VOLUME GROWTH, DE RHAM COHOMOLOGY, AND THE HIGSON COMPACTIFICATION

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Abstract. We construct a variant of (real) DeRham cohomology and apply it to prove that the integral cohomology of the Higson compactification of $\mathbb{R}^n$ has uncountably generated n^{th} integral cohomology.

1. Introduction

The Higson compactification of a non-compact complete Riemannian manifold was introduced by Higson in [Hig92] and modified by Roe in [Roe93]. Here is the construction from [Roe93].

We will say that a metric space is *proper* if closed balls of finite radius are compact.

Definition 1. *Let M and Z be proper metric spaces and let $\phi : M \to Z$ be a bounded continuous function. We define the* variation function $V_r\phi : M \to \mathbb{R}^+$ *by*

$$V_r\phi(x) = \sup\{d(\phi(y), \phi(x)) : y \in B(x; r)\}.$$

Definition 2. *Let M and Z be proper metric spaces. We let $C_h(M; Z)$ denote the space of all bounded continuous functions $\phi : M \to Z$ such that $V_r(\phi) \to 0$ at infinity. We can extend this definition to the case where Z is nonmetric by declaring that $\phi \in C_h(M; Z)$ if $f \circ \phi \in C_h(M; Z)$ for every continuous function $f : Z \to \mathbb{R}$. We will refer to elements of $C_h(M; Z)$ as* slowly oscillating *functions.*

Proposition 1. *$C_h(M; \mathbb{C})$ is a C^*-algebra, which we will denote by $C_h(M)$.*

Proof. This is Lemma 5.3 of [Roe93]. □

Definition 3. *The* Higson compactification *of M, which we denote by $\bar{M}$ is the maximal ideal space of $C_h(M)$. The* Higson corona *of M is $\nu M = \bar{M} - M$.*

The Higson compactification has the universal property that if Z is compact Hausdorff, a continuous function $f : M \to Z$ extends to a continuous function $\bar{f} : \bar{M} \to Z$ if and only if $f \in C_h(M; Z)$.

Remark. For the topologically inclined, the Higson compactification of M may also be constructed as follows. Let

$$e : M \to \prod_{f \in C_h(M;\mathbb{R})} \mathbb{R}$$

be defined by letting the f^{th} coordinate of $e(m)$ be $f(m)$. The closure of $e(M)$ in $\prod_{f \in C_h(M;\mathbb{R})} \mathbb{R}$ is the Higson compactification of M. One sees that $e(M)$ is precompact by noting that since the functions $f \in C_h(M; \mathbb{R})$ are bounded, $e(M)$ is contained in a product of closed intervals.

Date: July 22, 2007.

The Higson corona is interesting in its own right as a coarse invariant of proper metric spaces, but the compactification comes up naturally in attacks on the Novikov conjecture and the Gromov-Lawson conjecture. Typically, the proper metric space is the universal cover of a finite $K(\Gamma, 1)$. The Higson compactification is then the universal compactification in which fundamental domains shrink to points at infinity. In [Roe93], it was conjectured that if a space M is uniformly contractible, then the Higson compactification $\bar{M}$ of M has trivial cohomology. This would imply the rational Novikov conjecture for groups Γ such that $B\Gamma$ is a finite complex, and the Gromov-Lawson conjecture, which says that a closed Riemannian manifold with contractible universal cover cannot admit a metric of positive scalar curvature.

Unfortunately, in [Kee94], it was shown that the first integral cohomology of $\bar{M}$ is nontrivial for all noncompact M and in [DF97] it was shown that the Higson compactification of $\mathbb{R}^n$ has nonzero integral cohomology in all dimensions $1 \ldots n$, dimension n being the critical dimension for the Novikov and Gromov-Lawson conjectures.

Recently, Dranishnikov, Ferry, and Weinberger [DFW] have shown that if Γ is a group such that $B\Gamma$ is a finite complex and $E\Gamma$ has finite asymptotic dimension, then the cohomology of $\bar{M}$ is trivial with finite coefficients. In the same paper, it was shown that the rational Novikov conjecture for a group Γ with $B\Gamma$ a finite complex follows from the vanishing of the mod 2 cohomology of $\bar{E\Gamma}$. This means that integral vanishing results do not tell the entire story and that there is hope for using Higson's compactification to prove the Gromov-Lawson conjecture and an interesting general Novikov results.

