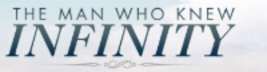
Life and Work of Ramanujan

National Mathematics Day
Maths Club, IISER Pune

Surya Teja Gavva
Rutgers University

Srinivasa Ramanujan



Srinivasa Ramanujan

Born December 22, 1887 in Pallipalayam, Erode and grew up in Kumbakonam
Parents: Komalatammal and Kuppuswamy Srinivasa Iyengar



Childhood

- Very religious upbringing
- Spent hours in temple lobby listening to drums while doing mathematics.
- He believed his discoveries came to him as visions from goddess Namagiri
- *"An equation for me has no meaning unless it expresses a thought of God."*

Kangeyam Primary School in Kumbakonam

C.V. Rajagopalachari, a friend of Ramanujan recounts:
Ramanujan asked

"Sir, if no banana is distributed to no student, will everyone still get a banana ?"

High School

- Outstanding student, won many academic awards and scholarships.
- He did mathematical exploration on his own, started noting down results in his notebooks.

Kumbakonam School

Once a senior at school posed to Ramanujan, who was in his fourth year at school, the following problem:

If $\sqrt{x} + y = 7$ and $x + \sqrt{y} = 11$, what are the values of x and y ?

SL Loney's Trigonometry

PLANE

TRIGONOMETRY

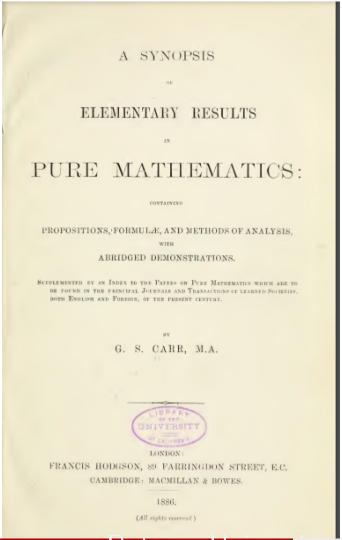
BY

S. L. LONEY, M.A.

LATE FELLOW OF SIDNEY SUSSEX COLLEGE, CAMBRIDGE,
PROFESSOR AT THE ROYAL HOLLOWAY COLLEGE.

CAMBRIDGE:
AT THE UNIVERSITY PRESS.
1893

G.S. Carr's: A Synopsis of Elementary Results, a book on Pure Mathematics



A SYNOPSIS

OF

ELEMENTARY RESULTS

IN

PURE MATHEMATICS:

CONTAINING

PROPOSITIONS, FORMULÆ, AND METHODS OF ANALYSIS,

WITH

ABRIDGED DEMONSTRATIONS.

SUPPLEMENTED BY AN INDEX TO THE PAPERS ON PURE MATHEMATICS WHICH ARE TO BE FOUND IN THE PRINCIPAL JOURNALS AND TRANSACTIONS OF LEARNED SOCIETIES, BOTH ENGLISH AND FOREIGN, OF THE PRESENT CENTURY.

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G. S. CARR, M.A.

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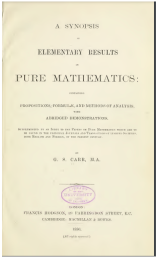
Carr's Synopsis



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- 4865 formulae without proofs, in algebra, trigonometry, analytical geometry and calculus.
- "*It was this book which awakened his genius. He set himself to establish the formulae given therein. As he was without the aid of other books, each solution was a piece of research so far as he was concerned.*" -P.V. Seshu Aiyar, R. Ramachandra Rao

Carr's Synopsis

Elementary identities, definite integrals, elliptic integrals/functions, infinite series, Hypergeometric series, Fourier series, continued fraction expansions

$$\sqrt{a \pm \sqrt{b}} = \sqrt{\frac{1}{2}\left(a + \sqrt{a^2 - b}\right)} \pm \sqrt{\frac{1}{2}\left(a - \sqrt{a^2 - b}\right)}$$

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} dx = (f(0) - f(\infty)) \log\left(\frac{b}{a}\right)$$

$$\left(\frac{2}{1 + \sqrt{1 - 4t}}\right)^n = 1 + nt + n \sum_{k=2}^{\infty} \frac{\Gamma(n + 2k)t^k}{\Gamma(n + k + 1)k!}$$

Ramanujan's obsession

- Ramanujan passed his Matriculation Examination in 1904 and joined the Government Arts College in Kumbakonam
- Pachaiyappa's College, Madras
- He got preoccupied with mathematics, neglected other subjects, and flunked out of college twice!

