Lecture Notes 8
Hahn-Banach Theorem

Dual space of a TVS V = cts linear f'nls on V
V^*

(itself a vector space),

V^* can have many interesting & important topologies, which all coincide if V is finite dim'l.

We will spend a lot of time on that later.

Dominated Extension

Thm Let $k=\mathbb{R}$ & $M \subseteq V$ a ~~subset.~~ subspace
Given $p: V \to \mathbb{R}$ s.t.

$$p(x+y) \leq p(x)+p(y) \ \& \ p(tx) = tp(x)$$
$$t \geq 0$$

& a linear function $f: M \to \mathbb{R}$
w/ $f(x) \leq p(x)$ on M

There exists a linear f'nl $\Lambda: V \to \mathbb{R}$
s.t. $\Lambda \equiv f$ on M

$$-p(-x) \leq \Lambda x \leq p(x) \quad \text{for all } x \in V.$$

(Assuming Axiom of Choice)

Proof: If $M=V$, $\lambda=f$ &

$-\lambda(x) = \lambda(-x) \leq p(-x)$

$\lambda(x) \geq -p(-x)$, completing proof.

So we may assume $M \neq V$.
We will first extend f to a subspace M' which is one dimension larger.
At the end of the proof, we will use the Axiom of Choice to extend all the way to V.

Let $v' \notin M$ & let

$$M' = \text{span}(M, v') = M + \mathbb{R}v'.$$

to extend f to f', linear on M', we need to define

$$f'(v') = \alpha,$$

where $\alpha = \sup\{ f(v) - p(v - v') \mid v \in M \}$

We need to check $f'(v+tv') = f(v) + t\alpha \leq p(v+tv')$ (*)
for all $t \in \mathbb{R}$, $t \neq 0$ (already know for $t=0$).
(the lower bound follows similarly)

We know

$f(v) - p(v - v') \leq \alpha$

$f(v) - \alpha \leq p(v - v')$

If $t>0$ $\quad t f(v) - t\alpha \leq t p(v - v')$

$f(tv) - t\alpha \leq p(tv - tv')$

Rescale $v \mapsto t'v$:

$$f(v) - t\alpha \le p(v - tv')$$

changing $t \mapsto -t$ we get

(which is ($\dagger$)) $f(v) + t\alpha \le p(v + tv')$ for $t < 0$,

Also, if $w \in M$

$$p(v+w) \le p(v - v') + p(v' + w)$$

$$\begin{array}{c} \ge \\ f(v+w) \\ \| \\ f(v)+f(w) \end{array}$$

since $v, w \in M$

$$p(v'+w) - f(w) \ge f(v) - p(v - v')$$

so taking sup over $v \in M$

$$\alpha \le p(v'+w) - f(w)$$

If $t > 0$ $\quad t\alpha \le p(tv' + tw) - f(tw)$

$w \mapsto w/t$ $\quad t\alpha \le p(w + tv') - f(w)$

$f(w) + t\alpha \le p(w + tv')$, which is ($\dagger$) for $t > 0$.

So ($\dagger$) is proven; we have extended f on M to f' on M', where $[M' : M] = 1$,

to extend up to V we use partial ordering and Hausdorff's maximality principle: extensions form a partially ordered family so there exists a maximal totally ordered subfamily

Hence we extend all the way to V. $\square$

Some Corollaries:

<u>Dominated Extension over $\mathbb{C}$</u> Let now $k = \mathbb{C}$.

If $M \subseteq V$ is a subspace of a vector space
f linear functional on M,
p seminorm on V, s.t. $|f(v)| \leq p(v)$ on M
then f extends to ${}^{\Lambda \text{ on}}V$ in a way
s.t. it is still bdd by $p(v)$:
$|\Lambda(v)| \leq p(v)$.

Proof
Let $u = \operatorname{Re} f$. Viewing V as a real vector space, u extends to a linear functional λ bdd by p.

