Klein's Quartic

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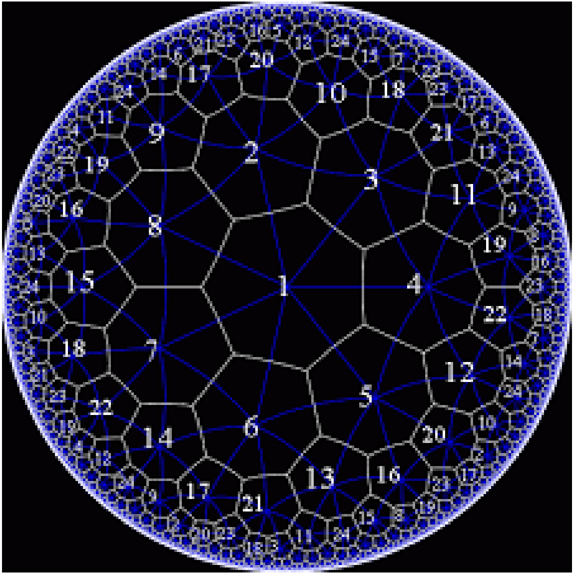
Klein Quartic

It's a very symmetric genus 3 projective curve. There can multiple ways to look at it. (Algebraic curve, Riemann surface, Hyperbolic surface, Tesselations, Modular curve, Shimura curve)

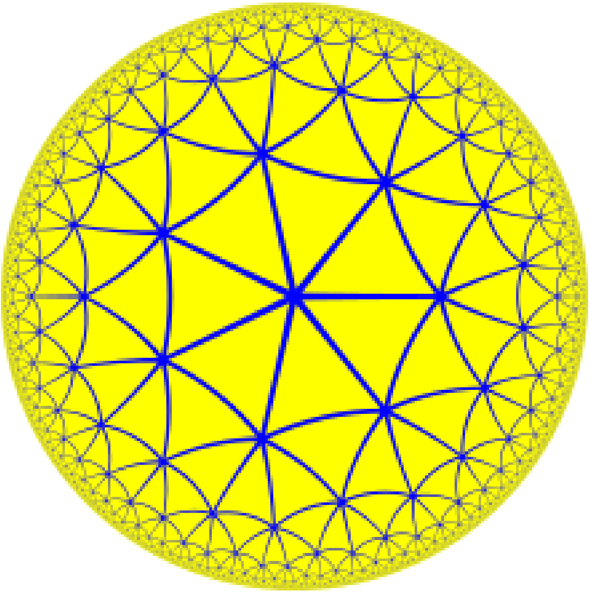
Introduction

- Made of 24 Heptagons, 3 Heptagons meeting at a vertex.
- Start from any edges, move along the edges -LRLRLRLR, you get back to the initial edge.
- All the 24 heptagon are equivalent (24 symmetries). 7 rotations of a fixed heptagon are also symmetries. Total $24 \times 7 = 168$ symmetries. (336 if you include reflections.)

