Midterm 1 Solution Guide

1. (10 points) Explain why the function $f(x) = |2x - 3|$ is not invertible on $\mathbb{R}$. If you use any theorems in your answer, define all relevant terms. Then give a domain (with reason) on which $f(x)$ is invertible.

   Solution. A function is invertible if and only if it is *injective* or *one-to-one*. This means that $f(a) = f(b)$ if and only if $a = b$. For this function, we can see that $f(2) = f(1)$, so it is not injective. Another way to show non-injectivity is to graph it, and show it fails the horizontal line test.

   In order to fix it, we need to restrict the domain so that we don't have this problem. The graph is the easiest way to see it: the 'vertex' of this absolute value function is at $x = 3/2$, where $f(x) = 0$. Therefore the domains $(-\infty, 3/2]$ or $[3/2, \infty)$ are both good options.

   See more here: https://www.desmos.com/calculator/bxgfysvarc.

2. (20 points) Let $f(x) = \dfrac{2x^2+6x+4}{x^2-x-6}$. Compute the following limits, if they exist. If the limit does not exist, explain why .

   (a) $\lim_{x\to 2} f(x)$

   (b) $\lim_{x\to -2} f(x)$

   (c) $\lim_{x\to 3} f(x)$

   (d) $\lim_{x\to \infty} f(x)$

   Solution. For all of these problems, we need to use factor the polynomials:

$$f(x) = \frac{2(x+2)(x+1)}{(x-3)(x+2)}.$$

   (a) We see that plugging in $x = 2$ just gives us the answer, because the function is continuous here. We get $f(2) = \frac{2(4)(3)}{(-1)(4)} = -6$.

   (b) We see that plugging in $x = -2$ gives us the form $\frac{2(0)(-1)}{(-5)(0)} = \frac{0}{0}$. This is an indeterminate form, so we need to simplify it by factoring something out. But we already have done this – we see that there's an $(x+2)$ in the top and the bottom. If we cancel that out, we get $\frac{2(-1)}{-5} = \frac{2}{5}$, which is our answer.

   (c) We see that plugging in $x = 3$ gives us the form $\frac{2(5)(4)}{(0)(5)} = \frac{40}{0}$. This is not an indeterminate form – it's just undefined! The limit does not exist.

   (d) We can use this trick we learned in class: if our function is rational, where the top and bottom are polynomials of the same degree, then $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$ are given by the quotient of the leading coefficients. That gives us $\frac{2}{1} = 2$.

3. (20 points) For each of the functions below, compute the derivative. You do not need to simplify the expression you get all the way, but it should be in a sensible form.

   (a) $f(x) = \sec\left(e^{x^2+1}\right)$

   (b) $g(x) = \dfrac{e^x}{\sqrt{x} + \sin x}$

   (c) $h(x) = \cos^5(7x)$

   (d) $k(x) = (x^2 - 1)^3(x^3 + 3x + 1)^2$

Solution.

(a) This is a double chain rule: the outside is $f(u) = \sec u$, the middle is $u(v) = e^v$, and the inside is $v(x) = x^2 + 1$. We have $f'(u) = \sec u \tan u$, $u'(v) = e^v$, and $v'(x) = 2x$. All together,

$$f'(x) = \sec(e^{x^2+1})\tan(e^{x^2+1}) \cdot e^{x^2+1} \cdot 2x$$

(b) This is a quotient rule without any chain rule:

$$g'(x) = \frac{(\sqrt{x} + \sin x) \cdot e^x - e^x \cdot (\frac{1}{2}x^{-1/2} + \cos x)}{(\sqrt{x} + \sin x)^2}$$

(c) This is actually a double chain rule again, remembering that $\cos^5(7x) = \cos(7x)^5$. The outside is $f(u) = u^5$, the middle is $u(v) = \cos(v)$, and the inside is $v(x) = 7x$. All together,

$$h'(x) = 5\cos^4(7x) \cdot (-\sin(7x)) \cdot 7$$

(d) This is a product rule, each of which has a little chain rule. The two main functions are $a(x) = (x^2 - 1)^3$ and $b(x) = (x^3 + 3x + 1)^2$. We can compute $a'(x) = 3(x^2 - 1)^2 \cdot 2x$ and $b'(x) = 2(x^3 + 3x + 1) \cdot (3x^2 + 3)$. Therefore the answer is

$$\begin{aligned} k'(x) &= a(x)b'(x) + a'(x)b(x) \\ &= (x^2 - 1)^3 \cdot 2(x^3 + 3x + 1) \cdot (3x^2 + 3) + 3(x^2 - 1)^2 \cdot 2x \cdot (x^3 + 3x + 1)^2 \end{aligned}$$

4. (10 points) Find the coordinates (x, y) on the curve $y = x^4 + 4x^3 - 8x^2 + 3$ where the tangent line is horizontal.

   Solution. First, we need to remember that 'horizontal tangent line' is the same thing as $y' = 0$. The derivative is the slope of the tangent line, after all, and horizontal means zero slope. Therefore let's take the derivative to get $y' = 4x^3 + 12x^2 - 16x$. Factoring this, we get $y' = 4x(x^2 + 3x - 4) = 4x(x + 4)(x - 1)$. This gives us x coordinates of $x = 0, -4, 1$. Reading the problem carefully, we need to also find the y-value, making the final answer the three points $(0, 3), (-4, -125), (1, 0)$.

5. (10 points) For each of the functions below, compute the *second* derivative. You do not need to simplify the expression you get all the way, but it should be in a sensible form.

   (a) $f(\theta) = \sin(\sin(\theta))$

   (b) $g(x) = \cot(e^x)$

   Solution.