If M is a complete Riemannian n-manifold, an element of $H^n(\bar{M})$ is represented by a map $\phi : \bar{M} \to K(\mathbb{Z}, n)$. The image of $\bar{M}$ is compact, so it lies in a finite skeleton L of $K(\mathbb{Z}, n)$, which we embed in a high-dimensional euclidean space and thicken up to a smooth compact codimension-zero manifold, N homotopy equivalent to L. The generator of $H^n(N; \mathbb{Z})$ is represented in De Rham cohomology by a closed n-form ω.

We will show that ϕ can be taken to be a C^∞ slowly oscillating function on $\mathbb{R}^n$ and that $\phi^*\omega$ is a well-defined element of the "slowly oscillating cohomology" of $\mathbb{R}^n$, depending only on the original cohomology class in $H^n(\bar{\mathbb{R}}^n; \mathbb{Z})$. We then show that an invariant involving volume growth of slowly oscillating classes can be used to detect uncountably many different classes in $H^n(\bar{\mathbb{R}}^n; \mathbb{Z})$. This all suggests the possibility that the integral cohomology groups $H^n(\bar{M}; \mathbb{Z})$ may posses the structure of a real vector space, which would imply the Novikov and Gromov-Lawson conjectures. Block and Weinberger, [BW92], encountered a phenomenon of this sort in their construction of "uniformly finite" homology.

2. Smoothing slowly oscillating functions

We begin with a useful lemma.

Lemma 1. *If $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous monotone increasing function with $\lim_{x\to\infty} \rho(x) = \infty$, then there is a continuous monotone C^∞ function $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ so that*

- *i. There is a $K > 0$ so that $\mu(x) < \rho(x)$ for all $x > K$.*
- *ii. $\lim_{x\to\infty} \mu(x) = \infty$ and so that*
- *iii. $\lim_{x\to\infty} \frac{d^k\mu}{dx^k}(x) = 0$ for all $k \geq 1$.*

Proof. Choose a sequence $a(n)$ so that $a(1) \geq 8$, $\rho(a(n)) > n+2$, and so that $a(n+1) > 2a(n)$ for all n. Let $\sigma(x)$ be the piecewise linear function interpolating the points $(a(n), n)$. The function σ is monotone increasing and $\sigma(x) < \rho(x) - 1$ for all x.

Let $\Psi : \mathbb{R} \to \mathbb{R}$ be a C^∞ function supported on $[-1,1]$ so that $\Psi(x) \geq 0$ for all x, $\int_{-\infty}^{\infty} \Psi(x)\, dx = 1$, and $\Psi(x) = \Psi(-x)$ for all x.

Let

$$\mu(x) = \int_{-2^n}^{2^n} \frac{1}{2^n} \Psi\left(\frac{x-t}{2^n}\right) \sigma(t)\, dt \quad \text{for} \quad \frac{a(n)+a(n+1)}{2} \leq x \leq \frac{a(n+1)+a(n+2)}{2}.$$

This function is C^∞ because it is equal to $\sigma(x)$ in a neighborhood of the points $\frac{a(n)+a(n+1)}{2}$, so the piecewise definition splices together to give a C^∞ function. Differentiation under the integral sign shows that all derivatives of μ go to 0 at infinity. For instance, the value of $\mu'(x)$ on $[\frac{a(n)+a(n+1)}{2}, \frac{a(n+1)+a(n+2)}{2}]$ is less than $\frac{n+2}{2^n}$ times the maximum value of $\Psi'(x)$ on $[-1,1]$. This defines $\mu(x)$ for $x \geq K = \frac{a(1)+a(2)}{2}$ and we extend to a monotone C^∞ function on $\mathbb{R}^+$. □

Lemma 2. *If $f : \mathbb{R}^n \to \mathbb{R}$ is a slowly oscillating function and $\epsilon > 0$ is given, f can be ϵ-approximated by a smooth function $\bar{f}$ so that all derivatives of $\bar{f}$ go to zero at infinity and so that $f - \bar{f}$ is slowly oscillating.*

Proof. Choose a monotone increasing function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ so that

i. $V_{\rho(|x|)} f(x) < \epsilon$ for all x
ii. $\lim_{x\to\infty} V_{\rho(|x|)} f(x) = 0$
iii. $\lim_{x\to\infty} \rho(x) = \infty$.

