1. Find a polynomial $P(x)$ such that $P(x)$ is divisible by $x^2 + 1$ and $P(x) + 1$ is divisible by $x^3 + x^2 + 1$.

2. Let n be any natural number. Prove that the fraction $(n^3 + 2n)/(n^4 + 3n^2 + 1)$ is in lowest terms.

3. Let $H(x)$ be a polynomial with integer coefficients. Suppose that a, b, c, d are distinct integers such that $H(a) = H(b) = H(c) = H(d) = 5$. Prove that there is no integer k such that $H(k) = 8$.

4. Let $f(x)$ be a polynomial with integer coefficients. Prove that if 100 is not a divisor of any of the number $f(1)$, $f(2)$, $\ldots$, $f(100)$ then f has no integer roots.

5. Let $p(x)$ be a polynomial with complex coefficients. Prove that there is a polynomial multiple $q(x)$ of $p(x)$ such that all of the exponents that appear with nonzero coefficients are multiples of 1000.

6. Suppose that $p(x)$ is a polynomial of degree n with integer coefficients. and let a_i be the coefficient of x^i. Suppose that b is a rational number that is a root of $p(x)$ and $b = r/s$ is a fraction in lowest terms. Prove that r divides a_0 and s divides a_n.

7. Suppose that $p(x)$ is a polynomial of degree n with integer coefficients. Suppose that a_0, a_n, and $f(1)$ are all odd. Prove that $p(x)$ has no rational roots.

8. Let $p(x)$ be a real polynomial of degree 4. Suppose that L_1 and L_2 are lines in the plane that each intersect the curve C given by $y = p(x)$ in four distinct points. Let x_1, x_2, x_3, x_4 be the x coordinates of the intersection of L_1 with C and y_1, y_2, y_3, y_4 be the x coordinates of the intersection of L_2 with C. Prove that $x_1+x_2+x_3+x_4 = y_1+y_2+y_3+y_4$.