Limits of polyhedra in Gromov-Hausdorff Space

S. C. Ferry

SUNY at Binghamton

November 20, 1995

Abstract. The main theorem of this paper is that compact metric spaces which are locally n-connected and which have cohomological dimension $\leq n$ for some n are precisely the spaces which are cell-like images of finite polyhedra. We show that this leads to a well-defined simple homotopy theory for such spaces. We also show that these spaces are precisely the compact metric spaces which are limits of polyhedra in Gromov's topological moduli spaces $\mathcal{M}(n,\rho)$ for some choice of ρ and n. In addition, we prove that every precompact subset of $\mathcal{M}(n,\rho)$ contains only finitely many simple homotopy types. In the final section, we discuss the problem of determining which metric spaces are limits of closed manifolds in $\mathcal{M}(n,\rho)$ for some n and ρ.

1. Introduction

Definition 1.1. A space X is said to be LC^k if for each point $x \in X$ and each neighborhood U of x, there is a neighborhood $V \subset U \subset X$ containing x so that $\pi_\ell(V) \to \pi_\ell(U)$ is the zero map for all $0 \leq \ell \leq k$ and for all choices of basepoint in V. X is said to be *weakly locally contractible* if X is LC^k for all k.

Definition 1.2. A metric space X is said to have *cohomological dimension* $\leq n$ if for each closed $A \subset X$, $\check{H}^{n+1}(X, A) = 0$.

Remark 1.3. It is an easy consequence of the definition that the cohomological dimension of a metric space is less than or equal to its covering dimension. The converse is true for finite-dimensional spaces. A nice explanation of this appears in [23]. The two notions of dimension diverge for spaces of infinite covering dimension – Dranishnikov [8], has produced spaces which have finite cohomological dimension and infinite covering dimension. In what follows, the word "dimension" will always mean covering dimension. We will use "cohomological dimension" or "cdim" when we wish to refer to cohomological dimension.

Definition 1.4.

(i) A compact metric space X is *cell-like* if X can be topologically embedded in the Hilbert cube Q in such a way that X contracts to a point inside of each of its neighborhoods. An argument using the Tietze extension theorem shows that if such an X is embedded into any ANR, then it contracts in each of its neighborhoods in that ANR.

(ii) A map $f : X \to Y$ is *proper* if $f^{-1}(K)$ is compact for each compact $K \subset Y$.

(iii) A map between metric spaces $q : X \to Y$ is *cell-like* if it is a proper surjection and $q^{-1}(y)$ is cell-like for each $y \in Y$. See [17] for general properties of cell-like maps.

Partially supported by NSF Grant DMS9305758

Here is our first main result.

THEOREM A. *Let X be a compact metric space which is* LC^n *and which has cohomological dimension $\leq n$. Then X is the cell-like image of a finite polyhedron.*

Before stating our other theorems, we recall the definitions of the Gromov-Hausdorff metric and some related concepts.

DEFINITION 1.5. If Z is a compact metric space and X and Y are closed subsets of Z, then the *Hausdorff distance* from X to Y in Z is

$$d_Z^H(X,Y) = \inf\{\epsilon > 0 \mid X \subset N_\epsilon(Y) \text{ and } Y \subset N_\epsilon(X)\}.$$

Here, $N_\epsilon(X)$ denotes the set of points in Z whose distance from X is less than ϵ. The *Gromov-Hausdorff distance* from X to Y is

$$d_{GH}(X,Y) = \inf_Z\{d_Z^H(X,Y) \mid X \text{ and } Y \text{ are embedded isometrically in } Z\}.$$

Let $\mathcal{CM}$ denote the set of isometry classes of compact Hausdorff spaces with the Gromov-Hausdorff metric. $\mathcal{CM}$ is a complete metric space ([14]).

We wish to study collections of topological manifolds and polyhedra in $\mathcal{CM}$. To insure that spaces in our class which are close together have similar algebraic-topological properties, we follow [2], [14], [20] by introducing the notion of a contractibility function.

DEFINITION 1.6. A function $\rho : [0,R) \to [0,\infty)$ with $\rho(0) = 0$ is a *contractibility function* if ρ is continuous at 0 and $\rho(t) \geq t$ for all t. A compact metric space X is *locally contractible with contractibility function* ρ if for each $r < R$, the ball $B_r(x)$ contracts to a point in $B_{\rho(r)}(x)$. Let $\mathcal{M}(\rho,n)$ denote the subset of $\mathcal{CM}$ consisting of isometry classes of compact metric spaces with Lebesgue covering dimension $\leq n$ which have contractibility function ρ.

Here are the statements of our other main theorems.

THEOREM B. *If $X \in \mathcal{CM}$, then $X \in \overline{\mathcal{M}(n,\rho)}$ for some n and ρ if and only if X is the cell-like image of some finite polyhedron K.*

COROLLARY. *If $X \in \mathcal{CM}$, then $X \in \overline{\mathcal{M}(n,\rho)}$ for some n and ρ if and only if there is a k such that* $\mathrm{cdim}(X) \leq k$ *and X is* LC^k.

DEFINITION 1.7. A subset $\mathcal{S} \subset \mathcal{CM}$ is said to be *precompact* if $\mathcal{S}$ has compact closure in $\mathcal{CM}$. Since $\mathcal{CM}$ is complete, $\mathcal{S}$ is precompact if and only if it has a finite cover by ϵ-balls for each ϵ.

THEOREM C. *If $\mathcal{S} \subset \mathcal{M}(n,\rho)$ is precompact for some n and ρ, then $\mathcal{S}$ contains only finitely many simple-homotopy types.*

REMARK 1.8. In [12], the author showed that f $n \neq 3$ and $\mathcal{S}$ is a precompact collection of topological n-manifolds in $\mathcal{M}(n,\rho)$ for some fixed n and ρ, then $\mathcal{S}$ contains only finitely many homeomorphism types of topological manifolds. The analogous result in dimension 3 would imply the 3-dimensional Poincaré Conjecture.

THEOREM D.