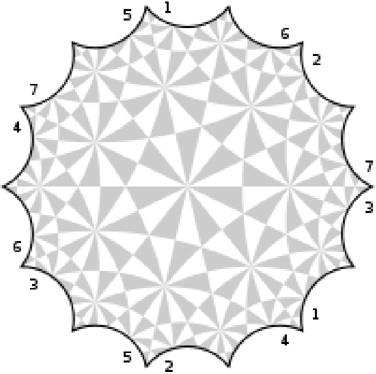
From Hyperbolic Tilings



From Hyperbolic Tilings

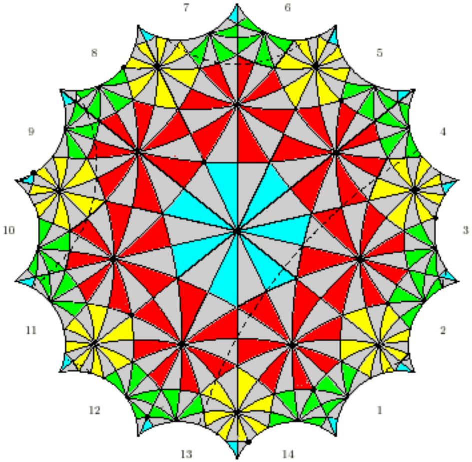


Quotient of (2, 3, 7) tiling



The fundamental domain is a 14-gon tiled by 336 $(2, 3, 7)$ triangles.
Area $=$

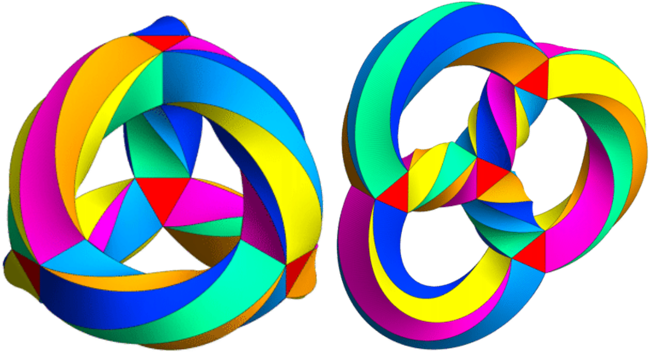
Folding/Identifications



connect edges 1 and 6
connect edges 3 and 8
connect edges 5 and 10
connect edges 7 and 12
connect edges 9 and 14
connect edges 11 and 2
connect edges 13 and 4

Topology

Euler's formula: $2 - 2g = V - E + F$



Symmetries

The group of symmetries is

$$G = \left\langle s, t; s^2 = t^7 = (st)^3 = \left(st^3\right)^4 = 1 \right\rangle$$

$$G = PSL_2(\mathbb{F}_7) \simeq GL_3(\mathbb{F}_2).$$

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \mathbb{Z}/7\mathbb{Z}, ab - cd = 1 \right\} / \{\pm \mathrm{Id}\}$$

$$1 \to \Gamma(7) \to \Gamma(1) \to G \to 1$$

Representation theory of the Group

c	$1A$	$2A$	$3A$	$4A$	$7A$	$7B$
$\#c$	1	21	56	42	24	24
χ_1	1	1	1	1	1	1
χ_3	3	-1	0	1	α	$\bar{\alpha}$
$\bar{\chi}_3$	3	-1	0	1	$\bar{\alpha}$	α
χ_6	6	2	0	0	-1	-1
χ_7	7	-1	1	-1	0	0
χ_8	8	0	-1	0	1	1

$$\alpha := \zeta + \zeta^2 + \zeta^4 = \frac{-1 + \sqrt{-7}}{2}$$

Klein Model

χ_3, is the character of a representation of G on 3-dimensional space V and $\bar{\chi}_3$ is the dual representation on V^*. If X, Y, Z are coordinates on V, the ring on invariant polynomials $\mathbb{C}[V^*]^G$ is generated by $\Phi_4, \Phi_6, \Phi_{14}, \Phi_{21}$. $\Phi_4, \Phi_6, \Phi_{14}$ are algebraically independent and Φ_{21}^2 is a polynomial in $\Phi_4, \Phi_6, \Phi_{14}$

$$\Phi_4 := X^3Y + Y^3Z + Z^3X$$

Klein Model

$$\Phi_6 := -\frac{1}{54}\begin{vmatrix} \partial^2\Phi_4/\partial X^2 & \partial^2\Phi_4/\partial X\partial Y & \partial^2\Phi_4/\partial X\partial Z \\ \partial^2\Phi_4/\partial Y\partial X & \partial^2\Phi_4/\partial Y^2 & \partial^2\Phi_4/\partial Y\partial Z \\ \partial^2\Phi_4/\partial Z\partial X & \partial^2\Phi_4/\partial Z\partial Y & \partial^2\Phi_4/\partial Z^2 \end{vmatrix}$$

