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Letter

# The relationship between two approximate symmetry frameworks

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ABSTRACT

The two most popular and practically useful approaches to calculate approximate symmetries of differential equations with a small parameter are the Baikov–Gazizov–Ibragimov (BGI) and Fushchich–Shtelen (FS) frameworks. It is proven that BGI approximate symmetries form a subset of FS approximate symmetries. Examples are provided.

## 1. Introduction

For a system of differential equations, a *symmetry* is a transformation of its solution set into itself (see, e.g., [1,2] and references therein). Symmetries that can be constructed systematically include discrete symmetries [3], such as reflections, other geometrical symmetries that can be found by inspection, such as rotational, translational, scaling, and Galilei symmetries, and more generally, infinitesimal Lie-type point, contact, higher, and nonlocal symmetries. For a system with independent variables $x = (x^1, \dots, x^n)$ and dependent variables $u = (u^1, \dots, u^m)$, a point symmetry generator has the form $X = \xi^i \partial_{x^i} + \eta^j \partial_{u^j}$ (the Einstein summation convention is implied throughout this paper), a point or contact or higher symmetry in the evolutionary form is generated by $\hat{X} = \varphi^j \partial_{u^j}$ where $\varphi^j$ are functions of independent variables, dependent variables and a finite number of derivatives. In the case of nonlocal symmetries, generators involve nonlocal variables (e.g., [1]). In many situations that arise in applications, a system of equations at hand is a perturbation of some well-known system $F_0 = 0$, having the form $F = F_0 + \epsilon F_1 = o(\epsilon)$ (denoted $F \approx 0$) in terms of a small parameter $\epsilon$. Examples include Euler vs. Navier-Stokes equations with small viscosity, MHD plasma models with small but nonzero magnetic permeability, nonlinear wave models in mechanics with a small viscoelastic contribution, linear or linearizable equations with a small nonlinear perturbation, etc. It is often the case that the exact symmetry algebra of the unperturbed equation $F_0 = 0$, that may be relatively rich, is significantly diminished for the perturbed equation $F \approx 0$.

In order to be able to recover, in some sense, at least some additional symmetries that hold for $F_0 = 0$ but do not exactly hold for the system $F \approx 0$, two main approaches have been introduced. In the approach of Baikov, Gazizov and Ibragimov (BGI) [4,5], the symmetry generator is split into two parts: $\hat{X} = \varphi_0^j \partial_{u^j} + \epsilon \varphi_1^j \partial_{u^j}$, with determining equations $\mathrm{pr}\, \hat{X}(F)\big|_{F\approx 0} \approx 0$. In an alternative approach by Fuschich and Shtelen [6], one introduces a series expansion of the dependent variable $u = v + \epsilon w + o(\epsilon)$, and splits the approximate system $F \approx 0$ into two sets of equations in $v$, $w$ corresponding to $O(\epsilon^0)$ and $O(\epsilon^1)$ terms; the latter system is treated as exact, and its symmetries are calculated. The two approaches have been known to yield different results, in particular, some symmetries of $F_0 = 0$ are *stable* (that is, arise as approximate symmetries) in the BGI approach, while different symmetries of $F_0 = 0$ may be stable in the FS framework (e.g., [7] and references therein). The algebraic form of the generators appears different, too: in the BGI approach, the number of functions defining symmetry components is doubled, while the dependencies remain the same; in the FS approach, the number of generator components is doubled for dependent variables, remains the same for independent variables, and their dependence includes two components $v$, $w$ of the expansion for $u$ *separately*. Symmetries obtained by either of the BGI and FS approaches have been demonstrated to yield useful approximate solutions of the governing equations (e.g., [4–8]).

An exact relationship between BGI and FS approximate symmetries has not been established since their introduction in 1989. Based on the results of comparative symmetry analysis, it has been observed that BGI approximate symmetries were 'almost always' included in the FS classification, and when a PDE system involves arbitrary components, FS approximate symmetries arise in a larger number of cases than do the BGI. It thus has been conjectured that FS is a more general approach.

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In this work we prove that for a regular PDE system, every BGI approximate symmetry is an FS approximate symmetry, whereas the converse is generally not true. This makes the FS approximate symmetry analysis the method of choice. We illustrate the result with two examples, the Boussinesq system with small advection and dispersion terms, and the equation of nonlinear hyperelastic shear wave propagation with a small quadratic term.

## 2. Basic notation

Let us introduce notation and briefly recall basic concepts related to jets and differential equations.

