Paul Erdős and the Probabilistic Method

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1 The Probabilistic Method

The Probabilistic Method is one of the most significant contributions of Paul Erdős. Indeed, Paul himself said, during his 80th birthday conference in Keszthely, Hungary, that he believes the method will live long after him. This has been the only time I have heard him making any comment about the significance and impact of his work. He has always been more interested in discussing new problems and results than in trying to assess their long time expected merits.

The method is a powerful technique with numerous applications in Combinatorics, Graph theory, Additive Number Theory and Geometry. The basic idea is very simple: Trying to prove that a structure with certain desired properties exists, one defines an appropriate probability space of structures and then shows that the desired properties hold in this space with positive probability. The amazing fact is that this simple reasoning can lead to highly nontrivial results. The results and tools are far too numerous to cover in a few pages, and my aim here is only to give a glimpse of the topic by describing a few examples of questions and results that illustrate the method. All of these have been initiated by Erdős, motivated by his questions and results. The fact that there is still an intensive ongoing work on all illustrates the influence and long term impact of his work. More material on the subject can be found in the books [6], [8], [23], [26].

2 Ramsey Numbers

Ramsey Theory is the study of the general phenomenon that every large structure, even if it looks totally chaotic, must contain a rather large well-organized substructure. This holds for many types of structures (though there are exceptions), and yields interesting applications in several mathematical areas. A detailed treatment of the subject can be found in [21].

Although several Ramsey-type theorems appeared earlier, the origin of Ramsey theory is usually credited to Frank Plumpton Ramsey, who proved in 1930 the fundamental theorem that edge colorings

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of finite or infinite graphs or hypergraphs satisfy such a theorem. The statement for finite graphs is as follows.

Let $H_1, H_2, \ldots, H_k$ be k finite, undirected, simple graphs. Then there is a finite number r such that in every edge coloring of the complete graph on r vertices by k colors, there is a monochromatic copy of H_i in color i for some $1 \leq i \leq k$. The smallest integer r that satisfies this property is called the (multicolor) *Ramsey number* of $H_1, \ldots, H_k$, and is denoted by $r(H_1, H_2, \ldots, H_k)$.

The determination or estimation of these numbers is usually a very difficult problem, which fascinated Erdős since the 30s. When each graph H_i is a complete graph K_t with $t > 2$ vertices, the only values that are known precisely are those of $r(K_3, K_m)$ for $m \leq 9$, $r(K_4, K_4)$, $r(K_4, K_5)$ and $r(K_3, K_3, K_3)$. The determination of the asymptotic behavior of Ramsey numbers up to a constant factor is also a very hard problem, and despite a lot of efforts by various researchers there are only a few infinite families of graphs for which this behavior is known.

In one of the first applications of the probabilistic method in Combinatorics, Erdős [12] proved that if $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$ then $R(K_k, K_k) > n$, that is, there exists a 2-coloring of the edges of the complete graph on n vertices containing no monochromatic clique of size k. This implies that $R(K_k, K_k) > 2^{k/2}$ for all $k \geq 3$. The proof is extremely short: the probability that a random two-edge coloring of K_n contains a monochromatic copy of K_k is at most $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$, and hence there is a coloring with the required property.

It is worth noting that although this argument seems trivial today, it was far from being obvious when published in 1947. In fact, several prominent researchers believed, before the publication of this short paper, that $R(K_k, K_k)$ may well be bounded by a polynomial in k. In particular, Paul Turán writes in [28] that he had conjectured for a while that $R(K_k, K_k)$ is roughly k^2, and that Erdős' result came to him as a big surprise, showing that this quantity behaves very differently than expected.

My own first meeting with Paul Erdős took place when I was finishing high school in the early 70s in Haifa, Israel. Paul had a special visiting position at the Technion, and I met him during one of his visits. A few months before that I have read his probabilistic lower bound for the Ramsey numbers $R(K_k, K_k)$, formulated as a counting argument without any mention of probability, and noticed that the argument can be used to provide several similar results. I (proudly) told Erdős about my observations, and he encouraged me to keep thinking about these problems, and gave me a book - The Art of Counting [15] - which has just been published at that time. This book contains selected publications of Erdős, and is the first serious mathematical book I have ever read. Reading it, and taking notes of much of its content, I quickly realized that it contains far more sophisticated extensions of the basic probabilistic lower bound proof of Erdős than the ones I observed. Paul, who surely knew well this fact, chose to suggest that I read the book and keep thinking about these problems, realizing that this is more stimulating than quickly pointing out the relevant references. Indeed, he always felt that young people interested in Mathematics should be encouraged, and I am convinced that this approach has been fruitful in many cases as it has been in mine.

