Lecture Notes 9

Banach-Alaoglu & Krein-Milman theorems

Banach-Alaoglu Applications of weak, weak*

Theorem Let V be a TVS, U any open nbd of 0. Then

$$K = \{ \Lambda \in V^* \mid |\Lambda v| \le 1 \text{ for all } v \in U \},$$
("polar of U")

is a convex compact subset of V^* under the weak* topology.

(Recall: weak* topology means all $v \in V$ induce cts linear functionals on V^*)

Pf K = Inverse image of a convex set $\{ z \in k \mid |z| \le 1 \}$ under linear map

$$\Lambda \mapsto \Lambda v,$$

hence convex itself.

For each $v \in V$, $\exists$ $c = c(v) > 0$ s.t. $c^{-1} v \in U$
(absorbing)

Thus for each $\Lambda \in K$ & $v \in V$

$$|\Lambda(c^{-1} v)| \le 1 \iff |\Lambda(v)| \le c(v)$$

Let $P = \{\text{functions } f: V \to k \mid |f(v)| \le c(v)\}$

$= $ Product of $\{|k| \le c(v)\}$ over all $v \in V$

Compact in product topology under Tychonov thm

(not box ~~sets are~~ topology)

So $\Lambda \in K \Rightarrow \Lambda \in P$.

We now claim that ~~the~~ the weak* top on $K \subseteq X^*$ agrees w/ the product topology on P (as a topological subspace).

Pf of claim: Let's compare what a local base looks like in each topology around an arbitrary point $\Lambda_0 \in K$.

In the weak* top, they are finite intersections of sets of the form

$$W_1 = \{\Lambda \in V^* \mid |(\Lambda - \Lambda_0)(v_i)| < \delta\}$$

In the P topology:

$$W_2 = \{f \in P \mid |(f - \Lambda_0)(v_i)| < \delta\}$$

for $v_1, \dots, v_n$ a finite set of points in V.

So for $K \subseteq P \cap V^*$, they agree. ✓

Another claim: K is closed in P

Pf of claim: let f_0 be in the closure of $K \subseteq P$. Then for any $\varepsilon > 0$

$$\{ f \in P \mid |f^{(x)} - f_0^{(x)}| < \varepsilon \} \text{ is}$$

an open subset of f_0, hence intersects K.

We need to show this f_0 is linear & $|f_0(x)| \leq 1$ $\forall x \in \mathcal{U}$.

Let now x, y, $\alpha x + \beta y$ be points in V. Then

$$\{ f \in P \mid |(f - f_0)(x)| < \varepsilon \} \cap \{ f \in P \mid |(f - f_0)(y)| < \varepsilon \} \cap \{ f \in P \mid |(f - f_0)(\alpha x + \beta y)| < \varepsilon \}$$

is also an open nbhd in P, intersecting K.

Let f be in the intersection w/ K.

Then

$$|f_0(\alpha x + \beta y) - \cancel{\alpha} f_0(x) - \beta f_0(y)| + f(\alpha x + \beta y) - \alpha f(x) - \beta(f(y))$$

"0" since f is lin

$$\leq |(f - f_0)(\alpha x + \beta y)| + |\alpha (f - f_0)(x)| + |\beta (f - f_0)(y)|$$

$$\leq (1 + |\alpha| + |\beta|)\varepsilon.$$

Take $\varepsilon \to 0$ (it may be chosen indep of α, β, x, y, but not f)

Then $f_0(\alpha x + \beta y) = \alpha f_0(x) + \beta f_0(y)$ p. 4/E

$\Rightarrow f_0$ is linear.

Now we prove the bound:

Let $x \in U$ & $\varepsilon > 0$,

Then by openness,

$$\{f \in P \mid |(f - f_0)(x)| < \varepsilon\}$$

intersects K, so ~~let f~~ be in this intersection

We have

$$|f(x)| \le |(f - f_0)(x)| + |f_0(x)|$$

$$\le \varepsilon + 1$$

$\varepsilon \to 0 \Rightarrow |f(x)| \le 1$ ✓

To conclude, we have shown

$K \subseteq P$ is closed,

P is cpt $\Rightarrow K$ is ~~closed~~ cpt,

in P-topology,

$\Downarrow$

K is compact, as a subspace of V in weak* topology.

this is because open sets in V restrict to open sets in subspace topology.