### 2.1. Jets (see, e.g., [9])

Let $\pi\colon\ E\to M^n$ be a locally trivial smooth vector bundle over a smooth manifold $M$, $\dim E = n+m$. Suppose $U\subset M$ is a coordinate neighborhood such that the bundle $\pi$ becomes trivial over $U$. Choose local coordinates $x^1,\,\dots,\,x^n$ in $U$ and $u^1,\,\dots,\,u^m$ along the fibers of $\pi$ over $U$. In local coordinates, a section $h$ of the bundle $\pi$ defines functions $u^1 = h^1(x^1,\dots,x^n)$, $\dots$, $u^m = h^m(x^1,\dots,x^n)$. Sections $h_1$, $h_2$ of $\pi$ represent the same $\infty$-jet at a point $x_0\in U$ if for $i=1,\dots,m$, the Taylor series of the functions $h_1^i$ and $h_2^i$ at $x_0$ coincide. Denote the $\infty$-jet of a section $h$ at a point $x_0\in M$ by $[h]^\infty_{x_0}$.

**Local coordinates.** It is convenient to treat a multi-index $\alpha$ as a formal sum of the form $\alpha=\alpha_1x^1+\ldots+\alpha_nx^n=\alpha_ix^i$, where all $\alpha_i$ are non-negative integers; $|\alpha|=\alpha_1+\ldots+\alpha_n$. One can introduce the following local coordinates on the set $J^\infty(\pi)$ of all $\infty$-jets of sections of $\pi$.

$$x^i([h]^\infty_{x_0}) = x^i(x_0)\,,\qquad u^i_\alpha([h]^\infty_{x_0}) = \frac{\partial^{|\alpha|}h^i}{(\partial x^1)^{\alpha_1}\dots(\partial x^n)^{\alpha_n}}\,(x_0)\,.$$

We call such local coordinates adapted and further consider only adapted coordinates.

**Functions and sections.** A function $f\colon\ J^\infty(\pi)\to\mathbb{R}$ is smooth if there is a non-negative integer $k\in\mathbb{N}_0$ such that in any adapted local coordinates, $f$ has a form of a real-valued $C^\infty$-function depending on independent variables $x^1,\dots,x^n$, dependent variables $u^1,\dots,u^m$ and derivatives $u^i_\alpha$ such that $|\alpha|\leqslant k$. Denote by $\mathcal{F}(\pi)$ the algebra of smooth functions on $J^\infty(\pi)$. In applied literature, smooth functions on jet manifolds are often called differential functions.

If $p\colon\ E_1\to M$ is a locally trivial smooth vector bundle over the same base as $\pi$, then by the induced bundle $\pi^*_\infty(p)$ we mean the bundle $\pi^*_\infty(p)\colon\ \pi^*_\infty(E_1)\to J^\infty(\pi)$, where $\pi^*_\infty(E_1)=\{(y;e)\in J^\infty(\pi)\times E_1\mid \pi_\infty(y)=p(e)\}$, $\pi^*_\infty(p)\colon\ (y;e)\mapsto y$. If both $\pi$ and $p$ become trivial over a coordinate neighborhood $U\subset M$, one can use the corresponding adapted local coordinates on $J^\infty(\pi)$ and coordinates along the fibers of $p$ as local coordinates on $\pi^*_\infty(E_1)$. We also call such coordinates on $\pi^*_\infty(E_1)$ adapted and assume that for each section of the induced bundle $\pi^*_\infty(p)$, there is $k\in\mathbb{N}_0$ such that its components in adapted local coordinates are $C^\infty$-functions of independent variables, dependent variables, and derivatives of orders $\leqslant k$.

**Evolutionary fields.** There is the natural projection $\pi_\infty\colon\ J^\infty(\pi)\to M$ defined by the formula $\pi_\infty([h]^\infty_{x_0})=x_0$. Denote by $\varkappa(\pi)$ the $\mathcal{F}(\pi)$-module of sections of the induced bundle $\pi^*_\infty(\pi)$. We call elements of this module *characteristics*, as they can be identified with symmetries of $J^\infty(\pi)$. More specifically, for each $\varphi\in\varkappa(\pi)$, the operators of total derivatives $D_{x^i}=\partial_{x^i}+u^j_{\alpha+x^i}\partial_{u^j_\alpha}$ $(i=1,\dots,n)$ commute with the corresponding evolutionary vector field

$$E_\varphi = D_\alpha(\varphi^i)\partial_{u^i_\alpha}\,.$$

Here $\varphi^1,\dots,\varphi^m$ are components of $\varphi$; $D_\alpha$ denotes the composition $D^{\alpha_1}_{x^1}\circ\ldots\circ D^{\alpha_n}_{x^n}$.

