Paul Erdős (1913 – 1996):
His Influence on the Theory of Computing

László Babai*

Abstract

Paul Erdős's *oeuvre* encompasses a multitude of areas of mathematics, including combinatorics, set theory, number theory, classical analysis, discrete geometry, probability theory, and more.

The *theory of computing* is conspicuously missing from this list. It is a field in which Erdős never took any interest. How, then, did Erdős become a household name in the theoretical computer science community? We address this question in this memorial.

1 Introduction

Paul Erdős, the mathematical prophet of the jet age, died of two successive heart attacks on September 20, 1996 while attending a workshop on graph theory in Warsaw.

The news reached most of the mathematical world within a day by e-mail. We gazed into our screens, dumbfounded and struggling not to believe. Erdős had been a constant of our lives, moving from meeting to meeting, dispensing mathematical problems to all whose brains were open and opening his brain to problems posed by others. His wry jokes about old age and stupidity have been around longer than most of us can remember; yet even at 80, he produced more papers per year than most of us do in a lifetime.

Two days before his last struggle with the S. F.[1], Erdős gave a splendid lecture at the Banach Center. Deviating from his habit in his countless "Problems and Results" lectures, he included technical details and ideas for proofs in his talk, to the delight of his audience. He had his ticket ready to move on to Lithuania for a number theory conference. We had come to believe that, somehow, he would just carry on indefinitely.

The sense of grief has been overwhelming. Erdős touched our hearts as well as our minds. The mourners include his countless disciples and many more; among them are all of us who have known him, or just observed, time and again, his frail figure seated in a conference lobby anywhere across the globe, his mind ready to link up with someone else's.

A mathematician of unique style and vision, Erdős will remain on the short list of those whose work defines the mathematics of our century. Erdős's interests covered a multitude of branches of mathematics. Foremost among them are number theory, finite and transfinite combinatorics, classical analysis (especially the theory of interpolation), and discrete geometry, but his work extends to many other fields, including probability theory, topology, group theory, complex functions, and more.

With over 1,500 papers to his name, Erdős was the second most prolific mathematician of all time, surpassed only by Euler in the volume of his oeuvre. Erdős considered mathematics a social activity; he wrote joint papers with more than 450 colleagues[2].

A major way in which Erdős exerted his influence was in the open problems he posed. As Ernst Straus observed on the occasion of Erdős's 70th birthday [115],

> "In this century, in which mathematics is so strongly dominated by 'theory constructors,' he has remained the prince of problem solvers and the absolute monarch of problem posers."

With an incessant flow of elementary questions, Erdős breathed new life into a number of fields, including such seemingly dormant areas as Euclidean plane geometry, and helped create entire new disciplines, such as combinatorial number theory, Ramsey theory, transfinite combinatorics, extremal set theory, and the study of random structures.

*Department of Computer Science, University of Chicago, 1100 E 58th St., Chicago IL 60637-1504. E-mail: laci@cs.uchicago.edu.

[1] S. F.: Supreme Fascist, Erdőshese for the God who sends us flu, misplaces our passports, and hides beautiful mathematical proofs from us.

[2] Contrary to common belief, Erdős was the sole author of a large number of papers, even in his 70s and 80s. He finished his last single-authored paper just two days before he passed away. The paper will appear in the proceedings of a conference held in Mátraháza, Hungary, 1995 (editor Vera T. Sós), to be published as a special issue of the journal *Combinatorics, Probability, and Computing*.

Erdős's excitement was contagious, it moved legions to attack his problems. Although Erdős never discussed the "big picture," it became evident to anyone with some experience with problems Erdős disseminated (whether his own or someone else's) that they were pieces in a large jigsaw puzzle; both the results and the requisite techniques are profoundly relevant to a large territory yet to be explored.

It will take a collective effort to assess Erdős's legacy. In this writing we focus on Erdős's influence on the *theory of computing* (TC), a field concerned with the intrinsic complexity of computational problems. What makes this task somewhat *paradoxical* is that Erdős never took any interest in the theory of computing [3], yet among the practitioners in the field, he is a household name.

The key to Erdős's impact on the theory of computing is the *relevance of his paradigms* on which many of his disciples were educated. Witness to this effect is the remarkable success of members of the Erdős school in the theory of computing.

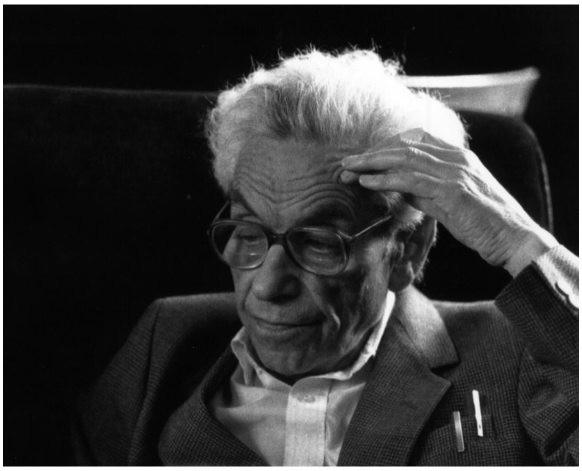


Figure 1: **Paul Erdős in an often observed position: sunken in an armchair in deep thought. 1991.** *Photo©George Paul Csicsery*

2 Freedom over convenience: a biographical sketch

2.1 Pronunciation, spelling

His often mispronounced name is Pál (Paul) Erdős, pronounced approximately as "air-dish", where the "i" in "dish" sounds like the "i" in "first." Note the long (Hungarian) umlaut on the "ő," often (even in Erdős's own papers) by mistake or out of typographical necessity replaced by "ö," the more familiar German umlaut which also exists in Hungarian. (We salute Donald E. Knuth for including the long Hungarian umlaut among his international characters in the TEX typesetting program; its code is `\H{o}`. Knuth's TEXbook uses Erdős's name to illustrate the use of the character, `Erd\H{o}s`.)

2.2 The wizard from Budapest

Paul Erdős was born on March 26, 1913, in Budapest, to Hungarian-Jewish parents. Erdős's birth was marred by tragedy: his sisters (of ages 5 and 3) contracted septic scarlet fever and died while Paul's mother was in the maternity ward. A year and a half later World War I broke out, and very soon Erdős's father was captured by the Russians and taken as POW to Siberia for *six years.* By the age of 4, Paul was able to multiply 4-digit numbers in his head. He would ask their visitors' date of birth and tell them how many seconds they lived.

Both of Erdős's parents were high school mathematics teachers, and Erdős received much of his early education from them. Erdős always remembered his parents with great affection.

Figure 2: **Paul Erdős at 13. This picture was published in the "Mathematical and Physical Monthly for Secondary Schools" in 1927, among the photos of the best problem solvers of the year. Founded in 1893, this periodical is generally credited with a large share of Hungarian students' success in mathematics. Erdős remained faithful to the Monthly and published several articles in it about problems in elementary plane geometry. At 14, László Lovász came across one of these articles, and was so enchanted, he read it "at least 20 times" [92, p.486].** *Photo courtesy: J. Bolyai Mathematical Society*

At 21, Erdős obtained his Ph. D. at Pázmány University, Budapest, formally under the great analyst Leopold Fejér (1934). But the subject of Erdős's thesis was number theory; he proved the existence of prime numbers between n and $2n$ belonging to certain arithmetic progressions. By the

[3] To my knowledge, Erdős never attended a STOC or FOCS conference. He is a coauthor of a FOCS'85 paper with Aharoni and Linial on integer programming duality. He coauthored several other papers on algorithmic themes, contributing mathematical ideas to the analysis but taking little interest in the TC motivation.

time Erdős graduated, his name was known among the leading number theorists of the time. Issai Schur called him "the wizard from Budapest." Louis Joel Mordell arranged a four-year fellowship for him to Manchester.

Erdős spent the years from 1934 to 1938 in Britain on the Manchester fellowship. "His wanderlust was already in evidence," remarks Béla Bollobás [24]: "from 1934 he hardly ever slept in the same bed for seven consecutive nights, frequently leaving Manchester for Cambridge, London, Bristol, and other universities." During the Manchester years he was mostly working on number theory but he also initiated work in combinatorics and Ramsey theory, most notably with Richard Rado (1906-89), his lifelong collaborator, a German-Jewish expatriate who had just escaped the Nazis. Notably, the Erdős–Ko-Rado theorem, one of the key results in *extremal set theory* was conceived during this period. The result became an instant classic upon publication 23 years later (!) [51]. At the same time Erdős maintained his collaboration with his friends in Budapest, working with Paul Turán in analysis and with T. Gallai (Grünwald) and E. Vázsonyi (Weiszfeld) on graph theory.

2.3 Safe but jobless in America. New disciplines born

In 1938, it was Erdős's turn to leave the Continent, soon to be dominated by Nazi Germany. In September, after a brief visit to Hungary, he moved to the U. S., and he did not return to the Old World for a decade.

In 1938-39 he held a fellowship at the Institute for Advanced Study with a stipend of $1,500 for the year. Even in 1995, more than 1300 papers later, Erdős remembered 38-39 as his best year. The crop included two seminal papers, with M. Kac [50] and A. Wintner [67], which established central limit theorems for a class of arithmetic functions. "Thus with a little impudence we would say that probabilistic number theory was born," Erdős wrote in 1995 [46, p. 105].

In spite of these and other major results, Erdős's fellowship at the Institute was not renewed, and subsequently he was without a job for considerable periods of time and survived on small loans from colleagues. His financial situation improved temporarily with a "research instructorship" at Purdue in 1943, but after 1945, he was without a job again. Meanwhile, the flow of ground-breaking results continued unabated. Erdős's paper "On the law of the iterated logarithm" [36] appeared in 1942; the first study of "inaccessible cardinals," fundamental to modern *set theory*, saw light in a paper by Erdős and Tarski in 1943 [64], and the Erdős-Stone theorem[62], which opened up the field of *extremal graph theory*, appeared in 1946.

Erdős's father died of heart attack in Budapest in 1942. During the war there was no postal service between the U. S. and Hungary; Erdős tried in vain to contact his mother through intermediaries.

Hitler's forces descended upon Hungary on March 19, 1944. Within weeks, the clockwork of the "Final Solution" engulfed Hungarian Jewry. Most of Erdős's relatives and many of his close friends perished "on the very eve of triumph over the barbarism[4]" (F. D. Roosevelt). As if by a miracle, Erdős's beloved mother survived.

In July 1948, Erdős met young and brilliant Atle Selberg at the Institute for Advanced Study, and from their brief encounter, an elementary proof of the Prime Number Theorem emerged [39], [109]. This result was prominently mentioned in Selberg's Fields Medal citation in 1950 and in Erdős's Cole Prize citation (A. M. S.) in 1951. It is a sad note on the history of number theory that a controversy over the genesis of this seminal work prevented these two great mathematicians from further collaboration.

2.4 Freedom and dignity with one small suitcase

Erdős left the U. S. in 1948, for the first time in a decade, and began what would become an unending journey around the globe. In Amsterdam in autumn he met a childhood acquaintance, Alfréd Rényi (1921-70), eight years Erdős's junior, who meanwhile had emerged as a mathematical genius working in a great variety of fields, including number theory, probability theory, orthogonal series, information theory, combinatorics, and applied mathematics. Erdős and Rényi began their influential collaboration with a paper "On consecutive primes," one of Erdős's lifetime favorite subjects.

