Modular Forms

Surya Teja Gavva
Rutgers University

Triangle Groups, Belyi Uniformization, and Modularity
Bhaskaracharya Pratishthana, Trimester II, Feb 2022

Modular Forms

Eichler -

> *"There are five elementary arithmetical operations: addition, subtraction, multiplication, division, and modular forms."*

Applications

- Reciprocity – Modularity, Functoriality
- Class Numbers, Complex Multiplication, CFT for $\mathbb{Q}(\sqrt{-D})$
- Quadratic Forms, Theta series
- Lattices, Elliptic Curves, Modular Curves
- Galois representations, congruences
- Finite Dimensionality and resulting identities
- Hecke Operators, Multiplicativity
- Trace Formulae, geodesics, regulators
- Periods, L-values
- Irrationality of $\zeta(3)$
- Borcherd's products, Monstrous Moonshine, Kac-Moody algebras
- Spectral Gap, Ramanujan Graphs
- Sphere Packing
- Equidistribution of integer points on sphere.
- Ruziewiecz's problem
- Elliptic genera — Spin manifolds, Representations of the cobordism ring.

Modular Forms

Modular forms are functions on the upper half plane $\mathbb{H}$ that transform nicely under the action of a discrete subgroup Γ.

Definition

A function $f : \mathbb{H} \to \mathbb{C}$ is called a modular form of weight k for Γ if

1. it is holomorphic
2. Modularity: $f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = f\left(\frac{az+b}{cz+d}\right)(cz+d)^{-k} = f(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$
3. Holomorphic at Cusps: $\left(f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(z)$ tends to a limit as $z \to i\infty$ for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

Congruence Subgroups

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1, \text{ and } a, b, c, d \in \mathbb{R} \right\}$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \right\}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N \right\}$$

Fourier Expansion

$f(z)$ has a Fourier expansion.

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z/\mu}$$

Noncongruence subgroups

There are lot of noncongruence subgroups of $SL_2(\mathbb{Z})$ (almost all of them!)
Consider type-II character subgroup of $\Gamma^0(p)$ of index p. There are $p+1$ of them out of which p are noncongruence.

Unbounded Denominators

There are 6 index-5 type II(A) character groups in $\Gamma^0(11)$. Among them, one is $\Gamma^1(11)$ and the other 5 are noncongruence. Moreover every one of these noncongruence subgroups satisfies the condition (UBD).

$$f_P = w^{-5} + w^{-4} - 3w^{-3} + 13w^{-2} + 20w^{-1} - 23 + \cdots ,$$

$$f_{Q+P} = w^{-5} + w^{-4} + \frac{23+\sqrt{5}+i(3+\sqrt{5})\sqrt{25+2\sqrt{5}}}{4} w^{-3} + \cdots ,$$

$$f_{Q+2P} = w^{-5} + w^{-4} + \frac{99-33\sqrt{5}+i(23+3\sqrt{5})\sqrt{25+2\sqrt{5}}}{44} w^{-3} + \cdots ,$$

$$f_{Q+3P} = w^{-5} + w^{-4} + \frac{99-33\sqrt{5}-i(23+3\sqrt{5})\sqrt{25+2\sqrt{5}}}{44} w^{-3} + \cdots ,$$

$$f_{Q+4P} = w^{-5} + w^{-4} + \frac{23+\sqrt{5}-i(3+\sqrt{5})\sqrt{25+2\sqrt{5}}}{4} w^{-3} + \cdots .$$

Unbounded Denominators

$\sqrt[5]{f_{P_i}}$ have unbounded deonominators

Existence of Modular Forms

We have $k < 0$

$$a_n e^{-2\pi n y} = \int_0^1 f(z) e^{-2\pi i n x} dx \ll y^{-k/2} \to 0 \text{ as } y \to 0$$

For $k = 0$, we have bounded holomoprhic function $f(z)$. Hence the only modular form of weight $k \leq 0$ are the constant functions.

(Why) Do they exist for higher levels?