Let μ be a monotone increasing smooth function less than ρ with derivatives going to 0 as in Lemma 1. Define $\bar{f}$ by

$$\bar{f}(\mathbf{x}) = \int_{\mathbb{R}^n} \left[\frac{1}{\mu(|\mathbf{x}|)}\right]^n \Psi\left(\frac{|\mathbf{x}-\mathbf{y}|}{\mu(|\mathbf{x}|)}\right) f(\mathbf{y})\, d\mathbf{y}$$

Here, Ψ is a smooth nonnegative C^∞ function supported on $[-1,1]$ and constant in a neighborhood of 0 such that $\int_{\mathbb{R}^n} \Psi(|\mathbf{x}|) = \mathbf{1}$. The function $\bar{f}$ approximates f because it averages f over balls of radius $\mu(\mathbf{x})$ where f is nearly constant. The derivatives of $\bar{f}$ go to zero because the derivatives of Ψ are bounded, $\mu(\mathbf{x})$, which appears in denominators, goes to infinity, and the derivatives of μ, which appear in numerators, go to zero. □

Lemma 3. *If f and $\bar{f}$ are as above, then $\bar{f}$ extends to a map $\bar{\mathbb{R}}^n \to \mathbb{R}^n$ and $f|\nu\mathbb{R}^n = \bar{f}|\nu\mathbb{R}^n$.*

□

Remark. It seems like this averaging process should work on a manifold of bounded geometry by pulling f back to the tangent space at $\mathbf{x}$ and averaging as above.

3. The cohomology of $\bar{\mathbb{R}}^n$

Let $h : \bar{\mathbb{R}}^n \times I \to K(\mathbb{Z}, n)$ be a homotopy. Since $\bar{\mathbb{R}}^n$ is compact, this homotopy lies in a skeleton L of $K(\mathbb{Z}, n)$. We embed L into $\mathbb{R}^\ell$, ℓ large, and let N be a regular neighborhood of L in $\mathbb{R}^\ell$. Choose $\epsilon > 0$ so that balls of radius 100ϵ centered at points of L are contained in N

and choose a sequence of points $0 = t_0 < t_1 < \ldots < t_k = 1$ so that $\mathrm{diam}(h(\mathbf{x} \times [t_i, t_{i+1}])) < \epsilon$ for $i = 0 \ldots k-1$ and $\mathbf{x} \in \bar{\mathbb{R}}^n$.

Approximate $h|\mathbb{R}^n \times t_i$ by a smooth function with derivatives going to zero by approximating its coordinate functions as in Lemma 2. Extend to $\bar{\mathbb{R}}^n$ and replace h by straight line homotopies between the approximations on the intervals $\bar{\mathbb{R}}^n \times [t_i, t_{i+1}]$.

Definition 4. *We will say that a smooth $k-$form α on $\mathbb{R}^n$ is* slowly oscillating *if*

$$\alpha = \sum_I a_I(x_1, \ldots, x_n)\, dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

where the a_I's are bounded C^∞ functions with all derivatives going to 0 at infinity. We will denote the vector space of slowly oscillating $k-$forms on M by $\Omega^k_{SO}(M)$.

Theorem 1 (Slowly oscillating Poincaré lemma)**.** *If $f, g : \bar{\mathbb{R}}^n \to L$ are homotopic and $\omega \in \Omega^k(N)$ is a smooth $k-$form, then $f^*\omega - g^*\omega = d\alpha$, where α is a smooth slowly oscillating $(k-1)-$form on $\mathbb{R}^n$.*

Proof. We begin by approximating f and g by smooth slowly oscillating maps as in Lemma 2. For ϵ small, the approximations are homotopic to the original maps by straight line homotopies in N. By breaking the homotopy from f to g into pieces as above, we can assume that there is a straight-line homotopy $h(t, \mathbf{x}) = t \cdot f(\mathbf{x}) + (1-t) \cdot g(\mathbf{x})$ connecting f and g.

