R_k(u_\mu, u_{\mu\nu}) = u_{\mu_0} u_{\mu_k} u_{\mu_0\mu_1} \cdots u_{\mu_{k-1}\mu_k}.$$

1°°. DI of AP(1,n) when $m > 1$:

$$R_k = R_k(u^r_\mu, u^1_{\mu\nu}),$$

$$S_{jk}(u^r_{\mu\nu}, u^1_{\mu\nu}) = u^1_{\mu_1\mu_2} \cdots u^1_{\mu_{j-1}\,\mu_j} u^r_{\mu_j\,\mu_{j+1}} \cdots u^r_{\mu_k\,\mu},$$

$$j = \overline{0,k}, \qquad k = \overline{1,n+1}, \qquad r = \overline{1,m}, \text{ no sum over } r.$$

2°. DI of $\mathrm{A}\widetilde{\mathrm{P}}(1,n) = \{\mathrm{AP}(1,n),\ D = x\partial + \lambda u^r \partial_{u^r}\}: \quad \lambda \neq 0,$

$$\frac{u^r}{u^1}, \qquad S_{jk}(u^r_{\mu\nu}, u^1_{\mu\nu})(u^1)^{k(2/\lambda - 1)}$$

$$R_k\left(u^r_\mu, u^1_{\mu\nu}\right)(u^1)^{(2k/\lambda-k-1)};$$

$\lambda = 0$:

$$u^r, \qquad S_{jk}\left(u^r_{\mu\nu}, u^1_{\mu\nu}\right)(u^1_{\alpha\alpha})^{-k},$$

$$R_r\left(u^r_\mu, u^1_{\mu\nu}\right)(u^1_{\alpha\alpha})^{-k}.$$

3°. DI of AC(1,n):

$$\lambda \neq 0, \qquad S_{jk}\left(\theta^r_{\mu\nu}, \theta^1_{\mu\nu}\right)(u^1)^{k(2/\lambda-1)}$$

$$\frac{u^r}{u^1}, \qquad R_k\left(\theta^r_\mu, \theta^1_{\mu\nu}\right)(u^1)^{[k(2/\lambda-1)-1]},$$

where $j = \overline{0,k}$, $k = \overline{1,n+1}$, $r = \overline{2,m}$, with no sum over r;

$$\theta^r_\mu = \frac{u^r_\mu}{u^r} - \frac{u^1_\mu}{u^1},$$

$$\theta^r_{\mu\nu} = \lambda u^r_{\mu\nu} + (1-\lambda)\frac{u^r_\mu u^r_\nu}{u^r} + \frac{\lambda}{1-n} g_{\mu\nu}\left(u^r_{\beta\beta} - \frac{u^r_\beta u^r_\beta}{u^r}\right),$$

$\lambda = 0, \quad \Rightarrow$

$$u^r, \quad \left(u^1_\alpha u^1_\alpha\right)^{-2k} S_{jk}\left(w^1_{\mu\nu}, w^r_{\mu\mu}\right),$$

$$R_k\left(u^r_\mu, w^1_{\mu\nu}\right)\left(u^1_\alpha u^1_\alpha\right)^{-2k+1}$$

where

$$w^r_{\mu\nu} = u^r_\alpha u^r_\alpha\left(u^r_{\mu\nu} + \frac{g_{\mu\nu}}{1-n}u^r_{\beta\beta}\right) - u^r_\beta(u^r_\mu u^r_{\beta\nu} + u^r_\nu u^r_{\beta\mu}),$$

$$j = \overline{0,k}, \qquad k = \overline{1,n+1}, \qquad r = \overline{2,m},$$

no sum over r.

Appendix 5

Differential invariants (DI) of Galilei algebras AG(1,n), A$\widetilde{\text{G}}$(1,n) and Schrödinger algebra ASch(1,n)

Following [42*] we give functional bases of second-order DI of algebras AG(1,n), A$\widetilde{\text{G}}$(1,n), and ASch(1,n).

1°. DI of AG(1,n) $= \{\partial_t, \partial_a, u\partial_u,\ J_{ab} = x_a\partial_b - x_b\partial_a,\ G_a = t\partial_a + \mu x_a u\partial_u\}$:

$S_j(\varphi_{ab}), \qquad \varphi = \ln u$

$M_1 = 2\mu\varphi_t + \varphi_a\varphi_a,$

$M_2 = \mu^2\varphi_{tt} + 2\mu\varphi_a\varphi_{at} + \varphi_a\varphi_b\varphi_{ab},$

$R_j(\theta_a, \varphi_{ab}),$

where

$$\theta_a = \mu\varphi_{at} + \varphi_a\varphi_{ab}, \qquad j = \overline{1,n}.$$

1°°. DI of A$\widetilde{\text{G}}$(1,n) $= \{\text{AG}(1,n), D = 2t\partial_t + x_a\partial_a + \lambda u\partial_u\}$:

$$\frac{M_2}{M_1^2}, \qquad \frac{R_j}{M_1^{3+j}}, \qquad \frac{S_j}{M_1^{1+j}}.$$

1°°°. DI of ASch(1,n) $= \left\{\text{A}\widetilde{\text{G}}(1,n), \Pi = tD - t^2\partial_t + \frac{\mu x^2}{2}u\partial_u, \quad \lambda = -\frac{n}{2}\right\}$:

$$\frac{N_2}{N_1^2}, \qquad \frac{\widehat{R}_j}{N_1^{3+j}}, \qquad \frac{\widehat{S}_j}{N_1^{1+j}},$$

$$N_1 = 2\mu\varphi_t + \varphi_{aa} + \varphi_a\varphi_a,$$
$$N_2 = \mu^2\varphi_{tt} + 2\mu\left(\frac{1}{n}\varphi_t\varphi_{aa} + \varphi_a\varphi_{at}\right) + \varphi_a\varphi_b\varphi_{ab} + \frac{1}{n}\varphi_a\varphi_a\varphi_{bb} + \frac{1}{2n}(\varphi_{bb})^2,$$

where

$$R_j = \sum_{\ell=0}^{j} R_\ell(\varphi_{aa})^{j-\ell}\frac{(-n)^\ell j!}{\ell!\,(j-\ell)!},$$
$$\widehat{S}_j = \sum_{\ell=0}^{j}\frac{(-n)^\ell(j-1)!(j+1)!}{(\ell+1)!\,(j-\ell)!}S_\ell(\varphi_{aa})^{j-\ell},$$

R_j, S_ℓ are given in (A.5.1).

2° DI of AG(1,n), $\mu = 0$:

$$M_1 = \varphi_t - \varphi_a\sigma_a, \qquad M_2 = \varphi_{tt} - \varphi_{at}\sigma_a,$$
$$R_j = R_j(\varphi_a, \varphi_{ab}), \qquad S_j = S_j(\varphi_{ab}) \qquad j = \overline{1,n}.$$

2°° DI of A$\widetilde{\text{G}}$(1,n), $\mu = 0$:

$$\frac{M_1^2}{M_2}, \qquad \frac{R_j}{M_1^{j+1}}, \qquad \frac{S_j}{M_1^{j+1}}.$$

2°°° DI of ASch(1,n), $\mu = 0$:

$$R_j M^{-\frac{1}{2}(j+1)}, \qquad S_j M^{-\frac{1}{2}(j+1)},$$

where

$$M = (\varphi_t - \sigma_a\varphi_a)^2 + (\varphi_{tt} - \varphi_{at}\sigma_a)(\lambda + \varphi_a\varphi_b r_{ab}),$$
$$\{r_{ab}\} = \{\varphi_{ab}\}^{-1}, \qquad \sigma_a = r_{ab}\varphi_{bt}.$$

Below we list DI in the case of complex scalar fields ψ.