In 1952 Erdős finally landed a secure job at Notre Dame University. He lost that job two years later to McCarthyist paranoia: the Immigration Service denied his reentry permit. Erdős coud have stayed in the States, but he chose freedom over convenience, attended the International Congress of Mathematicians in Amsterdam without the reentry permit, and was unable to return to the U. S. for nine years. In 1963, when he was finally readmitted to the U. S., he informed his audiences with his characteristic humor that "Sam[5] finally admitted me because he thinks I am too old and decrepit to overthrow him."

In autumn 1954, a job at the Technion saved Erdős from "starvation;" from then on he was listed in his passport as a resident of Israel, while maintaining his Hungarian citizenship. A few years later he became affiliated with the Mathematical Institute of the Hungarian Academy of Science, the only permanent affiliation he would maintain for the rest of his life.

Erdős's uncompromising view on freedom and dignity compelled him to take a voluntary exile from Hungary in 1973 after the Hungarian government denied visas to Israeli

[4] From a speech of March 24, 1944, warning of the menace facing Hungarian Jewry. Roosevelt's and Churchill's solemn words were not followed by action; the Allies did nothing to frustrate the Nazi genocide machine. Cf. [26, pp. 1095–1118], cited in some detail in [20, p. 51].

[5] "Sam:" the U. S. in Erdősese. "Joe" was Erdős's term for the Soviet Union and for Communist countries, referring to Joseph Stalin.

mathematicians, including invited speakers and old collaborators of Erdős, wishing to attend a conference held in Hungary to celebrate Erdős's 60th birthday. Erdős lifted his self-exile three years later, to attend the death-bed of his dear friend and major collaborator, Paul Turán (1910-76).

Erdős's lifestyle was legendary. He spent the second half of his life travelling from conference to conference with a small suitcase containing virtually all his earthly belongings. "Property is nuisance," he would declare, paraphrasing the French socialists who thought that property was sin. Erdős spent most of his time in the U. S., Canada, Hungary, Israel, the U. K. and the Netherlands but visited many other countries around the globe with fair frequency, including a number of visits to Australia. His mother accompanied him on his journey from 1964 until her death in Calgary in 1971, at the age of 91.

Erdős will also be remembered for his generosity, his kindness, and his caring for his fellow humans, as well as for his great interest in meeting mathematically gifted children and nurturing their talents.

Erdős received at least 15 honorary doctorates. He was a member of the national scientific academies of 8 countries, including the U. S. National Academy of Science (1979) and the Royal Society (1989). In 1984, Erdős shared the Wolf Prize with differential geometer Shiing-shen Chern, and promptly gave away the $50,000 award. Shortly before his death, Erdős renounced his honorary degree from the University of Waterloo over what he saw as an unfair treatment of a colleague.

3 Asymptotic thinking, combinatorial vision

3.1 The excitement of straight lines in the plane

A hallmark of much of Erdős's work is his unique *combinatorial vision* that revolutionized several fields of mathematics. Wherever he looked, he found elementary, yet often enormously difficult combinatorial questions.

Nothing serves as a better illustration of this point than the excitement his questions brought to the simplest concepts of Euclidean plane geometry: points, lines, triangles.

Consider a set of k points and t lines in the plane. What would Erdős ask about them? Many things, but perhaps the simplest question is this: what is the maximum number $f(k,t)$ of incidences between the points and the lines?

Erdős himself showed than $f(k,t) \geq c((kt)^{2/3} + m + n)$ by considering the points of a square grid and certain lines with many incidences. He went on to conjecturing that this bound is best possible, apart from the value of the constant c. This conjecture, along with at least five other conjectures of Erdős in plane geometry, was confirmed in a celebrated paper by Szemerédi and Trotter [118]. The paper was fittingly dedicated to Erdős's 70th birthday. For the proof "from the Book," see Székely [116].

Erdős proposed many problems on repeated distances. The simplest of these: what is the maximum number $g(n)$ of unit distances between n points in the plane. Erdős showed that $n^{1+c/\log\log n} < g(n) < O(n^{3/2})$ [37, 41]. The upper bound was improved by Spencer et al. to $O(n^{4/3})$ [114]. The gap between the upper and lower bounds remains large; it is conjectured that the lower bound is closer to the actual value.

Not all of the simplest and most natural problems of this kind were conceived by Paul Erdős. But problems fitting this description hardly ever escaped his attention, he embraced them (without ever explaining, why), raised them to high visibility, and if he did not solve them himself, he matched them to the right person.

As an example, consider the "k-set problem" in Euclidean plane geometry:

> Given a set S of n points in the plane in general position (no three on a line), and a parameter $0 \leq k \leq n$, what is the maximum possible number $f_k(n)$ of subsets T of S of size k that can be separated from $S \setminus T$ by a line?

This problem, raised by Gustavus Simmons in the late 60s, is a typically Erdős-style problem. Erdős carried it around the world, which by then included college freshman László Lovász, an *epsilon* with teeth, one of the "slower" child prodigies Erdős had mentored[6]. Lovász gave an $n^{2/3}$ upper bound for the case $n = 2k$ [88] (the case originally asked by Simmons); Simmons gave an $\Omega(n \log n)$ lower bound. In a subsequent paper, Erdős, Lovász, Simmons, and Straus [53] generalized these bounds to $\Omega(n \log k) \leq f_k(n) \leq O(n\sqrt{k})$. The rather large gap between the upper and lower bounds is still there, indicating the surprising difficulty of the problem. The only improvement has been a marginal reduction of the upper bound, by a factor of $\log^* k$ (Pach et al. [102]; the authors mention that actually a factor of $(\log k)^c$ can be shaved off).

3.2 Combinatorial number theory

A point illustrated by these examples and forcefully propagated by Erdős's myriad questions is his interest, not in exact numbers or formulae, but rather in *asymptotic orders of magnitude.*

[6] Lajos Pósa, Erdős's favorite child prodigy, made his mark on graph theory by his age of 14. His short paper [105] on Hamilton cycles dating from that time became a classic. A paper by Erdős and Pósa written in the same year (published in 1962) gives a *polynomial time algorithm* that decides, for each fixed k, if a given graph contains k vertex-disjoint cycles [55] and establishes that the collection of graphs having no k disjoint cycles is well-quasi-ordered under topological containment. This result became one of the starting points for Robertson and Seymour's seminal work establishing that the presence of any (fixed) graph minor can be tested in polynomial time and that the class of finite graphs is well-quasi-ordered under minors (cf. [119]). – In contrast, Pósa's his high school classmate László Lovász wrote his first significant papers "at the ripe old age of 17," Erdős comments [43].

Arguably, asymptotic thinking has its precursors in number theory, the subject of Erdős's first and foremost love affair. But it seems to be without precedent in combinatorics and in geometry. And even within number theory, Erdős's style brought about an entirely new field, *combinatorial number theory*. Here again, the simplest concepts dominate.

A *Sidon set* is a set S of integers such that each pair of numbers in S has a different sum. *Asymptotically how dense can an (infinite) Sidon set be?* This problem, dating to the 30s, is typical of Erdős's style. The gap between the lower and the upper bounds is still large: the trivial $O(n^{1/2})$ upper bound was improved by Erdős to $O((n/\log n)^{1/2})$ (infinitely often); and the $\Omega(n^{1/3})$ (greedy) lower bound has been improved to $\Omega((n \log n)^{1/3})$ [8]. It is of interest to note that this slight yet highly nontrivial improvement of the lower bound is due to Ajtai, Komlós, and Szemerédi, the trio well known for their joint as well as their separate contributions to the theory of computing.

It should also be noted that problems of combinatorial number theory often yield answers to problems on set systems; e. g. the well solved finite version of the Sidon set problem (Erdős–Turán [65]) is closely related to the construction of a set system used in an approximation-preserving reduction of chromatic number to maximum clique (Khanna, Linial, Safra [85]).

It is not entirely true that Erdős himself (in papers without coauthors) never worked on problems of computational interest. He in fact provided one of the early examples of a *lower bound in complexity theory.* His 1960 result [42] is presented in detail by Knuth [86, pp. 451–453].

The question was the number of multiplications required to calculate x^n, given x. This is equivalent to asking the minimum length of "addition chains" (straight line programs where addition is the only permitted operation) required to reach the number n, starting from 1. The problem so much fit in Erdős's style of additive number theory, he devoted the 3rd paper in his series "Remarks on number theory" to it.

The lower bound Erdős proved states that *for almost all k-digit integers, the minimum length is greater than* $k+(1-\epsilon)k/\log_2 k$. Observe that he did not indicate how one could *find* hard examples. Erdős added in a remark that his methods suffice to prove the same bound for "addition and subtraction chains," and mentioned that this problem is related to computing x^n by multiplications and divisions. He cited Péter Ungar as his source of the question. The lower bound settled the asymptotics of the nontrivial "extra term" in addition to the trivially necessary k steps, since an upper bound of $k+(1+o(1))k/\log_2 k$ had been known since 1939 (A. Brauer, cf. [86, loc. cit.]).

3.3 Master of patterns

If combinatorics is defined as *the art of finding regular patterns in structures under virtually no assumptions,* Erdős was certainly a master of the art. *"Sunflowers,"* a simple pattern among sets, were introduced by Erdős and Rado in 1960 [57] (they called them "Δ-systems"). A sunflower is a family of sets such that all pairwise intersections of these sets are the same. Erdős and Rado showed that large sunflowers occur in every sufficiently large family of sets of a given size. More precisely, they proved that among any family of $m > (k-1)^r r!$ distinct sets of size r, there exist k which form a sunflower. The problem they raised, whether C^r sets suffice to guarantee a sunflower with $k = 3$ "petals," remains open to this date.

Figure 3: **Ronald L. Graham, Paul Erdős, and Péter Frankl in Hakone, Japan, 1990.** Photograph by Jin Akiyama. *Photo courtesy: Peter Frankl*

Sunflowers give rise to a profound structure theory of large families of sets of a fixed size; this theory was developed in great depth especially by P. Frankl who used it, among other things, to give the first explicit construction of Ramsey graphs of superpolynomial size [71].

4 Why relevant?

4.1 Algorithms and complexity

Since the advent of complexity classes such as P and NP, *asymptotic thinking* has dominated complexity theory and the theory of algorithms, two central areas of the theory of computing. Moreover, *combinatorial objects* have gained great importance in both areas: a large part of the theory of algorithms deals with combinatorial structures (graphs, networks, etc.); combinatorial models, such as *Boolean circuits* and *branching programs*, have been the subject of many of the most significant developments in complexity theory [113]. The celebrated logarithmic depth sorting network of Ajtai, Komlós and Szemerédi [9] is a prime example in the theory of algorithms; in complexity theory, A. Razborov's superpolynomial lower bounds for *monotone circuits* provide striking examples [108]. It is noteworthy that most of the numerous joint papers by Ajtai, Komlós and Szemerédi concern problems of Erdős, and that Razborov's proof uses

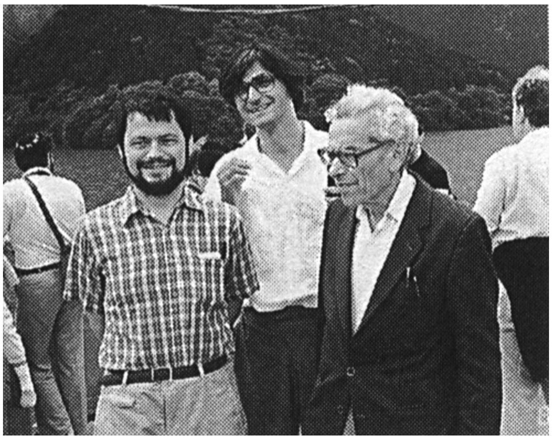


Figure 4: **Paul Erdős with Vojtech Rödl (left) and Zoltán Füredi, two dedicated solvers of Erdős's problems, in Hakone, Japan, 1990. In 1983, Rödl, together with Frankl (Figure 3) solved a 1000 dollar problem of Erdős [72], the largest sum ever claimed from Erdős, matched only by Szemerédi's 1000 dollar award. The problem Frankl and Rödl solved has erroneously been stated to have been valued by Erdős at \$500; the misunderstanding was due to the generosity of the solvers who accepted only half the award offered, in consideration of the state of the "Erdős bank." Frankl, Rödl, as well as Füredi have been intermittent contributors to the theory of computing.** Photograph by Jin Akiyama. *Photo courtesy: Peter Frankl*

Figure 5: **Paul Erdős with discrete geometer László Fejes Tóth in Oberwolfach, 1962.** Photograph by Branko Grünbaum. *Photo courtesy: Vera T. Sós.*

the sunflower theorem of Erdős and Rado, which Razborov rediscovered for his proof in 1985.