Returning to the asymptotics of Ramsey numbers, a particularly interesting example of an infinite family for which the behavior of the Ramsey number is known, is the following result of Kim and of

Ajtai, Komlós and Szemerédi.

Theorem 2.1 ([25], [1]) *There are two absolute positive constants c_1, c_2 such that*

$$c_1 m^2/\log m \leq r(K_3, K_m) \leq c_2 m^2/\log m$$

for all $m > 1$.

The upper bound, proved in [1], is probabilistic, and applies a certain random greedy algorithm. There are several subsequent proofs, all are based on probabilistic arguments. The lower bound is proved by a "semi-random" construction whose detailed analysis is subtle, relying on several large deviation inequalities. An alternative way of establishing the lower bound, which provides a better constant, appears in two recent papers, [20] and [7], that analyze the so called "triangle free process" suggested by Bollobás and Erdős. In this process one starts with a graph on n vertices with no edges, and keeps adding uniformly chosen random edges among those that do not create a triangle. At the end, all these chosen edges are colored red and the non-chosen edges are colored blue. Clearly the resulting coloring contains no red triangle, and a careful analysis shows that with high probability there is no blue clique K_m, for an appropriate choice of the initial size n.

It is worth noting that the question of obtaining a super-linear lower bound for $r(K_3, K_m)$ is mentioned already in [12], and Erdős has established in [13], by an elegant probabilistic construction, an $\Omega(m^2/\log^2 m)$ lower bound.

Even less is known about the asymptotic behavior of multicolor Ramsey numbers, that is, Ramsey numbers with at least 3 colors. The asymptotic behavior of $r(K_3, K_3, K_m)$, for example, has been very poorly understood for quite some time, and Erdős and Sós conjectured in 1979 (c.f., e.g., [10]) that

$$\lim_{m \mapsto \infty} \frac{r(K_3, K_3, K_m)}{r(K_3, K_m)} = \infty.$$

This has been proved in [5], where it is shown that in fact $r(K_3, K_3, K_m)$ is equal, up to logarithmic factors, to m^3. A more complicated, related result proved in [5], that supplies the asymptotic behavior of infinitely many families of Ramsey numbers up to a constant factor is the following.

Theorem 2.2 *For every $t > 1$ and $s \geq (t-1)! + 1$ there are two positive constants c_1, c_2 such that for every $m > 1$*

$$c_1 \frac{m^t}{\log^t m} \leq r(K_{t,s}, K_{t,s}, K_{t,s}, K_m) \leq c_2 \frac{m^t}{\log^t m},$$

where $K_{t,s}$ is the complete bipartite graph with t vertices in one color class and s vertices in the other.

The proof of the lower bound is probabilistic: each of the first three color classes is a randomly shifted copy of an appropriate $K_{t,s}$-free graph that contains a relatively small number of large independent sets, as shown by combining spectral techniques with character sum estimates.

3 Sum-free subsets

A set A of integers is called *sum-free* if there is no solution to the equation $a+b=c$ with $a,b,c \in A$. Erdős [14] showed that any set A of n positive integers contains a sum-free subset of size at least $n/3$. The proof is a short and simple yet intriguing application of the probabilistic method. It proceeds by choosing a uniform random x in $(0,1)$, and by observing that the set of all elements $a \in A$ satisfying $ax \mod 1 \in (1/3, 2/3)$ is sum-free and its expected size is $n/3$.

In [3] the authors showed that a similar proof gives a lower bound of $(n+1)/3$. Bourgain [9] has further improved this estimate to $(n+2)/3$. For quite some time it was not clear whether or not the constant $1/3$ can be replaced by a larger constant, until Eberhard, Green and Manners have proved in [11] that the constant $1/3$ is tight. Their proof is a sophisticated argument, which contains a crucial probabilistic ingredient. The problem of deciding whether or not every set of n nonzero integers contains a sum-free subset of cardinality at least $n/3 + w(n)$, where $w(n)$ tends to infinity with n, remains open. It will be extremely surprising if there is no such $w(n)$.

4 List coloring and Euclidean Ramsey Theory

The *list chromatic number* (or *choice number*) $\chi_\ell(G)$ of a graph $G=(V,E)$ is the minimum integer s such that for every assignment of a list of s colors to each vertex v of G, there is a proper vertex coloring of G in which the color of each vertex is in its list. This notion was introduced independently by Vizing in [29] and by Erdős, Rubin and Taylor in [19]. In both papers the authors realized that this is a variant of usual coloring that exhibits several new interesting properties, and that in general $\chi_\ell(G)$, which is always at least as large as the chromatic number of G, may be arbitrarily large even for graphs G of chromatic number 2.