(i) *If K_1 and K_2 are finite polyhedra and $\rho_1 : K_1 \to X$, $\rho_2 : K_2 \to X$ are cell-like maps, then there is a simple-homotopy equivalence $f : K_1 \to K_2$ so that $\rho_2 \circ f$ is homotopic to ρ_1.*

(ii) *If K_1 and K_2 are finite polyhedra, $\rho_1 : K_1 \to X_1$, $\rho_2 : K_2 \to X_2$ are cell-like maps, and $f : X_1 \to X_2$, then there is a homotopy $\bar{f}_t : K_1 \to K_2$, $0 \leq t < 1$, so that $\lim_{t\to 1} \rho_2 \circ \bar{f}_t = f \circ \rho_1$. If f is a weak homotopy equivalence, f_t is a homotopy equivalence for each $t \in [0,1)$ and setting $\tau(f) = \rho_{2\#}(\tau(f_0)) \in Wh(\mathbb{Z}\pi_1 X_2)$ extends the definition of Whitehead torsion to include weak homotopy equivalences between cell-like images of finite polyhedra.*

At the end of the paper, we discuss a program for determining which topological spaces are limits of closed topological manifolds in some $\mathcal{M}(n,\rho)$. If X is a weakly locally contractible homology n-manifold, $n \geq 6$, with finite cohomological dimension, this program, when implemented, will give an obstruction lying in $\pi_{n-1}(fiber(\mathcal{H}(X, G/TOP \times \mathbb{Z}) \to \mathcal{H}(X, \mathbb{L}(e))))$ which vanishes if and only if X is the cell-like image of a closed ANR homology manifold. Here, $\mathbb{L}(e)$ is the periodic L-theory spectrum of the trivial group..

2. THE PROOF OF THEOREM A

It is classical that covering dimension and cohomological dimension agree for cdim $= 1$. This implies that a space X which is LC^1 with $\mathrm{cdim}(X) = 1$ must be a 1-dimensional ANR. By results of Quinn, [21], such a space X has a mapping cylinder neighborhood in $\mathbb{R}^5$ and is therefore the cell-like image of a 5-dimensional polyhedron. Thus, we may assume that $n \geq 2$.

First, we need to show that our "cdim $\leq n$ and LC^n"space X is weakly locally contractible. We begin by quoting a theorem of Hurewicz. A modern reference for this result is [10] Corollary 3.3.

PROPOSITION 2.1 (HUREWICZ [16]). *Suppose that X is a compact LC^k metric space with $k \geq 1$ and that for each neighborhood U of $x \in X$ there is a neighborhood V of x in U with $\check{H}_{k+1}(V) \to H_{k+1}(U)$ trivial. Then X is LC^{k+1}.* ∎

The homology theory in this statement is Čech homology:

DEFINITION 2.2. If $X = \varprojlim K_i$ is a compact metric space, written as an inverse limit of finite polyhedra, we define $\check{H}_k(X)$ to be $\varprojlim H_k(K_i)$. In general, we define the Čech homology of a metric space to be the direct limit of the Čech homologies of its compact subsets.

PROPOSITION 2.3. *If X is a compact metric space with $\mathrm{cdim}(X) \leq n$, then $\check{H}_k(X) = 0$ for all $k > n$.*

PROOF: By a theorem of Alexandrov, we can write $X = \varprojlim K_i$, where the K_i's are finite polyhedra. For each i, we have a natural short exact sequence

$$0 \to \mathrm{Ext}(H^{k+1}(K_i), \mathbb{Z}) \to H_k(K_i, \mathbb{Z}) \to \mathrm{Hom}(H^k(K_i), \mathbb{Z}) \to 0.$$

Since $\check{H}^k(X) = \varinjlim H^k(K_i) = 0$ for $k > n$, we know that for each fixed k and i there is a $j(i) > i$ so that $H^k(K_i) \to H^k(K_{j(i)})$ is the zero map. It follows easily that the composition $H_k(K_i) \to H_k(K_{j(i)}) \to H_k(K_{j\circ j(i)})$ is zero on homology. This, in turn, shows that $\check{H}_k(X) = 0$.∎

An easy induction using Hurewicz' theorem now gives the following.

PROPOSITION 2.4. *If X is a compact* LC^n *metric space with* $\mathrm{cdim}(X) \le n$, *then X is weakly locally contractible.*∎

The principal tools used in proving Theorem A are theorems of R. D. Edwards and F. Quinn. We begin the proof with the statement of Edwards' theorem.

THEOREM (EDWARDS' RESOLUTION THEOREM [23]). *If X is a compact metric space with finite cohomological dimension n, then there exist a compact n-dimensional metric space Z and a cell-like map $\rho : Z \to X$.*

REMARK 2.5. In general, the space Z produced by Edwards' argument will not have good local properties. The point of our Theorem A is to show that when X is weakly locally contractible, then Z can be taken to be very nice indeed. Replacing the messy Z by a polyhedral one costs us some dimensions. For $n \ge 2$, the polyhedral Z produced by Theorem A will be $2n+1$-dimensional if X has cohomological dimension n. The author does not know if this can be improved.∎

If X is a space of cohomological dimension n as in the statement of Theorem A, let $\rho : Z \to X$ be a cell-like map with Z n-dimensional. As remarked above, Z need not have good local properties. By dimension theory, Z can be embedded in $\mathbb{R}^{2n+1}$. In fact, the set of embeddings is second category in the set of all maps $Z \to \mathbb{R}^{2n+1}$. This embedding of Z into $\mathbb{R}^{2n+1}$ can be taken to miss $\cup_{i=1}^{\infty} T_i^{(2)}$, where T_i is a sequence of triangulations of $\mathbb{R}^{2n+1}$ with mesh tending to 0 and $T_i^{(2)}$ is the 2-skeleton of T_i.

Form the adjunction space $\mathbb{R}^{2n+1} \cup_\rho X$. Since X is weakly locally contractible, an inductive argument as in pp 390-393 of [20] produces a compact manifold neighborhood M of Z in $\mathbb{R}^{2n+1}$ and a homotopy $r_t : M \cup_\rho X \to \mathbb{R}^{2n+1} \cup_\rho X$, $0 \le t \le 1$, such that r_0 is the inclusion, $r_1(M \cup_\rho X) = X$, and $r_t|X = id_X$ for all t. The idea here is to use the weak local contractibility of X to construct a deformation from $M \cup_\rho X$ to X in $\mathbb{R}^{2n+1} \cup_\rho X$.

NOTATION: Let $\bar{\rho} : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1} \cup_\rho X$ be the quotient map and let $\bar{r}_t : M \to \mathbb{R}^{2n+1} \cup_\rho X$ be the composition $r_t \circ \bar{\rho}$. In particular, we have $\bar{r}_1 : M \to X$ with $\bar{r}_1|Z = \rho$.

DEFINITION 2.6. If Y is a metric space and X is a closed subset of Y, we say that X has a *mapping cylinder neighborhood* in Y if there exist a space Z and a map $p : Z \to X$ so that the mapping cylinder $M(p)$ of p is homeomorphic to a closed neighborhood of X in Y. More precisely, we require that there be a map $j : M(p) \to Y$ so that $j|X = id$ and so that $j|(M(p) - Z)$ is open.

THE END THEOREM

By an *end problem*, we will mean a noncompact manifold M with compact boundary together with a map $p : M \to X$, where X is a compact LC^1 metric space. A *compact manifold completion* $(\bar{M}, \bar{p})$ of (M, p) is a compact manifold $\bar{M} \supset M$ with $\bar{M} - M \subset \partial\bar{M}$ and an extension of p to a map $\bar{p} : \bar{M} \to X$.