$\Downarrow$

K is cpt in V in weak* topology $\square$.

Strengthening

Suppose V is separable (contains a countable dense set)
Then

~~Lemma If V is separable, & $K \subseteq V^*$ weak~~

Thm If U is an open nbhd of 0 in a separable TVS V, & $\{\Lambda_n\} \subseteq V^*$ is a sequence s.t. $|\Lambda_n v| \leq 1$ for all $v \in U$, then $\{\Lambda_n\}$ has a convergent subsequence in the weak* topology whose limit is in V^*.

(the "polar" set K in the Banach-Alaoglu thm is "sequentially cpt" in the weak*-topology).

Pf Let $\{x_n\}$ be a countable dense subset of V, & let $\lambda_n(\Lambda) = \Lambda x_n$.

λ_n weak* cts by def'n,

Separates points, since x_n is dense (cts f'n determined by values on a dense set)

Make a metric:

$$d(\Lambda_1, \Lambda_2) = \sum_{n=1}^{\infty} 2^{-n} |\lambda_n(\Lambda_1 - \Lambda_2)|$$

λ_n is bdd if $\Lambda_1, \Lambda_2 \in K$. (sum converges)

$d(\Lambda_1, \Lambda_2) = 0 \Rightarrow \Lambda_1 = \Lambda_2$ since λ_n separate points

The result follows if we can show these metrics agrees, for the σ w/ the weak* top

K is metrizable, & cptness $\Rightarrow$ sequential compactness.

$d(\lambda_1, \lambda_2)$ is a cts f'n on K, since each $\partial_n(\lambda_1 - \lambda_2)$ is.

Hence ~~a~~ balls $\{\lambda_2 \mid d(\lambda_1, \lambda_2) < \varepsilon\}$ for λ_1, ε fixed are open.

Meanwhile metric is Hausdorff, ~~so~~ it ~~and a cpt finer cpt topology~~ mus and a Hausdorff space has no finer topology

Pf If it did, let F be a ~~cpt~~ closed hence cpt subset in the finer topology.
Any open cover in the courser one is an open cover in the finer one. So F is cpt in both, & a cpt subset of a Hausdorff space is closed

$\square$

<u>Thm</u> Let V be a locally convex TVS. Then weak bddness is equivalent to regular bddness.

<u>Pf</u>

Recall weak is courser than regular, so bdd in regular $\Rightarrow$ bdd on weak.

If $E \subseteq V$ is a (~~regular~~ weakly) bdd subset & U is a open nhd of 0 (in usual top), we need to show $E \subseteq tU$ for all suff. large t.

local convexity: we may assume $\exists\, U_1$ open nhd of 0 is balanced, convex, w/ $\overline{U_1} \subseteq U$.

Let $K = \{ \lambda \in V^* \mid |\lambda v| \le 1 \text{ for all } v \in U_1 \}$
polar of U_1

Now, claim: $\overline{U_1} = \{ v \in V \mid |\lambda v| \le 1 \text{ for all } \lambda \in K \}$ closed set, obvious by cts

Pf Need to show this set $\subseteq \overline{U_1}$ (other direction obvious). If $v_0 \in V - \overline{U_1}$, then last separation thm applied to $\overline{U_1}$ (closed, balanced, convex) implies $\lambda \in V^*$ w/ $|\lambda \overline{U_1}| \le 1$, $\lambda v_0 > 1$, so $v_0 \notin$ set on RHS.

E weakly bdd $\Rightarrow$ each open elt of the form contains $\{v \mid |\lambda v| < \varepsilon\}$,

$t^{-1}E$ for t sufficiently large.