3°. DI of AG(1,n) $= \{\partial_t, \partial_a,\ J_{ab} = x_a\partial_b - x_b\partial_a,\ G_a = t\partial_a + imx_aQ,\ \ I = \psi\partial_\psi + \psi^*\partial_{\psi^*}\ (Q \equiv \psi\partial_\psi - \psi^*\partial_{\psi^*})\}$:

$m \neq 0 \ \Rightarrow$

$$\phi + \phi^*, \qquad \left(\phi = \ln\psi, \qquad \operatorname{Im}\phi = \arctan\frac{\operatorname{Re}\psi}{\operatorname{Im}\psi}\right)$$

$$M_1 = 2im\phi_t + \phi_a\phi_a, \qquad M_1^*,$$

$$M_2 = -im^2\phi_{tt} + 2im\varphi_a\varphi_{at} + \phi_a\phi_b\phi_{ab}, \qquad M_2^*,$$

$$S_{jk} = S_{jk}(\varphi_{ab}, \varphi_{ab}^*),$$

$$R_j^1 = R_j(\theta_a, \phi_{ab}), \qquad R_j^2 = R_j(\theta_a^*, \phi_{ab}),$$

$$R_j^3 = R_j(\phi_a + \phi_a^*, \phi_{ab}), \qquad \theta_a = im\phi_{at} + \phi_b\phi_{ab}.$$

$3^{\circ\circ}$. DI of $\mathrm{A}\widetilde{\mathrm{G}}(1,n) = \{\mathrm{A}\widetilde{\mathrm{G}}(1,n), D = 2t\partial_t + x_a\partial_a - \frac{1}{2}nI\}, \qquad m \neq 0:$

$$\frac{M_1^*}{M_1}, \qquad \frac{M_2}{M_1^2}, \qquad \frac{M_2^*}{M_1^2},$$

$$R_j^\ell M_1^{-(3+j)}, \quad \ell = 1,2; \qquad R_j^3 M_1^{-(j+1)}, \qquad S_{jk} M_1^{-(j+1)},$$

and $\phi + \phi^*$ under $\lambda = 0$,

$$M_1 \exp\{\lambda(\phi + \phi^*)\} \text{ under } \lambda \neq 0.$$

$3^{\circ\circ\circ}$. DI of

$$\mathrm{ASch}(1,n) = \{\mathrm{A}\widetilde{\mathrm{G}}(1,n), \Pi = tD - t^2\partial_t + \tfrac{1}{2}mx_ax_aI\}, \quad \lambda = -\tfrac{1}{2}n, \quad m \neq 0:$$

$$N_1 \exp\left\{\frac{4}{n}(\phi + \phi^*)\right\}, \qquad \frac{N_1}{N_1^*}, \qquad \frac{N_2}{N_1^2}, \qquad \frac{N_2^*}{N_1^2}$$

$$R_j^\ell N_1^{-(3+j)}, \quad (\ell = 1,2); \qquad \widehat{R}_j^3 N_1^{-(j+1)}, \qquad \widehat{S}_{jk} N_1^{-(1+j)},$$

where

$$N_1 = 2im\phi_t + \phi_a\phi_a + \phi_{aa},$$

$$N_2 = -m^2\phi_{tt} + 2im\left(\phi_a\phi_{at} + \frac{1}{n}\phi_t\phi_{aa}\right)\phi_a\phi_b\phi_{ab} + \frac{1}{n}\phi_a\phi_a\phi_{bb} + \frac{1}{2n}\varphi_{aa}^2,$$

$$\widehat{S}_{jk} = \sum_{\ell=0}^{j}\sum_{r=0}^{k}\Big[S_{r\ell}(-n)^\ell C_k^r C_k^{\ell+1-r}(\phi_{aa})^{k-r}(\phi_{aa}^*)^{j-\ell-k+r} + $$

$$+ j(\phi_{aa})^k(\phi_{0a}^*)^{j-k-1}\Big],$$

$$\widehat{R}_j^\ell = \sum_{k=0}^{j} R_k^j(\varphi_{aa})^{j-k}\frac{(-n)^k j!}{k!(j-k)!}, \qquad \ell = 1,2,3.$$

4°. DI of AG(1,n), $m=0$:

$$\phi+\phi^*,\qquad M_1=\phi_t-\phi_a\sigma_a,\qquad M_1^*$$

$$M_2=\phi_{tt}-\phi_{at}\sigma_a,\qquad M_2^*,\qquad (\phi_{at}=\sigma_a,\phi_{ab}),$$

$$R_j^1=R_j(\phi_a,\phi_{ab}),\qquad R_j^2\,(\phi_a^*,\phi_{ab}),$$

$$R_j^3=R_j\,(\sigma_a-\sigma_a^*,\psi_{ab}),\qquad S_{jk}.$$

4°°. DI of $\mathrm{A\widetilde{G}}(1,n)$, $m=0$:

$$\frac{M_1}{M_1^*},\quad \frac{N_2}{M_1^2},\quad \frac{M_2^*}{M_1^2},\quad \frac{R_j^\ell}{M_1^{j+1}},\quad \frac{S_{jk}}{M_1^{j+1}}$$

and $M_1\exp\left\{\frac{2}{\lambda}(\phi+\phi^*)\right\}$ under $\lambda\neq 0$; $\phi+\phi^*$ under $\lambda=0$.

4°°°. DI if ASch(1,n), $m=0$:

$$\lambda=0\;\Rightarrow\;\phi+\phi^*,\qquad \frac{N_1^2}{N_2},\qquad \frac{N_1^{*2}}{N_2},\qquad \frac{(S_{jk})^2}{N_1^{j+1}},$$

$$\left(R_j^\ell\right)^2N_1^{-(j+1)},\qquad \ell=1,2,4;$$

$$\lambda\neq 0\;\Rightarrow\;N_1\exp\left\{\frac{4}{\lambda}(\phi+\phi^*)\right\},\qquad \frac{N_1}{N_1^*},$$

$$N_3\exp\left\{\frac{3}{\lambda}(\phi+\phi^*)\right\},\qquad \frac{(R_j^\ell)^2}{N_1^{j+1}},\quad (\ell=1,2,3),\qquad \frac{(S_{jk})^2}{N_1^{j+1}},$$

where

$$N_1=(\phi_t-\sigma_a\phi_a)^2+(\phi_{tt}-\sigma_a\phi_{at})\,(\lambda+\phi_a\phi_b r_{ab}),$$

$$\{r_{ab}\}=\{\phi_{ab}\}^{-1},$$

$$N_2=(\phi_t-\phi_c\sigma_c)\,\phi_a^*\phi_b^*r_{ab}^*-(\phi_t^*-\phi_c^*\sigma_c^*)\,\phi_a\phi_b r_{ab},$$

$$N_3=\tau_a\,(\phi_a^*-\phi_a)+\phi_t-\phi_t^*,$$

$$\tau_a=(\phi_b\phi_t+\lambda\phi_{bt})\,r_{ab},\qquad \{\widehat{r}_{ab}\}=\{\lambda\phi_{ab}+\phi_a\phi_b\}^{-1},$$

$$R_j^1=R_j(\phi_a,\phi_{ab}),\qquad R_j^2=R_j(\phi_a^*,\phi_{ab}),$$

$$R_j^3=R_j(\sigma_a-\sigma_a^*,\phi_{ab}),\qquad R_j^4=R_j(\rho_a,\phi_{ab}),$$

$$\left(\rho_a=(\phi_t-\sigma_b\phi_b)(\phi_c^*r_{ac}-\phi_c r_{ac}^*)-\phi_b\phi_d r_{bd}(\sigma_a-\sigma_a^*)\right);$$

$$S_{jk}=S_{jk}(\phi_{ab},\phi_{ab}^*).$$

Appendix 6

Compatibility and solutions of the overdetermined d'Alembert-Hamilton system

The system

$$\begin{aligned} \Box\omega &= F_1(\omega), \\ \omega_\mu\omega^\mu &= F_2(\omega), \end{aligned} \tag{A.6.1}$$

where $\omega_\mu = \partial\omega/\partial x^\mu$, $\mu = \overline{0,3}$, F_1, F_2 are arbitrary smooth functions, often arises from reduction of a Poincare invariant PDE to an ODE (see Paragraphs 1.4, 2.1, 5.12). Here we consider, following [86*, 122*, 130*], the compatibility of the overdetermined system (A.6.1) and describe its solutions.