Struggling Years

- Married to nine-year old Janaki in 1908
- Found a job as a clerk because of poverty to help his family
- Working on mathematics in all available free time!

Turning Point

- Prof. V. Ramaswamy Iyer of Indian Mathematical Society
- P.V. Seshu Aiyar at Presidency College
- R. Ramachandra Rao, Nellore District Collector
- Prof. C.L.T. Griffith of the Engineering College, Madras

- *"a short uncouth figure, stout, unshaved, not over-clean, with one conspicuous feature - shining eyes - walked in, with a frayed Notebook under his arm $\cdots$. He was miserably poor. He had run away from Kumbakonam to get leisure in Madras to pursue his studies. He never craved for any distinction. He wanted leisure, in other words, simple food to be provided for him without exertion on his part and that he should be allowed to dream on."* - R. Ramachandra Rao

- *"Ramanujan's methods were so terse and novel and his presentation was so lacking in clearness and precision, that the ordinary reader, unaccustomed to such intellectual gymnastics, could hardly follow him."* Prof. Seshu Iyer

Some Properties of Bernoulli's Numbers

$$1 + 2 + \cdots + n = \frac{1}{2}\left(n^2 + n\right)$$

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{3}\left(n^3 + \frac{3}{2}n^2 + \frac{1}{2}n\right)$$

$$1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}\left(n^4 + \frac{4}{2}n^3 + n^2\right)$$

$$\vdots$$

$$\sum n^m = \frac{1}{m+1}\left(B_0 n^{m+1} + \binom{m+1}{1}B_1^+ n^m + \binom{m+1}{2}B_2 n^{m-1} + \cdots + \binom{m+1}{m}B_m n\right)$$

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}, B_7 = 0, B_8 = -\frac{1}{30} \cdots$$

Bernoulli numbers

- $\dfrac{x}{e^x - 1} \equiv \sum_{n=0}^{\infty} \dfrac{B_n x^n}{n!}$
- Show up in sums over integers (discrete integrals) like zeta values $\zeta(n)$, Euler-Maclaurin summation formulae etc
- many interesting arithmetic properties

Ramanujan computed various Bernoulli numbers and illustrated some arithmetic properties of the numerators and denominators of B_k.

Magic

$$\sqrt{1+2\sqrt{(1+3\sqrt{(1+\cdots)\cdots})}} = ?$$

$$n(n+2) = n\sqrt{1+(n+1)(n+3)}$$

Ramanujan-Nagell Equation

$$x^2 + 7 = 2^n$$

$x = 1, 3, 5, 11, 181$ corresponding to $n = 3, 4, 5, 7, 15$ are the only solutions. Conjectured by Ramanujan, proved by Nagell.

Proof: Work in the ring of integers of $\mathbb{Q}[\sqrt{-7}]$ which has unique factorization.

English mathematicians

Prof. M.J.M. Hill, of University College, University of London, on Ramanujan's work

Mr. Ramanujan is evidently a man with a taste for Mathematics, and with some ability, but he has got on the wrong lines. He does not understand the precautions which have to be taken in dealing with divergent series, otherwise he could not have obtained the erroneous results you send me, viz

$$1 + 2 + 3 + \cdots + \infty = -1/12$$
$$1^2 + 2^2 + 3^2 + \cdots + \infty^2 = 0$$
$$1^3 + 2^3 + 3^3 + \cdots + \infty^3 = 1/120$$

Analytic Continuation of $\zeta(s)$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \ \mathrm{Re}(s) > 1$$

"Another analytic expression for $\zeta(s)$ that makes sense for any given s" like:

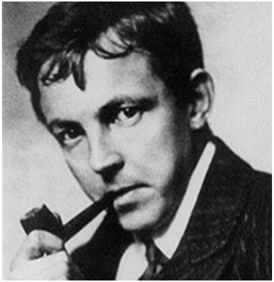
$$\zeta(s) = \frac{\pi^{s/2}}{s(s-1)\Gamma(s/2)} + \frac{\pi^{s/2}}{\Gamma(s/2)} \int_1^{\infty} \left(x^{s/2} + x^{(1-s)/2}\right) \left(\sum_{n=1}^{\infty} \mathrm{e}^{-\pi n^2 x}\right) \frac{dx}{x}$$

$$\zeta(-1) = -\frac{1}{12}$$

$$\zeta(-2) = 0$$

$$\zeta(-3) = \frac{1}{120}$$

G.H. Hardy



G.H. Hardy: Tract on Orders of infinity

"no definite expression has yet been found for the number of prime numbers less than any given number."