So for all $v \in E$ $|\lambda v| \le c$, $c = c(v)$, all $\lambda \in V^*$

K = convex, weak*-cpt

$\lambda \mapsto \lambda v$ weak* cts

equivalence of weak bddness

Baire version $\left[\begin{array}{l} \{\lambda x \mid \lambda \in \Gamma\} \\ \text{bdd in } Y \\ \text{for each } x \text{ in a cpt set } C \end{array} \right. \Rightarrow \left. \begin{array}{l} \exists \text{ bdd set} \\ \lambda(C) \subseteq \text{ fixed bdd set } B \\ \text{for all } \lambda \in \Gamma \end{array} \right]$

$\left(\begin{array}{l} \text{w/ domain } X^* \ \& \ Y = k = \text{scalars} \\ C = K \subseteq V^* \\ \Gamma = E \to (V^*)^* \end{array} \right)$

IMPLIES

that $\exists$ const $C > 0$ s.t. $C = C(V)$

$$\boxed{|\lambda v| \le C \text{ for all } v \in E, \lambda \in K.}$$

So $C^{-1}v \in \underline{U_1}$ by earlier claim.

$\Rightarrow E \subseteq C U_1 \subseteq tU_1$ for all $t \ge C$

(uses the fact $CU_1 \subseteq tU_1$ since U_1 is balanced) $\square$

Corollary ① If V is a (locally convex) normed TVS & E is a subset on which each cts lin ful is bdd, then $\exists\ C>0$ s.t. $\|e\| \le C$ for all $e \in E$.

weakboldness →

② If V is a TVS on which V^* separates points, & A & B are disjoint, nonempty, cpt, convex subsets, then $\exists\ \lambda \in V^*$ s.t.

$$\sup_{x \in A} \operatorname{Re} \lambda x < \inf_{y \in B} \operatorname{Re} \lambda b.$$

Pf ① immediate from thm

② B like separating thm (since locally cpt ⇒ V^* separates points, ~~then~~ ~~today~~ — itself a separation thm!)

A & B are cpt in the (weaker) weak topology, since cptness holds in weaker topologies.

Weak top Hausdorff when pts are separated, so A & B are (cpt ⇒) closed in this Hausdorff weak. top.

Weak top locally convex (open nbhds of 0 have form $\{|\lambda_i v| < \varepsilon\}$

Separation thm says there exists $\lambda \in (V_{weak})^*$ w/ our propts.

$\operatorname{Re} \lambda a \le c_1 < c_2 \le \operatorname{Re} \lambda b$.

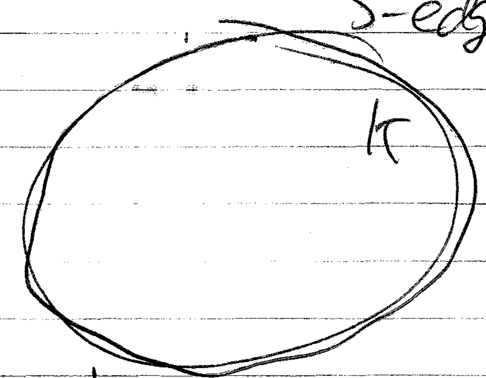
Since $V^* = (V_{weak})^*$ we are done, $\Box$. p.10/18

Back to geometry of Convex Sets

Definitions if $S \subseteq K \subseteq V$ = vector space, S is an *extreme set of* K if ~~for all~~

$$\begin{matrix} x \in K \\ y \in K \\ 0 < t < 1 \\ tx + (1-t)y \in S \end{matrix} \Longrightarrow \begin{matrix} x \in S \\ y \in S \end{matrix}$$

Picture

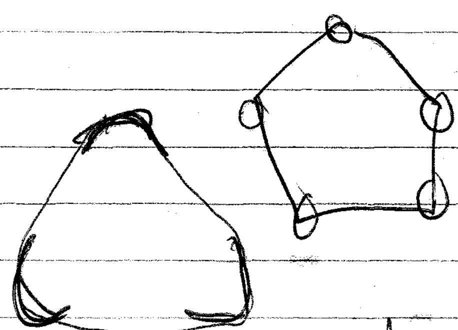


Extreme sets are closed under intersection (unless $S = \emptyset$ ...) then meaning is silly

no point in S is an internal pt of ~~one of~~ line segment whose end points lie in $K - S$.

