BISECTION METHOD

Input:

1) A continuous function and two end points $a$ and $b$ such that $f(a)$ and $f(b)$ have opposite signs i.e., $f(a)f(b) < 0$.
2) The number of iterations or the accuracy needed.

Output: An approximation to the root of $f(x)$ inside $[a, b]$.

Set $i = 0$, $a_i = a$ and $b_i = b$.
Set $c_i = \frac{a_i + b_i}{2}$
If $f(c_i) = 0$, then $c_i$ is a zero
Else check the signs of $f(a_i)f(c_i)$ and $f(c_i)f(b_i)$
If $f(a_i)f(c_i) < 0$, then $a_{i+1} = a_i, b_{i+1} = c_i$
If $f(c_i)f(b_i) < 0$, then $a_{i+1} = c_i, b_{i+1} = b_i$
and repeat with interval $[a_{i+1}, b_{i+1}]$

Error: The error after $n$ iterations $|x - x_n|$ is bounded by the length of the interval $|a_n - b_n|$.
We have $|a_n - b_n| = \frac{|b - a|}{2^n}$. So, the number of iterations to get an accuracy of $\varepsilon$ is $n$ such that $\frac{|b - a|}{2^n} < \varepsilon$, that is $n > \log_2 \frac{|b - a|}{\varepsilon}$

Convergence: ALWAYS converges and the rate of convergence is linear.

$|x - x_{n+1}| \sim \frac{1}{2}|x - x_n|$.
FALSE POSITION METHOD

Input:

1) A continuous function and two end points \(a\) and \(b\) such that \(f(a)\) and \(f(b)\) have opposite signs i.e., \(f(a)f(b) < 0\)
2) The number of iterations or the accuracy needed

Output: An approximation to the root of \(f(x)\) inside \([a, b]\)

Set \(i = 0, a_i = a\) and \(b_i = b\).
Set \(c_i = \frac{a_i f(b_i) - b_i f(a_i)}{f(b_i) - f(a_i)}\)
If \(f(c_i) = 0\), then \(c_i\) is a zero
Else check the signs of \(f(a_i)f(c_i)\) and \(f(c_i)f(b_i)\)
If \(f(a_i)f(c_i) < 0\), then \(a_{i+1} = a_i, b_{i+1} = c_i\)
If \(f(c_i)f(b_i) < 0\), then \(a_{i+1} = c_i, b_{i+1} = b_i\)
and repeat with interval \([a_{i+1}, b_{i+1}]\)

Convergence: ALWAYS converges and the rate of convergence is super-linear (if the root is not a multiple root, if the function is smooth–slower convergence for multiple roots).

\[|x - x_{n+1}| \sim C|x - x_n|\sqrt[2]{\frac{\sqrt{5} + 1}{2}}, \quad C = \frac{|f''(x)|}{2 f'(x)}\]
NEWTON’S METHOD

Input:

1) A differentiable function and a point $x_0$ ”close” enough to the root.
2) The number of iterations or the accuracy needed

Output: An approximation to the root of $f(x)$ near $x_0$

Convergence: Doesn’t always converge. Converges if $x_0$ is close enough to the root. and the rate of convergence is quadratic (if the root is not a multiple root-slower convergence for multiple roots) if the function is smooth.

$$|x - x_{n+1}| \sim C|x - x_n|^2, \quad C = \left| \frac{f''(x)}{2f'(x)} \right|$$

Algorithm:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
SECANT METHOD

Input:

1) A continuous function and points $x_0, x_1$ “close” enough to the root.
2) The number of iterations or the accuracy needed

Output: An approximation to the root of $f(x)$ near $x_0, x_1$

Convergence: Doesn’t always converge. Converges if $x_0, x_1$ are close enough to the root. and the rate of convergence is superlinear (if the root is not a multiple root-slower convergence for multiple roots) if the function is smooth.

\[
|x - x_{n+1}| \sim C |x - x_n|^{\frac{\sqrt{5+1}}{2}}, \quad C = \left| \frac{f''(x)}{2 f'(x)} \right|^{\frac{\sqrt{5}-1}{2}}
\]

Algorithm:

\[
x_{n+1} = x_n - \frac{f(x_n)}{f(x_n) - f(x_{n-1})}
\]
STEFFENSEN’S METHOD

**Input:**
1) A differentiable function and points $x_0$ ”close” enough to the root.
2) The number of iterations or the accuracy needed

**Output:** An approximation to the root of $f(x)$ near $x_0, x_1$

**Convergence:** Doesn’t always converge. Converges if $x_0$ is close enough to the root. and the rate of convergence is quadratic (if the root is not a multiple root-slower convergence for multiple roots).

**Algorithm:**

\[
h = f(x_n)
\]

\[
g(x_n) = \frac{f(x_n + h) - f(x_n)}{h}
\]

\[
x_{n+1} = x_n - \frac{f(x_n)}{g(x_n)}
\]
FIXED POINT METHOD

Input: A differentiable function $g$ and a point $x_0$

Output: An approximation to the fixed point of $g(x)$

Convergence: Doesn’t always converge.
Converges if
a) $g$ maps an interval $[a, b]$ to itself.
b) $g$ is ”contracting” that is $|g'(x)| < k, k < 1$

The convergence is linear.

$$|x - x_{n+1}| < k|x - x_n|,$$

Algorithm:

$$x_{n+1} = g(x_n)$$
AITKEN’S $\Delta^2$ METHOD

**Input:** A linearly convergent sequence $p_n$ with $\lim_{n \to \infty} \frac{p_{n+1} - p_n}{p - p_{n+1}} < 1$

**Output:** A sequence $\hat{p}_n$ which converges faster than $p_n$ to $p$

**Definition** The forward difference

\[
\Delta p_n = p_{n+1} - p_n
\]

\[
\Delta^2 p_n = \Delta(\Delta p_n) = p_{n+2} - 2p_{n+1} + p_n
\]

**Algorithm:**

\[
\hat{p}_n = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}
\]

\[
\hat{\hat{p}}_n = p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n}
\]