Quinn's End Theorem gives sufficient conditions for an end to admit a completion. In particular, Theorem 1.4 of [21] says that if $p : M \to X$ is a map such that p is *onto*, $0-\mathrm{LC}$, $1-\mathrm{LC}$, and *tame* (see below for definitions) then p admits a manifold completion.

Definition 2.7.

(i) $p : M \to X$ is *onto* if $p(M - K) = X$ for every compact $K \subset M$.
(ii) p is $0-\mathrm{LC}$ if for every $x \in X$, compact $K \subset M$, and neighborhood V of x in X, there is a compact $K' \supset K$ and an open neighborhood V' of x in V so that points in $p^{-1}(V') \cap (M - K')$ can be joined by arcs in $p^{-1}(V) \cap (M - K)$.
(iii) p is $1-\mathrm{LC}$ if, in addition, K' and V' can be chosen so that loops in $p^{-1}(V') \cap (M - K')$ contract in $p^{-1}(V) \cap (M - K)$.

Definition 2.8. The map $p : M \to X$ is *tame* provided that given any $\epsilon > 0$ and compactum $K \subset M$ there is a larger compactum $K' \supset K$ and a homotopy $h_t : M \to M$ such that $h_0 = id$, $h_t|K = id$ for all t, and $h_1(M - K) \subset K' - K$. In addition, we require that for each $m \in M$, $\mathrm{diam}(\{p \circ h_t(m) | \, 0 \leq t \leq 1\}) < \epsilon$.

Remark 2.9. We have stated a somewhat weakened version of Theorem 1.4 of [21]. In [21], the theorem is stated for locally compact X and for a more general class of control maps p. The interested reader is referred to that paper. That our end problem $r : M - Z \to X$ satisfies the $0-\mathrm{LC}$, and $1-\mathrm{LC}$ conditions will follow immediately from basic properties of cell-like maps. The tameness condition and the "onto" condition will require further discussion.∎

We wish to apply Quinn's end theorem to the end $\bar{r}_1| : M - Z \to X$ to produce a mapping cylinder neighborhood Q of X in $\mathbb{R}^{2n+1} \cup_\rho X$. The point is that if $\bar{r}_1|$ extends to $p : \bar{M} \to X$, with $\bar{M} = M \cup N$, then N has a neighborhood homeomorphic to $N \times [0,1]$ in $\bar{M}$ and $M(p) \cong N \times [0,1] \cup_p X$ is a mapping cylinder neighborhood of X in $\mathbb{R}^{2n+1} \cup_\rho X$.

The inverse image, call it P, of this mapping cylinder neighborhood in $\mathbb{R}^{2n+1}$ is the desired polyhedron. By Quinn's construction, the boundary of P is a codimension-1 PL submanifold of $\mathbb{R}^{2n+1}$. The composition

$$P \xrightarrow[CE]{\bar{\rho}|} Q = M(p) \xrightarrow{proj} X$$

is a cell-like map from a polyhedron onto X. Thus, the proof of Theorem A will be complete if we can show that the map $\bar{r}_1| : M - Z \to X$ is onto, $0-\mathrm{LC}$, $1-\mathrm{LC}$, and tame. The proof is an adaptation of the proof from [21] that codimension-3 1- LCC embedded ANRs have mapping cylinder neighborhoods. Our verification of conditions (ii) and (iii) of Definition 2.7 will rely on the following properties of cell-like maps.

Three properties of cell-like maps

Proposition 2.10 ([17], p. 506.). *Let X be an ANR, let Y be a compact metric space and let $q : X \to Y$ be a cell-like map. Suppose we are given a finite polyhedron L, a subpolyhedron L_0 of L and maps $f : L \to Y$ and $f_0 : L_0 \to X$ so that $q \circ f_0 = f \mid L_0$. Then for every $\epsilon > 0$, there is a map $\bar{f} : L \to X$ so that $\bar{f} \mid L_0 = f_0$ and $d(f, q \circ \bar{f}) < \epsilon$.*

$$\begin{tikzcd} L_0 \arrow[r, "f_0"] \arrow[d, hook] & X \arrow[d, "q"] \\ L \arrow[r, "f"] \arrow[ur, dashed, "\bar{f}"] & Y \end{tikzcd}$$

Corollary 2.11. *If $f : X \to Y$ is CE with X an ANR, then f is a weak homotopy equivalence.*

The second basic property of cell-like maps shows that the LC^k condition in the statement of Theorem A is necessary.

Proposition 2.12 ([17], p. ???). *Let X be a compact ANR and let $f : X \to Y$ be a cell-like map. Then Y is weakly locally contractible.*

Theorem (Vietoris-Begle). *If $f : X \to Y$ is a cell-like map between metric spaces, then $f^* : \check{H}^k(Y) \to \check{H}^k(X)$ is an isomorphism for all k.*

Returning to the proof of Theorem A, recall that we have an n-dimensional compactum $Z \subset \mathbb{R}^{2n+1}$, a cell-like map $\rho : Z \to X$, and a retraction $r_1 : M \cup_\rho X \to X$, where M is a compact PL manifold neighborhood of Z in $\mathbb{R}^{2n+1}$.

To see that $\bar{r}_1| : M - Z \to X$ is $0-\mathrm{LC}$, let K be a compact subset of M and let V be a neighborhood of x in X. Since X is LC^0, we can choose a neighborhood V' with $\bar{V}' \subset V$ so that any two points in V' can be connected by a path in V.

Let $\delta > 0$ be the minimum of *distance from $\bar{V}'$ to $X - V$* and *distance from K to X in* $\mathbb{R}^{2n+1} \cup_\rho X$. Choose K' so that $\mathrm{diam}(\{\bar{r}_t(m) | 0 \leq t \leq 1\})$ and $\mathrm{diam}(\{\bar{r}_1 \circ \bar{r}_t(m) | 0 \leq t \leq 1\})$ are less than $\delta/3$ for all $m \in M - K'$.

If $m_1, m_2 \in (\bar{r}_1^{-1}(V') \cap (M - K' - Z))$, we can join m_1 to m_2 in $\bar{r}_1^{-1}(V) \cap (M - K' - Z)$ by using the paths $\bar{r}_t(m_i)$ to get from m_i to $\bar{r}_1(m_i)$ and then going from $\bar{r}_1(m_1)$ to $\bar{r}_1(m_2)$ by a path in V. To complete this phase of the argument, we need to push this path, call it ω', off of X by a small move. By Proposition 2.10, we can find a path ω connecting m_1 and m_2 in $\mathbb{R}^{2n+1}$ whose image in $\mathbb{R}^{2n+1} \cup_\rho X$ is as close as we like to ω'. By simplicial approximation, we can assume that ω lies in $\cup T_i^{(2)}$, except for arbitrarily short paths near m_1 and m_2. This implies, in particular, that ω misses Z. The desired path from m_1 to m_2 is this ω, thought of as a path in $\mathbb{R}^{2n+1} \cup_\rho X$. The proof that $\bar{r}_1 : M - Z \to X$ is $1-\mathrm{LC}$ is entirely similar.