$$\Phi_{14} = \frac{1}{9}\begin{vmatrix} \partial^2\Phi_4/\partial X^2 & \partial^2\Phi_4/\partial X\partial Y & \partial^2\Phi_4/\partial X\partial Z & \partial\Phi_6/\partial X \\ \partial^2\Phi_4/\partial Y\partial X & \partial^2\Phi_4/\partial Y^2 & \partial^2\Phi_4/\partial Y\partial Z & \partial\Phi_6/\partial Y \\ \partial^2\Phi_4/\partial Z\partial X & \partial^2\Phi_4/\partial Z\partial Y & \partial^2\Phi_4/\partial Z^2 & \partial\Phi_6/\partial Z \\ \partial\Phi_6/\partial X & \partial\Phi_6/\partial Y & \partial\Phi_6/\partial Z & 0 \end{vmatrix}$$

$$\Phi_{21} = \frac{\partial\left(\Phi_4, \Phi_6, \Phi_{14}\right)}{14\partial(X, Y, Z)} = \frac{1}{14}\begin{vmatrix} \partial\Phi_4/\partial X & \partial\Phi_4/\partial Y & \partial\Phi_4/\partial Z \\ \partial\Phi_6/\partial X & \partial\Phi_6/\partial Y & \partial\Phi_6/\partial Z \\ \partial\Phi_{14}/\partial X & \partial\Phi_{14}/\partial Y & \partial\Phi_{14}/\partial Z \end{vmatrix}$$

Reduction mod 2 and 7

Take a lattice in V and reduce mod 2 and mod 7.
In particular, primes $\wp_2, \bar{\wp}_2$ above 2 in O_k, $k = \mathbb{Q}(\sqrt{-7})$, reduce modulo $\wp_2$ to get

$$G \simeq \mathrm{GL}_3\left(\mathbb{F}_2\right)$$

and reduce modulo the unique prime $\wp_7 = (\sqrt{-7})$of O_k above 7 to get

$$G \to \mathrm{GL}_2\left(\mathbb{F}_7\right)$$

The isomorphism $PSL_2(\mathbb{F}_7) \simeq GL_3(\mathbb{F}_2)$ is just mod 2 and mod 7 manifestions of isometries of the lattice.

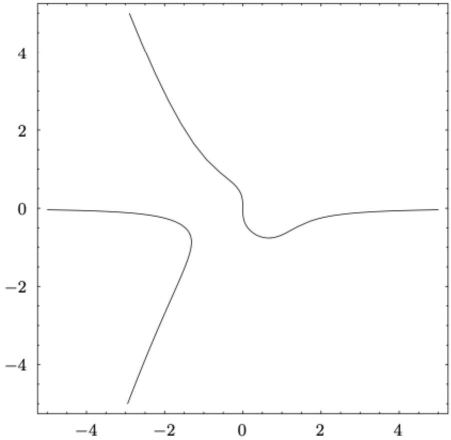
As Algebraic Curve

The projective algebraic curve $\mathcal{X}$ defined by

$$X^3 Y + Y^3 Z + Z^3 X = 0$$

- It is a smooth genus 3 curve, and has 168 symmetries
- 168 matches the maximum symmetries a genus 3 curve can have. (Hurwitz bound $84(g-1)$)
- Not hyperelliptic

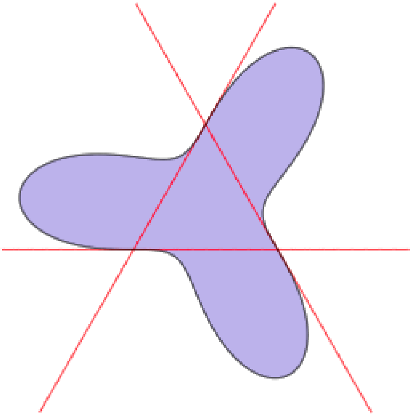
$X^3Y + Y^3Z + Z^3X = 0$



$x^3y + y^3 + x = 0$

The real points on $Z = 1$.

$X^3Y + Y^3Z + Z^3X = 0$



The real points on $X + Y + Z = 1$.

Why is it smooth?

As Algebraic Curve

Why is the genus 3?

Genus-Degree formula?