By means of an appropriate change of dependent variable, system (A.6.1) can be transformed to the form

$$\begin{aligned} \Box\,\omega &= F(\omega), \\ \omega_\mu\omega^\mu &= \lambda, \qquad \lambda = \text{const}. \end{aligned} \tag{A.6.2}$$

Lemma A.6.1. *Solutions of the system* (A.6.2) *satisfy the indentities*

$$\begin{aligned} &\omega_{\mu\nu_1}\omega_{\nu_1\mu} = -\lambda\dot{F}, \\ &\omega_{\mu\nu_1}\omega_{\nu_1\nu_2}\omega_{\nu_2\mu} = \frac{1}{2!}\lambda^2\ddot{F}, \ldots \\ &\omega_{\mu\nu_1}\omega_{\nu_1\nu_2}\cdots\omega_{\nu_n\,\mu} = \frac{(-1)^n}{n!}\lambda^n\frac{d^nF}{d\omega^n}, \end{aligned} \tag{A.6.3}$$

where $\omega_{\alpha\beta} = \dfrac{\partial^2 \omega}{\partial x^\alpha \partial x^\beta}$, $\alpha,\ \beta = \overline{0,3}$, $n \geq 1$, $\dot{F} \equiv \dfrac{dF}{d\omega}$.

Lemma A.6.2. *Solutions of the system* (A.6.2) *satisfy the equality*

$$\det\left(\omega_{\mu\nu}\right) = 0. \tag{A.6.4}$$

The proofs of these lemmas one can make straightforwardly.

Theorem A.6.1. *The necessary condition of compatibility of the overdetermined system* (A.6.2) *is*

$$F(\omega) = \begin{cases} 0, \\ \lambda(\omega + c_1)^{-1}, \\ 2\lambda(\omega + c_1)[(\omega + c_1)^2 + c_2]^{-1}, \\ 3\lambda[(\omega + c_1)^2 + c_2][(\omega + c_1)^3 + 3c_2(\omega + c_1) + c_3]^{-1}, \end{cases} \tag{A.6.5}$$

where c_1, c_2, c_3 *are arbitrary constants.*

Proof. By direct (and rather tiresome) verification one can be convinced of the following identity

$$\begin{gathered} 6(\omega_{\mu\nu_1}\omega_{\nu_1\nu_2}\omega_{\nu_2\nu_3}\omega_{\nu_3\mu}) - 8(\Box\omega)(\omega_{\mu\nu_1}\omega_{\nu_1\nu_2}\omega_{\nu_2\mu}) - \\ -3(\omega_{\mu\nu_1}\omega_{\nu_1\mu})^2 + 6(\Box\omega)^2(\omega_{\mu\nu_1}\omega_{\nu_1\mu}) - (\Box\omega)^4 = 24\det(\omega_{\alpha\beta}). \end{gathered} \tag{A.6.6}$$

Substituting (A.6.3), (A.6.4) into (A.6.6) we obtain the nonlinear ODE for

$$\lambda^3\dddot{F} + 4\lambda^2 F\ddot{F} + 3\lambda^2\dot{F}^2 + 6\lambda\dot{F}F^2 + F^4 = 0, \tag{A.6.7}$$

The general solution of Equation (A.6.7) is given in (A.6.5). Thus, the theorem is proved.

Remark A.6.1. Compatibility of the three-dimensional d'Alembert-Hamilton system has been investigated in detail by Collins [41], who essentially used geometrical methods which could not be generalized to higher dimensions.

Using Lie's method one can prove the following statement.

Theorem A.6.2. [86*, 130*] *System* (A.6.2) *is invariant under the 15-parameter conformal group* C(1,3) *iff*

$$F(\omega) = 3\lambda(\omega + c)^{-1}, \qquad \lambda > 0, \qquad c = \text{const}. \tag{A.6.8}$$

Remark A.6.2. Formula (A.6.8) can be obtained from (A.6.5) by putting $c_2 = c_3 = 0$. So Theorem A.6.2 demonstrates the close connection between compatibility of system (A.6.2) and its symmetry.

In conclusion, let us list the explicit form of exact solution of system (A.6.2), taking into account Theorem A.6.1.

Table A.6.1. Cases of compatibility of system (A.6.2) and corresponding solution.

N	λ	$F(\omega)$	$\omega = \omega(x)$
1	1	0	dx
2	1	ω^{-1}	$[(dx)^2 - (ax)^2]^{1/2}$
3	1	$2\omega^{-1}$	$[(dx)^2 - (ax)^2 - (bx)^2]^{1/2}$
4	1	$3\omega^{-1}$	$(x_\nu x^\nu)^{1/2}$
5	-1	0	$ax \cos h_1 + bx \sin h_1 + g_1$, $dx - ax \cos h_2 - bx \sin h_2 - g_2 = 0$
6	-1	$-\omega^{-1}$	$[(ax + h_1)^2 + (bx + h_2)^2]^{1/2}$
7	-1	$-2\omega^{-1}$	$[(ax)^2 + (bx)^2 + (cx)^2]^{1/2}$
8	0	0	h_1

In this table, h_1, g_1 are arbitrary smooth functions of $dx + cx$, and h_2, g_2 are arbitrary smooth functions of $\omega + cx$; a_μ, b_μ, c_μ, d_μ are arbitrary real constants satisfying conditions (2.1.27).

Appendix 7

Q-Conditional Symmetry of the Heat Equation

Here we consider in full detail, as a simple but non-trivial example, how to find and use Q-conditional symmetry of the one-dimensional heat equation*

$$u_0 = u_{11} \tag{A.7.1}$$

($u = u(x_0, x_1)$, $u_0 = \partial u/\partial x_0$, $u_1 = \partial u/\partial x_1$, and so on).

The definition of Q=symmetry is given in Sec. 5.7 (see Definition 5.7.3, p.328). The Lie-maximal invariance algebra of Equation (A.7.1) is written in (5.1.6). The problem of finding non-classical symmetry (in our terminology: Q-conditional symmetry) was first put forward by Bluman and Cole [131*]. However, in this important paper the authors did not give explicitly any operators which would differ from those of (5.1.6). Below we will present a complete investigation of this problem.

The general form of a first-order operator is

$$Q = A(x_0, x_1, u)\partial_0 + B(x_0, x_1, u)\partial_1 + C(x_0, x_1, u)\partial_u, \tag{A.7.2}$$

where A, B, C are some differentiable functions of x_0, x_1, u to be determined from the invariance condition (5.7.7). It will be noted that because of the imposed condition (5.7.6)

$$Qu = 0 \Longleftrightarrow Au_0 + Bu_1 = C, \tag{A.7.3}$$

there are really only two independent cases of operator (A.7.2).