$\boxed{\text{Example: } \quad , K \subseteq K \text{ is an extreme set in itself.}}$

Extreme Point Extreme set S w/ exactly one member



$\exists$ extreme points

Convex Hull of E = smallest convex set containing E $= \bigcap$ of all convex sets containing E.

closed convex hull = closure (of convex hull) p11/18

Famous Krein-Milman Thm Let V be a TVS on which V^* separates points. Then any cpt convex K subset of V is the closed convexhull of its extreme points.

Proof Observe that if $\lambda \in V^*$ has maximum value of $\operatorname{Re} \lambda$ equal to μ on an extreme set $S \subseteq K$, ~~then~~ & ~~if $\operatorname{Re} x \ldots \leq \mu$,~~ ~~then~~

Then $S_\lambda = \{v \in S \mid \operatorname{Re} \lambda v = \mu\}$

~~then $x \in K$, $y \in K$, $0<t<1$, $z = tx+(1-t)y$ $\Rightarrow$ $\operatorname{Re}\lambda z = \mu$ as $\operatorname{Re}\lambda x, \lambda y \leq \mu$~~

~~Since. $\operatorname{Re}\lambda z = t(\operatorname{Re}\lambda x) + (1-t)(\operatorname{Re}\lambda y)$~~

is also an extreme set.

This is because $x, y \in K$, $0<t<1$, $tx+(1-t)y \in S_\lambda \subseteq S$ $\Rightarrow$ $t \operatorname{Re}\lambda x + (1-t)\operatorname{Re}\lambda y = \mu$

$\Downarrow$ S extreme

$x, y \in S \Rightarrow \operatorname{Re}\lambda x, \operatorname{Re}\lambda y \leq \mu$

conclude $\operatorname{Re}\lambda x, \lambda y = \mu$

$\Rightarrow x, y \in S_\lambda$. Hence S_λ extreme

Claim every cpt extremeset S of K contains an extreme point.

Pf Use Hausdorff Maximality Principle to get maximal chain of smaller & smaller cpt extreme sets. By compactness (actually finite intersection property), the intersection M of such a chain is nonempty & also an extreme set. It is cpt, since the $\cap$ of closed sets is closed, & these are all contained in the cpt set S.

Let $\Lambda \in V^*$. Then M is cpt & $\mathrm{Re}\,\Lambda(M)$ has a maximal value μ. The set
$\{v \in S \mid \mathrm{Re}\,\Lambda v = \mu\}$ is closed, hence cpt,
& we already saw it is extreme. We therefore know, by the maximality of that chain, that it equals M.

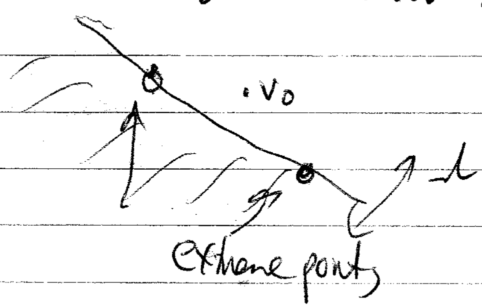
Thus all $\Lambda \in V^*$ have constant real part on M. So does $i\Lambda$, so all $\Lambda \in V^*$ are constant on M. But V^* separates points, so M is thus a singleton. ✓

Let H = convex hull of the extreme points. We must show $\bar{H} = K$. Obviously $H \subseteq K$ & thus $\bar{H} \subseteq K$ since K is convex & closed. So $\bar{H}$ is also cpt. We need to prove $K \subseteq \bar{H}$. Let $v_0 \in K \sim \bar{H}$ (we will show this is impossible). So $\{v_0\}$ & $\bar{H}$ are disjoint, nonempty, convex, cpt sets.