4.2 Computational geometry

Computational geometry has benefited from numeorus elementary geometric problems studied by Erdős.

Estimates on the number of certain configurations are helpful in analyzing the complexity of geometric algorithms. Naturally, such estimates may serve as lower bounds for the output size, but also, quite often, they lead to upper bounds on the complexity. The latter is especially typical in the case of algorithms which employ probabilistic techniques (a somewhat remote influence of Erdős), such as random sampling or randomized incremental insertions [29].

An important example is the k-set problem highlighted earlier, which has influenced the study of higher order Voronoi diagrams, ϵ-nets, and the analysis of the complexity of various algorithms for motion planning, range search, among others (cf. [79, 94, 111, 112, 34, 101]).

The Szemerédi-Trotter bound on point-line incidences was used by Matoušek to give a $O((kt)^{2/3} + k + t)$ upper bound for the complexity of *Hopcroft's problem*, asking, whether or not there is at least one incidence between k points and t lines in the plane [96].

In a significant recent development that could amount to a case study of the propagating effect of Erdős's problems, the $n^{4/3}$ upper bounds for the two separate problems of Erdős (point-line incidences and unit distances) turned out to be derivable from a common root (Clarkson et al. [28]), leading to a new method which not only gave simpler proofs, important extensions, and drastically reduced constants for both upper bounds, but also immediately found a number of algorithmic applications. The common root is a decomposition of the plane: Given n lines in the plane and a parameter $r < n$, one can partition the plane into $O(r^2)$ triangles so that no triangle intersects more than n/r lines. A number of other algorithms running in time close to $n^{4/3}$ have been found using this technique (cf. [1, 31, 95]).

4.3 Patterns: Ramsey theory, sunflowers

The aim of *Ramsey theory,* a large body of combinatorial theory created by Erdős, is to find "homogeneous" patterns in large systems of small sets (cf. [76]). Along with the Sunflower Theorem, another important pattern locator, Ramsey theory has been applied to proving lower bounds in models of computation where the critical restriction is on communication between processors (parallel RAMs, shallow circuits, communication complexity).

Examples include the first lower bound in multiparty communication complexity (Chandra, Furst, Lipton [27]). The result is based on a geometric Ramsey theorem by Tibor Gallai (1912–1992), Erdős's closest friend. Alon and Maass [16] prove a Ramsey-type result to obtain lower bounds for branching programs.

Fich et al. [70] use a simple case of the "Canonical Ramsey Theorem" of Erdős and Rado [56] to prove lower bounds in certain PRAM models. The Canonical Ramsey Theorem is particularly useful in applications because it does not restrict the number of colors used.

Grolmusz and Ragde [77] compare various PRAM models using an arsenal of combinatorial methods, including a significant reference to the Sunflower Theorem. Depth-2 circuits over $GF(2)$, another highly parallel model, is the subject of a paper by Alon et al. [15]; their tight $\Omega(n \log n)$ lower bound for computing an explicit set of linear functions again rests on the Sunflower Theorem.

Elementary geometry and Ramsey theory met in one of Erdős's earliest papers [63], written with fellow undergraduate and lifelong friend, George Szekeres, then a student of chemical engineering. They proved that sufficiently many points in the plane necessarily include k points which form a convex k-gon[7]. Later Erdős dubbed the question the "Happy End Problem:" proposed by Esther Klein and Erdős, the problem was first solved by Szekeres, who, in a remarkable *tour de force*, even rediscoved Ramsey's (then only three years old) theorem [107] for his solution, and subsequently married Klein. The result found its way into algorithmic research (cf. [34, Ch. 12.1]). Erdős and Szekeres also gave an explicit bound for graph Ramsey numbers. Their Ramsey bound is used for instance in the the approximate maximum clique algorithm by Boppana and Halldórsson [25].

Erdős's problem of *size-Ramsey numbers* motivated the study of structural consequences of expansion (see more on this in Section 6.1).

4.4 Fundamental parameters

The significance of Ramsey bounds goes far beyond such applications. They shed light on the relation between fundamental combinatorial parameters (clique number and independence number) relevant to virtually any model of computation. More broadly, the study of Ramsey-type questions has yielded fundamental structural insights in more areas than I could attempt to list.

I should mention here another basic combinatorial parameter: the *chromatic number* of graphs and hypergraphs (set systems). While the chromatic number of graphs has been considered for over a century in the limited context of planar graphs, the richness and fundamental nature of this concept, and its extension to set systems, has been espoused largely in the many papers of Erdős on "chromatic graph theory," an area of Erdős's creation (cf. [45]). Erdős's work may be largely responsible for what today every scholar of the theory of computing takes for granted: the eminent role played by the chromatic number, along with the clique number, in the theory of algorithms and in complexity theory. Consider the theories of interactive proofs, approximation algorithms, online algorithms, monotone circuits, you name it: the clique number and the chromatic number are among the prime targets.

Erdős studied finite and transfinite combinatorics hand in hand. Here is a gem buried in a long paper with András Hajnal, Erdős's number one collaborator [49]: *If the chromatic number of a graph G is $\geq \aleph_1$ then G contains a 4-cycle.* In fact, G contains a complete bipartite graph $K(m, \aleph_1)$ for every integer m. Contrast this with Erdős's earlier result that there exist finite graphs of arbitrarily large chromatic number and girth (girth = length of the shortest cycle) [40]. What the $\aleph_1$ result should do to our chromatic intuition, I am not sure, but I cannot cease to be fascinated by the result.

5 Randomization

What may be the most influential part of Erdős's work in all of discrete mathematics, including the theory of computing, is *randomization*. Two (interrelated) aspects of this subject should be distinguished: the *probabilistic method*, a non-constructive method of proof of the existence of certain objects; and the *analysis of the structure of random objects*.

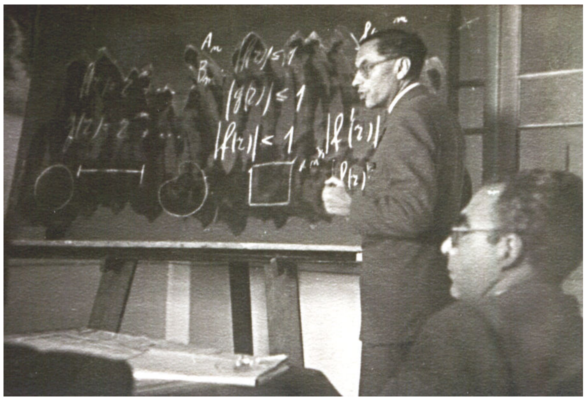


Figure 6: **Paul Erdős at a conference in Hungary in 1959. Probabilist Alfréd Rényi is looking on.** *Photo courtesy: Vera T. Sós.*

5.1 Random structures

The systematic analysis of random objects was initiated by two papers by Erdős and Rényi entitled "The evolution of random graphs" (1960-61) [58]. Richard M. Karp writes:

> "The Erdős-Rényi papers on random graphs exerted major influence on my work. The beautiful scenario of the successive stages in the evolution of random graphs, progressing in an essentially inevitable way, has stimulated me to find other stochastic processes, associated with algorithms, which unfold in the same kind of inevitability. Researchers have exhibited such

[7] One of the simplest, still open problem of Erdős asks whether a sufficiently large set of points in the plane necessarily contains an *empty* convex k-gon, i. e. a convex k-gon with no other points from the given set in its interior. This problem is open even for $k = 6$.

Figure 7: **Near the Mátraháza resort of the Hungarian Academy of Science in the 60s, a favorite hideout of Erdős and as a result the destination of mathematical pilgrimage while the master was in residence. Erdős enjoyed collaborating with several partners simultaneously on entirely different subjects, while also paying gentle attention to his mother. Squatting in the front: Alfréd Rényi. Standing, left to right: legendary sociologist Sándor Szalai who, among other things, rediscovered Ramsey's Theorem in the 50s (cf. [99]); function theorist Catherine Rényi, Alfréd's wife and his coauthor on a paper on the theory of trees; set theorist András Hajnal, Erdős's number one collaborator; Paul Erdős.** *Photo courtesy: Vera T. Sós*

> processes in connection with many problems related to graphs, Boolean formulas and other structures. Specific results related to random graphs have been applied to hashing, storage allocation, load balancing and other problems relevant to algorithms and computer systems."

One of the striking discoveries made by Erdős and Rényi was the sudden emergence of a giant component in a random graph as the number m of edges passes $n/2$. For $m < (1-\epsilon)n/2$, with high probability all components are of size $O(\log n)$; for $m > (1+\epsilon)n/2$, with high probability there will be a giant component, of size $\Omega(n)$, and all other components are of size $O(\log n)$.

The k-core of a graph is the largest subgraph with minimum degree $\geq k$. Pittel et al. [104] analyze the birth of a giant k-core with impressive accuracy. As an illustration as to how far these methods reach in computer science, Karp mentions that he, Alemany, and Thathachar [10] have recently used results of [104] to analyze the performance of a video-on-demand server.

A storage access scheme is analyzed via random graphs in Karp et al. [81].

Random processes occur in a number of other papers by Karp on the average case analysis of algorithms in combinatorial optimization (cf. [83, 84, 80, 82]).

In [80], Karp derives the structure of the transitive closure by analyzing a branching process related to a breadth-first search algorithm. A branching process related to depth-first search was previously analyzed by Ajtai, Komlós, and Szemerédi [7] to find long paths in random graphs; confirming a conjecture of Erdős, they prove that a random directed graph with $> (1+\epsilon)n$ edges contains, with high probability, a directed path of length $\Omega(n)$.

For other uses of the analysis of random structures, inspired by Erdős and Rényi, see Section 6.2.

5.2 Probabilistic proof of existence versus explicit construction

The *probabilistic method* establishes the existence of certain objects by selecting an object at random from a certain probability space and proving that the object has the desired properties with positive (usually overwhelming) probability. While Erdős was not the first to employ an idea of this type, it was he who recognized its vast scope and developed it into a powerful technique. Two results that demonstrate the power of the method are Erdős's exponential lower bound for the Ramsey number for graphs (1947) [38], and his proof of the existence of graphs of large chromatic number without short cycles (1959) [40]. Both results were quite surprising at the time. They also posed tantalizing *derandomization challenges*: the problem is to find explicit examples of such objects.

In complexity theory, the significance of *explicit constructions* cannot be overstated. In most models it is straightforward to show that random functions are hard to compute; what we need is explicit hard-to-compute functions. This is the essence of the P vs. NP problem, and of many other central problems in the area.