Riemann-Hurwitz formula: $C \to \mathbb{P}^1, [X, Y, Z] \to [X, Z]$

Affine part of the curve

If we just look at the Affine part of the curve, we get 24 cusps (punctures on the surface) corresponding to the points at infinity. We can think of them as the center of the 24 heptagons.

Symmetries in terms of the equation

Symmetries in terms of the equation

$$A = \begin{bmatrix} \omega^4 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{bmatrix}, \ B = \frac{-2}{\sqrt{7}} \begin{bmatrix} \sin 2\alpha & \sin 3\alpha & -\sin\alpha \\ \sin 3\alpha & -\sin\alpha & \sin 2\alpha \\ -\sin\alpha & \sin 2\alpha & \sin 3\alpha \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\omega^7 = 1, \alpha = \pi/7$$

1. A has order 7, B has order 2, and C has order 3
2. Quotient of $(2, 3, 7)$ triangle group. (More relations)

Special Points

- An orbit of 24 points, each of which is fixed by a subgroup of order 7. These are inflection points (zeros of Hessian Φ_6.)
- 56 points consisting of 28 pairs of points on the bitangents on the curve. They form one orbit and each of the points is fixed by a subgroup of order 3. Zeros of Φ_{14}
- 21 lines each fixed by an order 2 subgroup. Each line consists of 4 points on the curve. So 84 points in total. Zeros of Φ_{21}
- Rest of the orbits have size exactly $|G| = 168$

Quotients $\mathcal{X}/H$

For any subgroup $H \subset G$, we can consider the quotient $\mathcal{X}/H$.

- Quotients by subgroups of order $2, 3, 4$ give genus 1 curves; Any other subgroups give genus 0 curves, hence coverings $\mathcal{X} \to \mathbb{CP}^1$
- For $\langle h \rangle$ order 3 subgroup, we get

$$\mathcal{X}/\langle h \rangle = E_k : y^2 = 4x^3 + 21x^2 + 28x$$

  $\left(1 : e^{\pm 2\pi i/3} : e^{\mp 2\pi i/3}\right)$, branch points (fixed points of $\langle h \rangle$)
  $j = -15^3$, Complex multiplication by O_k.
- $\mathcal{X} \to \mathcal{X}/G \cong \mathbb{CP}^1$

$$(X, Y, Z) \to j = \frac{\Phi_{14}^3}{\Phi_6^7} = \frac{\Phi_{21}^2}{\Phi_6^7} + 1728$$

Mordel-Weil Lattice

Space of maps $M = \{f : \mathcal{X} \to E_k\}$. $L = M/E_k$ is a lattice with positive definite quadratic form $Q(f) = 2\deg(f)$.

$$\theta_L := \sum_{n=0}^{\infty} N_n q^n = \sum_{f \in L} q^{\frac{1}{2}\hat{Q}(f)}$$

θ_L is a modular form of weight 3 with quadratic character on $\Gamma_0(7)$.

$$\theta_L = \left(\sum_{\beta \in O_k} q^{\beta\bar{\beta}} \right)^3 - 6q \prod_{n=1}^{\infty} (1 - q^n)^3 \left(1 - q^{7n}\right)^3$$
$$= 1 + 42q^2 + 56q^3 + 84q^4 + 168q^5 + 280q^6 + 336q^7 + 462q^8 + \cdots$$

Rational Points on $\mathcal{X}$

By Falting's theorem, there are at most finite number of rational points.
But here we can use $\mathcal{X} \to E_k$, to show explicitly that $(1:0:0), (0:1:0), (0:0:1)$ are the only rational points.

- A descent argument via a 2-isogeny to show that the only rational points on E_k are $(0,0)$ and the point at infinity.
- Compute preimages of these points.