**Product rule.** Let $p$ be a locally trivial smooth vector bundle of rank $r$ over $M$. Denote by $P(\pi)$ the module of sections of the induced bundle $\pi^*_\infty(p)$. In coordinates, a section $F\in P(\pi)$ has components $F^1,\dots,F^r$. We say that $\Delta\colon\ P(\pi)\to P(\pi)$ is an *operator in total derivatives* if it has the form $\Delta^{k\,\alpha}_i D_\alpha$ in coordinates, i.e., for $F\in P(\pi)$, $k$-th component of $\Delta F\in P(\pi)$ is $\Delta^{k\,\alpha}_i D_\alpha F^i$, where $\Delta^{k\,\alpha}_i$ are components of $\Delta$ (and they are $C^\infty$-functions defined on the coordinate domain).

For an operator in total derivatives $\Delta\colon\ P(\pi)\to P(\pi)$ and $\varphi\in\varkappa(\pi)$, we denote by $E_\varphi\Delta\colon\ P(\pi)\to P(\pi)$ the operator that has the form $E_\varphi(\Delta^{k\,\alpha}_i)D_\alpha$ in adapted local coordinates. Since evolutionary fields commute with all total derivatives, $E_\varphi(\Delta^{k\,\alpha}_i D_\alpha F^i)=E_\varphi(\Delta^{k\,\alpha}_i)D_\alpha F^i+\Delta^{k\,\alpha}_i D_\alpha(E_\varphi F^i)$ and hence the product rule

$$E_\varphi(\Delta F)=(E_\varphi\Delta)F+\Delta(E_\varphi F)$$

holds for any $\varphi\in\varkappa(\pi)$, $F\in P(\pi)$.

### 2.2. Differential equations

Consider a differential equation $F=0$. In adapted local coordinates, this equation takes the form of the system

$$F^1=0\,,\quad\dots\,,\quad F^r=0\,.$$

In particular, $r$ can coincide with $m$. We assume that for each point $q\in\{F=0\}\subset J^\infty(\pi)$, the differentials $dF^1_q$, $\dots$, $dF^r_q$ are linearly independent. By *infinite prolongation* of an equation $F=0$, we mean the subset $\mathcal{E}\subset J^\infty(\pi)$ that is defined by the infinite system

$$D_\alpha(F^i)=0\,,\qquad |\alpha|\geqslant 0\,,\quad i=1,\dots,r\,.$$

We also assume that $\pi_\infty(\mathcal{E})=M$.

**Regularity assumptions.** We say that the infinite prolongation $\mathcal{E}$ of a differential equation $F=0$ is *regular* if for every function $f\in\mathcal{F}(\pi)$ that vanishes on $\mathcal{E}$, there is an operator in total derivatives $\Delta\colon\ P(\pi)\to\mathcal{F}(\pi)$ such that $f=\Delta F$. *In what follows, we consider only regular systems.*

**Infinitesimal symmetries.** An evolutionary field $E_\varphi$ on $J^\infty(\pi)$ is a symmetry of a differential equation $\{F=0\}\subset J^\infty(\pi)$ if there is an operator in total derivatives $\Delta\colon\ P(\pi)\to P(\pi)$ such that

$$E_\varphi F=\Delta F.$$

Let us recall that each point symmetry $\xi^i(x^1,\dots,x^n,u^1,\dots,u^m)\partial_{x^i}+\eta^i(x^1,\dots,x^n,u^1,\dots,u^m)\partial_{u^i}$ of an equation $F=0$ gives rise to its evolutionary symmetry $E_\varphi$ for $\varphi^i=\eta^i-u^i_{x^j}\xi^j$ $(i=1,\dots,m)$. The same applies to contact symmetries (provided that $p$ is a line bundle).

## 3. Perturbed differential equations

Suppose now that $F$ denotes a family of sections of $P(\pi)$ depending on a small parameter $\epsilon$. We consider perturbations of first order in $\epsilon$. It is convenient to use the approximate equality symbol $\approx$ for relations that hold up to $o(\epsilon)$. Then the approximate equation

$$F=o(\epsilon)\qquad\Leftrightarrow\qquad F\approx 0 \tag{3.1}$$

can be considered a perturbation of the unperturbed equation $F_0=0$, where $F_0=F|_{\epsilon=0}$. There is the section $F_1\in P(\pi)$ such that $F\approx F_0+\epsilon F_1$. Then one can rewrite equation (3.1) in the form $F_0+\epsilon F_1\approx 0$.