Theorem A.7.1. *The heat equation* (A.7.1) *is Q-conditionally invariant under operator* (A.7.2) *if and only if its coordinates are as follows:*

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* Some results herein were obtained in collaboration with R.E.Popovich.

Case 1:

$$A = 1, \quad B = W^1(x_0, x_1), \quad C = W^2(x_0, x_1)u + W^3(x_0, x_1) \quad \text{(A.7.4)}$$

and functions $\vec{W} = \vec{W}(x_0, x_1) = \{W^1, W^2, W^3\}$ *satisfy*

$$(\partial_0 + 2W_1^1 - \partial_{11})\,\vec{W} = \vec{F}, \qquad \vec{F} \equiv \{2W_1^2, 0, 0\}; \qquad \text{(A.7.5)}$$

Case 2:

$$A = 0, \quad B = 1, \quad C = v(x_0, x_1, u) \qquad \text{(A.7.6)}$$

and functions $v = v(x_0, x_1, u)$ *satisfies the PDE*

$$v_0 = v_{11} + 2vv_{1u} + v^2 v_{uu}. \qquad \text{(A.7.7)}$$

Proof. From the criterion of invariance

$$\mathcal{Q}(u_0 - u_{11})\Big|_{\substack{u_0 = u_{11} \\ Qu=0}} = 0, \qquad \text{(A.7.8)}$$

absolutely analogously to the standard Lie's algorithm, one finds the defining equation for the coordinates of operator (A.7.2) which can be reduced to (A.7.4)–(A.7.7). It is to be pointed out that unlike Lie's algorithm, in the cases considered above, the defining equations (A.7.5), (A.7.7) are nonlinear ones, which is a typical feature of Q-conditional invariance.

It goes without saying that Q-conditional invariance includes Lie's invariance in particular. So, in our case of the heat equation, we obtain infinitesimals (5.1.6) as simplest solutions of (A.7.5), (A.7.7):

$$\begin{aligned}
&A = 1, \quad \vec{W} = 0 \Rightarrow Q = \partial_0 \\
&A = v = 0, \quad B = 1 \Rightarrow Q = \partial_1, \qquad \text{(A.7.9)} \\
&A = 0, \quad B = 1, \quad v = -(x_1/2x_0)u \Rightarrow Q = G = x_0\partial_1 - \tfrac{1}{2}x_1 u \partial_u, \\
&A = 1, \quad W^1 = x_1/2x_0, \quad W^2 = W^3 = 0 \Rightarrow Q = D, \\
&A = 1, \quad W^1 = x_1/x_0, \quad W^2 = -(2x_0 + x_1^2)/4x_0^2, \quad W^3 = 0 \Rightarrow Q = \Pi.
\end{aligned}$$

Remark A.7.1. The system of defining equations (A.7.5) was first obtained by Bluman and Cole [131*]. Further investigation of system (A.7.5) was continued in [132*], where the question of linearization of the first two equations of (A.7.5) had been studied. The general solution of the problem of linearization of Equations (A.7.5), (A.7.7) will be given after a while.

Now let us list some concrete operators (A.7.2) of Q-conditional invariance of Equation (A.7.1) obtained as partial solutions of the defining equations

(A.7.5), (A.7.7). In the following table we also give corresponding invariant ansatze and the reduced equations.

Table A.7.1.

N	Operator Q	Ansatz $u =$	Reduced ODE
1.	$-x_1\partial_0 + \partial_1$	$= \varphi(x_0 + x_1^2/2)$	$\varphi'' = 0$
2.	$-x_1\partial_0 + \partial_1 + x_1^3\partial_u$	$= \varphi(x_0 + x_1^2/2) + \frac{1}{4}x_1^4$	$\varphi'' = -3$
3.	$x_1^2\partial_0 - 3x_1\partial_1 - 3u\partial_u$	$= x_1\varphi(x_0 + \frac{1}{6}x_1^2)$	$\varphi'' = 0$
4.	$x_1^2\partial_0 - 3x_1\partial_1 - -(3u + x_1^5)\partial_u$	$= x_1\varphi(x_0 + \frac{1}{6}x_1^2) + \frac{1}{15}x_1^5$	$\varphi = -15$
5.	$x_1\partial_1 + u\partial_u$	$= x_1\varphi(x_0)$	$\varphi' = 0$
6.	$\operatorname{cth} x_1\partial_1 + u\partial_u$	$= \varphi(x_0)\operatorname{ch} x_1$	$\varphi' - \varphi = 0$
7.	$\cot x_1\partial_1 - u\partial_u$	$= \varphi(x_0)\cos x_1$	$\varphi' + \varphi = 0$
8.	$\partial_1 - u\partial_u - \dfrac{u}{2x_0 - x_1}\partial_u$	$= (2x_0 - x_1)e^{-x_1}\varphi(x_0)$	$\varphi' - \varphi = 0$
9.	$\partial_1 - \sqrt{-2(x_0 + u)}\,\partial_u$	$= -x_0 - \frac{1}{2}[x_1 + \varphi(x_0)]^2$	$\varphi' = 0$
10.	$(x_0 + \frac{1}{2}x_1^2)\partial_0 - x_1\partial_1$	$= \varphi(x_0x_1 + \frac{1}{3!}x_1^3)$	$\varphi'' = 0$

Theorem A.7.2 *The Lie-maximal invariance algebra of system* (A.7.5) *is given by the operators*

$$\begin{aligned}
&\partial_0, \quad \partial_1, \quad G^{(1)} = x_0\partial_1 + \partial_{W^1} - \tfrac{1}{2}W^1\partial_{W^2} - \tfrac{1}{2}x_1W^3\partial_{W^3},\\
&D^{(1)} = 2x_0\partial_0 + x_1\partial_1 - W^1\partial_{W^1} - 2W^2\partial_{W^2},\\
&\Pi^{(1)} = x_0\big(x_0\partial_0 + x_1\partial_1 - W^1\partial_{W^1} - 2W^2\partial_{W^2} - \tfrac{5}{2}W^3\partial_{W^3}\big)+\\
&\qquad +x_1\big(\partial_{W^1} - \tfrac{1}{2}W^1\partial_{W^2}\big) - \tfrac{1}{2}\partial_{W^2} - \tfrac{1}{4}x_1^2W^3\partial_{W^3},\\
&X^{(1)} = (f_0 + f_1W^1 - fW^2)\partial_{W^3}, \qquad I^{(1)} = W^3\partial_{W^3},
\end{aligned} \tag{A.7.10}$$

where $f = f(x_0, x_1)$ *is an arbitrary solution of* (A.7.1), *that is,* $f_0 = f_{11}$.

Theorem A.7.3 *The Lie-maximal invariance algebra of system* (A.7.7) *is given by the operators*

$$\begin{aligned}
&\partial_0, \quad \partial_1, \quad D^{(2)} = 2x_0\partial_0 + x_1\partial_1 + u\partial_u, \quad D^{(3)} = u\partial_u + v\partial_v,\\
&G^{(2)} = x_0\partial_1 - \tfrac{1}{2}x_1\big(u\partial_u + v\partial_v\big) - \tfrac{1}{2}u\partial_v,\\
&\Pi^{(2)} = x_0\big(x_0\partial_0 + x_1\partial_1 - \tfrac{1}{2}u\partial_u - \tfrac{3}{2}v\partial_v\big) - \tfrac{1}{4}x_1^2\big(u\partial_u + v\partial_v\big) - \tfrac{1}{2}x_1^2u\partial_v,\\
&X^{(2)} = f\partial_u + f_1\partial_v, \qquad (f_0 = f_{11}).
\end{aligned} \tag{A.7.11}$$

One can get the proofs of these two theorems by means of the standard Lie's algorithm.

