Midterm 2 Solution Guide

1. (10 points) Using implicit differentiation, compute $\frac{dy}{dx}$ for the implicitly defined function

$$\sqrt{4x-y}+9=x^3+y^2$$

Then compute the equation of the tangent line at $(1,3)$.

Solution. First, we take the derivative of both sides, using y' for $\frac{dy}{dx}$:

$$\begin{aligned}\frac{1}{2}(4x-y)^{-1/2}(4-y') &= 3x^2+2y\cdot y'\\ \frac{2}{\sqrt{4x-y}}-\frac{y'}{2\sqrt{4x-y}} &= 3x^2+2y\cdot y'\end{aligned}$$

We need to move all the y' to one side and the other terms to the other:

$$\frac{2}{\sqrt{4x-y}}-3x^2=2y\cdot y'+\frac{y'}{2\sqrt{4x-y}}=y'\left(2y+\frac{1}{2\sqrt{4x-y}}\right)$$

The last thing to do is divide over. Some folks managed to rationalise the denominators before this step, leading to the second possibility:

$$y'=\frac{\frac{2}{\sqrt{4x-y}}-3x^2}{2y+\frac{1}{2\sqrt{4x-y}}}=\frac{4-6x^2\sqrt{4x-y}}{4y\sqrt{4x-y}+1}$$

We now need to find the tangent line at $(1,3)$. Plugging in, we get $y'(1,3)=-\frac{2}{13}$. Then the equation of the tangent line should be familiar to us:

$$y-3=-\frac{2}{13}(x-1).$$

2. (10 points) Find the derivative of the following functions.

   (a) $f(x) = \arctan(\ln(x))$

   (b) $g(x) = \log_3(\cos(x))$

Solution.

(a) It's important to remember the derivative of arctan, as I emphasised in class:

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$$

So, combining with the chain rule, we get

$$\frac{d}{dx}\arctan(\ln(x)) = \frac{1}{1+(\ln(x))^2}\cdot\frac{1}{x}$$

(b) We can recall that

$$\log_3(x) = \frac{\ln(x)}{\ln(3)}$$

so that we need to find the derivative of $\frac{\ln(\cos(x))}{\ln 3}$. With the chain rule,

$$\frac{d}{dx}\frac{\ln(\cos(x))}{\ln 3} = \frac{1}{\cos(x)\cdot\ln(3)}\cdot -\sin(x) = -\frac{\tan(x)}{\ln(3)}$$

3. (10 points) Consider the function $f(x) = \dfrac{x}{x^2 - x + 1}$.

   (a) Why does $f(x)$ have an absolute maximum and absolute minimum on the interval $[0, 3]$?

   (b) Find these values.

Solution.

(a) There were many bad answers for this problem, most of them overly vague and not getting to the point. Just going through the process of finding critical points and all that is what we do once we know that the *extreme value theorem* applies: $f(x)$ is a continuous function on a closed interval, so it achieves an absolute maximum and a minimum. It's continuous because the demoninator has no zeroes on $[0, 3]$ (and in fact, no zeroes at all). The absolute extrema occur (by Fermat's theorem) at the endpoints or at any critical points on the interior. Most people jumped straight to this part, which should be contained in (b).

(b) We need the derivative, which more than a few people took incorrectly:

$$\frac{d}{dx}\frac{x}{x^2 - x + 1} = \frac{(x^2 - x + 1)(1) - (x)(2x - 1)}{(x^2 - x + 1)^2} = \frac{1 - x^2}{(x^2 - x + 1)^2}$$
$$= \frac{(1 - x)(1 + x)}{(x^2 - x + 1)^2}$$

which gives the critical points $x = -1, 1$. The denominator is never zero, as we pointed out above. We can throw away $x = -1$ because it's not in $[0, 3]$ (which not everyone did). We only need to check $f(x)$ at $x = 1$ and the two endpoints:

$$f(0) = 0, \quad f(1) = 1, \quad f(3) = \frac{3}{7}$$

Since these are the only options, $(1, 1)$ is the maximum and $(0, 0)$ is the minimum.

4. (10 points) Let $g(x) = x^4(x-1)^4$.
   (a) Find all the critical points of $g(x)$.
   (b) What does the Second Derivative Test tell us about these critical points?
   (c) What does the First Derivative Test tell us about these critical points?

Solution.

(a) We can either use the product rule, or we can take a shortcut and use the chain rule:

$$g(x) = x^4(x-1)^4 = (x^2-x)^4$$
$$\implies g'(x) = 4(x^2-x)^3(2x-1) = 4x^3(x-1)^3(2x-1)$$

which makes the *three* critical points $x = 0, 1/2, 1$. Many people didn't find $1/2$ because they didn't factor $g'(x)$ properly.