Erdős's derandomization challenges inspired major efforts. Large chromatic graphs without short cycles and of size comparable to Erdős's were eventually constructed by Margulis [91] and by Lubotzky-Phillips-Sarnak [90] (1986) in their famous "Ramanujan graphs" paper, using a formidable algebraic arsenal. Frankl and Wilson (1979) [71, 73] made substantial progress on the problem of constructive Ramsey graphs, but their $n^{c \log n/\log\log n}$ bound is still far from exponential. The situation is even much worse for the bipartite version of this question; no explicit construction of size greater than cn^2 is known.

Paul Erdős not only set up derandomization challenges, but in a 1973 paper he wrote with J. Selfridge [60], he was also the first to invent an important *derandomization tool:* the *method of conditional expectations* (cf. [18, Chapter

15]). The method allows searching certain super-polynomial size sample spaces in polynomial time; this has led to the derandomization of a large class of parallel algorithms [23, 97] (cf. [98]).

Directly and indirectly, the probabilistic method also contributed to the development of the theory of randomized algorithms, such algorithms often being based on arguments of the abundance of witnesses.

5.3 The needle in the haystack

Usually, the probabilistic method establishes the *existence* of an object in cases that the object is actually present in abundance, i. e. when random choice leads to the desired object with probability approaching 1. A major development over the method was the *Lovász Local Lemma* ([89, Ex. 2.18]) which finds the *needle in the haystack*: it proves the existence of certain objects even if they are *exponentially rare*. While this result is correctly attributed to Lovász, it should be noted that it appeared in a paper by Erdős and Lovász [52], and was motivated by the following conjecture of Erdős:

> Let $A_1, \ldots, A_m \subseteq X$, $|A_i| = r$. Suppose none of the A_i intersects more than $f(r)$ of the A_j where $f(r)$ grows exponentially with r. Then X can be colored red and blue such that no A_i becomes monochromatic.

In other words, this hypergraph is 2-colorable, or has "property B," a property Erdős had investigated in many papers. The Erdős–Lovász paper establishes the validity of this conjecture with $f(r) = 2^{r-2}$ by showing that a random two-coloring has a *positive* chance of success (cf. [89, Ex. 13.43]). It is clear that this chance may be exponentially small.

This circumstance poses a special algorithmic challenge even for randomized algorithms: if the object we look for is present in abundance, the algorithmic problem is limited to verifying that we did indeed find the right object; on the other hand, objects guaranteed to exist by the Lovász Local Lemma (such as satisfying assigments to certain CNF formulas) seem to be very hard to find.

This difficulty was overcome, in cases like the Erdős problem stated above, by Beck [22], for smaller but still exponentially large $f(r)$. Beck's algorithm was subsequently simplified and parallelized by Alon [12].

* * *

For decades, hardly anyone other than Erdős recognized the significance of the probabilistic method. The situation changed with the 1974 publication of *Probabilistic Methods in Combinatorics* by Erdős and Joel Spencer [61]; the book had a major impact on combinatorics, and, mainly through Erdős's disciples, on the theory of computing.

6 What I learned from Erdős

I asked some of the prominent members of the Theory community about their experience with Paul Erdős and his work. I did not specifically ask what they *learned* through this experience; yet, several of the replies included explicit statements regarding that question.

6.1 Nick and Avi

Nick Pippenger and Avi Wigderson were never closely associated with Erdős.

> "I am not sure about the influence of Erdős *problems* on my career," Nick writes. "I don't think I have ever solved an Erdős problem (or even worked hard on one). His results had a much greater effect. I first met Paul when he gave at talk at MIT in 1973. I remember speaking with him about some results in my thesis (which involved using probabilistic arguments to show the existence of certain switching networks). But I thought of my use of probabilistic methods as growing out of Shannon's 1948 paper, rather than Erdős's 1947 paper. I do remember being very impressed by Paul's 1960 paper on independent sets in triangle-free graphs. *It taught me that one should not expect the probabilistic method to solve the whole problem – one should be prepared to add some additional ingredients!* [8]"

In fact, Nick *did* contribute to at least one problem directly descending from Erdős. The *size-Ramsey number* for the graph H is the minimum number of edges of a graph G such that no matter how we color the edges of G red and blue, H will appear in one of the colors. The *density version* of this concept, also promoted by Erdős, asks for a graph G such that no matter how we delete half the edges of G, the remaining graph will contain a copy of H. Friedman and Pippenger [74] prove, adapting a technique from Feldman and Pippenger [69], that the exists a graph G with $O(n)$ edges such that after deleting all but a δ fraction of the edges, the remaining graph will contain all trees of bounded degree with n vertices. The result builds and improves on previous work by Beck [21] and Alon and Chung [14]. It should also be mentioned that a key ingredient in all this work is a 1976 result of Pósa [106], probably the first result in the literature on structural consequences of expansion, stating that

> *if every subset X of size $\leq k$ in a graph G has $\geq 2|X| - 1$ neighbors outside X then G contains a path of length $3k - 2$.*

[8] Emphasis added (L. B.)

The main result of Pósa's influential paper[9] established the almost certain existence of Hamilton cycles in random graphs of average degree $O(\log n)$, in direct response to a question by Erdős and Rényi. This type of eventual linkup of the techniques used to attack separate questions of Erdős (as the Hamiltonicity of random graphs and size-Ramsey numbers) is quite typical, giving one the impression of that elusive jigsaw puzzle Paul was working on.

Let me now quote Avi's testimony.

> "I cannot say that my work was directly influenced, but I certainly loved many of his questions (mainly the asymptotic ones), the idea that one should propagate problems, rather than only solutions, and the *enourmous quantities I learned by collaborating with many members of the Erdős school.*[10]"

While several members of the "Erdős school" have made a successful move into the theory of computing, it seems the transition is not one-way. An increasing number of those who started from algorithmic and complexity theoretic backgrounds are taking serious interest in problems descending from Erdős. Avi works on explicit Ramsey constructions, Nick readily acknowledges that his work quoted above is "not exactly computer science" (although it *raises* algorithmic problems), much of the recent progress on Erdős's problems in discrete geometry has come from the computational geometry community. This may be another sign of the kinship between Erdős's world and the intrinsic problems of computation.

6.2 The Erdős school

It is beyond the scope of this writing to survey Erdős's disciples, or even just those who have made direct contributions to the theory of computing (TC). For some stories of interest to TC, I should refer to sections or passages of [20] on Fan Chung, Ron Graham, László Lovász, Vera Sós, and Joel Spencer.

The Turán family. The friendship and collaboration of Paul Erdős and Paul Turán (1910–76) started when they met in college. But, as Turán recalled with pride [120], their first "joint work" appeared in print even earlier, several years before they met: it was a solution to a problem in the High School Mathematical Monthly (cf. [93]) which they both had found (and no one else); their names were signed under the solution.

The joint work of Erdős and Turán encompasses a number of fields, including polynomials, interpolation theory, and a seminal series of articles on a subject of their creation, *statistical group theory* [66]. Properties of random permutations are analyzed to great depth in the series.

Turán married Vera Sós, a mathematician working in number theory. Notably, in high school, Sós had the most wonderful and inspiring mathematics teacher she could think of: Tibor Gallai, Erdős's closest friend and one of the initiators of several central concepts in combinatorial optimization. After marrying Turán, Sós saw Erdős quite frequently, but it wasn't until more than a decade later that Sós became one of Erdős's major collaborators in graph theory, especially in two areas she had initiated: Ramsey–Turán problems and anti-Ramsey problems.

Why are these details of interest to our subject? Look at the next generation: both Sós's elder son George Turán and Sós's nephew János Pach became theoretical computer scientists. G. Turán has been most active in computational learning theory and Boolean complexity theory (see e. g. his papers in COLT'93 and FOCS'94), Pach in discrete geometry and computational geometry. Pach also emerged as one of Erdős's most active collaborators during Erdős's last decade (in 1988 alone, four Erdős–Pach papers appeared).

János Pach grew up next door to the Turáns, often spending vacations with them and with Erdős. He was a keen observer of the great mathematicians. I recommend his warm personal account [100] to the reader's attention.

* * *

In the rest of this section I present stories of Alon, Ajtai, Komlós, and Szemerédi – four careers, each of which started under the determinant influence of Erdős (in combinatorics in three cases, and mostly in number theory in Szemerédi's case) and evolved naturally into a profile with a major TC component.

Noga Alon. The following is an essentially verbatim transcript of Noga's compelling account.

"I first saw Erdős as a first year undergraduate when he gave a talk at the Technion around 1975. We talked only briefly; I told him a few things I could prove using probabilistic arguments (at that time I was not aware of the fact that this was a well established method). He expressed interest, and I realized only later that all of this was very well known to him but he did not want to discourage me.

"In 1978 or 1979 I met him again, we talked quite a lot, and my M. Sc. thesis is on a question he told me (this is also my first paper [11]). In 1979 he also gave me 50 dollars for solving another question he mentioned (the maximum number of edges in graphs without two disjoint cycles under certain restrictions on the degrees). This turned out to follow quite easily from a result of Lovász, and I never wrote it down. I solved another $50 problem of Erdős two years ago [13].

"I did not work seriously on Erdős problems before meeting him, but of course I heard about him. I worked a lot on his problems before working in the theory of computing – I started working seriously in Computer Science only after I came to MIT in autumn 1983, and by that time I had already

[9] Pósa's solution is algorithmic and it led to a number of follow-up papers within the algorithmic community, see especially Angluin and Valiant [19].

[10] Emphasis added (L. B.)

done a lot of work in Extremal Combinatorics and Graph Theory.

"Erdős has surely been the driving force behind most of my own work, in Combinatorics as well as in Theoretical Computer Science[11]. My very first paper was on an extremal problem suggested by Erdős; the first serious book in Combinatorics I read was the collection of Erdős' papers published for his 60th birthday [44]. I actually took notes of most of the papers in this book, and I often find myself using results from there even in these days; an example (with a computational flavor) is a recent paper with Alon Orlitsky on repeated communication and Ramsey graphs [17], where the results of Erdős, Taylor and McEliece on the connection between the Shannon capacity of graphs and Ramsey graphs [54] are applied/extended.

"Much of my work in Theory uses probabilistic arguments or deals with derandomization, and Erdős was, of course, the real founder of these areas. Many of my papers in Combinatorics are about problems of Erdős, and many others (in Combinatorics and in theoretical CS) are directly motivated by such problems. In conclusion, I suppose I would have been a mathematician (and probably even one working in Combinatorics and in Theoretical Computer Science) even if I had never met Erdős, but my work would have surely been totally different. *To me, like to many others, Erdős-type combinatorial reasoning is at the heart of the theory of computation, and I cannot imagine how this area would have evolved without him.*[12]"

Miklós Ajtai. Ajtai has developed profound techniques to attack lower bound questions in complexity theory. Almost all his work uses random structures and randomization techniques.

Simultaneously with Furst, Saxe, and Sipser [75], Ajtai introduced random restrictions and obtained results slightly stronger than the FSS lower bounds for parity circuits [2]. Moreover, he proved very strong upper bounds on the difference between the number of even and odd inputs accepted by a bounded depth circuit; he used these bounds to prove a weaker version of Fagin's [68] model theoretic formulation of the NP≠coNP statement.

In a paper with Ron Fagin [5], Ajtai analyzed certain class of random directed graphs in order to show that directed graph reachability cannot be expressed by a monadic second order existential sentence (whereas undirected graph reachability can), establishing a formal gap between the complexities of the two problems.

In his paper with Yuri Gurevich [6], Ajtai demonstrates the existence of a monotone AC^0 function which cannot be computed by a positive AC^0 circuit (no negations). In [3] Ajtai proves a lower bound for a certain class of data structures. In [4] he proves that the pigeon hole principle has no bounded depth, polynomial size Frege proof. The proof of each of these remarkable results is based on the probabilistic proof of existence of certain objects which will be incorrectly classified in the given model.