For about ten years after the initial papers of Vizing and of Erdős, Rubin and Taylor there has been essentially no work on list coloring. Starting in the late 80s, the topic, motivated to a great extent by the many problems raised by Erdős and his collaborators in [19], received a considerable amount of attention. Paul Erdős himself told me in the early 90s that when they have written their paper, he thought that the topic is not very exciting, and was pleasantly surprised to see that it eventually stimulated so much activity. I view this as a sign showing that Paul was essentially unable to ask any non-interesting questions. When he asked a question, even if at first sight it seemed artificial or non-appealing (even to Paul himself !), almost always it eventually turned out to be interesting.

It is natural to extend the notion of list coloring to hypergraphs. A hypergraph H is an ordered pair (V,E) where V is a set of vertices and E is a collection of subsets of V, called edges. It is r-uniform if every edge contains exactly r vertices. Thus graphs are 2-uniform hypergraphs. The list chromatic number $\chi_\ell(H)$ of a hypergraph H is the minimum integer s such that for every assignment of a list of s colors to each vertex of H, there is a vertex coloring of H assigning to each vertex a color from its list, with no monochromatic edges.

An intriguing property of list coloring of graphs, which is not shared by ordinary vertex coloring, is the fact that the list chromatic number of any (simple) graph with a large average degree is large. Indeed, it is shown in [2] that the list chromatic number of any graph with average degree d is at least $\Omega(\log d)$. For $r \geq 3$, simple examples show that there is no nontrivial lower bound on the list chromatic number of an r-uniform hypergraph in terms of its average degree. However, such a result does hold for simple hypergraphs. Recall that a hypergraph is *simple* if every two of its distinct edges share at most one vertex. The following result is proved in [4].

Theorem 4.1 *For every fixed $r \geq 2$ and $s \geq 2$, there is a $d = d(r, s)$, such that the list chromatic number of any simple r-uniform hypergraph with n vertices and at least nd edges is greater than s.*

A similar result for the special case of d-regular 3-uniform simple hypergraphs has been obtained independently in [22]. A subsequent proof with a much better upper estimate for $d(r, s)$ appears in a recent paper of Saxton and Thomason [27].

The proof of the theorem is probabilistic. For the simpler case of graphs it shows that if $G = (V, E)$ is a graph with average degree $d > 10^s$, then when we assign to each vertex of G a randomly chosen list consisting of s colors among the colors $\{1, 2, \ldots, 2s - 1\}$, then with high probability there is no proper coloring of G assigning to each vertex a color from its list. The precise argument requires some work, and the result suggests an interesting algorithmic question: given a graph $G = (V, E)$ with minimum degree $d > 10^s$, can we find, deterministically and efficiently, lists of size s for each $v \in V$ so that there is no proper coloring of G assigning to each vertex a color from its list ? This problem is open, as is the simpler NP version of it, that is, that of exhibiting lists and providing a certificate that there is no proper coloring using them. Here the lists do not have to be found efficiently, and we only require that one will be able to check the certificate efficiently.

The last theorem has an interesting application in Euclidean Ramsey Theory - yet another subject initiated by Erdős and his collaborators. A well known problem of Hadwiger and Nelson is that of determining the minimum number of colors required to color the points of the Euclidean plane so that no two points at distance 1 have the same color. Hadwiger showed already in 1945 that 7 colors suffice, and Moser and Moser noted in 1961 that 3 colors do not suffice. These bounds have not been improved, despite a considerable amount of effort by various researchers, see [24, pp. 150-152] and the references therein for more on the history of the problem.

A more general problem is considered in [16], [17], [18], where the main question is the investigation of finite point sets K in the Euclidean space for which any coloring of an Euclidean space of dimension d by r colors must contain a monochromatic isometric copy of K. There are lots of intriguing conjectures that appear in these papers. One of them asserts that for any set K of 3 points which do not form an equilateral triangle the minimum number of colors required for coloring the plane with no monochromatic isometric copy of K is 3. The situation is very different for list coloring. A simple Corollary of the theorem above is the following.

Theorem 4.2 ([4]) *For any finite set X in the Euclidean plane and for any positive integer s, there is an assignment of a list of size s to every point of the plane, such that whenever we color the points of the plane from their lists, there is a monochromatic isometric copy of X.*

The examples described in this brief survey include applications of the Probabilistic Method of Paul Erdős in Graph Theory, Ramsey Theory, Additive Number Theory and Combinatorial Geometry. There have been recent results in the study of each of these examples, while the roots of all them lie in the work and questions of Paul. There is no doubt that the study and application of probabilistic arguments will keep playing a crucial role in the development of many mathematical areas in the future, providing further evidence for the profound influence of Erdős. The comment he made in his 80th birthday conference proved to be accurate: the Probabilistic Method does live, and will stay alive, long after him.

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