Similarly, if a characteristic $\varphi$ depends on a small parameter $\epsilon$, we denote by $\varphi_0,\varphi_1\in\varkappa(\pi)$ sections such that $\varphi\approx\varphi_0+\epsilon\varphi_1$ and call the corresponding evolutionary field $E_\varphi$ approximate.

## 4. Baikov-Gazizov-Ibragimov approximate symmetries

**Definition 1.** An approximate evolutionary field $E_\varphi$ is a BGI approximate symmetry of equation (3.1) if there exist operators in total derivatives $\Delta_0,\Delta_1\colon\ P(\pi)\to P(\pi)$ such that

$$E_\varphi F\approx\Delta F,\qquad\text{where}\ \ \Delta=\Delta_0+\epsilon\Delta_1,\ \varphi\approx\varphi_0+\epsilon\varphi_1. \tag{4.1}$$

We call a BGI approximate symmetry $E_\varphi$ *trivial* if $\varphi_0$ vanishes on the infinite prolongation of the unperturbed equation, i.e., if there is an operator in total derivatives $\nabla\colon\ P(\pi)\to\varkappa(\pi)$ such that $\varphi_0=\nabla F_0$. Since $E_{\varphi_0+\epsilon\varphi_1}=E_{\varphi_0}+\epsilon E_{\varphi_1}$, relation (4.1) takes the form $E_{\varphi_0}F_0+(E_{\varphi_0}F_1+E_{\varphi_1}F_0)\,\epsilon=\Delta_0F_0+(\Delta_0F_1+\Delta_1F_0)\,\epsilon+o(\epsilon)$. Analysis of the coefficients leads to the following

**Lemma 1.** *An approximate evolutionary field $E_\varphi$ is a BGI approximate symmetry of equation (3.1) if and only if there are operators in total derivatives $\Delta_0,\Delta_1\colon\ P(\pi)\to P(\pi)$ such that*

$$E_{\varphi_0}F_0=\Delta_0F_0\,,\qquad E_{\varphi_0}F_1+E_{\varphi_1}F_0=\Delta_0F_1+\Delta_1F_0\,.$$

## 5. Fushchich-Shtelen approximate symmetries

Let us consider the Whitney sum $\pi\oplus\pi\colon\ \pi^*(E)\to M$ and denote local coordinates along its fibers by $v^1,\dots,v^m$, $w^1,\dots,w^m$. Denote by $f$ a unique function $f\colon\ J^\infty(\pi\oplus\pi)\to J^\infty(\pi)$ such that $f([h]^\infty_{x_0})=[\pi^*(\pi)\circ h]^\infty_{x_0}$, where $\pi^*(\pi)\colon\ \pi^*(E)\to E$ is the induced bundle. In adapted local coordinates, $f$ is given by the formulae $u^i_\alpha=v^i_\alpha$. For a mapping $g$ that has $J^\infty(\pi)$ as its domain, we denote by $\widetilde{g}$ the pullback (composition) $f^*(g)=g\circ f$. We use the notation $\widetilde{D}_{x^i}$ for the corresponding total derivatives on $J^\infty(\pi\oplus\pi)$. For an operator in total derivatives $\Delta\colon\ P(\pi)\to P(\pi)$ having the form $\Delta^{k\,\alpha}_i D_\alpha$ in adapted local coordinates, we denote by $\widetilde{\Delta}$ the operator of the form $f^*(\Delta^{k\,\alpha}_i)\widetilde{D}_\alpha$. Note that here $f^*(\Delta G)=\widetilde{\Delta}\widetilde{G}$ for any $G\in P(\pi)$.