Let me quote Ajtai on the beginnings of his career.

"I was 16 when I first met Paul Erdős. My parents knew Rényi and he brought Erdős to us one day. When Erdős learned that I was interested in mathematics, he promptly asked me two questions. First, if we delete two opposite corners of a chessboard then the rest cannot be covered by dominoes each occupying two adjacent cells. This was easy, I solved it right away. The second one was more challenging: given $n + 1$ integers between 1 and $2n$, prove that there are two among them such that one divides the other. This one took me a day to solve. *The encounter, and the simplicity and elegance of the solutions impressed me very much and strengthened my resolve to become a mathematician.*[13]

"I learned about the random graph problems of Erdős and Rényi from Komlós and Szemerédi. All my work on random graphs is joint with them. After having studied these methods, it was natural to me to approach the computer science problems probabilistically; the methods used for random graphs were often directly applicable."

János Komlós. Ajtai is the logician of the accomplished Ajtai–Komlós–Szemerédi trio; Komlós is the probabilist, Szemerédi the number theorist. And each of the three "musketeers" has combinatorics in their blood, Erdős style. Together and separately, they have also made major contributions to the theory of computing.

Komlós was about 20 when he first met Paul Erdős. Komlós had a fresh neat little result, and Erdős happened to be in town. So Komlós gathered his courage, and told Erdős about the result. Erdős immediately recalled that an Israeli mathematician had proven the same result two decades earlier.

"In spite of my disappointment, I learned something important: always ask Uncle Paul, before you spend months on a problem."

Sadly, this recipe[14] cannot be used anyomore.

Back in the 60s, Erdős often took a few students out for lunch. Komlós was a frequent guest at the table. "As an orphan, I was particularly grateful for the free lunches." He worked a lot with Erdős, and even visited him at the Mátraháza resort (the result was a joint paper conceived and written up in a single day).

Komlós was deeply impressed by the Erdős–Rényi papers. "I think the fundamental articles of Erdős and Rényi were followed by silence, and we (AKS) gave random graphs a new boost from the mid 70s on. I am especially proud that

[11] Emphasis added (L. B.)

[12] Emphasis added (L. B.)

[13] Emphasis added (L.B.)

[14] Micha Sharir offers this story: "I started to work on Davenport-Schinzel sequences without knowing the real history of the problem, and only after a half year of work I learnt from Erdős about these older connections, which made a turning point in my research [110]."

we introduced the use of branching processes in the theory of random graphs [7]."

One of the startling successes of the AKS trio in the theory of computing was the construction of a logarithmic depth sorting network [9], using linear expanders as building blocks. This was the first constructive use of expanders. How did it happen? "By accident," says Komlós.

Ajtai reports that the original version of the what became the AKS sorting network was not a network at all, it was a parallel algorithm based on random sampling. It did not employ expanders. "In many respects, it was more natural than the final product," says Ajtai.

One afternoon József Beck presented Margulis's explicit linear expander construction at a seminar at the Mathematical Institute in Budapest. Fortuitously, Komlós decided to attend the seminar before seeing Ajtai and Szemerédi for another work session on their randomized sorting algorithm. Although he, as well as his collaborators, had been aware of expanders (Szemerédi had even written about them, see below), it was during Beck's lecture that the possibility of a connection dawned on Komlós. As the three began to work out the details, their surprise grew at each step as they gradually recognized the extent to which expanders were just the tools they needed. Not only did expanders eliminate the need for randomization from the algorithm, they even made it possible to turn it into a sorting network.

After the AKS sorting network, the algorithmic use of expanders spread rapidly. "But I believe, we were the first. Thank you, Jóska [Beck]."

Endre Szemerédi. When Erdős visited Budapest in 1963, Paul Turán introduced his student, number theorist András Sárközy, to Erdős. Sárközy brought his colleague Szemerédi along. That turned out to be a good idea. Between 1966 and 1970, a dozen Erdős–Szemerédi-Sárközy papers appeared, marking a new era in combinatorial number theory. At final count, Sárközy has more than 50 joint papers with Erdős, surpassing even Hajnal.

Feasting on Erdős's problems, Szemerédi emerged as one of the most formidable problem solvers of our time. His crowning achievement so far has been the proof, in "a masterpiece of combinatorial reasoning," [76, p. 46], that *a sequence of integers which does not contain a k-term arithmetic progression must have density zero* [117] (1975). The result confirmed a 1936 conjecture of Erdős and Turán and earned Szemerédi a $1000 award Erdős set for this problem, the largest amount Erdős ever had to pay[15].

A key lemma to this result, referred to as Szemerédi's *Regularity Lemma,* roughly says that the vertex set of any graph can be partitioned into a bounded number of nearly equal classes so that the graph has "nearly uniform density" between almost every pair of classes. This result turned out to be a major tool in graph theory [87].

[15] Cf. the caption of Figure 4.

Szemerédi was at Stanford in 1974. While he was working on what would become the Regularity Lemma, Donald Knuth proposed him another problem: to estimate the minimum number $f(n)$ of edges in a directed graph such that after the removal of any n vertices, the graph retains a directed path of length n. This is the directed graph analogue of a problem of Erdős discussed earlier in this section (in Nick's subsection), namely the "density version" of the size-Ramsey problem for paths, solved by Beck [21].

In contrast to the undirected case, the answer to Knuth's problem is *superlinear*:

$$\Omega(n \log n / \log\log n) < f(n) < O(n \log n). \qquad (1)$$

The result appeared in a joint paper by Erdős, Graham, and Szemerédi [48]. Both the lower and the upper bound use probabilistic arguments, plus a number of new ideas well worth studying. Indeed, the paper contains a cornucopia of ideas for use in TC.

Nick's advice applies to the upper bound in inequality (1): random graphs alone will not suffice. The construction uses *expanders* as building blocks, anticipating their "accidental" emergence in the AKS sorting network six years later.

Separator ideas obtained via depth-reduction using certain nested partitions (motivated by Szemerédi's thoughts around the Regularity Lemma) were the main tool for the lower bound.

A ground breaking paper by L. G. Valiant, on graph theoretic methods to obtain superlinear lower bounds on complexity, appeared in 1977 [121]. One of the approaches discussed by Valiant relies on lower bounds for digraphs that are dense in long paths, the question studied in the Erdős-Graham-Szemerédi paper. Valiant adapted the depth-reduction technique to his parameters.

Nine years later Szemerédi made a key contribution to a major result in the theory of computing: the *separation of deterministic and nondeterministic linear time* [103]. The combinatorial core of the lower bound is a "segregator theorem" for a class of directed graphs called "k-pushdown graphs," which model the information flow on multitape Turing machines. A (nontrivial) modification of the Stanford idea of depth-reduction via nested partitions worked splendidly for this problem.

The moral appears to be that Erdős's problems, such as the occurrence of arithmetic progressions or size-Ramsey numbers, seem to be probing segments of the mathematical universe quite close to those where attempts to understand the inherent obstacles to efficient computation have navigated over the past two decades.

Summary. In my view, the key to Paul Erdős's impact on theory of computing, an area in which he never took direct interest, is in the profound *relevance of his paradigms* on which many of his disciples were educated.

It is clear that Erdős's *asymptotic thinking* and *combinatorial vision* has had a great effect on the work of many theoretical computer scientists. Erdős's search for *regular patterns* and organizing principles in highly irregular structures has become relevant to the analysis of the power of various (often combinatorial) models of computation, the power of the models themselves being compared in terms of asymptotic orders of magnitude. His incessant flow of elementary questions on *combinatorial extrema* in number theory, geometry, and combinatorics stimulated insights and techniques applicable to many areas of theoretical computer science; *computational geometry* should be mentioned as an area especially enriched by this stimulus.

Probabilistic methods, pioneered by Erdős, have been used very successfully in proving lower bounds in complexity theory and showing the existence of certain structures of computational relevance. Probabilistic proofs of existence have posed important *derandomization challenges* which in turn inspired major efforts. Erdős himself, in a paper with Selfridge, met a class of these challenges with the introduction of an efficient derandomization tool.

The *analysis of random structures,* pioneered by Erdős and Rényi, has been a major source of inspiration for the *average case analysis* of algorithms, and contributed to the analysis of randomized algorithms as well as to some lower bound techniques.

It seems apparent that the structures and phenomena Erdős concentrated on lie at the foundations of the theory of computing, and as time passes, more of this relevance will unfold.

Finally, Erdős's inimitable style of communicating directions of study in terms of interminable sequences of easily stated open problems helped focus attention on these structures and phenomena more effectively than any philosophical pronouncements or project statements could have.

The relevance of Erdős's paradigms was gradually recognized by the theory community over the past quarter century, as the asymptotic analysis of algorithms and the notion of complexity classes took root, and methods borrowed from recursion theory gave way to combinatorial arguments and models. This process was facilitated by the interest several of Erdős's disciples took in the theory of computing, recognizing on their side that the methods developed to study Erdős's problems equipped them with new approaches to models of computation.

Acknowledgments. I wish to thank Miklós Ajtai, Noga Alon, Richard M. Karp, Donald E. Knuth, János Komlós, János Pach, Nicholas Pippenger, Micha Sharir, Endre Szemerédi, and Avi Wigderson for contributing their thoughts to this article. I am particularly grateful to Micha Sharir for his detailed comments on discrete and computational geometry which I used liberally in the corresponding sections (however, I am solely responsible for any errors there and elsewhere in the paper).

I wish to thank all those who provided photographs. My special thanks are due to Peter Kimmel for temporarily suspending his work on communication complexity and instead struggling with the complexities of electronic photo technology to get these pictures into the article. His help has been invaluable.

Short preliminary versions of this article appeared in SIGACT News 27/2 (1996) pp. 62–65, and SIAM News 30/1 (1997) p. 3. I am grateful to SIAM editor Gail Corbette for her perceptive suggestions from which the present article has also benefited.

References

[1] P. K. Agarwal: *Intersection and Decomposition Algorithms for Planar Arrangements.* Cambridge University Press, New York, 1991.

[2] M. Ajtai: Σ_1^1-formulae on finite structures. *Annals of Pure and Appl. Logic* **24** (1983), 1-48.

[3] M. Ajtai: A lower bound for finding predecessors in Yao's cell-probe model. *Combinatorica* **8** (1988), 235-247.

[4] M. Ajtai: The complexity of the pigeonhole principle. *Combinatorica* **14** (1994), 417-433.

[5] M. Ajtai, R. Fagin: Reachability is harder for directed than for undirected graphs, *Journal of Symbolic Logic* **55** (1990), 113-150.

[6] M. Ajtai, Yu. Gurevich: Monotone versus Positive. *J. ACM* **34** (1987), 1004–1015.

[7] M. Ajtai, J. Komlós, E. Szemerédi: The longest path in a random graph. *Combinatorica* **1** (1981), 1-12.

[8] M. Ajtai, J. Komlós, E. Szemerédi: A dense infinite Sidon sequence. *Europ. J. Comb.* **2** (1981), 1–11.

[9] M. Ajtai, J. Komlós, E. Szemerédi: Sorting in $c \log n$ parallel steps. *Combinatorica* **3** (1983), 1-19.

[10] Juan Alemany, R. M. Karp, J. Thathachar: Probabilistic analysis of video-on-demand service. (Work in progress.)

[11] N. Alon: On the number of subgraphs of prescribed type of graphs with a given number of edges. *Israel J. Math.* **38** (1981), 116-130.

[12] N. Alon: A parallel algorithmic version of the local lemma. *Random Structures and Algorithms* **2** (1991), 367-378.