In the previous thm we showed $\exists\, \lambda \in V^*$ s.t. $\sup_{v\in \bar{H}} \operatorname{Re}\lambda v < \operatorname{Re}\lambda v_0$. Actually $\operatorname{Re}\lambda(\bar{H})$ is cpt, so we infact have the stronger statement:

$$\operatorname{Re}\lambda v < \operatorname{Re}\lambda v_0 \quad \forall v \in \bar{H}.$$

Now, K is a compact extreme set of K,
Hence $K_\lambda = \{v \in K \mid \operatorname{Re}\lambda v = \mu\}$ is (μ = max of $\operatorname{Re}\lambda$ on K) also a cpt extreme set.
Hence it has an extreme point (claim on last page) but extreme points have ~~even large strict~~
$\operatorname{Re}\lambda v < \operatorname{Re}\lambda v_0 \le \mu$, a contradiction.



Another version of Krein-Milman: Let V = locally convex TVS, E set of extreme points of a cpt set K in X. Then
$K \subseteq$ closed convex hull of E.

Pf locally convex $\Rightarrow V^*$ separates points.
The only place convexity of K was used was to prove $\bar{H} \subseteq K$, & hence get ~~get~~ that $\bar{H} \subseteq K$ & by compactness $\bar{H} = K$ learn $\bar{H}$ is hence cpt.
~~we~~ If ~~$\bar{H}$~~ H is not cpt, we cannot use that previous thm to separate v_0 from $\bar{H}$. However, that result was

already established for locally convex spaces in this thm, proven earlier. $\square$

[Prior thm If A & B are disjoint, nonempty, convex sets in a locally convex TVS, A cpt, B closed, then $\exists\, \lambda \in V^*$ & $c_1, c_2 \in \mathbb{R}$ s.t. $\operatorname{Re} \lambda a < c_1 < c_2 < \operatorname{Re} \lambda b$ $a \in A$ $b \in B$.]

Notion of "Total boundedness" – a subset A of a metric space X is totally bdd if forall $\varepsilon > 0$ $A \subseteq$ finite union of open balls of radius ε. (of course, as $\varepsilon \to 0$ you take more balls).

TVS version of total bddness (doesn't use metric spaces)

$A \subseteq V$ is totally bdd if forall open nbhds U of 0 $\exists$ a finite set S s.t.

$$A \subseteq S + U.$$

(of course this agrees w/ metric space def'n when V is metrizable).

Back to Convexity now.

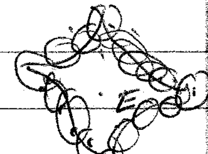
Theorem Let V = locally convex TVS
H = convex hull of a totally bdd set E.
Then H is totally bdd also.

Pf Let U' = open nbhd of 0. Then U' has a convex open nbhd U s.t. $U+U \subseteq U'$.

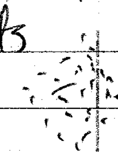
E totally bdd $\Rightarrow$ $\exists$ finite set F s.t.
$E \subseteq F+U$.

~~Let~~

Let H' = convex hull of F



F = (discrete) set of points



So H' looks like ~~a~~ ~~...~~ an approx to H.

Intuitively, H' is cpt.

Formally, H' = convex linear combo of a finite number of points $\{f_1, \dots, f_n\}$:

$$H' = \Big\{ \sum_{j \le n} t_j f_j \;\Big|\; \sum_{j \le n} t_j = 1,\ t_j \ge 0 \Big\}$$

So $[0,1]^n \to H'$ onto, continuously
$(t_1, \dots, t_n) \in$
so image H' is cpt.

Let $v \in H$ = convex hull of E.

$$H = \{\text{all } \textstyle\sum t_j e_j \mid \sum t_j = 1,\ t_j \ge 0\}.$$

So $v = \sum_{j \le m} t_j e_j$ for some finite set $\{e_1, \dots, e_m\} \subseteq E$.

$E \subseteq F+U$, so $\exists$ pts $\{d_1, \dots, d_m\} \in F$ s.t.
$e_j - d_j \in U$, $j \le m$.

Write $v = v_1 + v_2$, $v_1 = \sum t_j d_j$, $\underbrace{\sum t_j (e_j - d_j)}_{v_2}$.

Now $v_2 \in U$ by local convexity of U,

$v_1 \in H'$ = convex hull of F

So $H \subseteq H' + U$.