Eisenstein Series

For an even integer $k \geq 4$, the non-normalized weight k Eisenstein series is the function

$$G_k(z) = \sum_{m,n\in\mathbb{Z}}^{*} \frac{1}{(mz+n)^k}$$

It has the Fourier expansion

$$G_k(z) = 2\zeta(k) + 2 \cdot \frac{(2\pi i)^k}{(k-1)!} \cdot \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

Follows from Poisson summation or the using the Fourier formula

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{m=1}^{\infty} \left(\frac{1}{z+m} + \frac{1}{z-m} \right)$$

G_2

$$G_2(\tau) = 2\zeta(2) - 8\pi^2 \sum_{n=1}^{\infty} \sigma(n) q^n, \quad q = e^{2\pi i \tau}, \sigma(n) = \sum_{\substack{d|n \\ d>0}} d$$

$$(G_2 \mid \gamma)(\tau) = G_2(\tau) - \frac{2\pi i c}{c\tau + d}$$

Theta Series

$$\theta(\tau) = \sum_{t \in \mathbf{Z}} e^{2\pi i t^2 \tau}$$

Poisson summation implies

$$\theta(-1/(4\tau)) = \sqrt{-2i\tau}\,\theta(\tau)$$

$$\theta(\gamma(\tau))^4 = (c\tau + d)^2 \theta(\tau)^4 \text{ for } \gamma = \pm \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \gamma = \pm \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$$

Cusp Forms

Subspace where the functions vanish at all the cusps.

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

Weight 12 cusp form of level 1.

$$\Delta(z) = \sum_{n \geq 1} \tau(n) q^n = q \prod_{n \geq 1} (1 - q^n)^{24} = \eta(z)^{24}$$

Ramanujan Conjectures

1. $\tau(mn) = \tau(m)\tau(n)$ if $\gcd(m, n) = 1$ (meaning that $\tau(n)$ is a multiplicative function)
2. $\tau\left(p^{r+1}\right) = \tau(p)\tau\left(p^{r}\right) - p^{11}\tau\left(p^{r-1}\right)$ for p prime and $r > 0$.
3. $|\tau(p)| \leq 2p^{11/2}$ for all primes p.

Finite Dimensionality

Finite Dimensionality of $M_k(\Gamma)$ can proved from several type of arguments: Riemann-Roch, Trace-Formulae (Eichler-Shimura, Petersson.)

$$\dim M_k(\Gamma_0(q), \chi) = \frac{k-1}{12}\left[\Gamma_0(1) : \Gamma_0(q)\right] + O(\sqrt{qk})$$

```
sage: M=ModularForms(Gamma0(13), 6)
sage: M
Modular Forms space of dimension 7 for Congruence Subgroup Gamma0(13) of weight 6 over Rational Fiel
sage: M=ModularForms(Gamma0(13), 7)
sage: M
Modular Forms space of dimension 0 for Congruence Subgroup Gamma0(13) of weight 7 over Rational Fiel
sage: M=ModularForms(Gamma0(13), 8)
sage: M
Modular Forms space of dimension 9 for Congruence Subgroup Gamma0(13) of weight 8 over Rational Fiel
sage: M=CuspForms(Gamma0(13), 8)
sage: M
Cuspidal subspace of dimension 7 of Modular Forms space of dimension 9 for Congruence Subgroup Gamma
sage: M=CuspForms(Gamma0(13), 10)
sage: M
Cuspidal subspace of dimension 9 of Modular Forms space of dimension 11 for Congruence Subgroup Gamm
```

Ring of Modular Forms of Level 1

$$\mathcal{M}\left(\mathrm{SL}_2(\mathbf{Z})\right) = \bigoplus_{k \in \mathbf{Z}} \mathcal{M}_k\left(\mathrm{SL}_2(\mathbf{Z})\right)$$

$$\mathcal{M}\left(\mathrm{SL}_2(\mathbf{Z})\right) = \mathbf{C}\left[E_4, E_6\right], \quad \mathcal{S}\left(\mathrm{SL}_2(\mathbf{Z})\right) = \Delta \cdot \mathcal{M}\left(\mathrm{SL}_2(\mathbf{Z})\right)$$

Ring of Modular Forms

```
sage: M=ModularFormsRing(Gamma0(1))
sage: M
Ring of modular forms for Modular Group SL(2,Z) with coefficients in Rational Field
sage: M.generators()
[(4,
  1 + 240*q + 2160*q^2 + 6720*q^3 + 17520*q^4 + 30240*q^5 + 60480*q^6 + 82560*q^7 + 140400*q^8 + 181
 (6,
  1 - 504*q - 16632*q^2 - 122976*q^3 - 532728*q^4 - 1575504*q^5 - 4058208*q^6 - 8471232*q^7 - 170478
```

```
[sage: M=ModularFormsRing(Gamma0(11))
[sage: M.generators()
[(2,
  1 + 12*q^2 + 12*q^3 + 12*q^4 + 12*q^5 + 24*q^6 + 24*q^7 + 36*q^8 + 36*q^9 + O(q^10)),
 (2, q - 2*q^2 - q^3 + 2*q^4 + q^5 + 2*q^6 - 2*q^7 - 2*q^9 + O(q^10)),
 (4, 1 + O(q^10))]
```

Hecke Operators

Different descriptions: Double Cosets, Lattices (sublattices), Hecke correspondences etc. Explicitly for $M_k(\Gamma_0(N), \chi)$

$$T(n)f(z) = \frac{1}{n}\sum_{ad=n}\chi(a)a^k \sum_{0\leqslant b<d} f\left(\frac{az+b}{d}\right)$$

$T_n, (n, q) = 1$ are commuting normal operators.