[13] N. Alon: Bipartite graphs. *Combinatorica* **16** (1996), 301-311.

[14] N. Alon, F. R. K. Chung: Explicit construction of linear sized tolerant networks. *Discrete Math.* **72** (1988), 15–19.

[15] N. Alon, M. Karchmer and A. Wigderson: Linear circuits over $GF(2)$. *SIAM J. Comput.* **19** (1990), 1064-1067.

[16] N. Alon, W. Maass: Meanders, Ramsey Theory and lower bounds for branching programs. In: *Proc. 27th IEEE FOCS*, 1986, pp. 410–417.

[17] N. Alon, A. Orlitsky: Repeated communication and Ramsey graphs. *IEEE Trans. on Info. Theory* **41** (1995), 1276-1289.

[18] N. Alon, J. H. Spencer: *The Probabilistic Method.* Wiley, New York, 1992.

[19] D. Angluin, L. G. Valiant: Fast probabilistic algorithms for Hamiltonian circuits and matchings. In: *Proc. 9th ACM STOC*, 1977, pp. 30–41.

[20] L. Babai: In and out of Hungary: Paul Erdős, His Friends, and Times. In [30], Vol. 2, pp. 7–95.

[21] J. Beck: On size-Ramsey numbers of paths, trees, and circuits, I. *J. Graph Theory* **7** (1983), 115–129.

[22] J. Beck: An algorithmic approach to the Lovász Local Lemma I. *Random Structures and Algorithms* **2** (1991), 343–365.

[23] B. Berger, J. Rompel: Simulating $(\log n)^c$-wise independence in NC. *Journal of the ACM* **38** (1991), 1026–1046.

[24] B. Bollobás: A Life of Mathematics – Paul Erdős, 1913-1996. *FOCUS*, Math. Assoc. of Amer. **16/6** (Dec. 1996), pp. 1, 4–6.

[25] R. Boppana, M. M. Halldórsson: Approximating maximum independent sets by excluding subgraphs. *BIT* **32** (1992), 180–196.

[26] Randolph L. Braham: *The Politics of Genocide. The Holocaust in Hungary.* Columbia University Press, New York 1981.

[27] A. K. Chandra, M. L. Furst, R. J. Lipton: Multi-party protocols. In: *Proc. 15th ACM STOC*, 1983, pp. 94–99.

[28] K. Clarkson, H. Edelsbrunner, L. Guibas, M. Sharir, E. Welzl: Combinatorial complexity bounds for arrangements of curves and spheres. *Discrete Comput. Geom.* **5** (1990), 99–160.

[29] K. Clarkson and P. Shor: Applications of random sampling in computational geometry, II. *Discrete Comput. Geom.* 4 (1989), pp. 387–421.

[30] *Combinatorics: Paul Erdős Is Eighty,* Volumes 1 and 2 (D. Miklós, V. T. Sós, T. Szőnyi, eds.). Bolyai Society Math. Studies 1 and 2, Budapest 1993 and 1996. Distributed by the A. M. S.

[31] M. de Berg, M. Overmars, O. Schwarzkopf: Computing and verifying depth orders. *SIMA J. Comp.* **23** (1994), 437–446.

[32] G. Berzsenyi: In memoriam: Paul Erdős (1913–1996). *Quantum*, Nov-Dec. 1996, pp. 40–41.

[33] D. de Caen, L. A. Székely: On dense bipartite graphs of girth eight and upper bounds for certain configurations in planar point-line systems. *J. Combin. Theory* Ser. A. 1997, to appear.

[34] H. Edelsbrunner: *Algorithms in Combinatorial Geometry.* EATCS Monographs on Theor. Comp. Sci. 10. Springer Verlag, Heidelberg 1987.

[35] H. Edelsbrunner, R. Seidel: Voronoi diagrams and arrangements. *Discrete Comput. Geom.* **1** (1986), 25–44.

[36] P. Erdős: On the law of the iterated logarithm. *Annals of Math.* **43** (1942), 419-436.

[37] P. Erdős: On sets of distances of n points. *Amer. Math. Monthly* **53** (1946), 248–250.

[38] P. Erdős: Some remarks on the theory of graphs. *Bull. A. M. S.* **53** (1947), 292-294.

[39] P. Erdős: On a new method in elementary number theory which leads to an elementary proof of the prime number theorem. *Proc. Nat. Acad. Sci.* **35** (1949), 374-384.

[40] P. Erdős: Graph theory and probability. *Canad. J. Math.* **11** (1959), 34-38.

[41] P. Erdős: On sets of distances of n points in euclidean space. *Publ. Math. Inst. Hung. Acad. Sci.* **5** (1960), 165–168.

[42] Remarks on number theory, III. On addition chains. *Acta Arithm.* **6** (1960), 77–81.

[43] P. Erdős: Child prodigies. In: *Proc. Washington State Univ. on Number Theory* (J. H. Jordan, W. A. Webb, eds.), Washington State Univ., Pullman, Wash., 1971, pp. 1–12.

[44] P. Erdős: *The Art of Counting* (Joel Spencer, ed.). M. I. T. Press, 1973.

[45] P. Erdős: Some new and old problems on chromatic graphs. In: *Combinatorics and applications* (Proc. conf. Calcutta, 1982) Indian Statist. Inst., Calcutta, 1984, pp. 118-126.

[46] P. Erdős: On some of my favourite theorems. In [30], Vol. 2, pp. 97–132.

[47] P. Erdős: List of Publications, in [30], Vol. 1, pp. 471–527 and Vol. 2, pp. 507–517.

[48] P. Erdős, R. L. Graham, E. Szemerédi: On sparse graphs with dense long paths. In: *Computers and mathematics with applications.* Pergamon, Oxford, 1976, 365-369.

[49] P. Erdős, A. Hajnal: On chromatic number of graphs and set-systems. *Acta. Math. Hungar.* **17** (1966), 61-99.

[50] P. Erdős, M. Kac: The Gaussian law of errors in the theory of additive number theoretic functions. *Amer. J. Math.* **62** (1940), 738–742.

[51] P. Erdős, Chao Ko, R. Rado: Intersection theorems for systems of finite sets. *Quarterly J. Math.* **12** (1961), 313–320.

[52] P. Erdős, L. Lovász: Problems and results on 3-chromatic hypergraphs and some related questions. In: *Infinite and Finite Sets* (A. Hajnal and V. T. Sós, eds.), Proc. Conf. in honor of P. Erdős's 70th birthday, Coll. Math. Soc. J. Bolyai 10, Bolyai – North-Holland 1974, pp. 609–627.

[53] P. Erdős, L. Lovász, G. Simmons, and E. Straus: Dissection of graphs of planar point sets. In: *A survey of combinatorial theory (ded. to R.C. Bose on his 70th birthday, Proc. Internat. Sympos. Combinatorial Math. and its Appl., Colorado State Univ., Fort Collins, CO 1971),* North Holland, Amsterdam, 1973, pp. 139-149.

[54] P. Erdős, R. J. McEliece, H. Taylor: Ramsey bounds for graph products. *Pacific J. Math.* **37** (1971), 45-46.

[55] P. Erdős, L. Pósa: On the maximal number of disjoint circuits of a graph, *Publ. Math. Debrecen* **9** (1962), 3–12.

[56] P. Erdős, R. Rado: A combinatorial theorem. *J. London Math. Soc.* **25** (1950), 249–255.

[57] P. Erdős, R. Rado: Intersection theorems for systems of sets. *J. London Math. Soc.* **35** (1960), 85–90.

[58] P. Erdős, A. Rényi: On the evolution of random graphs, *Magyar Tudományos Akadémia Matematikai Kutatóintézetének Közleményei* **5** (1960), 17-61.

[59] P. Erdős, A. Rényi: On the evolution of random graphs, *Bull. Inter. Stat. Inst. Tokyo* **38** (1961), 343-347.

[60] P. Erdős, J. Selfridge: On a combinatorial game. *J. Combinatorial Theory, Ser. B* **14** (1973), 298–301.

[61] P. Erdős, J. H. Spencer: *Probabilistic Methods in Combinatorics.* Akadémiai Kiadó, Budapest, 1974.

[62] P. Erdős, A. H. Stone: On the structure of linear graphs. *Bull. Amer. Math. Soc.* **52** (1946), 1087-1091.

[63] P. Erdős, G.Szekeres: A combinatorial problem in geometry. *Compositio Math.* **2** (1935), 463-470.

[64] P. Erdős, A. Tarski: On families of mutually exclusive sets. *Annals of Math.* **44** (1943), 315-329.

[65] P. Erdős, P. Turán: On a problem of Sidon in additive number theory and on some related problems. *J. London Math. Soc.* **16** (1941), 212-215.

[66] P. Erdős, P. Turán: On some problems of a statistical group theory, I – VII. *Z. Wahrsch.* **4** (1965), 175–186; *Acta Math. Hung.* **18** (1967), 151–164; 309–320; **19** (1968), 413-435; *Per. Math. Hung.* **1** (1971), 5–13; *J. Indian Math. Soc.* **34** (1970-71), 175–192; *Per. Math. Hung.* **2** (1972), 149-163.

[67] P. Erdős, A. Wintner: Additive arithmetical functions and statistical independence. *Amer. J. Math.* **61** (1939), 713–721.

[68] R. Fagin: Generalized first-order spectra and polynomial time recignizable sets. In: *Complexity of Computation* (R. M. Karp, ed.), A. M.,S., Providence, R. I. 1974, pp.43–73.

[69] P. Feldman, N. Pippenger: Wide-Sense Nonblocking Networks. *SIAM J. Disc. Math.* **1** (1988), 158-173.

[70] F. E. Fich, F. Meyer auf der Heide, P. Ragde, A. Wigderson: One, two, three ... infinity: lower bounds for parallel computation. In: *Proc. 17th ACM STOC*, 1985, pp. 48–58.

[71] P. Frankl: A constructive lower bound for some Ramsey numbers. *Ars Combinatoria* **3** (1977), 297–302.

[72] P. Frankl, V. Rödl: Hypergraphs do not jump. *Combinatorica* **4** (1984), 149–159.

[73] P. Frankl and R. M. Wilson: Intersection theorems with geometric consequences. *Combinatorica* **1** (1981), 357–368.

[74] J. Friedman, N. Pippenger: Expanding graphs contains all small trees. *Combinatorica* **7** (1987), 71–76.

[75] M. L. Furst, J. B. Saxe, M. Sipser: Parity, circuits, and the polynomial time hierarchy. *Math. Systems Theory* **17** (1984), 13–27.

[76] R. L. Graham, B. L. Rothschild, J. H. Spencer: *Ramsey Theory*, 2nd ed. Wiley, New York, 1990.

[77] V. Grolmusz, P. Radge: Incomparability in parallel computation. In: *28th IEEE FOCS*, 1987, pp. 89–98.

[78] A. Hajnal, E. Szemerédi: Proof of a conjecture of Erdős. In: *Combinatorial Theory and Its Appl.* (P. Erdős, A. Rényi, V. T. Sós, eds.), Coll. Math. Soc. J. Bolyai 4, North-Holland, 1970, Vol. 2, pp. 601–623.

[79] D. Haussler, E. Welzl: ϵ-nets and simplex range queries. *Discrete Comput. Geom.* **2** (1987), 127–151.

[80] R. M. Karp: The Transitive Closure of a Random Digraph. *Random Structures and Algorithms* **1** (1990) 73–94.

[81] R. M. Karp, M. Luby, F. Meyer auf der Heide: Efficient PRAM Simulation on a Distributed Memory Machine. *Algorithmica* **16** (1996), 517-542.