The "onto" condition of Quinn's End Theorem is *not* automatically satisfied.

Example 2.13. Let $[-1,1] = [-1,1] \times \{0\} \subset \mathbb{R}^2$ and let $r : \mathbb{R}^2 \to [-1,1]$ be a retraction. Let $\mu : \mathbb{R}^2 \to [0,1]$ be a map so that $\mu^{-1}(0) = [-1,1]$. Then $s(x) = (1 - \mu(x)) \cdot r(x)$ is a retraction from $\mathbb{R}^2$ to $[-1,1]$ so that $s(\mathbb{R}^2 - [-1,1]) \neq [-1,1]$.

Fortunately, the weaker condition that $\bar{r}_1(M-K-Z)$ be dense in X for all compact $K \subset M$, suffices for the proof of the End Theorem. With a little work, one could always replace such a "dense" end by an "onto" end without disturbing the the LC -0 and LC -1 conditions. At the cost of another dimension – which doesn't matter for the main results of this paper – we can use an easy trick to alter our construction to produce "onto" ends. Let $j : Z \subset \mathbb{R}^{2n+1}$ and $r_1 : M \cup_\rho X \to X$ be an embedding and retraction as above. Including into $\mathbb{R}^{2n+1} \times \mathbb{R} = \mathbb{R}^{2n+2}$, we can define $r_1^* : M \times \mathbb{R} \to X$ be $\bar{r}_1 \circ proj$. The restriction $r_1^*|(M \times [-1,1] - (j(Z) \times \{0\}))$ is an "onto" end replacing $\bar{r}_1$. It is easy to check that the LC -0 and LC -1 conditions are undisturbed by this modification.

Tameness

If $p : M \to X$ is an end problem which is $0-$ LC and $1-$ LC, we will say that p is *homologically tame* if for every open $V \subset X$, open V' with $V' \supset \bar{V}$, and compact $K \subset M$, there exists a compact K' with $K \subset K' \subset M$ so that

$$H_*(p^{-1}(V), p^{-1}(V) \cap K) \to H_*(p^{-1}(V'), p^{-1}(V') \cap K')$$

is zero. In §5 of [21], Quinn proves that a $0-$LC and $1-$LC end which is homologically tame is tame. Thus, it suffices to verify homological tameness for our map $\bar{r}_1| : M - Z \to X$.

Given open V, V' with $V \subset \bar{V} \subset V' \subset X$, and a codimension-0 PL submanifold M_1 of M with $\overline{M - M_1}$ compact, we need to find a codimension-0 submanifold M_2 of M_1 so that $\overline{M_1 - M_2}$ is compact and

$$(*) \qquad H_*(\bar{r}_1|^{-1}(V), \bar{r}_1|^{-1}(V) \cap \overline{M - M_1}) \to H_*(\bar{r}_1|^{-1}(V'), \bar{r}_1|^{-1}(V') \cap \overline{M - M_2})$$

is the zero map. We can rewrite the left side of $(*)$:

$$\begin{aligned} H_*(\bar{r}_1|^{-1}(V), \bar{r}_1|^{-1}(V) \cap \overline{M - M_1}) &\cong H_*(\bar{r}_1^{-1}(V) \cap M_1 - \rho^{-1}(V), \bar{r}_1^{-1}(V) \cap \partial M_1) \\ &\cong \check{H}_c^{n-*}(\bar{r}_1^{-1}(V) \cap M_1, \rho^{-1}(V)) \\ &\cong \check{H}^{n-*}(\bar{r}_1^{-1}(\bar{V}') \cap M_1, \rho^{-1}(\bar{V}') \cup \bar{r}_1^{-1}(\bar{V}' - V) \cap M_1) \end{aligned}$$

where the passage from the first to second lines uses Alexander duality and the cohomology groups in the second and third lines are Čech cohomology. Similarly, we can rewrite the right hand side of $(*)$ as

$$\begin{aligned} H_*(\bar{r}_1|^{-1}(V'), \bar{r}_1|^{-1}(V') \cap \overline{M - M_2}) &\cong H_*(\bar{r}_1^{-1}(V) \cap M_2 - \rho^{-1}(V'), \bar{r}_1^{-1}(V') \cap \partial M_2) \\ &\cong \check{H}_c^{n-*}(\bar{r}_1^{-1}(V') \cap M_2, \rho^{-1}(V')) \\ &\cong \check{H}^{n-*}(\bar{r}_1^{-1}(\bar{V}') \cap M_2, \rho^{-1}(\bar{V}') \cup \bar{r}_1^{-1}(\bar{V}' - V') \cap M_2). \end{aligned}$$

We therefore need to check that the inclusion-induced map

$$\check{H}^{n-*}(\bar{r}_1^{-1}(\bar{V}') \cap M_1, \rho^{-1}(\bar{V}') \cup (\bar{r}_1^{-1}(\bar{V}' - V) \cap M_1)) \longrightarrow \check{H}^{n-*}(\bar{r}_1^{-1}(\bar{V}') \cap M_2, \rho^{-1}(\bar{V}') \cup (\bar{r}_1^{-1}(\bar{V}' - V') \cap M_2))$$

is zero for some choice of M_2. Choose V'' open with $\bar{V}' \subset V'' \subset X$. Then we have

$$\check{H}^{n-*}(\bar{r}_1^{-1}(\bar{V}') \cap M_1, \rho^{-1}(\bar{V}') \cup (\bar{r}_1^{-1}(\bar{V}' - V) \cap M_1)) = \\ \check{H}^{n-*}(\bar{r}_1^{-1}(\bar{V}'') \cap M_1, \rho^{-1}(\bar{V}'') \cup (\bar{r}_1^{-1}(\bar{V}'' - V) \cap M_1))$$

so it suffices to check that

$$\check{H}^{n-*}(\bar{r}_1^{-1}(\bar{V}'') \cap M_1, \rho^{-1}(\bar{V}'') \cup (\bar{r}_1^{-1}(\bar{V}'' - V) \cap M_1)) \longrightarrow \\ \check{H}^{n-*}(\bar{r}_1^{-1}(\bar{V}') \cap M_2, \rho^{-1}(\bar{V}') \cup (\bar{r}_1^{-1}(\bar{V}' - V') \cap M_2))$$