Petersson inner product on $M_k(\Gamma_0(q), \chi)$

$$(f, g\rangle = \int_{\mathbb{H}/\Gamma} f(z)\overline{g(z)}y^k d\mu(z).$$

Poincare Series

$$E_k(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \bar{\chi}(\gamma)(cz+d)^{-k}$$

$$P_m(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \bar{\chi}(\gamma)(cz+d)^{-k} e(m\gamma z)$$

Poincare series span $S_k(\Gamma_0(N), \chi)$

Petersson Trace Formula

$$\sum_{f\in H_k(q,\chi)}^{h} \lambda_f(n)\overline{\lambda_f(m)}$$

$$= \delta(m,n) + 2\pi i^{-k} \sum_{\substack{c>0 \\ c\equiv 0 (\bmod q)}} c^{-1} S_\chi(m,n;c) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

Atkin-Lehner (Newforms/Oldforms)

Can the eigenvalues of T_n for an eigenform f determine f?
Multiplicity one: True only for the subspace of newforms.
(Orthogonal to oldforms coming from lower levels)

For $d \mid N/M$,

$$\alpha_d : S_k(\Gamma_1(M)) \to S_k(\Gamma_1(N)) : \quad f(\tau) \mapsto f(d\tau)$$

$$\bigoplus_{d|(N/M)} S_k(\Gamma_1(M)) \to S_k(\Gamma_1(N))$$

Elliptic Curves

$$E : y^2 = x^3 - x$$

The L-function matches with a Hecke L-function

$$L(E, s) = L(s, \chi) = L(s, f)$$

where

$$f(z) = \frac{1}{4} \sum_{\alpha \in \mathbf{Z}[i]} \rho(\alpha)\alpha e\left(z|\alpha|^2\right).$$

This is the CM case.

Computations

$M_k(\Gamma_1(N))$ are computable. That is we want compute arbitrary Fourier coefficients of a basis of forms given k and N and the required precision.

Computations

```
sage: M=ModularForms(Gamma0(2022), 8, prec=10)
sage: M
Modular Forms space of dimension 2370 for Congruence Subgroup Gamma
sage: M=ModularForms(Gamma0(11), 4, prec=10)
sage: M
Modular Forms space of dimension 4 for Congruence Subgroup Gamma0(1
sage: M.basis()
[
q + 3*q^3 - 6*q^4 - 7*q^5 - 8*q^6 + 14*q^7 + 4*q^8 + 14*q^9 + O(q^1
q^2 - 4*q^3 + 2*q^4 + 8*q^5 - 5*q^6 - 4*q^7 - 10*q^8 + 8*q^9 + O(q^
1 + O(q^10),
q + 9*q^2 + 28*q^3 + 73*q^4 + 126*q^5 + 252*q^6 + 344*q^7 + 585*q^8
]
```

Weight 2 Modular Symbols

The group $\mathcal{M}_2$ is the free abelian group on symbols $\{\alpha, \beta\}$ with

$$\alpha, \beta \in \mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$$

modulo the relations, (for all $\alpha, \beta, \gamma \in \mathbb{P}^1(\mathbb{Q}), g \in \Gamma$)

1. $\{\alpha, \beta\} = -\{\beta, \alpha\}$ (2-term relation)
2. $\{\alpha, \beta\} = \{\alpha, \gamma\} + \{\gamma, \beta\}$ (3-term relation)
3. $\{g\alpha, g\beta\} = \{\alpha, \beta\}$ for all $g \in \Gamma$ (Γ-action)
4. $\{\alpha, g\alpha\} \in H_1(X_\Gamma; \mathbb{Z})$
5. $\{\alpha, g\alpha\} = \{\beta, g\beta\}$

$\{\alpha, \beta\}$ is a class of geodesic from α to β.