H' is cpt, so $\exists$ finite set S such that
$$H' \subseteq S + U.$$

So $H \subseteq S + U + U \subseteq S + U'$.

U' = arbitrary open nbhd of 0,

so H totally bdd. $\square$

Thm V = Frechet space, H = convex hull of cpt set K. Then
$$\overline{H} \text{ is cpt.}$$

Pf H is totally bdd by last result. (cpt $\Rightarrow$ totally bdd, of course)

~~Frechet space~~

~~Let $\{U_\alpha\}$ be an open cover of $\overline{H} = \{v \in V \text{ s.t. } \forall \varepsilon > 0\ \exists h \in H \text{ w/ } d(h,v) < \varepsilon\}$~~

~~(Recall: Frechet spaces are complete metric spaces).~~

~~It has a finite subcover over H~~

~~$\forall \varepsilon > 0$, $H \subseteq S + B_0(\varepsilon)$~~

To show $\overline{H}$ is cpt, we need to show only that it is sequentially cpt, i.e. that all convergent sequences have convergent subsequences (this is equivalent to cptness in a complete metric space), like a Frechet space for example.

Let x_n be a subsequence, ~~& write H~~ let $\varepsilon > 0$, & write $H \subseteq S_\varepsilon + B_0(\frac{\varepsilon}{2})$, where S_ε is a finite set (depending on ε).

Then $\overline{H} \subseteq \overline{S_\varepsilon + B_0(\frac{\varepsilon}{2})} \subseteq S_\varepsilon + B_0(\varepsilon)$.

($\Rightarrow \overline{H}$ is totally bdd)

So there an infinite subsequence in one $s + B_0(\varepsilon)$ for some $s + B_0(\varepsilon)$,
$= B_s(\varepsilon)$,

Now repeat using $\overline{H} \cap \overline{B_s(\varepsilon)}$ also totally bdd instead

⋮

get a Cauchy sequence lying in $\overline{H}$. It has a limit since we are in a complete space. Since $\overline{H}$ is closed, the limit is in $\overline{H}$. Thus $\overline{H}$ is sequentially compact.

$\square$

~~Now, if If V $\mathbb{R}^n$~~

Lemma If v lies in the convex hull of a subset E in $\mathbb{R}^n$, it lies furthermore in the convex hull of a subset having at most $n+1$ points

Pf Convex Hull of E = all convex combinations of points of E.

Suppose $v \in E$ is $\sum_{j \le r} t_j e_j \qquad \sum t_j = 1,\ t_j \geq 0$

& $r > n+1$.

Then we can reduce r to $r-1$ changing points using linear dependence! $\exists\, \alpha_j,\ j \le r+1$ not all zero w/ $\sum \alpha_j = 0$ & $\sum \alpha_j e_j = 0$.

Pick $m \le r+1$ so that

$$\left|\frac{\alpha_i}{t_i}\right| \le \left|\frac{\alpha_m}{t_m}\right| \quad \text{for all } i \le r+1,$$

Define $c_i = t_i - \alpha_i \frac{t_m}{a_m}$,

so that $c_i \geq 0$

$$\sum c_i = \sum \left(t_i - \alpha_i \frac{t_m}{a_m}\right)$$
$$= \sum t_i \quad (\text{as } \textstyle\sum \alpha_i = 0),$$
$$= 1$$

so $x = \sum c_j x_j$ but $c_m = 0$, so we used one fewer point. □.

<u>Corollary of lemma</u> If $V = \mathbb{R}^n$ & H = convex hull of a cpt subset K, then H is cpt.

<u>Pf</u> We use mapping argument again, to map $\Sigma \times \overbrace{K \times \cdots \times K}^{n+1 \text{ copies}}$
($\Sigma_1 \subseteq \mathbb{R}^{n+1}$, $\Sigma_1 = \{(t_1, \dots, t_{n+1}) \mid t_j \geq 0, \sum t_j = 1\}$)
onto H. This is a cts map so image of a cpt set like $\Sigma \times K^{n+1}$ is cpt also.
□.