is zero. By the Vietoris-Begle Theorem, this is the same as checking that

$$\check{H}^{n-*}(r_1^{-1}(\bar{V}'') \cap \bar{\rho}(M_1), \bar{V}'' \cup (r_1^{-1}(\bar{V}'' - V) \cap \bar{\rho}(M_1))) \longrightarrow \\ \check{H}^{n-*}(r_1^{-1}(\bar{V}') \cap \bar{\rho}(M_2), \bar{V}' \cup (r_1^{-1}(\bar{V}' - V') \cap \bar{\rho}(M_2)))$$

is zero. But if M_2 is a sufficiently small neighborhood of Z, the homotopy r_t deforms $\bar{\rho}(M_2)$ into X by a homotopy (rel X) keeping $r_1^{-1}(\bar{V}') \cap \bar{\rho}(M_2)$ inside of $r_1^{-1}(\bar{V}'') \cap \bar{\rho}(M_1)$ and $r_1^{-1}(\bar{V}' - V') \cap \bar{\rho}(M_2)$ inside of $r_1^{-1}(\bar{V}'' - V) \cap \bar{\rho}(M_1)$. This shows that the induced map on cohomology is zero, as desired. This completes the proof of Theorem A. ∎

REMARK 2.14.

(i) The "Borsuk Conjecture" says that if X is a compact ANR, then X is homotopy equivalent to a finite polyhedron. This was proved by West in [24]. Our Theorem A generalizes this by showing that every weakly locally contractible compact metric space with finite cohomological dimension is weak homotopy equivalent to a finite polyhedron. If we replace cohomological dimension by topological dimension and drop the "weak," this becomes the Borsuk Conjecture for finite-dimensional ANRs.

(ii) The argument above uses an Alexander duality isomorphism for noncompact manifolds with boundary which says that if M is an orientable noncompact manifold with boundary ∂M and X is a closed subset of M with $\partial M \cap X = \emptyset$, then $H_k(M - X, \partial M) \cong \check{H}_c^{n-k}(M, X)$.[1] To see this isomorphism in the case where M is PL, we start by proving the analogous result for a compact orientable PL manifold P containing a closed subset X. We write $X = \cap N_i$, where $\{N_i\}$ is a nested sequence of compact PL manifolds meeting ∂P regularly. Let Q be a codimension-0 submanifold of ∂P with $Q \cap X = \emptyset$. Then

$$\begin{aligned} H_k(P - X, Q) &\cong \varinjlim H_k(P - \mathring{N}_i, Q) \\ &\cong \varinjlim H^{n-k}(P - \mathring{N}_i, \partial(P - \mathring{N}_i) - \mathring{Q}) \\ &\cong \varinjlim H^{n-k}(P, (\partial P - \mathring{Q}) \cup N_i) \\ &\cong \check{H}^{n-k}(P, (\partial P - \mathring{Q}) \cup X). \end{aligned}$$

[1]The argument on page 286 of [21] uses an incorrect form of this duality.

Returning to the noncompact case, if we write $M = \cup P_i$, where $\{P_i\}$ is a nested sequence of compact PL manifolds with boundary and $Q_i = \partial P_i \cap \partial M$ is a submanifold of ∂P_i, we have

$$\begin{aligned} H_k(M - X, \partial M) &\cong \varinjlim H_k(P_i - (P_i \cap X), Q_i) \\ &\cong \varinjlim \check{H}^{n-k}(P_i, \overline{(\partial P_i - Q_i)} \cup (P_i \cap X)) \\ &\cong \varinjlim \check{H}^{n-k}(M, \overline{(M - P_i)} \cup (P_i \cap X)) \\ &\cong \check{H}^{n-k}_c(M, X). \end{aligned}$$

3. The proof of theorem B

To prove Theorem B, we need to show

(i) If X is the cell-like image of a finite polyhedron, then $X \in \overline{\mathcal{M}(n,\rho)}$ for some n and ρ.
(ii) If $X \in \overline{\mathcal{M}(n,\rho)}$ for some n and ρ, then X is weakly locally contractible.
(iii) If $X \in \overline{\mathcal{M}(n,\rho)}$ for some n and ρ, then $\mathrm{cdim}(X) \leq k$ for some k.

Proofs of these facts already appear in papers of Borsuk, Moore, and Petersen. For the first, here is the statement of Theorem 1 from [19].

Definition 3.1. If $\rho : [0, R) \to \mathbb{R}$ is a contractibility function, we will say that a space X is in class $\mathrm{LGC}^n(\rho)$ (LGC stands for locally geometrically contractible) if for every $\epsilon > 0$ and map $\alpha : \partial\Delta^k \to X$, $0 \leq k \leq n$, with $\mathrm{diam}(\alpha(\partial\Delta^k)) < t < R$, there is a map $\bar{\alpha} : \Delta^k \to X$ extending α with $\mathrm{diam}(\bar{\alpha}(\Delta^k)) < \rho(t)$.

Theorem. *If M^n is a closed n-dimensional manifold and $f : M \to X$ is a cell-like map, then there is a contractibility function ρ and a continuous path $w : [0,1] \to \mathrm{LGC}^n(\rho)$ so that $w(t)$ is homeomorphic to M for $0 \leq t < 1$ and $w(1) = X$.*

As observed in [19], the proof given is valid for M a compact ANR. This proves part (i). Part (iii) is a consequence of Theorem 2 from the same paper. We quote:

Theorem. *If X is a compact metric space such that there are compact ANR's $X_i \in \mathcal{M}(n,\rho)$ so that $\lim_{i\to\infty} X_i = X$ in $\mathcal{CM}$, then X is the cell-like image of a compact, n-dimensional metric space. More precisely, there exist a subsequence $\{X_{i_j}\}$ of $\{X_i\}$ and maps $f_{i_j} : X_{i_j} \to X_{i_{j-1}}$, so that there is a cell-like map $p : Z \to X$, where $Z = \varprojlim(X_{i_j}, f_{i_j})$.*

It is an immediate consequence of the Vietoris-Begle Theorem and the fact that cohomological dimension is less or equal to covering dimension that the cell-like image of an n-dimensional metric space has cohomological dimension $\leq n$. This proves part (*iii*). In [20], Petersen observed the closely related fact that if X is a limit of spaces in $\mathcal{M}(n,\rho)$ for some n and ρ, then every finite-dimensional subset of X has dimension $\leq n$. The reader should be warned, however, that the theorem on p. 393 of [20] is incorrect. See [19] and [12] for details.

Finally, we need to know that if X is a limit of spaces in $\mathcal{M}(n, \rho)$, then X is LC^n. This is proven in §16 of [1].

PROPOSITION 3.2. *If $X = \lim\ X_i$ where each X_i is in $\mathcal{M}(n, \rho)$ for some fixed ρ and n, then X is in $\mathrm{LGC}^k(\bar{\rho})$ for all k if $\bar{\rho}$ is any contractibility function with $\bar{\rho}(t) > \rho(t)$ for all $t \in (0, R)$.* ∎

This completes the proof of Theorem B.