Elements of $\varkappa(\pi\oplus\pi)$ can be regarded as ordered pairs of sections of the induced bundle $(\pi\oplus\pi)^*_\infty(\pi)$, $E_{(\mu_0;\mu_1)}=D_\alpha(\mu^i_0)\partial_{v^i_\alpha}+D_\alpha(\mu^i_1)\partial_{w^i_\alpha}$. For each $G\in P(\pi)$, the relation $E_{(\widetilde{\varphi}_0;\mu_1)}\widetilde{G}=f^*(E_{\varphi_0}G)$ holds for any $\varphi_0\in\varkappa(\pi)$ and any section $\mu_1$ of the induced bundle $(\pi\oplus\pi)^*_\infty(\pi)$. Let us denote by $\delta$ the evolutionary field with the characteristic $(w;0)$

$$\delta=E_{(w;0)}=w^i_\alpha\partial_{v^i_\alpha}\,.$$

The approximate mapping $u^i_\alpha\approx v^i_\alpha+\epsilon w^i_\alpha$ allows one to relate solutions of approximate equation (3.1) to solutions of the system of differential equations, which we call FS system of approximate equation (3.1)

$$\widetilde{F}_0=0\,,\qquad \delta\widetilde{F}_0+\widetilde{F}_1=0\,.\tag{5.1}$$

Let us denote by $\widetilde{\mathcal{E}}$ the infinite prolongation of system (5.1). We say that the FS system *defines an FS-covering* if $f(\widetilde{\mathcal{E}})=\mathcal{E}$, where $\mathcal{E}$ is the infinite prolongation of the unperturbed equation $F_0=0$. By FS symmetries of approximate equation (3.1), we mean evolutionary symmetries of FS system (5.1). We say that an FS symmetry $E_{(\mu_0;\mu_1)}$ is *trivial* if $\mu_0$ has the form $f^*(\varphi_0)$, where $\varphi_0\in\varkappa(\pi)$ vanishes on $\mathcal{E}$.

**Remark 1.** If an unperturbed equation $F_0=0$ can be written in an extended Kovalevskaya form, its FS system defines an FS-covering for any perturbing term $\epsilon F_1$. In the general case, the situation is more complicated. If $A\colon\ P(\pi)\to\mathcal{F}(\pi)$ is an operator in total derivatives such that $F_0\in\ker A$, then on approximate solutions of (3.1), $\epsilon AF_1=A(F_0+\epsilon F_1)\approx 0$, and hence, among solutions to $F_0=0$, those that do not satisfy the constraint $AF_1=0$ cannot be perturbed. According to second Noether's theorem (e.g., [2]), such constraints can be non-trivial for Lagrangian gauge systems. To obtain an FS-covering, it is necessary to choose $F_1$ such that $AF_1|_{\mathcal{E}}=0$ for any operator of this type.

## 6. Correspondence between BGI and FS approaches

**Informal idea.** One can informally say that an approximate evolutionary field $E_\varphi$ generates the flow given by the equation $u_\tau\approx\varphi_0+\epsilon\varphi_1$. Substituting

$$u=v+\epsilon w,\tag{6.1}$$

we obtain the system $v_\tau=\widetilde{\varphi}_0$, $w_\tau=\delta\widetilde{\varphi}_0+\widetilde{\varphi}_1$ corresponding to the evolutionary field with the characteristic $(\widetilde{\varphi}_0;\delta\widetilde{\varphi}_0+\widetilde{\varphi}_1)$.

**Theorem 1.** *Suppose the FS system of a perturbed equation $F\approx 0$ defines an FS-covering. An approximate evolutionary field $E_\varphi$ is a BGI approximate symmetry of the equation $F\approx 0$ if and only if $(\widetilde{\varphi}_0;\delta\widetilde{\varphi}_0+\widetilde{\varphi}_1)$ is the characteristic of an FS symmetry.*

**Proof.** Let us denote by $X$ the evolutionary vector field with the characteristic $(\widetilde{\varphi}_0;\delta\widetilde{\varphi}_0+\widetilde{\varphi}_1)$. Since $X$ and $\delta$ are evolutionary symmetries of $J^\infty(\pi\oplus\pi)$, their commutator is also an evolutionary symmetry of $J^\infty(\pi\oplus\pi)$. Thus, $[X,\delta]$ is completely determined by its characteristic having the form $(\widetilde{\varphi}_1;\,\dots)$. Using this observation, we find

$$X(\delta\widetilde{F}_0+\widetilde{F}_1)=\delta(X\widetilde{F}_0)+E_{(\widetilde{\varphi}_1;\,\dots)}\widetilde{F}_0+E_{(\widetilde{\varphi}_0;\,\dots)}\widetilde{F}_1\,.$$