Operators (A.7.10), (A.7.11) can be used to find exact solutions of Equations (A.7.5), (A.7.7). In particular, using the formula of generating solutions at the expense on invariance under $\Pi^{(2)}$

$$v^{II}(x_0, x_1, u) = (1-\theta x_0)^{3/2} \exp\left\{\frac{\theta x_1^2}{4(1-\theta x_0)}\right\} v^{I}(x_0', x_1', u') +$$
$$+\frac{\theta}{1-\theta x_0}\cdot\frac{x_1 u}{2}; \qquad x_0' = \frac{x_0}{1-\theta x_0}, \qquad x_1' = \frac{x_1}{1-\theta x_0}, \tag{A.7.12}$$
$$u' = (1-\theta x_0)^{1/2} \exp\left\{-\frac{1}{4}\frac{\theta x_1^2}{1-\theta x_0}\right\}, \qquad (\theta = \text{const})$$

one can construct new solutions of Equations (A.7.7) starting from known ones.

Solutions of Equations (A.7.5), (A.7.11) can be obtained by the use of reduction on subalgebras of the invariance algebras (A.7.10), (A.7.11). For example, using the subalgebra $<\partial_0 + aI^{(1)}>$ of the algebra (A.7.10) we find the following solution of the system (A.7.5)

$$\begin{aligned} W^1 &= \frac{c_1^2 - c_3^2}{-c_1\tan(c_1x+c_2) + c_3\tan(c_3x+c_4)},\\ W^2 &= -c_1c_3\frac{c_1\tan(c_3x+c_4) - c_3\tan(c_1x+c_2)}{-c_1\tan(c_1x+c_2)+c_3\tan(c_3x+c_4)},\\ W^3 &= (\varphi_{xx} - W^1\varphi_x - W^2\varphi)e^{at}, \end{aligned} \tag{A.7.13}$$

where $c_1, \ldots, c_4$ are arbitrary constants, $\varphi = \varphi(x)$, $\varphi_{xx} = a\varphi$.

Theorem A.7.4. *The system* (A.7.5) *is reduced to the system of disconnected heat equations*

$$\vec{z}_0 = \vec{z}_{11}, \qquad (\vec{z} = \vec{z}(x_0, x_1) = \{z^1, z^2, z^3\}) \tag{A.7.14}$$

with the help of the nonlocal transformations

$$W^1 = -\frac{z_{11}^1 z^2 - z^1 z_{11}^2}{z_1^1 z^2 - z^1 z_1^2}, \tag{A.7.15}$$

$$W^2 = -\frac{z_{11}^1 z_1^2 - z_1^1 z_{11}^2}{z_1^1 z^2 - z^1 z_1^2}, \qquad W^3 = z_{11}^3 + W^1 z_1^3 - W^2 z^3.$$

Expressions (A.7.15) result in (after using the corresponding operator (A.7.2), (A.7.4)) the ansatz

$$u = z^1\varphi(\omega) + z^3, \qquad \omega = \frac{z^2}{z^1} \tag{A.7.16}$$

(z^1, z^2, z^3 are solutions of (A.7.14)), and the reduced equation is $\varphi'' = 0$. This means that

$$u = c_1 z^1 + c_2 z^2 + c_3 z^3. \tag{A.7.17}$$

So, we get just the well-known supersosition principle for the heat equation.

Letting $W^2 = W^3 = 0$ we get from (A.7.5) the Burger's equation

$$W_0^1 + 2W^1 W_1^1 = W_{11}^1. \tag{A.7.18}$$

Using Hopf-Cole transformation (see Sec. 5.1) one obtains solutions of Equation (A.7.18) in the form

$$W^1 = -\partial_1 \ln f = -\frac{f_1}{f}, \qquad (f_0 = f_{11}). \tag{A.7.19}$$

This results in the operator

$$Q = f\partial_0 - f_1\partial_1. \tag{A.7.20}$$

Q-conditional symmetry of Equation (A.7.1) under the operator Q (A.7.20) leads to the following statement.

Theorem A.7.5. *If function f is an arbitrary solution of the heat equation* (A.7.1) *and u is the general integral of the ODE*

$$f_1 dx_0 + f dx_1 = 0, \tag{A.7.21}$$

then u satisfies Equation (A.7.1).

Proof. We note that Equation (A.7.21) is a perfect differential equation and therefore its general solution, $u(x_0, x_1) = c$, possesses the following property

$$u_0 = f_1, \qquad u_1 = f. \tag{A.7.22}$$

Having used (A.7.22) we obtain

$$u_0 - u_{11} = f_1 - f_2 \equiv 0,$$

and the theorem is proved.

Theorem A.7.5 may be considered as another algorithm of generating solutions of Equation (A.7.1). Indeed, even starting from a rather trivial solution of the heat equation, $u = 1$, we get the chain of quite interesting solutions

$$1 \to x_1 \to x_0 + \frac{x_1^2}{2!} \to x_0 x_1 + \frac{x_1^3}{3!} \to \cdots, \tag{A.7.23}$$

and among them the solutions

$$\frac{x_1^{2m}}{(2m)!}+\frac{x_0}{1!}\frac{x_1^{2m-2}}{(2m-2)!}+\frac{x_0^2}{2!}\frac{x_1^{2m-4}}{(2m-4)!}+\cdots+\frac{x_0^{m-1}}{(m-1)!}\frac{x_1^2}{2!}+\frac{x_0^m}{m!}, \tag{A.7.24}$$

$$\frac{x_1^{2m+1}}{(2m+1)!}+\frac{x_0}{1!}\frac{x_1^{2m-1}}{(2m-1)!}+\frac{x_0^2}{2!}\frac{x_1^{2m-3}}{(2m-3)!}+\cdots+\frac{x_0^{m-1}}{(m-1)!}\frac{x_1^2}{3!}+\frac{x_0^m}{m!}\frac{x_1}{1!}, \tag{A.7.25}$$

where $m = 1, 2, 3, \ldots$.

It will be also noted thay supposing function v in (A.7.7) to be independent of x_1, and denoting

$$v = \frac{1}{w(x_0, u)} \tag{A.7.26}$$

we get instead of (A.7.7) the following remarkable nonlinear heat equation

$$w_0 = \partial_u\left(w^{-2}w_u\right). \tag{A.7.27}$$

One easily see that the operator

$$Q = w(x_0, u)\partial_1 + \partial_u \tag{A.7.28}$$

sets the connection between Equations (A.7.27) and (A.7.1):

$$\begin{aligned} w_0 - \partial_u(w^{-2}w_u) &= \frac{1}{u_1}\partial_1\left(\frac{u_0 - u_{11}}{u_1}\right), \\ u_0 - u_{11} &= \frac{1}{w}\int\left[w_0 - \partial_u(w^{-2}w_u)\right]du \end{aligned} \tag{A.7.29}$$

by means of the change of variables

$$w(x_0, u) = \frac{\partial x_1(x_0, u)}{\partial u}, \qquad \frac{\partial u(x_0, x_1)}{\partial x_1} = \frac{1}{w(x_0, u)}. \tag{A.7.30}$$

This result has been obtained differently in [133*, 134*].