4. The proof of Theorem C

The proof of Theorem C will be an argument by contradiction. Suppose that $\mathcal{S}$ is a precompact subset of $\mathcal{M}(n, \rho)$ for some fixed n and ρ. Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of spaces in $\mathcal{S}$ with no two simple-homotopy equivalent. By precompactness, we may assume that $\lim_{i \to \infty} X_i = X$ for some $X \in \mathcal{CM}$. By Proposition 3.2, X is weakly locally contractible.

We will obtain a contradiction by using a theorem of T. A. Chapman to prove that there is an $N > 0$ so that X_i and X_j are simple-homotopy equivalent for all $i, j > N$. Here is the theorem of Chapman which we will use.

THEOREM (THEOREM 1′ OF [4]). *Let Z be a compact metric ANR and let $p : Z \to X$ be a map from Z to an LC^1 compact metric space X. There is an $\epsilon_Z > 0$ so that if $f : Y \to Z$ is a homotopy equivalence from another compact ANR to Z with a homotopy inverse $g : Z \to Y$ and homotopies $k_t : f \circ g \simeq id_Z$ and $h_t : g \circ f \simeq id_Y$ so that for each $z \in Z$ and $y \in Y$, $\mathrm{diam}(\{p \circ k_t(z)\}) < \epsilon_Z$ and $\mathrm{diam}(\{p \circ f \circ h_t(y)\}) < \epsilon_Z$, then $\tau(f) \in ker(p_{\#} : Wh(\mathbb{Z}\pi_1 Z) \to Wh(\mathbb{Z}\pi_1 X))$.*

REMARK 4.1.

(i) The epsilon in Chapman's theorem depends on Z with its given metric. Chapman's theorem is remarkable for the fact that there is no local hypothesis on the map p.
(ii) Simple-homotopy theory was first developed by Whitehead for homotopy equivalences between finite polyhedra. If $f : K \to L$ is a homotopy equivalence between finite polyhedra, the theory gives an obstruction $\tau(f) \in Wh(\mathbb{Z}\pi_1 L)$ which vanishes if and only if there is a homotopy commuting diagram

$$\begin{tikzcd} N(K) \arrow[r, "h"] & N(L) \\ K \arrow[u, hook] \arrow[r, "f"] & L \arrow[u, hook] \end{tikzcd}$$

where $N(K)$ and $N(L)$ are closed regular neighborhoods of K and L in some high-dimensional euclidean space and h is a PL homeomorphism. Standard references for simple-homotopy theory include [7] and [18]. In [5], Chapman extended the theory to include homotopy equivalences between compact ANRs.

Chapman's theorem says that if a homotopy equivalence from X to Y has small tracks when projected to Z, then the torsion of that homotopy equivalence lies in the kernel of the induced map from $Wh\mathbb{Z}\pi_1 X$ to $Wh\mathbb{Z}\pi_1 Z$. To apply this theorem to our finiteness problem, we need to know that spaces in $\mathcal{M}(n,\rho)$ which are close together are homotopy equivalent "with small homotopies."

PROPOSITION 4.2.

(i) *If X is a compact n-dimensional metric space and Y is a compact metric space in* $\mathrm{LGC}^{n-1}(\rho)$*, then for every $\epsilon > 0$ there is a $\delta > 0$ so that if d is a metric on $Z = X \coprod Y$ so that $d^H_Z(X,Y) < \delta$, then there is a map $f : X \to Y$ with $d(x, f(x)) < \epsilon$ for all $x \in X$.*

(ii) *If X and Y are in $\mathcal{M}(n,\rho)$, then for every $\epsilon > 0$ there is a $\delta > 0$ so that if d is a metric on $Z = X \coprod Y$ so that $d^H_Z(X,Y) < \delta$, then there are maps $f : X \to Y$, $g : Y \to X$, and homotopies $h_t : id_X \simeq g \circ f$, $k_t : id_Y \simeq f \circ g$ so that $d(f(x), x) < \epsilon$, $d(g(y), y) < \epsilon$, $d(h_t(x), x)$, and $d(k_t(y), y) < \epsilon$ for all x, y, and t.*

PROOF: Part (i) is the proposition on page 390 of [20]. The second part is the theorem on page 392 of the same paper. Closely related results appear in [1].∎

We can now complete the proof of Theorem C. If $\{X_i\}$ is a sequence of spaces in $\mathcal{M}(n,\rho)$ converging to $X \in \mathcal{CM}$, so $X \in \mathrm{LGC}(2\rho)$. As in [14], we can find a metric on

$$Z = (\coprod_{i=1}^{\infty} X_i) \coprod X$$

so that $\lim X_i = X$ in the Hausdorff metric on Z.

By part (i) of the above, given any $\delta > 0$, there is an $N > 0$ so that each X_i with $i \geq N$ admits a map $p_i : X_i \to X$ with $d(x, p_i(x)) < \delta$ in this metric. Moreover, this N can be chosen so that if $i, j \geq N$ there are maps $f_{ij} : X_i \to X_j$ and $g_{ij} : X_j \to X_i$ with homotopies $k^{ij}_t : id \simeq f_{ij} \circ g_{ij}$, $h^{ij}_t : id \simeq g_{ij} \circ f_{ij}$ so that $d(f_{ij}(x), x) < \delta$, $d(g_{ij}(x), x) < \delta$, $d(h^{ij}_t(x), x)$, and $d(k^{ij}_t(y), y) < \delta$ for all x, y, and t.

Let $\epsilon_X > 0$ be the number in Chapman's theorem with X replacing Z. If we choose $\delta < \frac{\epsilon_X}{5}$, the conditions of Chapman's theorem are satisfied with respect to the control map $p_j : X_j \to X$. It follows that the torsion of f_{ij} is in the kernel of $(p_j)_\#$. But for δ small, p_j induces an isomorphism on π_1 and, therefore, an isomorphism of Whitehead groups – if α is a loop in X, we can take a fine subdivision of α and choose points in X_j δ-close to the vertices. The points in X_j corresponding to adjacent vertices will be no more than 2δ apart and can be connected by small arcs using the LC^0 condition in X_j. This gives a loop α' in X_j whose image under p_j is close to α. The LC^0 condition in X gives us paths from the vertices of α to the corresponding vertices of $p_j(\alpha')$ and the LC^1 condition lets us fill in to get a homotopy from α to $p_j(\alpha')$. This shows that p_j induces an epimorphism on π_1. A similar argument using the LC^2 condition shows that p_j induces a monomorphism, as well. For $n \geq 2$, this proves that X_i and X_j are simple homotopy equivalent for $i, j \geq N$, a contradiction which completes the proof of Theorem C. For $n = 1$, the Whitehead groups are trivial and the theorem is true by default. The reader who would like to see more details of this argument is referred to pages 390–393 of [20].