First Letter to Hardy, 16th January 1913

Dear Sir,
I beg to introduce myself to you as a clerk in the Accounts Department of the Port Trust Office at Madras on a salary of only 20 per annum. I am now about 23 years of age. I have had no University education but I have undergone the ordinary school course. After leaving school I have been employing the spare time at my disposal to work at Mathematics. I have not trodden through the conventional regular course which is followed in a University course, but I am striking out a new path for myself. I have made a special investigation of divergent series in general and the results I get are termed by the local mathematicians as 'startling'.

Just as in elementary mathematics you give a meaning to a^n when n is negative and fractional to conform to the law which holds when n is a positive integer, similarly the whole of my investigations proceed on giving a meaning to Eulerian Second Integral for all values of n. My friends who have gone through the regular course of University education tell me that $\int_0^\infty x^{n-1}e^{-x}dx = \Gamma(n)$ is true only when n is positive. They say that this integral relation is not true when n is negative. Supposing this is true only for positive values of n and also supposing the definition $n\Gamma(n) = \Gamma(n+1)$ to be universally true, I have given meanings to these integrals and under the conditions, I state the integral is true for all values of n negative and fractional. My whole investigations are based upon this and I have been developing this to a remarkable extent so much so that the local mathematicians are not able to understand me in my higher flights.

First Letter to Hardy, 16th January 1913

Very recently I came across a tract published by you styled Orders of Infinity in page 36 of which I find a statement. that no definite expression has been as yet found for the number of prime numbers less than any given number. I have found an expression which very nearly approximates to the real result, the error being negligible. I would request you to go through the enclosed papers. Being poor, if you are convinced that there is anything of value I would like to have my theorems published. I have not given the actual investigations nor the expressions that I get but I have indicated the lines on which I proceed. Being inexperienced I would very highly value any advice you give me. Requesting to be excused for the trouble I give you.

I remain, Dear Sir,

Yours truly

S. Ramanujan

V Theorems on summations of series; e.g.

(1) $$\frac{1}{1^3}\cdot\frac{1}{2} + \frac{1}{2^3}\cdot\frac{1}{2^2} + \frac{1}{3^3}\cdot\frac{1}{2^3} + \frac{1}{4^3}\cdot\frac{1}{2^4} + \&c$$
$$= \frac{1}{6}(\log 2)^3 - \frac{\pi^2}{12}\log 2 + \left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right)$$

(2) $$1 + 9\cdot\left(\frac{1}{4}\right)^4 + 17\cdot\left(\frac{1\cdot5}{4\cdot8}\right)^4 + 25\cdot\left(\frac{1\cdot5\cdot9}{4\cdot8\cdot12}\right)^4 + \&c = \frac{2\sqrt{2}}{\sqrt{\pi}\cdot\{\Gamma(\frac{3}{4})\}^2}$$

(3) $$1 - 5\cdot\left(\frac{1}{2}\right)^3 + 9\cdot\left(\frac{1\cdot3}{2\cdot4}\right)^3 - \&c = \frac{2}{\pi}$$

(4) $$\frac{1^{13}}{e^{2\pi}-1} + \frac{2^{13}}{e^{4\pi}-1} + \frac{3^{13}}{e^{6\pi}-1} + \&c = \frac{1}{24}.$$

(5) $$\frac{\coth\pi}{1^7} + \frac{\coth 2\pi}{2^7} + \frac{\coth 3\pi}{3^7} + \&c = \frac{19\pi^7}{56700}.$$

(6) $$\frac{1}{1^5\cosh\frac{\pi}{2}} - \frac{1}{3^5\cosh\frac{3\pi}{2}} + \frac{1}{5^5\cosh\frac{5\pi}{2}} - \&c = \frac{\pi^5}{768}.$$