Modular Symbols

These symbols generate relative homology $H_1(X_\Gamma, \partial X_\Gamma, \mathbb{Z})$

To get $H_1(X_\Gamma, \mathbb{Z})$ we need to consider $\mathcal{S}_2$, the part of $\mathcal{M}_2$ which lies in the kernel of the boundary map to the free group on cusps.

$$\{\alpha, \beta\} \in \mathcal{M}_2 \to \{\beta\} - \{\alpha\}$$

Modular Symbols and Modular Forms

Modular symbols are dual to Modular Forms. We have a pairing

$$S_2(\Gamma) \times H_1(X_\Gamma, \mathbb{R}) = S_2(\Gamma) \times (\mathcal{S}_2 \otimes \mathbb{R}) \to \mathbb{C}$$

$$\langle f, \{\alpha, \beta\}\rangle \to 2pii \int_\alpha^\beta f(z)dz$$

Hecke Operators

The pairing respects Hecke action. For $\Gamma_0(N)$, the operators are given by

$$T_p(\{\alpha,\beta\}) = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\{\alpha,\beta\} + \sum_{r \bmod p} \begin{pmatrix} 1 & r \\ 0 & p \end{pmatrix}\{\alpha,\beta\}$$

We have

$$\langle T_n f, \{\alpha,\beta\}\rangle = \langle f, T_n\{\alpha,\beta\}\rangle$$

Weight k Modular Symbols

$\mathcal{M}_2 := \mathbb{Q}$-vector space on symbols $\{\alpha, \beta\}$ modulo the 2 term, 3-terms relations. $\mathcal{M}_k := \mathbb{Q}[X, Y]_{k-2} \otimes_{\mathbb{Q}} \mathcal{M}_2$

$$(gP)(X, Y) := P\left(g^{-1}\begin{bmatrix} X \\ Y \end{bmatrix}\right)$$

$$g\{\alpha, \beta\} := \{g\alpha, g\beta\}$$

$$g(P\{\alpha, \beta\}) = gP\{g\alpha, g\beta\}$$

$$\mathcal{M}_k(\Gamma) := \mathcal{M}_k/(P\{\alpha, \beta\} - g(P\{\alpha, \beta\}))$$

Weight k cuspidal symbols

$$\mathcal{B}_2 := \mathbb{Q}\text{-vector space on symbols } \{\alpha\} \text{ for } \alpha \in \mathbb{P}^1(\mathbb{Q})$$
$$\mathcal{B}_k := \mathbb{Q}[X, Y]_{k-2} \otimes_{\mathbb{Q}} \mathcal{B}_2$$
$$\mathcal{B}_k(\Gamma) := \mathcal{B}_k/(x - gx)$$

$$\partial(P\{\alpha, \beta\}) = P\{\beta\} - P\{\alpha\}$$
$$\mathcal{S}_k(\Gamma) := \ker(\partial)$$

Modular Symbols and Modular Forms

We have the pairing

$$\left(S_k(\Gamma) \oplus \bar{S}_k(\Gamma)\right) \times \mathcal{M}_k(\Gamma) \to \mathbb{C}$$

$$\langle (f_1, f_2), P\{\alpha, \beta\} \rangle = \int_\alpha^\beta f_1(z) P(z, 1) dz + \int_\alpha^\beta f_2(z) P(\bar{z}, 1) d\bar{z}$$

Shokurov

The pairing

$$\langle \cdot, \cdot \rangle : S_k(\Gamma) \oplus \bar{S}_k(\Gamma) \times \mathcal{S}_k(\Gamma) \otimes_{\mathbb{Q}} \mathbb{C} \to \mathbb{C}$$

is a non-degenerate pairing of complex vector spaces.

Manin Symbols

Restrict to $\{\alpha, \beta\}$ which are unimodular. They generate all modular symbols. Let

$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \quad \text{and} \quad J = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Then $\mathcal{M}_k(\Gamma_0(N))$ is the $\mathbb{Q}$-vector space generated by

$$x = X^i Y^{k-2-i}(c : d) \in \mathbb{P}^1,$$

modulo the relations

$$\begin{aligned} x + xS &= 0 \\ x + xR + xR^2 &= 0 \\ x - xJ &= 0 \end{aligned}$$

Main Symbols

To write any $\{\alpha, \beta\}$ in terms of unimodular pairs, write $\{\alpha, \beta\} = \{\alpha, 0\} + \{0, \beta\}$, and then express $\{0, \alpha\}$ in terms of continued fraction convergents of α

$$\frac{4}{7} = 0 + \frac{1}{1 + \frac{1}{1 + \frac{1}{3}}}$$

The convergents are

$$\frac{b_{-2}}{a_{-2}} = \frac{0}{1}, \quad \frac{b_{-1}}{a_{-1}} = \frac{1}{0}, \quad \frac{b_0}{a_0} = \frac{0}{1}, \quad \frac{b_1}{a_1} = \frac{1}{1}, \quad \frac{b_2}{a_2} = \frac{1}{2}, \quad \frac{b_3}{a_3} = \frac{4}{7}$$