Define $\Lambda(v) = \lambda(v) - i\lambda(iv)$, $\operatorname{Re} \Lambda = \lambda$

to be a complex linear functional on V, ~~ago~~
again thought of as a complex vector space
Let $v \in V$, let
$$\alpha^{-1} = \frac{\Lambda(v)}{|\Lambda(v)|} \in \mathbb{C}, \ |\alpha| = 1.$$

Then
$$\begin{aligned}\Lambda(\alpha v) &= |\Lambda(v)| \\ &\parallel \\ &\operatorname{Re} \Lambda(\alpha v) \\ &\parallel \\ &\lambda(\alpha v) \leq p(\alpha v) = p(v). \quad \square\end{aligned}$$

<u>Another Corollary</u> If V is a normed vector space & $v_0 \in V$, $\exists\, \Lambda \in V^*$ ($=$ cts lin. f'nl) such that

$$\Lambda v_0 = \|v_0\|. \quad \text{and} \quad |\Lambda v| \le \|v\|$$

<u>Proof</u> on span $v_0 = \mathbb{C} v_0$, let
$f(tv_0) = t\|v_0\|$,
a linear f'nl dominated by (semi) norm $\|\cdot\|$.
Use previous theorem over $\mathbb{C}$. $\square$.
(cts: $|\Lambda v| \to 0$ as $\|v\| \to 0$.
or: obviously bdd)

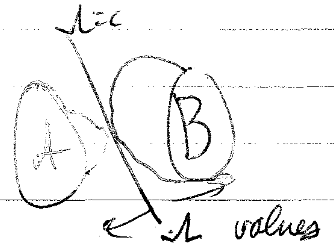
Now we use topology:

<u>Separation Theorem</u> Let A, B be disjoint, nonempty convex subsets of a TVS V. Then

(1) A open $\Rightarrow \exists\, \Lambda \in V^*$, $c \in \mathbb{R}$ s.t.

$$\mathrm{Re}\,\Lambda a < c \le \mathrm{Re}\,\Lambda b$$

for all $a \in A, b \in B$.



(2) A cpt, B closed, V locally convex
$\Rightarrow \exists\, \Lambda \in V^*$, $c_1, c_2 \in \mathbb{R}$ s.t.
$\mathrm{Re}\,\Lambda a < c_1 < c_2 < \mathrm{Re}\,\Lambda b$, $\forall a \in A, b \in B$.

Proof First we prove if $K=\mathbb{R}$, then reduce the case $K=\mathbb{C}$ to this.

First we do the first case, when A is open.

Let $a_0 \in A, b_0 \in B$

$x_0 = b_0 - a_0$

$C = A - B + x_0$,

so C is a convex, open nbd of 0.

~~union of open sets~~

~~$A - b + x_0$, $b \in B$~~,

Let p be C's Minkowski functional

$x_0 \notin C \Rightarrow p(x_0) \geq 1$,

We define $f(tx_0) = t$ on $\operatorname{span}(x_0) = M$

Then $f \leq p$ on $\operatorname{span}(x_0) = M$

since

$f(tx_0) = t$

$p(tx_0) = tp(x_0) \geq t$,

By Extension thm, f extends to a linear functional Λ on V, $\Lambda \leq p$.

$|\Lambda| \leq 1$ on $\underbrace{C \cap (-C)}_{\text{open nbd of } 0}$,

so Λ is cts (bdd on open set containing 0).

We now compute how Λ behaves on A & B,

$\Lambda(a-b+x_0) < 1$

$\parallel$ since $a-b+x_0 \in C$

$\Lambda(a)-\Lambda(b)+1$

so $\Lambda(a) < \Lambda(b)$.

Now, $\Lambda(A)$ & $\Lambda(B)$ are convex subsets of $\mathbb{R}$, disjoint.

Λ: nonconstant lin fl, onto, open mapping

so $\Lambda(A)$ open. If $c =$ ~~right~~ $\sup \Lambda(A)$ we are done.

2nd part We proved way back that given a compact subset K & closed C of a TVS, $\exists$ an open nbhd U of 0 s.t. $K+U$ & $C+U$ are disjoint.

Shrinking U, we may assume U is convex (since V is locally convex)

so $A+U$ & B are disjoint

$\left(\begin{array}{l} A=K \\ B=C \end{array} \right)$

so

$\Lambda(A+U)$ open disjoint from $\Lambda(B)$

$\Lambda(A)^c$ compact, this gives separation

$[\Lambda(A)]_\varepsilon \subset \Lambda(A+U) \mid \Lambda(B)$

Now, if $K = \mathbb{C}$ instead of $\mathbb{R}$, view everything as a real space get $\Lambda_{\mathbb{R}}$.

Make $\Lambda_{\mathbb{C}}(v) = \Lambda_{\mathbb{R}}(v) - i\Lambda_{\mathbb{R}}(iv)$ cts linear fl.