(7) $$\frac{1}{\ldots} + \frac{1}{\ldots}$$

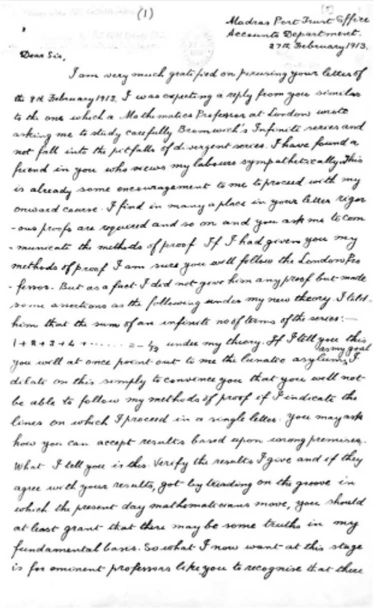
Formulae in the Letter

$$1-\frac{3!}{(1!2!)^3}x^2+\frac{6!}{(2!4!)^3}x^4-\cdots=\left(1+\frac{x}{(1!)^3}+\frac{x^2}{(2!)^3}+\cdots\right)\left(1-\frac{x}{(1!)^3}-\frac{x^2}{(2!)^3}+\cdots\right).$$

$$1-5\left(\frac{1}{2}\right)^3+9\left(\frac{1\cdot 3}{2\cdot 4}\right)^3-13\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^3+\cdots=\frac{2}{\pi}$$

$$\int_0^\infty \frac{dx}{(1+x^2)(1+r^2x^2)(1+r^4x^2)\cdots}=\frac{\pi}{2(1+r+r^3+r^6+r^{10}+\cdots)}$$

$$R(q):=\frac{1}{1+\frac{q}{1+\frac{q^2}{1+\frac{q^3}{1+\cdots}}}}\to R\left(e^{-2\pi}\right)=e^{2\pi/5}\cdot\left(\sqrt{\frac{5+\sqrt{5}}{2}}-\frac{1+\sqrt{5}}{2}\right)$$



(1)

Madras Port Trust Office
Accounts Department.
27th February 1913.

Dear Sir,

I am very much gratified on perusing your letter of the 8th February 1913. I was expecting a reply from you similar to the one which a Mathematics Professor at London wrote asking me to study carefully Bromwich's Infinite series and not fall into the pitfalls of divergent series. I have found a friend in you who views my labours sympathetically. This is already some encouragement to me to proceed with my onward course. I find in many a place in your letter rigorous proofs are required and so on and you ask me to communicate the methods of proof. If I had given you my methods of proof I am sure you will follow the London Professor. But as a fact I did not give him any proof but made some assertions as the following under my new theory. I told him that the sum of an infinite no of terms of the series:— $1+2+3+4+\cdots = -1/12$ under my theory. If I tell you this you will at once point out to me the lunatic asylum as my goal. I dilate on this simply to convince you that you will not be able to follow my methods of proof if I indicate the lines on which I proceed in a single letter. You may ask how you can accept results based upon wrong premises. What I tell you is this. Verify the results I give and if they agree with your results, got by treading on the groove in which the present day mathematicians move, you should at least grant that there may be some truths in my fundamental basis. So what I now want at this stage is for eminent professors like you to recognise that there

Hardy's Reaction

"These formulas defeated me completely. I had never seen anything in the least like this before... They could only be written down by a mathematician of the highest class. They must be true because no one would have the imagination to invent them."

His reply:

I was exceedingly interested by your letter and by the theorems which you state. You will however understand that, before I can judge properly of the value of what you have done, it is essential that I should see proofs of some of your assertions. Your results seem to me to fall into roughly three classes:
(1) there are a number of results that are already known, or easily deducible from known theorems;
(2) there are results which, so far as I know, are new and interesting, but interesting rather from their curiosity and apparent difficulty than their importance;

(3) there are results which appear to be new and important...

Ramanujan: *"I have found a friend in you who views my labours sympathetically. ... I am already a half starving man. To preserve my brains I want food and this is my first consideration. Any sympathetic letter from you will be helpful to me here to get a scholarship either from the university of from the government."*

Cambridge

- Hardy invited Ramanujan to Cambridge
- They had very productive collaboration and Ramanujan published 30 papers during his visit.