If we suppose v from (A.7.7) to have the form

$$v = \varphi(x_0, x_1)u, \tag{A.7.31}$$

then (A.7.7) is reduced to the Burger's equation for φ

$$\varphi_0 = 2\varphi\varphi_1 + \varphi_{11}, \tag{A.7.32}$$

and one may say that operator

$$Q = \partial_1 + \varphi u\partial_u \tag{A.7.33}$$

sets the connection between Equations (A.7.32) and (A.7.1) via the substitution

$$\varphi = f_1/f \tag{A.7.34}$$

Letting

$$v = \varphi(x_0, x_1)u + h(x_0, x_1) \tag{A.7.35}$$

and substituting it into (A.7.7) one finds the Burger's equation (A.7.32) for function φ and the following equation for h

$$h_0 = 2h\varphi_1 + h_{11}. \tag{A.7.36}$$

Having made the change of variables

$$h = -(f_1/f)g + g_1 \tag{A.7.37}$$

we reduce (A.7.32), (A.7.36) to two disconnected heat equations

$$f_0 = f_{11} \qquad g_0 = g_{11}. \tag{A.7.38}$$

Now we see how to linearize Equation (A.7.7) in the general case. Let us introduce the notations

$$S_1(x_0, x_1, u, v) = v_0 - \left(v_{11} + 2vv_{1u} + v^2 v_{uu}\right). \tag{A.7.39}$$

After changing the variables to

$$v = -\frac{z_1}{z_u}, \qquad z = z(x_0, x_1, u) \tag{A.7.40}$$

we get

$$S_1(x_0, x_1, u, v) = -\frac{1}{z_u}(\partial_1 + v\partial_u)S_2(x_0, x_1, u, z), \tag{A.7.41}$$

where

$$S_2(x_0, x_1, u, z) = z_0 - z_{11} + 2\frac{z_1}{z_u}z_{1u} - \frac{z_1^2}{z_u^2}z_{uu}. \tag{A.7.42}$$

Having applied the hodograph transformation

$$y_0 = x_0, \qquad y_1 = x_1, \qquad y_2 = z, \qquad R = u \tag{A.7.43}$$

we get

$$S_2(x_0, x_1, u, z) = -\frac{1}{R_2}(R_0 - R_{11}), \tag{A.7.44}$$

where

$$R = R(y_0, y_1, y_2).$$

So, we see that Equations (A.7.5), (A.7.7) are reduced to the heat equations (A.7.14), (A.7.44), respectively.

Appendix 8

On Nonlocal Symmetries of Nonlinear Heat Equation*

We will show how to find and use nonlocal symmetries of the nonlinear heat equation to construct its exact solutions.

The complete group classification of nonlinear one-dimensional heat equation

$$u_0 = \partial_1(F(u)u_1), \tag{A.8.1}$$

($u = u(x_0, x_1)$), $F(u)$ is an arbitrary differentiable function, $u_0 = \partial u/\partial x_0$, and so on) was made by Ovsyannikov (see [161]). His results may be summarized as follows.

Theorem A.8.1. *Depending on the function $F(u)$, the maximal invariance algebra (MIA) of Equation* (A.8.1) *is determined by the following basis elements:*

1) $F(u)$ is an arbitrary function, then MIA is three-dimensional,

$$\partial_0 = \frac{\partial}{\partial x_0}, \qquad \partial_1 = \frac{\partial}{\partial x_1}, \qquad D_1 = 2x_0\partial_0 + x_1\partial_1 \tag{A.8.2}$$

2) $F(u) = \lambda u^k$, then MIA is four-dimensional,

$$\partial_0, \qquad \partial_1, \qquad D_1, \qquad D_2 = 2x_1\partial_1 + \frac{2}{k}u\partial_u \tag{A.8.3}$$

3) $F(u) = \lambda e^u$, then MIA is four-dimensional,

$$\partial_0, \qquad \partial_1, \qquad D_1, \qquad D_3 = x_1\partial_1 + 2\partial_u \tag{A.8.4}$$

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* Results in this appendix were obtained with the collaboration of V.A.Tychinin and T.K.Amerov.

4) $F(u) = \lambda u^{-4/3}$, then MIA is five-dimensional,

$$\partial_0, \quad \partial_1, \quad D_1, \quad D_4 = x_1\partial_1 - \frac{3}{2}u\partial_u, \quad \Pi = x_1^2\partial_1 - 3x_1 u\partial_u. \quad (A.8.5)$$

It is also known (see [135*], for example) that the chain of transformations

$$u(x_0, x_1) = \frac{\partial v(x_0, x_1)}{\partial x_1}, \quad (A.8.6)$$

$$x_0 = t, \qquad x_1 = w(t, x), \qquad v = x; \quad (A.8.7)$$

$$\frac{\partial w(t, x)}{\partial x} = z(t, x) \quad (A.8.8)$$

does not go out from the class of equations (A.8.1). It means that having made transformations (A.8.6)–(A.8.8) one gets from (A.8.1) another equation

$$z_t = \partial_x(F^*(z)z_x), \quad (A.8.9)$$

where

$$F^*(z) = z^{-2}F(z^{-1}). \quad (A.8.10)$$

Below we use transformations (A.8.6)–(A.8.8) to construct nonlocal ansatze which reduce Equation (A.8.1) to ODEs. We also give nonlocal formulae of generating solutions and superposition of solutions of Equation (A.8.1) for some nonlinearities $F(u)$.

First of all we consider the equation

$$u_0 = \partial_1(u^{-2}u_1). \quad (A.8.11)$$

As it follows from (A.8.10) this equation is reduced by means of (A.8.6)–(A.8.8) to the linear heat equation

$$z_t = z_{xx} \quad (A.8.12)$$

whose maximal invariance algebra (MIA) is 6-dimensional, and it is given in (5.1.6). The difference between MIAs of these two (equivalent in a sense) Equations (A.8.11) and (A.8.12) says that nonlinear Equation (A.8.11) possessess a non-Lie symmetry which corresponds to Lie symmetry of the heat equation (A.8.12) generated by Galilei and projective operators (compare MIAs (A.8.3) and (5.1.6)). It is obvious that this non-Lie symmetry of equation (A.8.11) cannot be obtained with the help of Lie's method.

Let us make use of this fact and construct ansatze for Equation (A.8.11) which correspond to those of Equation (A.8.12) invariant under operators

$$\begin{aligned} G &= t\partial_x - \tfrac{1}{2}xz\partial_z, \\ \Pi &= t(t\partial_t + x\partial_x - \tfrac{1}{2}z\partial_z) - \tfrac{1}{4}x^2 z\partial_z. \end{aligned} \quad (A.8.13)$$

Using transformations (A.8.6)–(A.8.8) we find

$$\begin{aligned} u(x_0,x_1) &= [x_0x_1 + x_1h(\omega)]^{-1}, \qquad \omega = \tau + x_0^2, \\ &\exp\{x_0\tau + \tfrac{2}{3}x_0^3\}\varphi(\omega) = x_1; \end{aligned} \tag{A.8.14}$$

$$\begin{aligned} u(x_0,x_1) &= \frac{2(x_0^2+1)}{x_1[2(x_0^2+1)^{1/2}h(\omega) - \tau x_0]}, \qquad \omega = \frac{\tau}{\sqrt{1+x_0^2}}, \\ &\exp\left\{\lambda \arctan x_0 - \frac{x_0\tau}{4(1+x_0^2)}\right\}\varphi(\omega) = x_1(1+x_0^2)^{1/4}. \end{aligned} \tag{A.8.15}$$

In (A.8.14), (A.8.15) $\tau = \tau(x_0,x_1)$ is a function parameter, $h = \dot{\varphi}/\varphi$. Ansatze (A.8.14), (A.8.15) reduced (A.8.11) to the Riccati equation

$$\dot{h} + h^2 = \omega, \tag{A.8.16}$$

$$\dot{h} + h^2 = -\tfrac{1}{4}\omega^2 + \lambda, \tag{A.8.17}$$

respectively, or in terms of φ we have

$$\ddot{\varphi} - \omega\varphi = 0,$$

$$\ddot{\varphi} + (\tfrac{1}{4}\omega^2 - \lambda)\varphi = 0.$$

Solutions of these equations are expressed in terms of Bessel functions.