Descent Argument

FLT $n = 7$

There is a map from the Fermat curve $\mathcal{F} : A^7 + B^7 + C^7 = 0$ to the Klein quartic given by

$$(A, B, C) \to (X, Y, Z) = (A^3C, B^3A, C^3B)$$

$\mathcal{X}$ mod 2

Compute the number of points over the finite field $\mathbb{F}_{2^m} = N_{2^m}$

m	1	2	3	4	5	6	7	8	...
$\#(\mathcal{X}(\mathbb{F}_{2^m}))$	3	5	24	17	33	38	129	257	...

which implies

$$Z(\mathcal{X}/\mathbb{F}_2, T) = \exp\left(\sum_{n=1}^{\infty} \#\mathcal{X}(\mathbb{F}_{q^n}) \frac{T^n}{n}\right) = \frac{1 + 5T^3 + 8T^6}{(1-T)(1-2T)}$$

Extremal properties. For instance, highest number of $\mathbb{F}_{2^3}$ points among all genus 3 curves. (Coming from 24 inflection points)

$\mathcal{X} \mod 7$

The curve is singular. For instance $(2, 4, 1)$ is a singular point.

$\mathcal{X}$ mod p

Different models like

$$X'^4 + Y'^4 + Z'^4 + 3\alpha \left(X'^2 Y'^2 + X'^2 Z'^2 + Y'^2 Z'^2 \right) = 0$$

have good reduction in char 7 over large enough extension of $\mathbb{Q}$

As Modular Curve $X(7)$

$$G \cong \mathrm{PSL}_2(\mathbb{F}_7) = \Gamma(1)/\Gamma(7)$$

$\Gamma(7)$ acts on the upper half-plane $\mathbb{H}$, and we have $Y(7) = \mathbb{H}/\Gamma(7)$ and $X(7) = \mathbb{H}^*/\Gamma$. $Y(7)$ has 24 cusps. $X(7)$ is the compactification.

- $X(1)$ parametrizes elliptic curves
- $X(N)$ parametrizes elliptic curves with a level N structure: (Isomorphism of N-torsion elements with $(\mathbb{Z}/N) \times \boldsymbol{\mu}_N$)

$X(7) \to X(1)$

The inverse images of $i, e^{2\pi i/3}, \infty$ are precisely the orbits of 84, 56 and 24 special points.

Modular Forms

Consider weight 2 modular forms on $\Gamma(7)$.

$$f\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^2 f(\tau)$$

$f(\tau)d\tau$ is $\Gamma(7)$ invariant.

Modular Forms

$$\mathbf{x}(\tau) = -\sum_{\xi \equiv 1 (\bmod \mathfrak{f})} \mathrm{Re}(\xi) q^{\xi\bar{\xi}/7}$$

$$\mathbf{y}(\tau) = \sum_{\xi \equiv 2 (\bmod f)} \mathrm{Re}(\xi) q^{\xi\bar{\xi}/7}$$

$$\mathbf{z}(\tau) = \sum_{\xi \equiv 4 (\bmod \mathfrak{f})} \mathrm{Re}(\xi) q^{\xi\bar{\xi}/7}$$

$f = (\sqrt{-7})$ be the unique prime ideal above 7 in $\mathbb{Z}[\frac{-1+\sqrt{7}}{2}]$

$$\mathbf{x} = q^{4/7}\left(-1 + 4q - 3q^2 - 5q^3 + 5q^4 + 8q^6 - 10q^7 + 4q^9 - 6q^{10} \cdots\right)$$
$$\mathbf{y} = q^{2/7}\left(1 - 3q - q^2 + 8q^3 - 6q^5 - 4q^6 + 2q^8 + 9q^{10} \cdots\right)$$
$$\mathbf{z} = q^{1/7}\left(1 - 3q + 4q^3 + 2q^4 + 3q^5 - 12q^6 - 5q^7 + 7q^9 + 16q^{10} \cdots\right)$$

Uniformization, Ramanujan q-series

$$a(\tau) = -\sum_{n\in\mathbb{Z}} (-1)^n q^{(14n+5)^2/56}$$

$$b(\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{(14n+3)^2/56}$$

$$c(\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{(14n+1)^2/56}$$

a, b, c satisfy the equation $X^3Y + Y^3Z + Z^3X = 0$

Embedding $X(7)$

The map $X(7) \to \mathbb{P}^2$ is given by $\tau \to [\mathbf{x} : \mathbf{y} : \mathbf{z}] = [a : b : c]$