The limitation of his knowledge was as startling as its profundity. Here was a man who could work out **modular equations, and theorems of complex multiplications***, to orders unheard of, whose mastery of continued fractions was, on the formal side at any rate, beyond that of any mathematician in the world, who had found for himself* **the functional equation of the zeta-function***, and the dominant terms of many of the most famous problems in the analytic theory of numbers, and he had never heard of a* **doubly periodic function or of Cauchys theorem***, and had indeed but the vaguest idea of what a function of a* **complex variable** *was. His ideas of what constituted a mathematical proof were of the most shadowy description. All his results, new or old, right or wrong, had been arrived at by a process of mingled argument, intuition and induction, of which he was entirely unable to give a coherent account.*

Cambridge

- Littlewood
  "it was extremely difficult because every time some matter, which it was thought that Ramanujan needed to know, was mentioned, Ramanujan's response was an avalanche of original ideas which made it almost impossible for Littlewood to persist in his original intention"
- E.W. Barnes
- P.C. Mahalanobis

Hardships in Cambridge

- Racism
- World War I
- Unavailability of Vegetarian Food, malnutrition
- Several Health issues

Despite hardships, his ground-breaking work got him elected Fellow of the Royal Society.

Highly Composite Numbers

A highly composite number is a positive integer with more divisors than any smaller positive integer has.

1, **2**, 2, **3**, 2, **4**, 2, 4, 3, 4, 2, **6**, 2, 4, 4, 5, 2, 6, 2, 6, 4, 4, 2, **8**, 3, 4, 4, 6, 2, 8, 2, 6, 4, 4, 4, **9**, 2, 4, 4, 8, 2, 8, 2, 6,

6, 4, 2, **10**, 3, 6, 4, 6, 2, 8, 4, 8, 4, 4, 2, 12 $\cdots$ ..

$$2, 4, 6, 12, 24, 36, 48, 60, 120, 180, 240, 360, 720, 840, 1260, , \cdots$$

$$\lim_{x\to\infty} \frac{Q(x)}{\ln x} = \infty$$

Partition

Partition number $p(n)$ is the number of ways of writing n as a sum of positive integers.

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1 \rightarrow p(4) = 5$$

$$1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, \cdots \ldots$$

Asymptotics and Circle Method

Hardy-Ramanujan

$$p(n) \sim \frac{1}{4n\sqrt{3}} \cdot e^{\pi\sqrt{\frac{2n}{3}}}$$

Circle method is one of the most fundamental tools in analytic number theory today!

Ideas

$$\sum_{n=0}^{\infty} p(n)q^n = \left(1+q+q^2+q^3+\cdots\right)\left(1+q^2+q^4+q^6+\cdots\right)$$
$$\left(1+q^3+q^6+q^9+\cdots\right)\left(1+q^4+q^8+q^{12}+\cdots\right)\cdots$$
$$= \prod_{k=1}^{\infty}\left(\frac{1}{1-q^k}\right) =: F(q)$$

- $p(n) = \frac{1}{2\pi i}\oint_C \frac{F(q)}{q^{n+1}}dq$
- Singularities at $e^{2\pi i h/k}$. (Farey points, Ford circles)
- Analyse using modularity transformations of the $\prod_{k=1}^{\infty}(1-q^k)$

Congruences

$$p(5k+4) \equiv 0 \pmod{5}$$
$$p(7k+5) \equiv 0 \pmod{7}$$
$$p(11k+6) \equiv 0 \pmod{11}$$

Ramanujan's proofs

$$\sum_{k=0}^{\infty} p(5k+4)q^k = \frac{5\left\{(1-q^5)(1-q^{10})(1-q^{15})\cdots\right\}^5}{\left\{(1-q)(1-q^2)(1-q^3)\cdots\right\}^6} =: 5\frac{(q^5)_\infty^5}{(q)_\infty^6},$$

$$\sum_{k=0}^{\infty} p(7k+5)q^k = 7\frac{(q^7)_\infty^3}{(q)_\infty^4} + 49q\frac{(q^7)_\infty^7}{(q)_\infty^8}.$$

- Now combinatorial proofs are known.

Roger-Ramanujan Identities

- The number of partitions of n such that the adjacent parts differ by at least 2 is the same as the number of partitions of n such that each part is congruent to either 1 or 4 modulo 5.
- The number of partitions of n such that the adjacent parts differ by at least 2 and such that the smallest part is at least 2 is the same as the number of partitions of n such that each part is congruent to either 2 or 3 modulo 5.