$\Lambda_{\mathbb{R}} = \mathrm{Re}\,\Lambda_{\mathbb{C}}$.

Rest follows. $\square$

<u>Corollary</u> Let $A_{"\{v\}"}$ & $B_{"\{w\}"}$ be singletons. Then $\exists \Lambda \in V^*$ s.t. $\Lambda(v) \neq \Lambda(w)$. $\square$

<u>Corollary</u> If $\begin{cases} V \text{ locally convex} \\ M \subseteq V \text{ subspace} \\ x_0 \in V - \overline{M}. \end{cases}$ Then $\exists \Lambda \in V^*$ s.t. $\Lambda x_0 = 1$, $\Lambda x = 0$ on M.

<u>Pf</u> Take $A = \{x_0\}$, $B = \overline{M}$

get $\Lambda \in V^*$ s.t. Λx_0 disjoint from $\Lambda \overline{M}$ & hence ΛM.

ΛM must $= \{0\}$, since it is a subspace.

Rescale Λ so $\Lambda x_0 = 1$. $\square$

Thus $\overline{M} = \underline{\bigcap \ker \Lambda}$ over Λ vanishing on M.

A good way to show something is in $\overline{M}$ (if all Λ vanishing on M vanish on it).

Corollary If f is acts lin'fal on a subspace M of a locally convex TVS V, $\exists \Lambda \in V^*$ extending f.

Pf. Trivial if $f \equiv 0$.
Else $M_0 = \ker f$ on $M \subsetneq M$
& $\exists x_0 \in M$ w/ $f(x_0) = 1$.
$x_0 \notin \overline{M_0}$ since f is cts.

$$\left(\overline{M_0}^{\text{in } M} = \overline{M_0} \cap M \right)$$
$\uparrow$ in X, since subspace topology.

We have now some $\Lambda \in V^*$ trivial on $\overline{M_0}$,
& w/ $\Lambda x_0 = 1$.
Then Λ extends f. $\square$

Weak vs. Strong topologies.

If τ_1 & τ_2 are 2 topologies on a space & all open sets in τ_1 are open in τ_2,
τ_1 is weaker than τ_2
τ_2 — stronger — τ_1 finer = stronger.

This is strange terminology (linguistically). Think of "Stronger" like stronger vision - can resolve/see finer objects.

We need this lemma later, which for $n=1$ was used implicitly a moment ago (& is obvious there).

<u>Lemma</u> Let V = vector space
$\lambda_1, \ldots, \lambda_n$ linear functionals
$N = \bigcap_{j \leq n} \ker \lambda_j$.

TFAE

① λ = linear comb $\sum c_j \lambda_j$

② $\exists\, C > 0$ s.t. $|\lambda v| \leq C \cdot \max |\lambda_j v|$

③ λ vanishes on N

<u>Pf</u> ① ⇒ ② ⇒ ③ obvious.
~~If~~ ③ holds, we use the map

$$\pi : V \to k^n$$
$$\pi(v) = (\lambda_1(v), \ldots, \lambda_n(v))$$

③ : $\pi(v) = 0 \Rightarrow \lambda(v) = 0$
$\pi(v) = \pi(v') \Rightarrow \lambda(v) = \lambda(v')$
$\lambda = F \circ \pi$, F a linear fn'l on k^n
so F gives form in ①. $\square$

Given a family of linear fnls on V, we have a topology on V the weakest in which all these are cts.
It is gen by $f^{-1}(U)$, U open in k.

<u>Theorem</u> Let V = vector space
V' = ~~set of~~ separating set of linear fnls on V.
Give V the V'-weak-topology

Then V is a locally convex space
& $V^* = V'$ (dual).

<u>Proof</u> Since separates points
$$\{0\} = \bot - \bigcup_{\lambda \in V'} [V - \ker \lambda] \quad \text{is closed,}$$
Basic open sets are of form
$$U = \{ x \mid |\lambda_i x| < r_i , \ i \leq n \},$$
which is balanced & convex,
So locally convex.

U of this form satisfies $\frac{1}{2}U + \frac{1}{2}U \subseteq U$
by convexity; conversely any $u \in U$
$= \frac{1}{2}u + \frac{1}{2}u$.
so, $\frac{1}{2}U + \frac{1}{2}U = U$.
$\Rightarrow$ addition is cts.