The connection between solutions of Equations (A.8.11) and (A.8.12), as follows from (A.8.6)–(A.8.8), is

$$u(x_0,x_1) = \left[\frac{\partial z(x_0,\tau)}{\partial \tau}\right]^{-1}, \tag{A.8.18}$$

where $\tau = \tau(x_0,x_1)$ is a functional parameter to be determined from the condition

$$z(x_0,\tau) = x_1. \tag{A.8.19}$$

Now consider the following question. Since Equation (A.8.12) is linear, then function

$$\overset{2}{z}(t,x) = Q\overset{1}{z}(t,x) \tag{A.8.20}$$

will be its solution provided $\overset{1}{z}$ is a given solution and Q is any symmetry operator from (5.1.6). How to obtain analogous results for Equation (A.8.11)? To this end one has to use the connection between the two Equations (A.8.11) and (A.8.12) given by (A.8.6)–(A.8.8). So, choosing in (A.8.20) as Q operator ∂_x we find corresponding formula of generating solutions for Equation (A.8.11)

$$\overset{2}{u}(x_0,x_1) = -[\overset{1}{u}(x_0,\tau)]^3 \left[\frac{\partial u(x_0,\tau)}{\partial \tau}\right]^{-1}, \tag{A.8.21}$$

where $\overset{2}{u}$ is a new solution and $\overset{1}{u}$ is a given one; τ is to be determined from the condition

$$\overset{1}{u}(x_0,\tau) = x_1^{-1}. \tag{A.8.22}$$

For example, starting from solution

$$\overset{1}{u}(x_0,\tau) = \frac{\sqrt{x_0}}{x_1}\left(-\ln\sqrt{x_0}\,x_1\right)^{-1/2}, \tag{A.8.23}$$

we obtain with the help of (A.8.21), (A.8.22) the new parametrical solution

$$\overset{1}{u}(x_0,\tau) = x_0^{3/2}\tau\left(\ln\tau - \tfrac{1}{2}\right)^{-1}, \quad \ln\tau = (x_0x_1\tau)^2. \tag{A.8.24}$$

When $Q = \partial_\tau$, then formulae (A.8.18)–(A.8.20) result in

$$\overset{1}{u}(x_0,x_1) = \frac{[\overset{1}{u}(x_0,\tau)]^5}{2[\overset{1}{u}_\tau(x_0,\tau)]^2 - [\overset{1}{u}(x_0,\tau)]^2\overset{1}{u}_0(x_0,\tau)}, \tag{A.8.25}$$

where $\tau = \tau(x_0,x_1)$ is to be determined from the condition

$$\overset{1}{u}_\tau(x_0,\tau) + x_1[\overset{1}{u}(x_0,\tau)]^3 = 0. \tag{A.8.26}$$

In the same way one can construct the explicit form of formulae (A.8.20) for Equation (A.8.11) for any symmetry operator from MIA of the heat equation (5.1.6)

Formulae (A.8.3) mean that Equation (A.8.11) is not invariant under the Galailei transformations while the heat equation (A.8.12) is, the transformation having the form

$$t' = t, \qquad x' = x + 2at, \qquad z' = \exp\{-a(x+at)\}z \tag{A.8.27}$$

(a is an arbitrary constant). However, having used the nonlocal connection (A.8.18), (A.8.19) between the two equationz one can write down corresponding Galilean transformations for Equation (A.8.11) and afterwards the formula of generating solutions, the latter having the form

$$\overset{2}{u}(x_0,x_1) = \frac{\overset{1}{u}(x_0,\tau)}{ax_1\overset{1}{u}(x_0,\tau) + x_1/\tau}, \tag{A.8.28}$$

where $\tau = \tau(x_0,x_1)$ is a function parameter to be determined from the equations

$$\begin{gathered}\tau_1 = \left[ax_1\overset{1}{u}(x_0,\tau) + x_1/\tau\right], \\ [\overset{1}{u}(x_0,\tau)]^2\tau_0 = \tau_{11}/\tau_1^2 + 2a\overset{1}{u}(x_0,\tau).\end{gathered} \tag{A.8.29}$$

It will be noted that the first equation of (A.8.29) contains x_0 as a parameter (so this equation may be considered as an ODE with separate variables) and the second equation serves for sharpening dependence τ on x_0. The following example shows the efficiency of formulae (A.8.28), (A.8.29). Choosing as $\overset{1}{u}$ the simplest solution of Equation (A.8.11), $\overset{1}{u}(x_0, x_1) = 1$, we get, with the help of (A.8.28), (A.8.29), the new highly non-trivial solution

$$-\ln\left[(x_1u)^{-1} - a\right] + a\left[(x_1u)^{-1} - a\right]^{-1} = \ln x_1 + a^2x_0. \qquad \text{(A.8.30)}$$

This is a parametric solution.

Next we will consider how to construct formula of superposition of solutions for Equation (A.8.11) analogous to that of linear superposition $\overset{3}{z}(t,x) = \overset{1}{z}(t,x) + \overset{2}{z}(t,x)$ of the heat equation (A.8.12) ($\overset{1}{z}, \overset{2}{z}$ are given solutions, $\overset{3}{z}$ is the new one). Having used once more formulae (A.8.18), (A.8.19) one finds

$$\frac{1}{\overset{3}{u}(x_0, x_1)} = \frac{1}{\overset{1}{u}(x_0, \overset{1}{\tau})} + \frac{1}{\overset{2}{u}(x_0, \overset{2}{\tau})}, \qquad \text{(A.8.31)}$$

where $\overset{k}{\tau} = \overset{k}{\tau}(x_0, x_1)$, $k = 1, 2$ are functional parameters to be determined from the conditions

$$\begin{aligned} &\overset{1}{u}(x_0, \overset{1}{\tau})d\overset{1}{\tau} = \overset{2}{u}(x_0, \overset{2}{\tau})d\overset{2}{\tau}, \qquad \overset{1}{\tau} + \overset{2}{\tau} = x_1, \\ &\overset{k}{\tau}_0 = \overset{k}{\tau}_{11}\left[\overset{k}{\tau}_1\overset{k}{u}(x_0, \overset{k}{\tau})\right]^{-2}, \qquad k = 1, 2. \end{aligned} \qquad \text{(A.8.32)}$$

Note, letting $u(x_0, x_1) = 1/w(x_0, x_1)$ we rewrite Equation (A.8.11) and formulae (A.8.31), (A.8.32) as

$$w_0 = w^2w_{11}; \qquad \text{(A.8.33)}$$

$$\overset{3}{w}(x_0, x_1) = \overset{1}{w}(x_0, \overset{1}{\tau}) + \overset{2}{w}(x_0, \overset{2}{\tau}); \qquad \text{(A.8.34)}$$

$$\begin{aligned} &\frac{d\overset{1}{\tau}}{\overset{1}{u}(x_0, x_1)} = \frac{d\overset{2}{\tau}}{\overset{2}{w}(x_0, \overset{2}{\tau})}, \qquad \overset{1}{\tau} + \overset{2}{\tau} = x_1, \\ &\overset{k}{\tau}_0 = \overset{k}{\tau}_{11}\left[\overset{k}{w}(x_0, \overset{k}{\tau})/\overset{k}{\tau}_1\right]^2, \qquad k = 1, 2. \end{aligned} \qquad \text{(A.8.35)}$$

For example, starting from two simple solutions of Equation (A.8.33)

$$\overset{1}{w}(x_0, x_1) = x_1, \qquad \overset{2}{w}(x_0, x_1) = 2x_1 \qquad \text{(A.8.36)}$$

we find by means of (A.8.34), (A.8.35) another solution

$$\overset{3}{w}(x_0, x_1) = \pm e^{-2x_0}\left[1 - 2x_1e^{2x_0} \pm \sqrt{1 - 2x_1e^{2x_0}}\right], \qquad \text{(A.8.37)}$$

and one may compare (A.8.36) with (A.8.37).