[82] R. M. Karp, A. Rinnooy Kan, R. Vohra: Average Case Analysis of a Heuristic for the Assignment Problem. *Math. of Operations Research* **19/3** (1994), 513-522.

[83] R. M. Karp, M. Sipser: Maximum Matchings in Sparse Random Graphs. In: *Proc. 22nd IEEE FOCS*, 1981, pp. 364–375.

[84] R. M. Karp, R. E. Tarjan: Linear Expected-Time Algorithms for Connectivity problems. *J. Algorithms* **1** (1980), 374–393.

[85] S. Khanna, N. Linial, S. Safra: On the hardness of approximating the chromatic number. In: *Proc. 2nd Israel Symp. on Theory and Computing Sys. ISTCS'93*, IEEE C. S. Press 1993, pp. 250–260.

[86] D. E. Knuth: *The Art of Computer Programming, Vol. 2: Seminumerical Algorithms.* 2nd ed. Addison-Wesley, Reading, MA, 1981.

[87] J. Komlós, M. Simonovits: Szemerédi's Regularity Lemma and its Applicationis in Graph Theory. In: [30], Vol. 2, pp. 295–352.

[88] L. Lovász: On the number of halving lines. *Annales Univ. R. Eötvös, Sect. Math.* **14** (1971), 107–108.

[89] L. Lovász: *Combinatorial Problems and Exercises.* Akadémiai Kiadó, Budapest, and North-Holland, Amsterdam, 1979.

[90] A. Lubotzky, R. Phillips, P. Sarnak: Ramanujan graphs. *Combinatorica* **8** (1988), 261–277.

[91] G. A. Margulis: Explicit group theoretic construction of combinatorial schemes and their application to the construction of expanders and concentrators (in Russian). *Probl. Inform. Transmission* **24** (1988) 51–60. English translation: Pleanum Press, New York, pp. 39–46.

[92] *Mathematical and Physical Journal for Secondary Schools, Centennial Special Issue* (in Hungarian). J. Bolyai Mathematical Society and Eötvös Physical Society, Budapest, December 1993 and April 1994. *To order by e-mail, write to:* `h3341sza@ella.hu`.

[93] *Mathematical and Physical Journal for Secondary Schools, Centennial Special Issue in English.* J. Bolyai Mathematical Society and Eötvös Physical Society, Budapest, 1994. *To order by e-mail, write to:* `h3341sza@ella.hu`.

[94] J. Matoušek: Construction of ϵ-nets. *Discrete Comput. Geom.* 5 (1990), 427–448.

[95] J. Matoušek, Reporting points in halfspaces. *Comput. Geom. Theory Appl.* **2** (1992), 169–186.

[96] J. Matoušek: Geometric range searching. *ACM Comput. Surv.* **26** (1994), 421–461.

[97] R. Motwani, J. Naor, M. Naor: The probabilistic method yields deterministic parallel algorithms. *J. Computer and System Sci.* **49** (1994), 478–516.

[98] R. Motwani, P. Raghavan: *Randomized algorithms.* Cambridge University Press, 1995.

[99] J. Pach: Delicate symmetry. *Computers Math. Applic.* **17** (1989), 117–124.

[100] J. Pach: Two places at once: a remembrance of Paul Erdős. *Mathematical Intelligencer* **19/2** (1997), to appear.

[101] J. Pach, P. K. Agarwal: *Combinatorial Geometry.* Wiley, New York 1995.

[102] J. Pach, W. Steiger, E. Szemerédi: An upper bound on the number of planar k-sets. *Discrete and Computational Geometry*, **7** (1992), 109–123. (Ext. abstract: FOCS'89.)

[103] W. Paul, N. Pippenger, E. Szemerédi, W. T. Trotter: On determinism versus nondeterminism and related problems. In: *24th IEEE FOCS*, 1983, pp. 429–438.

[104] B. Pittel, J. Spencer, N. Wormald: Sudden emergence of a k-core in a random graph. *J. Combinat. Theory (Ser. B)* **67** (1996), 111–151.

[105] L. Pósa: A theorem concerning Hamiltonian lines. *Publ. Math. Inst. Hungar. Acad. Sci.* **7** (1962), 225–226.

[106] L. Pósa: Hamiltonian circuits in random graphs. *Discrete Math.* **14** (1976), 359–364.

[107] F. P. Ramsey: On a problem of formal logic. *Proc. London Math. Soc.* **30** (1930), 264–286.

[108] A. Razborov: Lower bounds for the monotone complexity of some Boolean functions. *Dokl. Akad. Nauk SSSR* **281** (1985), 798–801 (in Russian). English translation: *Soviet Math. Dokl.* **31** (1985), 354–357.

[109] A. Selberg: An elementary proof of the prime number theorem. *Annals of Math.* **50** (1949), 305–319.

[110] M. Sharir: Improved upper bounds on the length of Davenport–Schinzel sequences. *Combinatorica* **8** (1988), 117–124.

[111] M. Sharir: On k–sets in arrangements of curves and surfaces. *Discrete Comput. Geom.* **6** (1991), 593–613.

[112] M. Sharir: Almost tight upper bounds for lower envelopes in higher dimensions. *Discrete Comput. Geom.* **12** (1994), 327–345.

[113] M. Sipser: The History and Status of the P versus NP Question. In: *Proc. 24th ACM STOC*, 1992, pp. 603–618.

[114] . J.H. Spencer, E. Szemerédi and W.T. Trotter: Unit distances in the Euclidean plane. In: *Graph Theory and Combinatorics* (Proc. Cambridge Conf. on Combinatorics, B. Bollobás, ed.), Academic Press, 1984, pp. 293–308.

[115] E.G. Straus: Paul Erdős at 70. *Combinatorica* **3** (1973), 245–246.

[116] L. A. Székely: Crossing numbers and hard Erdős problems in discrete geometry. *Combinatorics, Probability, and Computing*, 1997, to appear.

[117] E. Szemerédi: On sets of integers containing no k elements in arithmetic progression. *Acta Arithm.* **1975**, 199–245.

[118] E. Szemerédi, W.T. Trotter, Jr.: Extremal problems in discrete geometry. *Combinatorica* **3** (1983), 381–392.

[119] C. Thomassen: Embeddings and Minors. Chap. 5 of the *Handbook of Combinatorics* (R.L. Graham, M. Grötschel, L. Lovász, eds.) Elsevier 1995, Vol. 1, pp. 301–349.

[120] P. Turán: Foreword to "The Art of Counting" [44].

[121] L.G. Valiant: Graph theoretic arguments in low-level complexity. In: *Math. Foundations of Computer Science* **6** (1977), 162–176.

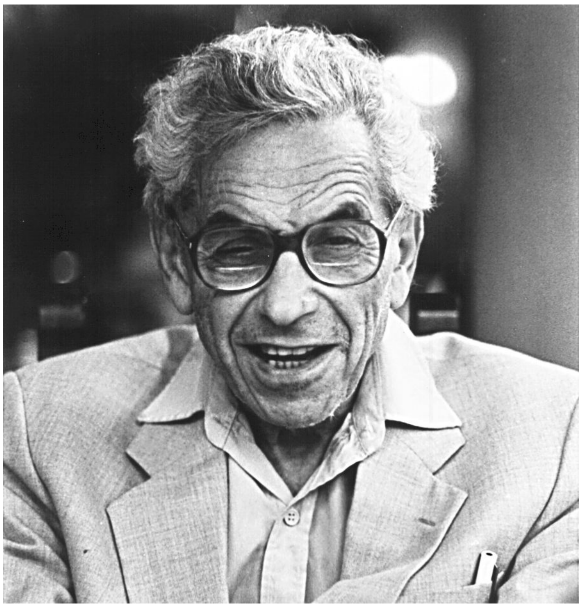


Figure 8: **Paul Erdős in 1991.** *Photo©George Paul Csicsery*

Appendix

Volume 2 of [30] contains a unique article by Paul Erdős, on his favorite *theorems*. The same volume contains a 90-page biographic article on Paul Erdős by this writer. Volumes 1 and 2 together contain the most complete list of publications of Erdős currently in print.

A video documentary on Erdős, entitled *"N Is a Number – a portrait of Paul Erdős,"* by George Paul Csicsery, is available from the *Mathematical Association of America*.

For more pointers, including a number of newspaper obituaries, you may consult the Erdős home page of the Theory group at the University of Chicago at `http://www.cs.uchicago.edu/ /groups/theory/erdos.html`.

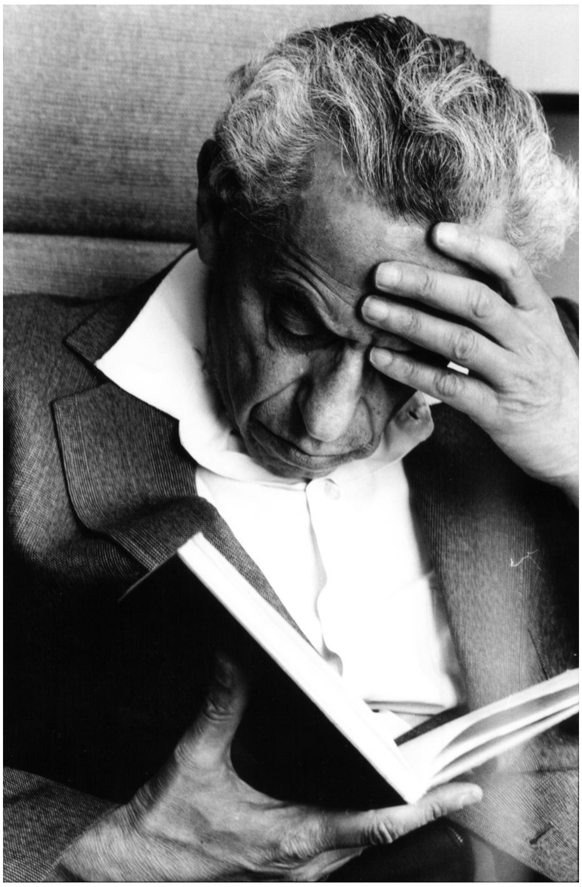


Figure 9: **Paul Erdős immersed in study. New Hamburg, Ontario, 1979.** *Photo©J. Adrian Bondy*

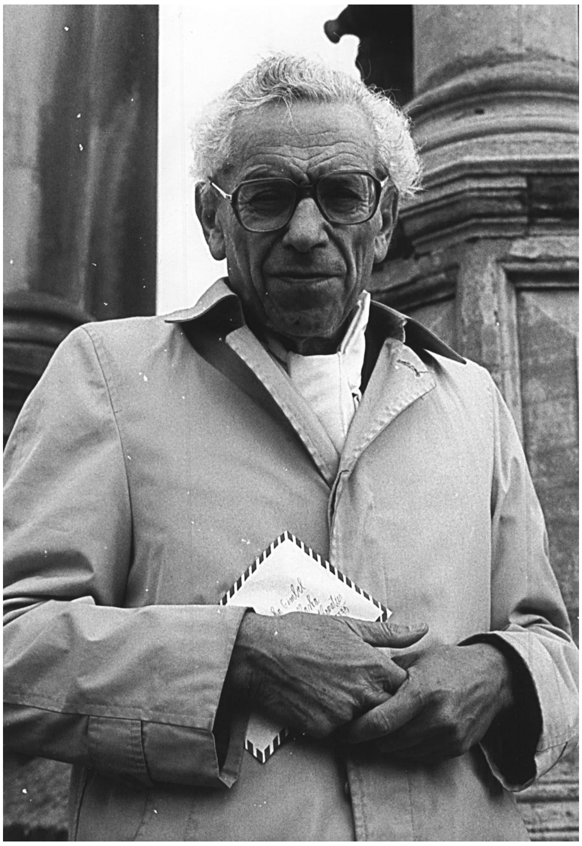


Figure 10: **Erdős at Cambridge, 1988, at a conference in honor of his 75th birthday. He holds a letter addressed to graph theorist John Gimbel; one can see Erdős's distinctive handwriting quite clearly.** *Photo©J. Adrian Bondy*