**1)** Let $E_\varphi$ be a BGI approximate symmetry of the equation $F\approx 0$. There are operators in total derivatives $\Delta_0,\Delta_1\colon\ P(\pi)\to P(\pi)$ from Lemma 1. Since $X\widetilde{F}_0=E_{(\widetilde{\varphi}_0;\dots)}\widetilde{F}_0=f^*(E_{\varphi_0}F_0)=f^*(\Delta_0F_0)$, the relation $X\widetilde{F}_0=\widetilde{\Delta}_0\widetilde{F}_0$ holds and we can apply the product rule:

$$\begin{aligned}
&X(\delta\widetilde{F}_0+\widetilde{F}_1)=\\
&=\delta(\widetilde{\Delta}_0\widetilde{F}_0)+E_{(\widetilde{\varphi}_1;\,\dots)}\widetilde{F}_0+E_{(\widetilde{\varphi}_0;\,\dots)}\widetilde{F}_1=\\
&=(\delta\widetilde{\Delta}_0)\widetilde{F}_0+\widetilde{\Delta}_0(\delta\widetilde{F}_0)+E_{(\widetilde{\varphi}_1;\,\dots)}\widetilde{F}_0+E_{(\widetilde{\varphi}_0;\,\dots)}\widetilde{F}_1=\\
&=(\delta\widetilde{\Delta}_0)\widetilde{F}_0+\widetilde{\Delta}_0(\delta\widetilde{F}_0+\widetilde{F}_1)-\widetilde{\Delta}_0\widetilde{F}_1+E_{(\widetilde{\varphi}_1;\,\dots)}\widetilde{F}_0+E_{(\widetilde{\varphi}_0;\,\dots)}\widetilde{F}_1=\\
&=(\delta\widetilde{\Delta}_0)\widetilde{F}_0+\widetilde{\Delta}_0(\delta\widetilde{F}_0+\widetilde{F}_1)+f^*(-\Delta_0F_1+E_{\varphi_1}F_0+E_{\varphi_0}F_1)=\\
&=(\delta\widetilde{\Delta}_0)\widetilde{F}_0+\widetilde{\Delta}_0(\delta\widetilde{F}_0+\widetilde{F}_1)+f^*(\Delta_1F_0)=\\
&=(\delta\widetilde{\Delta}_0+\widetilde{\Delta}_1)\widetilde{F}_0+\widetilde{\Delta}_0(\delta\widetilde{F}_0+\widetilde{F}_1)\,.
\end{aligned}$$

Both $(\delta\widetilde{\Delta}_0+\widetilde{\Delta}_1)\widetilde{F}_0$ and $\widetilde{\Delta}_0(\delta\widetilde{F}_0+\widetilde{F}_1)$ vanish on the infinite prolongation $\widetilde{\mathcal{E}}$ of the FS system. Thus, $X\widetilde{F}_0$ and $X(\delta\widetilde{F}_0+\widetilde{F}_1)$ vanish on the infinite prolongation of the FS system and hence $X$ is an FS symmetry.

**2)** Suppose now that $X$ is an FS symmetry. Then $f^*(E_{\varphi_0}F_0)$ vanishes on $\widetilde{\mathcal{E}}$ due to the relation $X\widetilde{F}_0=f^*(E_{\varphi_0}F_0)$. Since the FS system defines an FS-covering, $E_{\varphi_0}F_0$ vanishes on the infinite prolongation $\mathcal{E}$ of the unperturbed equation. Hence, there exists an operator in total derivatives $\Delta_0\colon\ P(\pi)\to P(\pi)$ such that $E_{\varphi_0}F_0=\Delta_0F_0$. The relation $X\widetilde{F}_0=\widetilde{\Delta}_0\widetilde{F}_0$ and the product rule yield

$$X(\delta\widetilde{F}_0+\widetilde{F}_1)=(\delta\widetilde{\Delta}_0)\widetilde{F}_0+\widetilde{\Delta}_0(\delta\widetilde{F}_0+\widetilde{F}_1)+f^*(-\Delta_0F_1+E_{\varphi_1}F_0+E_{\varphi_0}F_1).$$

The terms $(\delta\widetilde{\Delta}_0)\widetilde{F}_0$ and $\widetilde{\Delta}_0(\delta\widetilde{F}_0+\widetilde{F}_1)$ vanish on $\widetilde{\mathcal{E}}$. Then $f^*(-\Delta_0F_1+E_{\varphi_1}F_0+E_{\varphi_0}F_1)$ also vanishes on $\widetilde{\mathcal{E}}$ because $X$ is an FS symmetry. Since the FS system defines an FS-covering, $-\Delta_0F_1+E_{\varphi_1}F_0+E_{\varphi_0}F_1$ vanishes on $\mathcal{E}$. So, there is an operator in total derivatives $\Delta_1\colon\ P(\pi)\to P(\pi)$ such that $\Delta_1F_0=-\Delta_0F_1+E_{\varphi_1}F_0+E_{\varphi_0}F_1$. This result and Lemma 1 complete the proof.