$\Phi_6, \Phi_{14}, \Phi_{21}$

$$\Phi_6(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = q - 24q^2 + 252q^3 \cdots$$

$$\Phi_{14}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \Delta^2 E_2 = \Delta^2 \left(1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n} \right)$$
$$= q^2 + 192q^3 - 8280q^4 \cdots$$

$$\Phi_{21}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \Delta^3 E_3 = \Delta^3 \left(1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n} \right)$$
$$= q^3 - 576q^4 + 22140q^5 \cdots$$

Belyi map

The map $C \to C/G$ is a Belyi map– ramified at $0, 1728$ and ∞. Dessin d'enfant is the 1-skeleton of the heptagonal tiling above.

As a Shimura Curve

What is Shimura Curve? A quaternion algebra B over $\mathbb{Q}$ such that $B \otimes_{\mathbb{Q}} \mathbb{R} = M_2(\mathbb{R})$. Then the element of norm ± 1 in a maximal order $\mathcal{O}$ of B form a discrete subgroup of $SL_2(\mathbb{R})$ with cocompact quotients.

- $\mathcal{O}[\tau, 1]^T$ is a lattice in $\mathbb{C}^2$, and the quotient $\mathbb{C}^2/\Lambda$ is an abelian variety with endomorphism ring $\mathcal{O}$.
- The units of norm 1 just act by changing basis. And hence $\mathcal{H}/\Gamma$ parametrizes abelian surfaces with quaternionic multiplication.

We can generalize to algebras over number fields K such that for some real embedding v,

$$B \otimes_K K_v \simeq M_2(\mathbb{R})$$

As a Shimura Curve

$$c = \zeta + \zeta^{-1} = 2\cos(2\pi/7)$$

$$\mathbf{i}^2 = \mathbf{j}^2 = c, \quad \mathbf{ij} = -\mathbf{ji}, \mathbf{j}' = \frac{1}{2}\left(1 + c\mathbf{i} + \left(c^2 + c + 1\right) j\right)$$

The norm one elements in $\mathbb{Z}[c]\mathbf{1} + \mathbb{Z}[c]\mathbf{i} + \mathbb{Z}[c]\mathbf{j}' + \mathbb{Z}[c]\mathbf{ij}'$ is the triangle group $\Delta(2,3,7)$. The Fuchsian group corresponding to $\mathcal{X}$ is the subgroup

$$\{a_1\mathbf{1} + a_2\mathbf{i} + a_3\mathbf{j}' + a_4\mathbf{ij}' \mid \ a_i \in \mathbb{Z}[c], a_2, a_3, a_4 = 0 \bmod \ (2-c)\}$$

Spectral Theory

The Laplacian-Betrami operator (div grad)

$$\Delta f = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} \mid g^{ij} \partial_j f \right)$$

is self-adjoint with spectrum given below. (Dual to length spectrum on the surface)

Eigenvalue	Numerical value	Multiplicity
λ_0	0	1
λ_1	2.67793	8
λ_2	6.62251	7
λ_3	10.8691	6
λ_4	12.1844	8
λ_5	17.2486	7
⋮	⋮	⋮

Problems

Understand

1. algebraic geometry of the curve
2. The arithmetic aspects:
3. The analytic aspects:
4. as hyperbolic surface
5. as a Riemann surface
6. the symmetries and representation theory

Visualizing the surface

Greg Egan

- http://www.gregegan.net/SCIENCE/KleinQuartic/KleinQuartic.html
- http://www.gregegan.net/SCIENCE/KleinQuartic/KleinQuarticEq.html

Jos Leys

- https://www.youtube.com/watch?v=YhVsqnXhVWc
- https://www.youtube.com/watch?v=6SZ8ONJlw7I

Tim Hutton

- https://www.youtube.com/watch?v=noLJ_ktAxQE

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- https://www.gregegan.net/SCIENCE/KleinQuartic/KleinQuarticEq.html