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} = 1+q+q^2+q^3+2q^4+2q^5+3q^6+\cdots$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty} = 1+q^2+q^3+q^4+q^5+2q^6+\cdots$$

Ramanujan's Tau function

$$\Delta(z) = q \prod_{n\geq 1} (1 - q^n)^{24} = \sum_{n\geq 1} \tau(n) q^n$$

Ramanujan observed that

- $\tau(n)$ is multiplicative: $\tau(mn) = \tau(m)\tau(n)$ if $\gcd(m, n) = 1$
- $\tau\left(p^{r+1}\right) = \tau(p)\tau\left(p^r\right) - p^{11}\tau\left(p^{r-1}\right)$
- $|\tau(p)| \leq 2p^{11/2}$
- $L(s) = \sum_{n\geq 1} \frac{\tau(n)}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - \tau(p)p^{-s} + p^{11-2s}}$

Ramanujan Tau function

$\Delta(z)$ is a modular form. It's a holomorphic cusp form of weight 12 and level 1.

$$\Delta\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^{12}\Delta(\tau), a, b, d, c\in\mathbb{Z} \text{ with } ad-bc=1$$

- Related to discriminant $\Delta=-16\left(4A^3+27B^2\right)$ of elliptic curves.
- Fundamental in number theory, geometry, physics etc

163

$$e^{\pi\sqrt{163}} = 262537412640768743.99999999999925\ldots$$

Comes from the theory of complex multiplication.

$$j(\tau) = \frac{1}{q} + 744 + 196884q + \cdots$$

$$j\left(\frac{1+\sqrt{-d}}{2}\right) \approx -e^{\pi\sqrt{d}} + 744$$

$d = 163$ is a Heegner number – trivial class group.

Heegner points

$$e^{\pi\sqrt{19}} \approx 96^3 + 744 - 0.22$$
$$e^{\pi\sqrt{43}} \approx 960^3 + 744 - 0.00022$$
$$e^{\pi\sqrt{67}} \approx 5280^3 + 744 - 0.0000013$$
$$e^{\pi\sqrt{163}} \approx 640320^3 + 744 - 0.00000000000075$$

$$e^{\pi\sqrt{22}} - 24 \approx (6 + 4\sqrt{2})^6 + 0.00011\ldots$$
$$e^{\pi\sqrt{37}} + 24 \approx (12 + 2\sqrt{37})^6 - 0.0000014\ldots$$
$$e^{\pi\sqrt{58}} - 24 \approx (27 + 5\sqrt{29})^6 - 0.0000000011\ldots$$

Modular equations and Approximations to π

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \frac{[1103 + 26390n]}{396^{4n}}$$

$$\left(9^2 + \frac{19^2}{22}\right)^{\frac{1}{4}} = 3.14159265262\ldots$$

Some definite integrals

$$\int_0^\infty x^{s-1} \{\phi(0) - x\phi(1) + x^2\phi(2) - \cdots\} \, dx = \frac{\pi\phi(-s)}{\sin s\pi}$$

$$\int_0^\infty x^{s-1} \left\{\lambda(0) - \frac{x}{1!}\lambda(1) + \frac{x^2}{2!}\lambda(2) - \cdots\right\} = \Gamma(s)\lambda(-s)$$

Taxicab Number

"I remember once going to see him when he was ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavourable omen. "No," he replied, "it is a very interesting number; **it is the smallest number expressible as the sum of two cubes in two different ways.***"*

Fermat, Euler, Ramanujan

$$X^3 + Y^3 = Z^3 + T^3$$

$$\alpha^2 + \alpha\beta + \beta^2 = 3\lambda\gamma^2$$
$$\implies \left(\alpha + \lambda^2\gamma\right)^3 + (\lambda\beta + \gamma)^3 = (\lambda\alpha + \gamma)^3 + \left(\beta + \lambda^2\gamma\right)^3$$