(b) One should address the actual critical points found in part (a) for this problem: a surprising number of people did not. Nonetheless, partial credit was given for correctly identifying what the Second Derivative Test does: it classifies critical points as maxima (concave down), minima (concave up), or inconclusive ($g''(x) = 0$).

For us, we can compute (though it takes some time with the product and chain rules)

$$g''(x) = x^2(x-1)^2\left(8(x^2-x) + 12(2x-1)^2\right)$$

Put this way, it's pretty clear that $g''(0) = g''(1) = 0$, telling us nothing about the critical points. However, $g''(1/2)$ is comprised of three terms: the first two are always positive, and the last term is $(8(-1/4)+0) < 0$, so the whole $g''(1/2)$ is negative. This corresponds to $x = 1/2$ being a local *maximum*.

(c) We need to check the intervals of increasing and decreasing to see how $g'(x)$ changes (or doesn't) as it passes through the critical points. We need test points on the intervals $(-\infty, 0), (0, 1/2), (1/2, 1), (1, \infty)$. We can compute:

$$g'(-1) < 0, \quad g'(1/4) > 0, \quad g'(3/4) < 0, \quad g'(2) > 0.$$

It's easier to get the signs than the actual numbers. This means that $x = 0, 1$ are local minima and $x = 1/2$ is (still) a local maximum.

5. (10 points) Let x and y be positive numbers.
   (a) Find x, y such that $2x+3y = 10$ and the quantity xy^2 is as large as possible.
   (b) Can you find x, y as above so that xy^2 is minimal? Why or why not?

Solution.

(a) We need to set up a one-variable problem, and we can use the substitution

$$2x + 3y = 10 \implies x = \frac{10 - 3y}{2} = 5 - \frac{3}{2}y$$

So we have to deal with the function

$$f(y) = \left(5 - \frac{3}{2}y\right) \cdot y^2 = 5y^2 - \frac{3}{2}y^3$$

Taking the derivative,

$$f'(y) = 10y - \frac{9}{2}y^2 = y\left(10 - \frac{9}{2}y\right)$$

This gives us two critical points: $y = 0$ and $y = \frac{20}{9}$. The point $y = 0$ is not positive, so we throw it out. We can them compute that the corresponding x value is $x = \frac{5}{3}$.

(b) The interval we are working on is $y \in (0, 10/3)$, because $x > 0$ requires that $y < 10/3$ from the above equation. This interval is not closed, so the extreme value theorem doesn't apply and we aren't required to have a minimum or a maximum. We can see that xy^2 gets really small as you make y or x small – but it can't actually achieve a minimum. Another explanation is that the function only has two critical points, and $y = 0$ isn't in our domain $(0, 10/3)$.

6. (10 points) Let $f(x) = \sqrt{x}$.

   (a) What does $f(x)$ need to satisfy on the interval $[1, 4]$ in order for the Mean Value Theorem to apply?

   (b) Find a number c such that $c \in (1, 4)$ satisfies the Mean Value Theorem for $f(x)$ on $[1, 4]$.

Solution.

(a) The Mean Value Theorem requires that $f(x)$ needs to be continuous on $[1, 4]$ and differentiable on $(1, 4)$. Luckily, it is.

(b) The Mean Value Theorem says that there is a $c \in (1, 4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1} = \frac{2 - 1}{3} = \frac{1}{3}$$

We can also compute that $f'(x) = \dfrac{1}{2\sqrt{x}}$, so we need to solve

$$\frac{1}{2\sqrt{c}} = \frac{1}{3} \implies \sqrt{c} = \frac{3}{2} \implies c = \frac{9}{4}$$

and this is in the interval $(1, 4)$, as promised.

7. (10 points) Find the limit $\lim_{x\to 0}(1-2x)^{3/x}$. If you use any rules or theorems for computing limits, be sure to indicate where and why these techniques are used.

Solution. What if we just try plugging in? We get 1^∞, which is definitely indeterminate. We can't do anything with this directly, but by the trick from class we can take the natural log of the problem and solve it instead:

$$\lim_{x\to 0}(1-2x)^{3/x} = L \implies \lim_{x\to 0}\ln\left((1-2x)^{3/x}\right) = \lim_{x\to 0}\frac{3\ln(1-2x)}{x} = \ln L.$$

Can we do this limit now? Plugging in, we get $\frac{0}{0}$, which means we can apply L'Hôpital's rule. It is important to notice this explicitly!

$$\ln L = \lim_{x\to 0}\frac{\frac{-6}{1-2x}}{1} = \lim_{x\to 0}\frac{-6}{1-2x} = -6$$

with the last step by direct computation. We now compute that $L = e^{-6}$.