   (a) The first derivative follows from the chain rule: $f'(\theta) = \cos(\sin(\theta)) \cdot \cos(\theta)$. To do another derivative, it'll be the chain rule and the product rule, but nothing too bad:

   $$\begin{aligned} f''(\theta) &= \cos(\sin(\theta)) \cdot (-\sin(\theta)) + \cos(\theta) \cdot (-\sin(\sin\theta) \cdot \cos\theta) \\ &= -\sin(\theta)\cos(\sin(\theta)) - \cos^2(\theta)\sin(\sin(\theta)) \end{aligned}$$

   (b) This one is a bit more unpleasant. We have the first derivative using the chain rule, $g'(x) = -\csc^2(e^x) \cdot e^x$. Then for the second derivative, it's the product rule and a double-chain rule for $-\csc^2(e^x) = -(\csc(e^x)^2)$. The outside is $f(u) = -u^2$, the middle is $u(v) = \csc(v)$, and the inside is $v(x) = e^x$. Thus the derivative of this part is $-2\csc(e^x) \cdot (-\csc(e^x)\cot(e^x)) \cdot e^x$. All together,

   $$g''(x) = -2\csc(e^x) \cdot (-\csc(e^x)\cot(e^x)) \cdot e^x \cdot e^x + -\csc^2(e^x) \cdot e^x$$

6. (10 points) Use the definition of the derivative to compute the equation of the tangent line to $f(x) = \dfrac{-1}{3x+2}$ at $x = 1$. You may use other techniques to check your answer, but these will receive no points.

Solution. We have to use the actual definition, which is either

$$\lim_{x\to 1} \frac{f(x) - f(1)}{x-1} \text{ or } \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

Let's use the second one. We have to solve

$$\lim_{h\to 0} \frac{\frac{-1}{3(1+h)+2} - \frac{-1}{5}}{h} = \lim_{h\to 0} \frac{\frac{-1}{3h+5} + \frac{1}{5}}{h}$$

To solve this, we need to combine that top difference by finding a common denominator:

$$\frac{-1}{3h+5} + \frac{1}{5} = \frac{-5}{5(3h+5)} + \frac{3h+5}{5(3h+5)} = \frac{3h}{5(3h+5)}$$

and thus we need to solve

$$\lim_{h\to 0} \frac{\frac{3h}{5(3h+5)}}{h} = \lim_{h\to 0} \frac{3h}{5h(3h+5)}$$

At this point we can cancel out the h, which was the whole problem! Then we get

$$\lim_{h\to 0} \frac{3}{5(3h+5)} = \frac{3}{5(0+5)} = \frac{3}{25}$$

To get the equation of the tangent line, we also need $f(1) = \frac{-1}{5}$, which let us write

$$y - f(1) = f'(1)(x-1) \implies y + \frac{1}{5} = \frac{3}{25}(x-1)$$

7. (10 points) If possible, find the values of a and c that make the function $f(x)$ below continuous:

$$f(x) = \begin{cases} \dfrac{\sin(ax)}{x} & \text{if } x > 0 \\ c & \text{if } x = 0 \\ \dfrac{\sqrt{x+1}-1}{x} & \text{if } x < 0 \end{cases}$$

If impossible, explain why. Be sure to explain your reasoning in either case.

Solution. It is possible. In order for $f(x)$ to be continuous, the only difficulty is at $x = 0$, as the function is continuous (where defined) for $x < 0$ and $x > 0$. We therefore need

$$\lim_{x\to 0^+} f(x) = f(0) = \lim_{x\to 0^-} f(x)$$

The first one of these is probably the easiest: we showed in class that

$$\lim_{x\to 0} \frac{\sin(ax)}{x} = \lim_{x\to 0} a \cdot \frac{\sin(ax)}{ax} = a$$

If this is true for the two-sided limit, it's true for the one-sided limit $x \to 0^+$. Okay, so that means that $a = c$. But what should c be? Well, this depends on the other limit. If we plug in, we get

$$\lim_{x\to 0^-} \frac{\sqrt{x+1}-1}{x} = \frac{0}{0}$$

so that's an indeterminate form. This means we might be able to get an answer out of it if we can manipulate this, and what we want to do is multiply by the conjugate $\sqrt{x+1}+1$ on top and bottom:

$$\frac{\sqrt{x+1}-1}{x} \cdot \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} = \frac{x+1-1}{x(\sqrt{x+1}+1)} = \frac{x}{x(\sqrt{x+1}+1)}$$

Now we can cancel out the x and be done!

$$\lim_{x\to 0^-} \frac{\sqrt{x+1}-1}{x} = \lim_{x\to 0^-} \frac{1}{\sqrt{x+1}+1} = \frac{1}{2}$$

This means that we must have $a = c = 1/2$.

8. (10 points) Prove that the equation $\cos(2x) - \sin(3x) = x^2$ has at least one solution on the interval $(0, \pi/6)$.

Solution. For this, we need to use the intermediate value theorem. We write a new function $f(x) = \cos(2x) - \sin(3x) - x^2$ and notice that a solution to the original equation is the same as $f(x) = 0$. Moreover, $f(x)$ is continuous on all of $\mathbb{R}$, and in particular on $[0, \pi/6]$, so we can use the IVT. So we compute:

$$f(0) = 1 - 0 - 0 = 1, \quad f(\pi/6) = \cos(\pi/3) - \sin(\pi/2) - (\pi/6)^2 = -\frac{1}{2} - \frac{\pi^2}{36}.$$

The endpoint $f(\pi/6)$ is mysterious but definitely negative, and $f(0) = 1$ is definitely positive. Therefore on $(0, \pi/6)$ we can find a zero of $f(x)$, and that solution c so that $f(c) = 0$ answers the original problem.