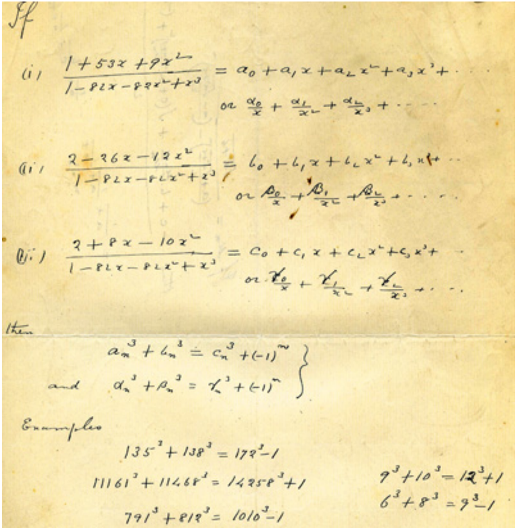
Fermat's Cubic surface

$$X^3 + Y^3 = Z^3 + 1$$

$$x^3 + y^3 + z^3 = w^3$$

$$x = 3n^2 + 5nm - 5m^2$$
$$y = 4n^2 - 4nm + 6m^2$$
$$z = 5n^2 - 5nm - 3m^2$$
$$w = 6n^2 - 4nm + 4m^2$$

Fermat's Cubic surface



If

(i) $$\frac{1+53x+9x^2}{1-82x-82x^2+x^3} = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$
or $$\frac{\alpha_0}{x} + \frac{\alpha_1}{x^2} + \frac{\alpha_2}{x^3} + \cdots$$

(ii) $$\frac{2-26x-12x^2}{1-82x-82x^2+x^3} = b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots$$
or $$\frac{\beta_0}{x} + \frac{\beta_1}{x^2} + \frac{\beta_2}{x^3} + \cdots$$

(iii) $$\frac{2+8x-10x^2}{1-82x-82x^2+x^3} = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots$$
or $$\frac{\gamma_0}{x} + \frac{\gamma_1}{x^2} + \frac{\gamma_2}{x^3} + \cdots$$

then

$$a_n^3 + b_n^3 = c_n^3 + (-1)^n$$
and $$\alpha_n^3 + \beta_n^3 = \gamma_n^3 + (-1)^n$$

Examples

$$135^3 + 138^3 = 172^3 - 1$$
$$11161^3 + 11468^3 = 14258^3 + 1$$
$$791^3 + 812^3 = 1010^3 - 1$$
$$9^3 + 10^3 = 12^3 + 1$$
$$6^3 + 8^3 = 9^3 - 1$$

Fermat's Cubic surface

$$\frac{1+53t+9t^2}{1-82t-82t^2+t^3} = x_0 + x_1 t + x_2 t^2 + \dots$$

$$\frac{2-26t-12t^2}{1-82t-82t^2+t^3} = y_0 + z_1 t + y_2 t^2 + \dots$$

$$\frac{-2-8t+10t^2}{1-82t-82t^2+t^3} = z_0 + z_1 t + z_2 t^2 + \dots$$

$$x_n = \frac{1}{85}\left[(64+8\sqrt{85})\left(\frac{83+9\sqrt{85}}{2}\right)^n + (64-8\sqrt{85})\left(\frac{83-9\sqrt{85}}{2}\right)^n - 43(-1)^n\right]$$

$$y_n = \frac{1}{85}\left[(77+7\sqrt{85})\left(\frac{83+9\sqrt{85}}{2}\right)^n + (77-7\sqrt{85})\left(\frac{83-9\sqrt{85}}{2}\right)^n + 16(-1)^n\right]$$

$$z_n = \frac{1}{85}\left[(93+9\sqrt{85})\left(\frac{83+9\sqrt{85}}{2}\right)^n + (93-9\sqrt{85})\left(\frac{83-9\sqrt{85}}{2}\right)^n - 16(-1)^n\right]$$

$$\implies x_n^3 + y_n^3 + z_n^3 = (-1)^n.$$

Return to India

- Ramanujan fell seriously ill in 1917
- He sailed to India on 27 February 1919

Mock Theta functions

Dear Hardy,

I am extremely sorry for not writing you a single letter up to now. I discovered very interesting functions recently which I call Mock θ*-functions. . they enter into mathematics as beautifully as the ordinary theta functions...*

Ramanujan, January 12, 1920

Mock Theta functions

"a mock theta function is a function defined by a q series convergent when $|q| < 1$ for which we can calculate asymptotic formulae, when q tends to a rational point, $e^{2\pi ir/s}$ of the unit circle, of the same degree of precision as those furnished for the ordinary theta functions by the theory of linear transformations".