Remarks and extensions

Theorem C could also be proven using the machinery of [21]. For i and j large, the homotopy equivalence f_{ij} has a controlled torsion lying in the controlled Whitehead group of X, which vanishes. There is a forgetful homomorphism from the controlled Whitehead group to the ordinary Whitehead group taking controlled torsions to ordinary torsions, so the homotopy equivalence f_{ij} is simple. This yields a better result than Theorem C, since it shows that the controlled torsion, not just the ordinary torsion, vanishes.

The proof via Chapman's theorem has the advantage of accessibility. Short finite-dimensional proofs that CE maps between finite polyhedra are simple-homotopy equivalences appear in [6] and [17]. These proofs generalize to recover the α-Approximation Theorem of [11]. See [13] or the references in [4]. Chapman's Theorem $1'$ is an immediate consequence of this α-Approximation Theorem. See [4] for details.

Chapman's paper gives a second approach to certain consequences of Theorem A, as well. After Moore [19] showed that limits of polyhedra in Gromov-Hausdorff space were spaces of finite cohomological dimension which were weakly locally contractible, it became natural to ask whether every such space has the weak homotopy type of an a finite polyhedron. This reduces immediately to the question of whether the geometric realization of the singular complex of such a space is homotopy equivalent to a finite complex.

If we form $\bar{\rho} : M \to M \cup_\rho X$ as in the proof of Theorem A and a retraction $\bar{r}_1 : M \cup_\rho X \to X$, we have a homotopy equivalence $|\mathcal{S}(M)| \cong |\mathcal{S}(M \cup_\rho X)|$ and a retraction $|\bar{r}_1| : |\mathcal{S}(M \cup_\rho X)| \to |\mathcal{S}(X)|$. This shows that $|\mathcal{S}(X)|$ is a finitely dominated CW complex. Theorem $2'$ of [4] shows that controlled finitely dominated CW complexes with control maps inducing isomorphisms on π_1 have the homotopy types of finite complexes. While $|\mathcal{S}(X)|$ does not appear to be controlled finitely dominated, the homotopy equivalent subcomplex $|\mathcal{S}_\delta(X)|$ consisting of simplices of diameter $\leq \delta$ is controlled finitely dominated over X. This shows that $|\mathcal{S}(X)|$ has the homotopy type of a finite complex.

5. The proof of Theorem D

The proof of part (i) of Theorem D is an application of Theorem C together with Proposition 2.10. If $\rho_1 : K_1 \to X$ and $\rho_2 : K_2 \to X$ are CE maps from finite complexes to X, Proposition 2.10 applies to the diagram

$$\begin{tikzcd} & K_2 \arrow[d, "\rho_2"] \\ K_1 \arrow[ur, dashed, "f"] \arrow[r, "\rho_1"'] & X \end{tikzcd}$$

to produce a map f making the diagram ϵ-commute for ϵ as small as we like. Since X is weakly locally contractible, this implies that the diagram homotopy commutes. The maps ρ_i are weak homotopy equivalences by Corollary 2.11, so f induces isomorphisms on homotopy and is a homotopy equivalence.

To see that f is simple requires a bit more geometry. Reversing the roles of K_1 and K_2 produces a map $g : K_2 \to K_1$ so that $\rho_1 \circ g$ is ϵ-close to ρ_2. The map ρ_1 is 2ϵ-close to $\rho_1 \circ g \circ f$, so the two maps are homotopic by weak local contractibility of X. Lifting this homotopy rel the identity map and $g \circ f$ on the ends, we have a homotopy from $g \circ f$ to id which projects to a small homotopy in X. Symmetry gives a similar homotopy from $f \circ g$ to id. Applying Chapman's Theorem $1'$ or the results of [21] to this homotopy equivalence as in the proof of Theorem C shows that f is simple. The argument also shows that *any* $f : K_1 \to K_2$ making the diagram ϵ-commute for small ϵ is simple.

For part (ii), if $f : X_1 \to X_2$ is a weak homotopy equivalence between weakly locally contractible compacta with finite cohomological dimension, we can find finite polyhedra K_1 and K_2 with CE maps to X_1 and X_2. Lifting in the diagram

$$\begin{tikzcd} & & K_2 \arrow[d, "\rho_2"'] \\ K_1 \arrow[r, "\rho_1"] \arrow[urr, dashed, "\bar f"] & X_1 \arrow[r, "f"] & X_2 \end{tikzcd}$$

produces a homotopy equivalence $\bar{f} : K_1 \to K_2$. We define the torsion of f in $Wh(\mathbb{Z}\pi_1 X_2)$ to be $(\rho_2)_\#(\tau(f))$. An easy application of part (i) shows that this is well-defined.

Remark 5.1. This argument also applies to define the Whitehead torsion of any shape morphism $f : X_1 \to X_2$ which induces isomorphisms on homotopy groups. The point is that by weak local contractibility the shape morphism $f \circ \rho_1$ is represented by a map, so we can follow the same procedure as above, lifting to get a map $\bar{f} : K_1 \to K_2$ and setting $\tau(f) = (\rho_2)_\#(\tau(\bar{f}))$.

6. Homology manifolds

We now consider limits of closed topological n-manifolds in $\mathcal{M}(n, \rho)$. As in [15], one see that such limits are weakly locally contractible homology manifolds with cohomological dimension n. Two questions suggest themselves.

Question 6.1. Is every weakly locally contractible homology manifold with cohomological dimension n a limit of closed ANR homology manifolds in some $\mathcal{M}(n, \rho)$?

Question 6.2. Is every weakly locally contractible homology manifold with cohomological dimension n the cell-like image of a closed ANR homology manifold?

We ask these questions with "closed ANR homology manifold" rather than "topological manifold" because of examples in [3].

Even with this modification, both questions are false as stated. In [9], Dranishnikov and the author produce examples of nonhomeomorphic closed topological manifolds M_1 and M_2 which admit CE maps ρ_1 and ρ_2 onto the same compactum X. Forming $M(\rho_1) \cup_X M(\rho_2)$ and doubling along $M_1 \cup M_2$ gives a closed weakly locally contractible homology manifold with cdim $= n$ which admits no resolution. If such a resolution existed, it could be taken to be the identity near $M_1 \cup M_2$, so the inverse image $M(\rho_1) \cup_X M(\rho_2)$ would be a cobordism from M_1 to M_2. By the material in the previous section, this cobordism would be an s-cobordism and M_1 would be homomorphic to M_2, a contradiction. A similar argument using the α-Approximation Theorem shows that this is a counterexample to Question 6.2, as well. Thus, Questions 6.1 and 6.2 should be modified to ask what the obstructions are to approximating such spaces by closed ANR homology manifolds.