8. (10 points) Use an appropriate linearization to estimate $\frac{1}{1+(1.01)^2}$. Be explicit about the function $f(x)$ and the point $x = a$ that you choose.

Solution. The suspicious decimal 1.01 is pretty close to $a = 1$ in the equation $f(x) = \frac{1}{1+x^2}$. There are other options, but this is by far the easiest choice. We need to now compute $f(1)$ and $f'(1)$. We have $f(1) = 1/2$ and

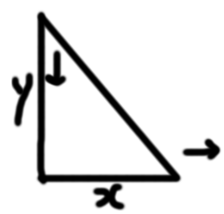
$$f'(x) = \frac{-2x}{(1+x^2)^2} \implies f'(1) = \frac{-2}{2^2} = -\frac{1}{2}$$

This means that

$$f(1.01) = f(1) + (1.01 - 1)f'(1) = \frac{1}{2} + \frac{-1}{100} \cdot \frac{1}{2} = \frac{99}{200}$$

9. (10 points) A ladder with length 10 feet is leaning against a vertical wall. The bottom of the ladder is slipping away from the wall at a rate of 5 feet per second. How fast is the top of the ladder sliding down the wall when the bottom of the ladder is 8 feet away from the wall? As part of your solution, draw a picture with explicit variable names for the various quantities involved.

   Solution. Let x denote the horizontal distance from the ladder to the wall and y the vertical distance from the ladder to the floor:



   These variables are related by the equation

$$x^2 + y^2 = 10^2 = 100.$$

   Taking derivatives, we have that

$$2x \cdot x' + 2y \cdot y' = 0$$

   so all that remains is to plug in numbers. At $x = 8$, we can solve that $y = 6$. Therefore:

$$2 \cdot 8 \cdot 5 + 2 \cdot 6 \cdot y' = 0 \implies y' = -\frac{80}{12} = -\frac{20}{3} \text{ ft/s}$$

10. (10 points) Consider $g(x) = \dfrac{3}{x^2 - 9}$. Find the intervals where $g(x)$ is increasing and decreasing. Find the intervals where $g(x)$ is concave up and concave down. Find the critical points, inflection points, and the vertical and horizontal asymptotes of $g(x)$. If you think a sketch would help, feel free to draw one, but a sketch alone is not a solution.

Solution. First, we can address asymptotes. Because this is a rational function whose denominator has a higher degree than the numerator, we have that $\lim_{x\to\pm\infty} g(x) = 0$, so $y = 0$ is the horizontal aymptote. For vertical asymptotes, we have that the denominator equals zero at $x = \pm 3$.

Now, $g'(x) = \dfrac{-6x}{(x^2-9)^2}$. We have a critical point at $x = 0$, and $g'(x)$ is not defined at $x = \pm 3$, where the vertical asymptotes are. We need to check $(-\infty, -3), (-3, 0), (0, 3), (3, \infty)$ for the sign of $g'(x)$. We can compute that

$$g'(-5) = \frac{30}{(25-9)^2} > 0, \quad g'(-1) = \frac{6}{(1-9)^2} > 0,$$
$$g'(1) = \frac{-6}{(1-9)^2} < 0, \quad g'(5) = \frac{-30}{(25-9)^2} < 0$$

so that gives us the intervals of increasing and decreasing. We also see that $x = 0$ is a local maximum.

Finally, we compute that

$$g''(x) = \frac{(x^2-9)^2(-6) - (-6x)(2(x^2-9)(2x))}{(x^2-9)^4}$$

which is a bit difficult to get a handle on. Luckily, we can cancel out an $x^2 - 9$ from the top and bottom terms right away, and simplify from there:

$$g''(x) = \frac{(x^2-9)(-6) - (-6x)(2(2x))}{(x^2-9)^3} = \frac{-6x^2 + 54 + 24x^2}{(x^2-9)^2} = \frac{18x^2 + 54}{(x^2-9)^3}$$

This is never zero, so there are no points of inflection. However, the concavity can still change through the vertical asymptotes, so we need to check $(-\infty, -3), (-3, 3), (3, \infty)$. We have

$$g''(-5) = \frac{18(25) + 54}{(25-9)^3} > 0, \quad g''(0) = \frac{54}{(-9)^3} < 0, \quad g''(5) = \frac{18(25) + 54}{(25-9)^3} > 0$$

This means that we're mostly concave up, except between the asymptotes. This also reconfirms (via the Second Derivative Test) that $x = 0$ is a local maximum.