- Holomorphic part of harmonic weak Maass forms
- Appear in invariants of three manifolds, Lie superalgebras, CFTs, Umbral moonshine, counting degeneracies of quantum black holes etc etc

Theta functions

$$\sum_{-\infty}^{\infty}(-)^n q^{6n^2+4n} = 1 - q^2 - q^{10} + q^{16} + q^{32} - \cdots$$

$$\phi(q) = 1 + \frac{q}{1+q^2} + \frac{q^4}{(1+q^2)(1+q^4)} + \ldots$$

Tragedy

On April 26, 1920, at the age of 32, and three days after the last entry in his notebook, he died.

Notebooks

Ramanujan's Notebooks, Part I, Bruce C. Berndt (1985)

Ch.	Subject	# of results
1.	Magic Squares	43
2.	Sums related to the Harmonic Series or the Inverse Trigonometric Function	68
3.	Combinatorial Analysis and Series Inversions	86
4.	Iterates of the Exponential Function and an Ingenious Formal Technique	50
5.	Eulerian Polynomials and Numbers, Bernoulli Numbers and the Riemann Zeta-Function	94
6.	Ramanujan's Theory of Divergent Series	61
7.	Sums of Powers, Bernoulli Numbers and the Γ Function	110
8.	Analogues of the Gamma Function	108
9.	Infinite Series Identities, Transformations, and Evaluations Ramanujan's Quarterly Reports	139

Ramanujan's Notebooks, Part II, Bruce C. Berndt (1989)

Ch.	Subject	# of results
10.	Hypergeometric Series I	116
11.	Hypergeometric Series II	103
12.	Continued Fractions	113
13.	Integrals and Asymptotic Expansions	92
14.	Infinite Series	87
15.	Asymptotic Expansions Modular Forms	94

Ramanujan's Notebooks, Part III, Bruce C. Berndt (1991)

Ch.	Subject	# of results
16.	q- Series and Theta Functions	134
17.	Fundamental Properties of Elliptic Function	162
18.	The Jacobi Elliptic Functions	135
19.	Modular Equations of Degree 3, 5 and 7 and Associated Theta Function Identities	185
20.	Modular Equations of Higher and Composite Degrees	173
21.	Eisenstein Series	45

Ramanujan's Notebooks, Part IV, Bruce C. Berndt (1994)

Ch.	Subject	# of Results
22.	Elementary Results	47
23.	Number Theory	108
24.	Theory of Prime Numbers	24
25.	Theta Function and Modular Equations	86
26.	Inversion Formulas for Lemniscate and other functions	10
27.	q Series	9
28.	Integrals	63
29.	Special Functions	139
30.	Partial Fraction Expansions	15
31.	Elementary and miscellaneous analysis	36
	16 Chapters of the First Notebook	54

Ramanujan's Notebooks, Part V, Bruce C. Berndt (1997)

Ch.	Subject	# of Results
32.	Continued Fractions	73
33.	Ramanujan's Theories of Elliptic Functions to Alternative Bases	62
34.	Class Invariants and Singular Moduli	196
35.	Values of Theta-Functions	24
36.	Modular Equations and Theta-Function Identities in Notebook 1	87
37.	Infinite Series	53
38.	Approximations and Asymptotic Expansions	46
39.	Miscellaneous Results in the First Notebook	24

Legacy

- Ramanujan Conjectures (Langland's Program)
- Circle Method
- Ramanujan theta functions
- Mock Modular Forms
- Roger-Ramanujan Identities
- Ramanujan Graphs
- Ramanujan-Nagell equation
- Rapidly converging approximations to π
- 1729 and K3 surface etc etc

His work influenced many mathematicians and continues to inspire a lot of new mathematics.

Dyson: *"Whenever I am angry or depressed, I pull down the Collected Papers from the shelf and take a quiet stroll in Ramanujan's garden. I recommend this therapy to all of you who suffer from headaches or jangled nerves. And Ramanujan's papers are not only a good therapy for headaches. They also are full of beautiful ideas which may help you to do more interesting mathematics.*

References

- GH Hardy: Ramanujan, Twelve Lectures on Subjects Suggested by His Life and Work
- Bruce C. Berndt and Robert A. Rankin: Ramanujan: Letters and Commentary
- Collected Papers of Srinivasa Ramanujan, Ed. G.H. Hardy, P.V. Seshu Iyer and B.M. Wilson,
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