It is not hard to conjecture the answer to this question. If X is such a space and $p : M \to X$ is a CE map from a codimension zero submanifold of $\mathbb{R}^{2n+2}$ to X as in the proof of Theorem A, M is a controlled Poincaré duality space over X and we have a controlled surgery problem

$$\begin{array}{c} M \\ {\scriptstyle p}\big\downarrow \\ X. \end{array}$$

Assuming a version of Ranicki's total surgery obstruction ([22]) for controlled surgery over X, we expect a fibration sequence of spectra

$$\mathcal{S}\begin{pmatrix} M \\ \downarrow \\ X \end{pmatrix} \to \mathcal{H}(M, \mathbb{L}(e)) \to \mathcal{H}(X, \mathbb{L}(e))$$

and a total surgery obstruction $\theta\begin{pmatrix} M \\ \downarrow \\ X \end{pmatrix} \in \pi_{n-1}\mathcal{S}\begin{pmatrix} M \\ \downarrow \\ X \end{pmatrix}$. Here, $\mathbb{L}(e)$ is the periodic L-theory spectrum of the trivial group. Since M is n-dimensional, $\pi_{n-1}\mathcal{H}(M, \mathbb{L}(e)) \cong \pi_{n-1}\mathcal{H}(M, G/TOP \times \mathbb{Z}) \cong \mathcal{H}(X, G/TOP \times \mathbb{Z})$. This last uses the fact that the Vietoris-Begle theorem is true for homology theories which are bounded below. Thus, our putative total surgery obstruction will live in the $(n-1)^{st}$ homotopy group of the fiber of the map

$$\mathcal{H}(X, G/TOP \times \mathbb{Z}) \to \mathcal{H}(X, \mathbb{L}(e))$$

and will vanish if and only if X can be resolved to a closed ANR homology manifold. Assuming the existence of such a theory, we have:

(Conjectural) Corollary. *If X is a weakly locally contractible homology manifold with cohomological dimension n and $H_*(X;\mathbb{Z})$ has no odd torsion, then X admits a resolution by a closed ANR homology manifold.*

The point is that the L-theory spectrum is nearly a product of Eilenberg-MacLane spectra. One can use this to show that if $H_*(X;\mathbb{Z})$ lacks odd torsion, then $\mathcal{H}(X, G/TOP \times \mathbb{Z}) \to \mathcal{H}(X, \mathbb{L}(e))$ is a homotopy equivalence and the obstruction group vanishes.

The total surgery obstruction suggests that there should be two classes of counterexamples to Questions 6.1 and 6.2. The first class is detected by the failure of any resolution to have a suitable tangent bundle, while the second is analogous to Quinn's resolution obstruction. See [3] for references. The example constructed above is of the first kind. One would expect examples of the second kind to be constructed analogously to the examples in [3] with double mapping cylinder singularities like the ones above replacing the mapping cylinder constructions of [3]. The sequence above also suggests that weakly locally contractible homology manifolds with finite cohomological dimension should have most of the rational attributes of topological manifolds, including an appropriate theory of rational characteristic classes. An interesting question in this regard is whether the potential $\varprojlim^1$ term in $H(X, \mathbb{L}(e))$ is ever realized. Realization of this $\varprojlim^1$ would presumably lead to some very strange examples of nonresolvable weakly locally contractible homology manifolds with finite cohomological dimension.

Bibliography

1. K. Borsuk, *On some metrizations of the hyperspace of compact sets*, Fundamenta Mathematicae **41** (1955), 168-202.
2. ———, *Theory of Retracts*, PWN-Polish Scientific Publishers, Warsaw, 1966.
3. J. Bryant, S. Ferry, W. Mio, and S. Weinberger, *Topology of homology manifolds*, Ann. Math. (to appear).
4. T.A. Chapman, *Invariance of torsion and the Borsuk Conjecture*, Canadian J. Math. **32** (1980), 1333–1341.
5. ———, *Simple homotopy theory for ANRs*, Top. and its App. **7** (1977), 165–174.
6. ———, *Cell-like mappings*, Springer Lecture Notes in Mathematics **482** (1973), 230–240.
7. M. M. Cohen, *A course in simple homotopy theory*, Springer-Verlag, New York, 1973.
8. A. N. Dranishnikov, *On a problem of P.S. Alexandroff*, Mathematics of the USSR-Sbornik **63** (1989), 539-545.
9. A. N. Dranishnikov and S. C. Ferry, *Cell-like images of topological manifolds and limits of manifolds in Gromov-Hausdorff space (preprint).*
10. J. Dydak and J. Segal, *Local n-connectivity of decomposition spaces*, Top. and App. **18** (1984), 43–58.
11. S. C. Ferry, *The homeomorphism group of a compact Q-manifold is an ANR*, Ann. of Math. **106** (1977), 101–120.
12. ———, *Topological finiteness theorems in Gromov-Hausdorff space*, Duke Journal of Math. **74** (1994), 95–106.
13. ———, *Polyhedral and TOP Topology*, Oxford University Press (to appear).
14. M. Gromov, J. Lafontaine, and P. Pansu, *Structures metriques pour les Varietes Riemanniannes*, Cedic/Fernand Nathan (1981).
15. K. Grove, P. Petersen, and J. Wu, *Geometric finiteness theorems via controlled topology*, Invent. Math. **99** (1990), 205–213; *Correction to geometric finiteness theorems in controlled topology*, Invent. Math. **104** (1991), 222–223.
16. W. Hurewicz, *Homotopie, Homologie, und lokaler Zusammenhang*, Fund. Math. **35**, 467–485.
17. R.C. Lacher, *Cell-like mappings and their generalizations*, Bul. Amer. Math. Soc. **83** (1977), 495–552.
18. J. Milnor, *Whitehead torsion*, Bul. Amer. Math. Soc. **72** (1966), 358–426.
19. T. Moore, *Gromov-Hausdorff convergence to non-manifolds*, to appear in the J. of Geometric Analysis.
20. P. Petersen V, *A finiteness theorem for metric spaces*, J. Differential Geom. **31** (1990), 387–395.
21. F. Quinn, *Ends of Maps, I*, Ann. Math. **110** (1979), 275–331.
22. A. Ranicki, *The total surgery obstruction*, Springer Lecture Notes in Mathematics **763** (1979), 275–316.
23. J. J. Walsh, *Dimension, cohomological dimension, and cell-like mappings*, Shape Theory and geometric topology, SLN 870, Springer Verlag, 1981, pp. 105–118.
24. J. West, *Mapping Hilbert cube manifolds to ANR's: A solution to a conjecture of Borsuk*, Ann. Math. **106** (1977), 1–18.

Department of Mathematical Sciences, SUNY at Binghamton, Binghamton, NY 13901