**Remark 2.** For any $\varphi_0\in\varkappa(\pi)$ and any operator in total derivatives $\nabla\colon\ P(\pi)\to\varkappa(\pi)$, the following conditions are equivalent: $\varphi_0=\nabla F_0\ \Leftrightarrow\ \varphi_0-\nabla F_0=0\ \Leftrightarrow\ f^*(\varphi_0-\nabla F_0)=0\ \Leftrightarrow\ \widetilde{\varphi}_0=f^*(\nabla F_0)$. Thus, (non)trivial BGI approximate symmetries correspond to (non)trivial FS symmetries.

Theorem 1 allows one to define stable symmetries of unperturbed equations in the following way.

**Definition 2.** A symmetry $E_{\varphi_0}$ of an unperturbed equation $F|_{\epsilon=0}=0$ is *stable* if there is a section $\mu_1$ of the bundle $(\pi\oplus\pi)^*_\infty(\pi)$ such that $(\widetilde{\varphi}_0;\,\mu_1)$ is a characteristic of an FS symmetry. This property is commonly referred to as *symmetry stability in the FS sense.*

We use more suitable indices for local coordinates on jets in the examples below.

**Example 1.** Let us consider the following perturbed system of equations (Boussinesq's equations)

$$u_t + \eta_x + \epsilon\Big(uu_x - \frac{1}{2}u_{xxt}\Big) \approx 0\,, \qquad \eta_t + u_x + \epsilon\Big(u_x\eta + u\eta_x - \frac{1}{6}u_{xxx}\Big) \approx 0\,.$$

Here $\pi, p\colon\ \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ are the projection onto the second factor, $u^1 = u$, $u^2 = \eta$. The FS system reads

$$v_t + h_x = 0\,, \quad h_t + v_x = 0\,, \quad w_t + \zeta_x + vv_x - \frac{1}{2}v_{xxt} = 0\,,$$
$$\zeta_t + w_x + v_x h + vh_x - \frac{1}{6}v_{xxx} = 0\,,$$

where $v^1 = v$, $v^2 = h$, $w^1 = w$, $w^2 = \zeta$; the function $f$ has the form $f\colon\ u_\alpha = v_\alpha$, $\eta_\alpha = h_\alpha$; $\delta = w_\alpha \partial_{v_\alpha} + \zeta_\alpha \partial_{h_\alpha}$. The symmetries $\partial_u$ and $\partial_\eta$ of the unperturbed system are stable since the FS system admits the symmetries

$$X_1 = \partial_v - tv_x\partial_w - th_x\partial_\zeta + \dots\,,$$
$$X_2 = \partial_h - \frac{xv_x}{2}\,\partial_w + \frac{h - xh_x}{2}\,\partial_\zeta + \dots \tag{6.2}$$

Let us note that these symmetries are not equivalent to point ones. Nevertheless, they generate flows on $J^\infty(\pi \oplus \pi)$. For instance, the flow generated by $X_2$ is defined by

$$\frac{dx}{d\tau} = 0\,,\ \ \frac{dt}{d\tau} = 0\,,\ \ \frac{dv}{d\tau} = 0\,,\ \ \frac{dh}{d\tau} = 1\,,\ \ \frac{dw}{d\tau} = -\frac{xv_x}{2}\,,\ \ \frac{d\zeta}{d\tau} = \frac{h - xh_x}{2}\,,$$
$$\frac{dv_x}{d\tau} = 0\,,\ \ \frac{dh_x}{d\tau} = 0\,,\ \ \dots$$

The corresponding group of (invertible) transformations is given by

$$x' = x\,, \quad t' = t\,, \quad v' = v\,, \quad h' = h + \tau\,, \quad w' = w - \tau\frac{xv_x}{2}\,,$$
$$\zeta' = \zeta + \tau\frac{h - xh_x}{2} + \frac{\tau^2}{4}\,, \quad \dots$$

This observation does not contradict Bäcklund's theorem, but shows that its analogue is not valid in the case of infinite jets. Indeed, these transformations preserve only infinite order tangency. They are produced by two-sided invertible differential operators. One can derive such symmetries systematically, seeking evolutionary fields with characteristics of a particular form.