Let us consider the equation

$$u_0 = \partial_1\left(u^{-2/3}u_1\right). \tag{A.8.38}$$

This equation is transformed through the use of (A.8.6)–(A.8.8) into the equation

$$z_t = \partial_x\left(z^{-4/3}z_x\right) \tag{A.8.39}$$

which is remarable by its local symmetry propertites (see Theorem A.8.1, case 4). Without going into details we list below Lie ansatze for Equation (A.8.39) and then transform some of them with the help of (A.8.6)–(A.8.8) into ansatze for Equation (A.8.38), acting in much the same way as when considering Equations (A.8.11), (A.8.12).

Ansatze for Equation (A.8.30):

$$\begin{aligned}
&1)\quad z = x^{-3}\varphi(\omega), \qquad \omega = t;\\
&2)\quad z = x^{-3}\varphi(\omega), \qquad \omega = at + \frac{1}{x};\\
&3)\quad z = t^{3/4}x^{-3}\varphi(\omega), \qquad \omega = a\ln t + \frac{1}{x};\\
&4)\quad z = (x^2+1)^{-3/2}\varphi(\omega), \qquad \omega = t + \lambda\arctan x;\\
&5)\quad z = (x^2-1)^{-3/2}\varphi(\omega), \qquad \omega = t + \lambda\,\mathrm{arcth}\, x;\\
&6)\quad z = t^{3/4}(x^2+1)^{-3/2}\varphi(\omega), \qquad \omega = \ln t + \lambda\arctan x;\\
&7)\quad z = t^{3/4}(x^2-1)^{-3/2}\varphi(\omega), \qquad \omega = \ln t + \lambda\,\mathrm{arcth}\, x.
\end{aligned} \tag{A.8.40}$$

Ansatze for Equation (A.8.38):

$$\begin{aligned}
&1)\quad u = \left[\varphi^1(x_0)x_1^2 + \varphi^2(x_0)\right]^{-3/2};\\
&2)\quad \left[x_1 + \varphi^1(x_0)\right]\left[\dot\varphi^2(x_0)\right]^{3/4} = -\tau\dot\varphi^3(\omega) + \varphi^3(\omega),\\
&\qquad \omega = \varphi^2(x_0) + \tau, \qquad -\frac{\tau_1}{\tau} = u;\\
&3)\quad \left[x_1 + \varphi^1(x_0)\right]\left[\dot\varphi^2(x_0)\right]^{3/4} = \int [\dot\varphi^3(\tau)]^{3/2}\varphi^4(\omega)d\tau,\\
&\qquad \omega = \varphi^2(x_0) + \varphi^3(\tau), \qquad \tau_1 = u.
\end{aligned} \tag{A.8.41}$$

After substitution of ansatze 1)–3) (A.8.41) into Equation (A.8.38) we get, respectively

$$\begin{aligned}
1)\qquad &\dot\varphi^1 + 4(\varphi^1)^2 = 0,\\
&\dot\varphi^2 - 2\varphi^1\varphi^2 = 0;
\end{aligned} \tag{A.8.42}$$

2) $\dot{\varphi}^1 = \lambda_1(\dot{\varphi}^2)^{1/4}$,

$\ddot{\varphi}^2 = \lambda_2(\dot{\varphi}^2)^2$,

$3(\ddot{\varphi}^3)^{-1/3} + \lambda_3\dot{\varphi}^3 + \frac{3}{4}\lambda_2\varphi^3 - \lambda_1 = 0$;

3) $\dot{\varphi}^1 = 0$,

$\ddot{\varphi}^2 = \lambda_2(\dot{\varphi}^2)^2$,

$2\dddot{\varphi}^3\dot{\varphi}^3 - 3(\ddot{\varphi}^3)^2 = 2\lambda_1(\dot{\varphi}^3)^4$,

$(\varphi^4)^{-4/3}\ddot{\varphi}^4 - \frac{4}{3}(\varphi^4)^{-7/3}(\dot{\varphi}^4)^2 + 3\lambda_1(\varphi^4)^{-1/3} + \frac{3}{4}\lambda_2\varphi^4 - \dot{\varphi}^4 = 0$,

where $\lambda_1, \lambda_2, \lambda_3$ are arbitrary constants.

Having integrated system 2) from (A.8.42) under $\lambda_2 = 0$, we obtain the following parametric solution of Equation (A.8.38)

$$\begin{aligned} u^{1/3} &= \frac{-c\left(\frac{5}{4}c_1^3 x_1 + c_2 x_0\right)}{\tau(\tau - 4c_3 x_0)}, \\ (\tau + c_3 x_0)(\tau - 4c_3 x_0)^4 &= \left(\tfrac{5}{4}c_1^3 x_1 + c_2 x_0\right)^4, \end{aligned} \tag{A.8.43}$$

where c_1, c_2, c_3 are constants of integration.

In conclusion we consider such case of transformations (A.8.6)–(A.8.8) when they are symmetry transformations of Equation (A.8.1). One sees from (A.8.10) that it meets the case when

$$z^{-2}F(z^{-1}) = F(z). \tag{A.8.44}$$

This functional equation has solution

$$F(z) = z^{-1}f(\ln z), \tag{A.8.45}$$

where f is an arbitrary differentiable even function. So, formulae (A.8.6)–(A.8.8) determine nonlocal symmetry transformations of the equation

$$u_0 = \partial_1\left[f(\ln u)u_1/u\right], \qquad (f(\alpha) = f(-\alpha)). \tag{A.8.46}$$

Using this fact we construct formula of generating solutions of Equation (A.8.46). It looks like

$$\overset{2}{u}(x_0, x_1) = \frac{1}{\overset{1}{u}(x_0, \tau)}, \tag{A.8.47}$$

where the functional parameter $\tau = \tau(x_0, x_1)$ is to be determined from the equations

$$\tau_1 = \frac{1}{\overset{1}{u}(x_0, \tau)}, \qquad \tau_0 = \frac{f(\ln \tau_1)\tau_{11}}{\tau_1}. \tag{A.8.48}$$

For example, starting from the solution

$$\overset{1}{u}(x_0, x_1) = \frac{x_0}{1 + \cos x_1}, \tag{A.8.49}$$

of equation

$$u_0 = \partial_1 \left(\frac{u_1}{u} \right) \tag{A.8.50}$$

we find, with the help of (A.8.47), (A.8.48), the new solution of Equation (A.8.50)

$$\overset{2}{u}(x_0, x_1) = \frac{2x_0}{x_0^2 + x_1^2}, \tag{A.8.51}$$

It will be noted that solutions (A.8.49) and (A.8.51) essentially differ from each other on their properties, such as boundedness, periodicity, and analyticity behavior at infinity, at zero, and so on. It is a distinguishing feature of nonlocal transformations that they may essentially change properties of solutions unlike those of Lie's solutions.

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