If α, β scalars, $|\beta-\alpha|<r$

$x \in U$ $\quad y-x \in rU$, then

$x \in sU$ for some s $\quad \beta y-\alpha x=(\beta-\alpha)y+\alpha(y-x)$

$$\lambda_i(\beta y-\alpha x)=(\beta-\alpha)\lambda y+\alpha\lambda(y-x)$$

$$|\lambda_i(\beta y-\alpha x)| \le \underset{+r_i}{r}\cdot\binom{r_i}{+s}+|\alpha|\cdot r\cdot r_i$$

If r is small

$r(s+r+|\alpha|)<1$

so $\beta y-\alpha x \in V$.

So mult is cts.

$\Rightarrow$ TVS,

Now, to prove $V'=V^*$.

obviously $V'\subseteq V^*$,

If $\lambda \in V^*$

$|\lambda v|<1$ on some set

$U=\{x \mid |\lambda_i x|<r_i\}$

$$\Rightarrow |\lambda v| \le C\cdot\max|\lambda_i v|$$

$\Rightarrow$ λ a lin comb of λ_i

$\Rightarrow \lambda \in V'$. $\square$.

This is a good example of a separating family of seminorms! $p_\lambda(v) := |\lambda v|$.

Weak Topology of a TVS fundamental notion.

$V =$ TVS

assume dual V^* separates points (eg if V locally convex)

the Weak topology on V is the weakest topology so that each $\lambda \in V^*$ is cts.

(it is weaker than the given topology).

Notation: $V_w = V$ endowed w/ weak topology.

Original top is sometimes called "strong"

Last thm says $V_w^* = V^*$ — so spaces have same dual! (equivalent to $(V_w)_w = V_w$ — proved earlier (notion of weakest))

(Does not happen on $\mathbb{R}^n$!)

In general, V & V^* are not isomorphic.

~~Weak open nhd~~ gen by $\{v \mid |\lambda_i v| < r_i \text{ for } i \le n\}$ finite collection $\{\lambda_i\} \le$

Weak Convergence $v_n \to 0 \iff \lambda v_n \to 0$ for all $\lambda \in V^*$.

Weakly Bounded $E \subseteq V$ is weakly bdd if each set of the form

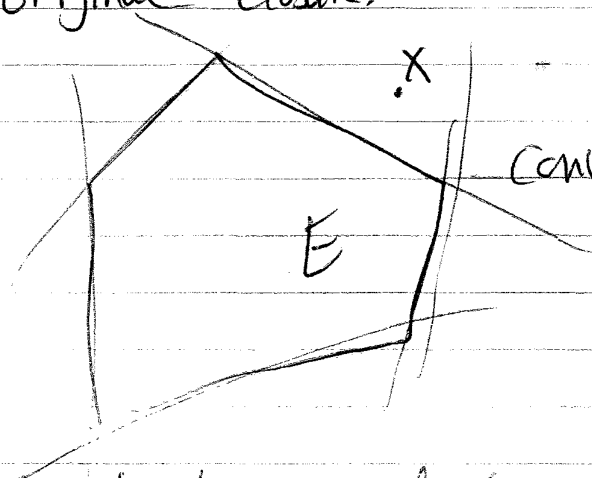
$\{v \mid |\lambda_i v| < r_i\} \supseteq tE$ for t small ~~large~~ enough.

That means for all $\Lambda \in V^*$

$\Lambda(E)$ is bdd,

$\therefore$ Weakly bdd means each $\Lambda \in V^*$ is bdd on E.

Theorem If V is locally convex & $E \subseteq V$ is convex, then its weak closure equals its original closure.



convexity matches with linear functionals.

Pf Need to show weak closure is contained in $\overline{E}$ (a priori, might be bigger in weaker topology).

Let $x \in V - \overline{E}$. We will show it is not in the weak closure.

The separation thm shows $\exists\ \Lambda \in V^*$ & $c \in \mathbb{R}$ s.t.

$\operatorname{Re} \Lambda x < c < \operatorname{Re} \Lambda v$, $\operatorname{Re} \Lambda(v-x) > 0$

for all $v \in \overline{E}$.

~~If the TVS is ... we are done since ...~~

So basic open nbd $\left[\{\,|\Lambda v| < \varepsilon/2\,\} + x\right] \cap E = \emptyset$.