Therefore we have

$$\{0, 4/7\} = \{0, \infty\} + \{\infty, 0\} + \{0, 1\} + \{1, 1/2\} + \{1/2, 4/7\}$$

Example: Weight 2 symbols on $\Gamma_0(11)$

```
sage: set_modsym_print_mode ('modular')
sage: M = ModularSymbols(Gamma0(11), 2)
sage: M.basis()
({Infinity, 0}, {-1/8, 0}, {-1/9, 0})
sage: S=M.cuspidal_submodule()
sage: S.integral_basis()
({-1/8, 0}, {-1/9, 0})
```

Example: Weight 2 symbols on $\Gamma_0(11)$

```
sage: set_modsym_print_mode ('modular')
sage: M = ModularSymbols(Gamma0(11), 2)
sage: M.basis()
({Infinity, 0}, {-1/8, 0}, {-1/9, 0})
sage: S=M.cuspidal_submodule()
sage: S.integral_basis()
({-1/8, 0}, {-1/9, 0})
```

Example: Weight 6 forms on $\Gamma_0(7)$

```
sage: M = ModularSymbols(Gamma0(7), 6)
sage: M.dimension()
8
sage: M.basis()
(X^4*{0, Infinity},
 Y^4*{Infinity, 0},
 X^4*{0, 1},
 16*X^4*{-1/2, 0} + 32*X^3*Y*{-1/2, 0} + 24*X^2*Y^2*{-1/2, 0} + 8*X*Y^3*{-1/2, 0} + Y^4*{-1/2, 0},
 81*X^4*{-1/3, 0} + 108*X^3*Y*{-1/3, 0} + 54*X^2*Y^2*{-1/3, 0} + 12*X*Y^3*{-1/3, 0} + Y^4*{-1/3, 0},
 256*X^4*{-1/4, 0} + 256*X^3*Y*{-1/4, 0} + 96*X^2*Y^2*{-1/4, 0} + 16*X*Y^3*{-1/4, 0} + Y^4*{-1/4, 0},
 625*X^4*{-1/5, 0} + 500*X^3*Y*{-1/5, 0} + 150*X^2*Y^2*{-1/5, 0} + 20*X*Y^3*{-1/5, 0} + Y^4*{-1/5, 0},
 1296*X^4*{-1/6, 0} + 864*X^3*Y*{-1/6, 0} + 216*X^2*Y^2*{-1/6, 0} + 24*X*Y^3*{-1/6, 0} + Y^4*{-1/6, 0})
```

Manin Symbols

```
sage: M = ModularSymbols(Gamma0(11), 2)
sage: M
Modular Symbols space of dimension 3 for Gamma_0(11) of weight 2 with sign 0 over Rational Field
sage: M.manin_generators()
[(0,1),
 (1,0),
 (1,1),
 (1,2),
 (1,3),
 (1,4),
 (1,5),
 (1,6),
 (1,7),
 (1,8),
 (1,9),
 (1,10)]
sage: [M.manin_generators()[i] for i in M.manin_basis()]
[(1,0), (1,8), (1,9)]
sage: [x.modular_symbol_rep() for x in M.basis()]
[{Infinity, 0}, {-1/8, 0}, {-1/9, 0}]
```

Computing Modular Forms

The dimension of $\mathcal{S}_2$ is 2. Therefore there is one dimensional space of cusp forms.

```
[sage: M = ModularSymbols(Gamma0(11), 2)
[sage: S=M.cuspidal_submodule()
[sage: S.T(2).matrix()
 [-2  0]
 [ 0 -2]
[sage: S.T(3).matrix()
 [-1  0]
 [ 0 -1]
```

More examples

$M_2(SL_2(\mathbb{Z})$

More examples

$M_2(\Gamma 0(2))$

More examples

$M_4(SL_2(\mathbb{Z}))$

More examples

$M_4(\Gamma 0(2))$

References

- https://wstein.org/books/modform/modform/index.html
- ON MODULAR FORMS FOR SOME NONCONGRUENCE SUBGROUPS OF SL2(Z), CHRIS A. KURTH AND LING LONG
- https://www.williamstein.org/edu/Fall2003/252/references/magma/ModSym.pdf