We adapt the argument from p. 28 of [Fla63]. If U is an open domain in euclidean space, the author constructs a map

$$K : \Omega^{k+1}(I \times U) \to \Omega^k(U)$$

so that

$$Kd + dK = j_1^* - j_0^*.$$

where j_0 and j_1 are inclusions into the bottom and top of $I \times U$. The formula for K on monomials is $K(a(t, \mathbf{x}))\, d\mathbf{x}^{\mathbf{H}} = 0$ and $K(a(t, \mathbf{x})\, dt \wedge d\mathbf{x}^{\mathbf{H}}) = \left(\int_0^1 a(t, \mathbf{x})\, dt\right) d\mathbf{x}^{\mathbf{H}}$.

Writing $f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_\ell(\mathbf{x}))$, $g(\mathbf{x}) = (g_1(\mathbf{x}), \ldots, g_\ell(\mathbf{x}))$, and $\omega = \sum a_H(\mathbf{z})\, dz^{\mathbf{H}}$, we have

$$h^*(\omega) = \sum_{H=(i_1, \ldots, i_k)} a_H(h(t, \mathbf{x}))\, dh_{i_1} \wedge \ldots dh_{i_k}$$

and

$$dh_{i_j} = (f_{i_j} - g_{i_j})\, dt + \sum_{k=1}^{n} \frac{\partial h_{i_j}}{\partial x_k}\, dx_k$$

The functions $a_H(\mathbf{z})$ are bounded so the functions $\int_0^1 a_H(h(t, \mathbf{x}))\, dt$ are slowly oscillating. The functions $(f_{i_j} - g_{i_j})$ and $\frac{\partial h_{i_j}}{\partial x_k}$ are slowly oscillating, so $\alpha = Kh^*(\omega)$ is a slowly oscillating form. Since $d\alpha = g^*\omega - f^*\omega$, the proof is complete.

□

Definition 5. *We define the* slowly oscillating cohomology *of* $\mathbb{R}^n$ *to be the quotient*

$$H^k_{SO}(\mathbb{R}^n) = \frac{\ker d : \Omega^k_{SO} \to \Omega^{k+1}_{SO}}{\operatorname{im} d : \Omega^{k-1}_{SO} \to \Omega^k_{SO}}$$

It follows immediately from the Poincaré Lemma that there is a well-defined map

$$H^k(\bar{\mathbb{R}}^n;\, \mathbb{Z}) \to H^k_{SO}(\mathbb{R}^n)$$

obtained by taking a form ω on a regular neighborhood of a skeleton of $K(\mathbb{Z};\, k)$ and pulling it back to a slowly oscillating form on $\mathbb{R}^n$.

If ω is a slowly oscillating smooth n-form on $\mathbb{R}^n$, we can define a map $\gamma(\omega) : \mathbb{R}^+ \to \mathbb{R}$ by $\gamma(\omega)(r) = \int_{B(0;\, r)} \omega$. If two such forms represent the same cohomology class, we have

$$\int_{B(0;\, r)} \omega_1 - \omega_2 = \int_{B(0;\, r)} d\alpha = \int_{S(0;\, r)} \alpha$$

where α is slowly oscillating, so $\gamma(\omega_1) - \gamma(\omega_2)$ is $O(r^{n-1})$. This construction allows us to distinguish uncountably many cohomology classes in $H^n(\bar{\mathbb{R}}^n;\, \mathbb{Z})$.

Example 1. *Let* $c : T^2 \to S^2$ *be a smooth map that crushes the wedge of two circles to a point. Let* $e : \mathbb{R}^2 \to T^2$ *be the covering map and let* $f : \mathbb{R}^2 \to S^2$ *be the map* $c \circ e(x^\alpha, y^\alpha)$ *with* $0 < \alpha < 1$*. Let* ω *be the volume form on* S^2*, so the integral of* $f^*\omega$ *over a square of side* $m^{1/\alpha}$ *is* $4\pi m^2$ *with a possible variation of* $O(m^{1/\alpha})$ *depending on the choice of representative in the cohomology class.*

Now consider $f_t(x, y) = c \circ e(tx^\alpha, ty^\alpha)$*, with* $t \in \mathbb{R}$ *a constant. The integral of* $f_t^*\omega$ *over the same square of side* $m^{1/\alpha}$ *is approximately* t^2 *times the value of the earlier integral, so the variation obtained using different values of* t *is* $O(m^2)$*. This means that for* $1/2 < \alpha < 1$*, we obtain uncountably many different elements of* $H^2(\bar{\mathbb{R}}^2;\, \mathbb{Z})$ *by varying* t*.*

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