~~$x_n \in E$ s.t. $\Lambda x_n \to \Lambda x$, but this is impossible. $\square$~~

(Rudin's argument here is incomplete)

Corollaries In a locally convex TVS

1. A subspace is closed $\Leftrightarrow$ weakly closed
2. A convex subset is dense $\Leftrightarrow$ weakly dense.
3. In a metrizable locally convex TVS $x_n \to x$ weakly $\Rightarrow \exists$ sequence $y_n \to x$ (strongly) w/ y_n a finite lin comb of x_n.

$y_n = \sum c_{n,m} x_m$

$\sum_m c_{n,m} = 1$

for a.a. m, $c_{n,m} = 0$.

Pf 1, 2 immediate.

3

Let C = set of all such "convex linear combinations" of x_m

= convex hull = smallest convex set containing $\{x_m\}$

Let K = weak closure of C.

$x \in K$.

Therefore $x \in \overline{C}$. (previous thm)

Since metrizable, a limit of points $y_n \in C$. $\square$

Approximation Theorem

Consider $C([0,1])$, Banach space under sup norm, ~~and~~ a sequence $f_n \to f$ pointwise, $|f_n| \le 1$.

~~Then thi~~

Linear functional δ_y, $y \in [0,1]$

$\delta_y(f) = f(y)$ is cts.

family $\{\delta_y\}$ ~~es~~ separates points.

$\Rightarrow$ $f_n \to f$ weakly

$\Rightarrow$ a convex comb of $f_n \to f$ uniformly! (sup norm).

Big Definition Weak* - a topology on dual space.

$V \hookrightarrow (V^*)^*$ by

$\lambda_v(\Lambda) = \Lambda(v)$.

image separates points in V^*.

The top on V^* induced by $\operatorname{Im}(V)$ is the weak* topology on V^*.

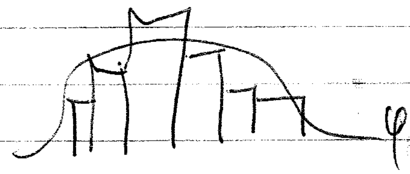
If $\operatorname{Im} V = (V^*)^*$, then "Reflexive".

A lot of the rest of the course is about this configuration.

EG Sample real #s in $[0,1]$ on $C([0,1])$ again

$x_1, \dots, x_n$

Empirical Histogram



$$\mu_E = \frac{1}{n}\sum_{j\le n}\delta_{x_j}$$

$$\int_0^1 \mu_E(x) f(x)\,dx = \frac{1}{n}\sum f(x_j)$$

Theoretical curve $\mu_T(x) = \varphi(x)$ that you think fits it.

Often one tries to prove that the distribution of $\{x_1, \dots, x_n, \dots\}$ follows the law μ_T by showing

$$\mu_E \to \mu_T \text{ in weak}^*$$

i.e. $\frac{1}{n}\sum f(x_j) \to \int_0 \varphi(x) f(x)\,dx$

For example

let $x_j = j\alpha - [j\alpha]$, for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

actually x_j are uniformly distributed over $[0,1)$

Need to show

$\frac{1}{n}\sum_{j=1}^{n} f(j\alpha) \to \int_0^1 f(x)\,dx$ p. 18/18

for any cts function f on $\mathbb{R}/\mathbb{Z}$ (~~slike~~ $C([0,1])$)

Weyl's insight
weak* convergence holds if it holds on a basis.

Fourier thy gives basis $e^{2\pi i k x}$, $k \in \mathbb{Z}$

Show $\frac{1}{n}\sum_{j=1}^{n} e^{2\pi i k j \alpha} \to \int_0^1 e^{2\pi i k x}\,dx = \begin{cases} 1 & k=0 \\ 0, & \text{---} \end{cases}$

If $k=0$, $\Sigma = 1$

$k \neq 0$, $\Sigma = \frac{e^{2\pi i k\alpha}}{n} \cdot \frac{e^{2\pi i k n\alpha} - 1}{e^{2\pi i k\alpha} - 1}$

$|\quad| \leq \frac{1}{n} \cdot \frac{2}{|e^{2\pi i k\alpha} - 1|} \to 0$

as $n \to \infty$.

This is the first example of
Ergodic Thy.

Eg. odds a power of 7 starts w/ an "8" digit is ~~$\frac{\log 9}{\log 8}$~~ $\boxed{\log_{10}(9/8)}$