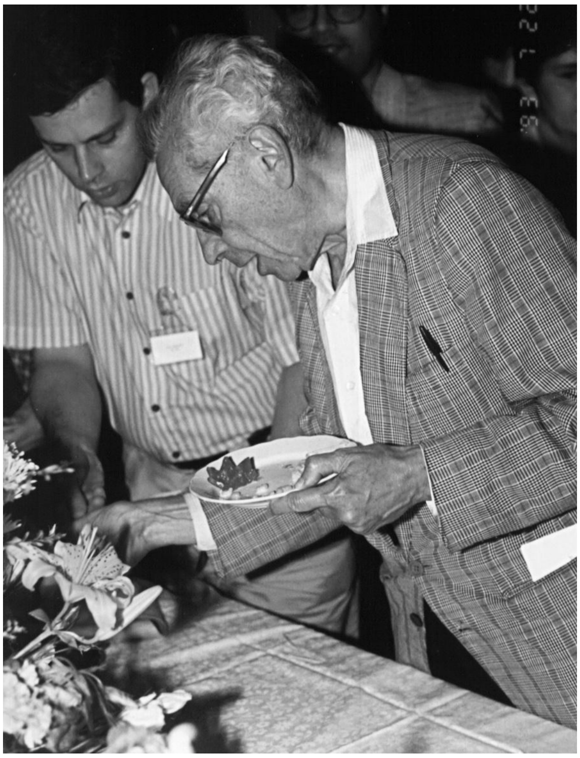


Figure 12: **Paul Erdős trying to grab a bite at the banquet of the conference held to honor his 80th birthday in Keszthely, Hungary, 1993. Not an easy task among hundreds of hungry admirers, but he gets some help from Péter L. Erdős (no relation).** *Photograph by L. Babai*

Figure 11: **Erdős at W. T. Tutte's home in Westmontrose, Ontario, 1985. GO and pingpong were Erdős's favorite games.** *Photo©J. Adrian Bondy*

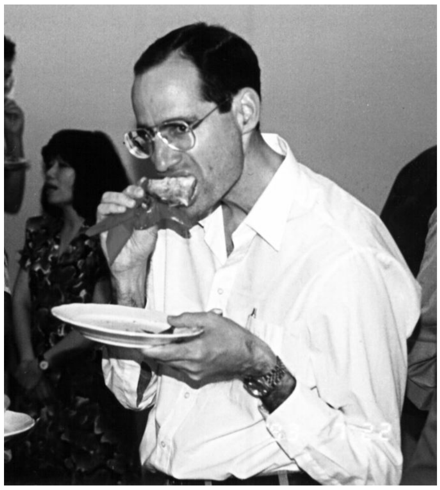


Figure 13: **Noga Alon did all right at the buffet. Fan R. K. Chung in the background. "Erdős 80" conference.** *Photograph by L. Babai*

Figure 14: **Noam Nisan celebrates Erdős's 80th birthday at the barock palace of count Festetich, Keszthely. "Erdős 80" conference.** *Photograph by L. Babai*

Figure 15: **A moving scene at the "Erdős 80" banquet: Old pal George Szekeres playing Mozart, accompanied on the piano by analyst Miklós Laczkovich (background) who had just squared the circle.** *Photograph by L. Babai*

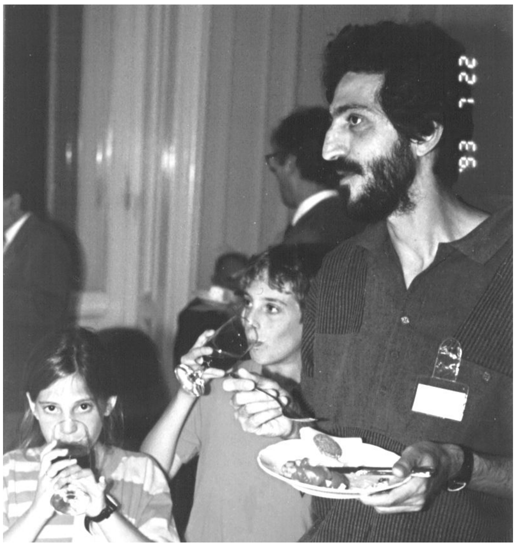
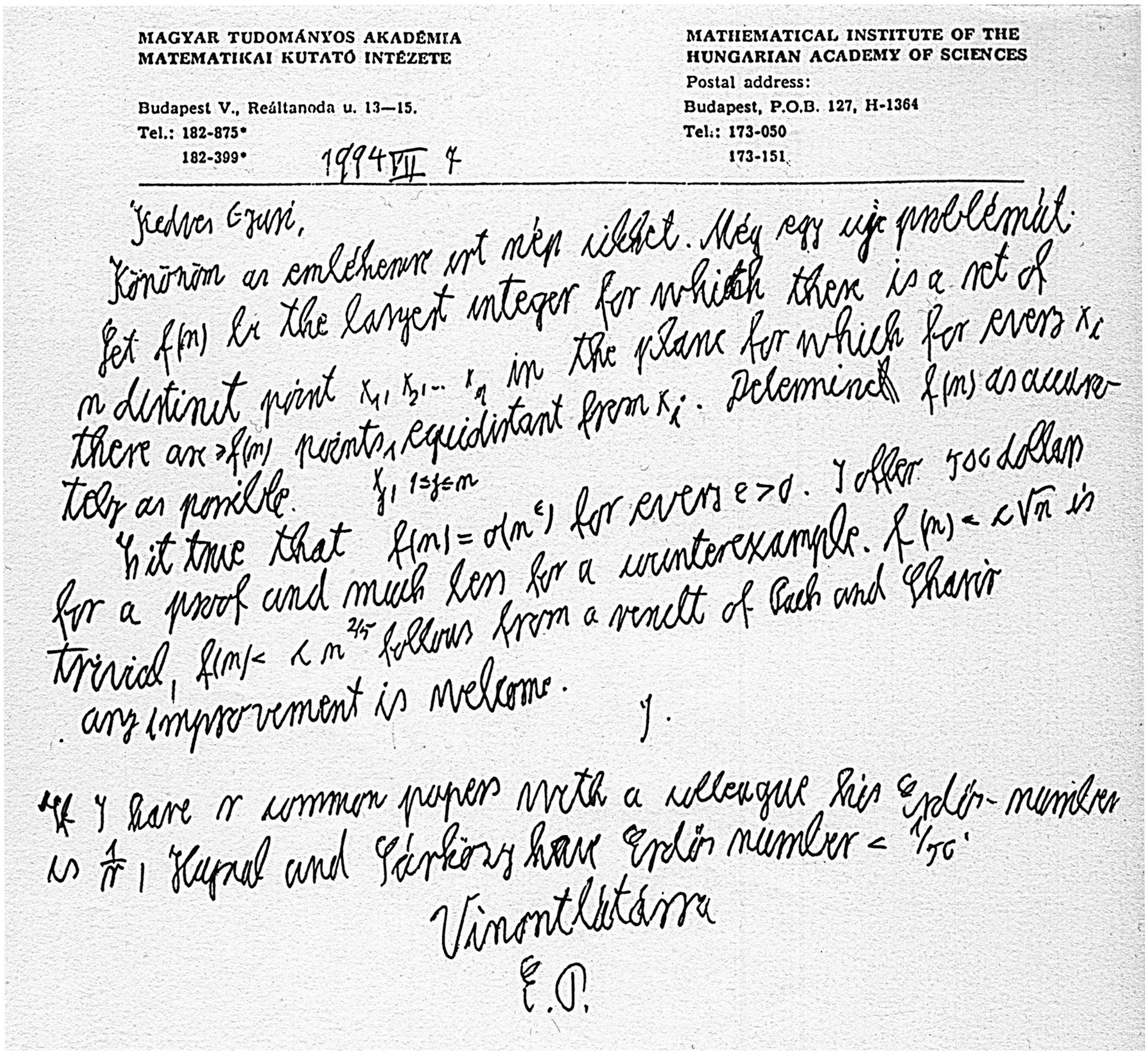


Figure 16: **Einat, Eyal, and Avi Wigderson at the banquet. (Edna was there, too.)** *Photograph by L. Babai*

Figure 17: **Erdős listens to the violin. Invisible in the background: Esther Klein, wife of Szekeres, initiator of the "happy end problem." The Erdős–Szekeres paper of 1935 which solved the problem represented a milestone in the development of Erdős's "combinatorial vision."** *Photograph by L. Babai*



MAGYAR TUDOMÁNYOS AKADÉMIA
MATEMATIKAI KUTATÓ INTÉZETE

Budapest V., Reáltanoda u. 13—15.
Tel.: 182-875*
182-399*

MATHEMATICAL INSTITUTE OF THE
HUNGARIAN ACADEMY OF SCIENCES
Postal address:
Budapest, P.O.B. 127, H-1364
Tel.: 173-050
173-151

1994 VII 4

Kedves Gyuri,
Köszönöm az emlékemre írt szép cikket. Még egy új problémát:
Let f(n) be the largest integer for which there is a set of n distinct points $x_1, x_2, \ldots x_n$ in the plane for which for every x_i there are $\geq f(n)$ points x_j, $1 \leq j \leq n$ equidistant from x_i. Determine f(n) as accurately as possible.
Is it true that $f(n) = o(n^\epsilon)$ for every $\epsilon > 0$. I offer 500 dollars for a proof and much less for a counterexample. $f(n) < c\sqrt{n}$ is trivial, $f(n) < cn^{2/5}$ follows from a result of Pach and Sharir. Any improvement is welcome.

*If I have r common papers with a colleague his Erdős-number is $\frac{1}{r}$, Hajnal and Sárközy have Erdős number < 1/50.

Viszontlátásra
E.P.

Figure 18: **A 1994 letter by Erdős to George Berzsenyi, a dedicated educator who believes in nurturing mathematical talent through problem solving. Berzsenyi is an editor of several mathematics journals for students, including *Quantum*. He shared the problem provided by Erdős with his readers in his "Math Investigations" column [32].** The greeting (in Hungarian) displays Erdős's typical humour: "Dear Gyuri, Thank you for the nice article you wrote *to my memory*." (Reference to an earlier article by Berzsenyi which also included some open problems by Erdős.) Then Erdős switches to English and states a new problem: "Let $f(n)$ be the largest integer for which there is a set of n distinct points $x_1, x_2, \ldots, x_n$ in the plane for which for every x_i there are $\geq f(n)$ points x_j equidistant fom x_i. Determine $f(n)$ as accurately as possible. Is it true that $f(n) = o(n^\epsilon)$ for every $\epsilon > 0$? I offer 500 dollars for a proof and much less for a counterexample. $f(n) < c\sqrt{n}$ is trivial, $f(n) < cn^{2/5}$ follows from a result of Pach and Sharir. Any improvement is welcome." While the letter is in many ways typical of Erdős's style, there are two unusual points in it: first, Erdős almost always offered the same amount for a proof and a dispoof of his conjectures. Second, the relevant result is due to Clarkson et al. [28], a lapse of memory highly uncharacteristic for Erdős. *Facsimile courtesy: George Berzsenyi*