Both FS symmetries (6.2) give rise to BGI approximate symmetries. For example, the characteristic of $X_1$ reads $(1, 0\,; -tv_x, -th_x)$. It has the form $(\widetilde{\varphi}_0; \delta\widetilde{\varphi}_0 + \widetilde{\varphi}_1)$ for $\varphi_0 = (1, 0)$, $\varphi_1 = (-tu_x, -t\eta_x)$. Accordingly, the corresponding BGI approximate symmetry can be written in the form $(1 - \epsilon tu_x)\partial_u - \epsilon t\eta_x\partial_\eta + \dots$ This approximate vector field gives rise to the point approximate symmetry $\epsilon t\partial_x + \partial_u$.

**Example 2.** Let $a$, $b$, and $c$ be real numbers such that $c \neq 0$, $a^2 + b^2 > 0$. Consider the perturbed wave equation

$$u_{tt} \approx c^2(u_{xx} + u_{yy}) + \epsilon\Big(a(u_x^2 + u_y^2)_x + b(u_x^2 + u_y^2)_y\Big) \tag{6.3}$$

The PDE (6.3) arises in the description of shear Love-type waves in a nonlinear elastic solid [10], and corresponds to the following FS system:

$$v_{tt} = c^2(v_{xx} + v_{yy})\,,$$
$$w_{tt} = c^2(w_{xx} + w_{yy}) + a(v_x^2 + v_y^2)_x + b(v_x^2 + v_y^2)_y\,.$$

Here both $\pi, p\colon\ \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ are the projection onto the second factor. The scaling symmetry $u\partial_u + \dots$ of the unperturbed equation is stable since the FS system admits the evolutionary symmetry $X = v\partial_v + 2w\partial_w + \dots$ with the characteristic $(v; 2w)$. Here $\varphi_0 = u$, $\widetilde{\varphi}_0 = v$, $\delta\widetilde{\varphi}_0 = w$. The characteristic $(v; 2w)$ does not have the form $(\widetilde{\varphi}_0; \delta\widetilde{\varphi}_0 + \widetilde{\varphi}_1) = (v; w + \widetilde{\varphi}_1)$ because there is no $\varphi_1 \in \varkappa(\pi)$ such that $\widetilde{\varphi}_1 = w$. Thus, the $X$ does not correspond to any BGI approximate symmetry, and is thus unstable in the BGI sense.

## 7. Conclusion

Theorem 1 establishes an important relationship between BGI and FS symmetries, in particular, that a point or a local BGI symmetry necessarily has an FS counterpart. Example 1 demonstrates such a correspondence. The converse is generally not true: an FS symmetry may not correspond to any local BGI symmetry. This situation is illustrated by Example 2.

From the computational point of view, since FS symmetries include BGI approximate symmetries, for a given PDE system, FS symmetries are the primary object to be calculated. At the same time, it may also be convenient to seek BGI approximate point symmetries because in the case when they correspond to higher-order FS symmetries, the BGI calculations may be technically simpler. We also note that when the unperturbed equations $F_0 = 0$ are linear, to be tractable, the computations of FS and BGI approximate symmetries of the perturbed system may require the use of simplifying ansätze.

In the future work, among others, it is important to address the following questions. First, it is of interest to calculate further examples and consider the correspondences between the two approaches to approximate symmetries (e.g., [5,7,11–13]). Second, in applications, such as for example shallow water-type and nonlinear anisotropic mechanical models, there exist systems with multiple unrelated small parameters; it is important to develop an optimal approach to calculate approximate symmetries of such systems. Third, it is essential to clarify the correspondence between BGI and FS symmetry approaches when a PDE system contains a parameter expansion with several terms involving increasing powers of small parameters. In particular, for point FS and BGI approximate symmetries, the expansion (6.1) can be extended to include additional small parameters, and the FS-BGI correspondence is similar to that stated in Theorem 1. Finally, it is important to develop a general approach and study detailed examples of local and *nonlocal* approximate symmetry classifications introducing potentials to exact and approximate, local and nonlocal conservation laws, and using other differential coverings [9].

## CRediT authorship contribution statement

**Kostya Druzhkov:** Writing – original draft, Visualization, Methodology, Investigation, Formal analysis, Conceptualization. **Alexei Cheviakov:** Writing – review & editing, Supervision, Project administration, Investigation, Funding acquisition, Conceptualization.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

No data was used for the research described in the article.

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