Lecture Notes 6 - Baire Category

We start with recalling that a subset of a topological space is <u>nowhere dense</u> if its closure has an empty interior.

"<u>First Baire Category</u>" subsets that are countable unions of nowhere dense sets.

"<u>Second Baire Category</u>" subsets that are not in the first Baire Category.

Easy
<u>Proposition</u> (a) Any subset of a First Category set is also first category
[obvious, since a subset of a nowhere dense set is also nowhere dense]

(b) Countable unions of first category Sets are also in the first category.

(c) Closed sets with nonempty interior are first category.

(d) The image of a set under a homeomorphism has the same category.

Now we come to the main theorem due to Baire.

Baire Category Theorem In any complete metric space or locally compact Hausdorff space, if $U_1, U_2, \ldots$ is a countable collection of open dense sets, then $\bigcap_{n\geq 1} U_n$ is also dense.

Proof We need to show that each nonempty open set S intersects $\bigcap_{n\geq 1} U_n$.

Consider $S \cap U_1$, which is an open set & nonempty (because U_1 is dense).

(actually we do not really need the closures $\overline{B_n}$ here — B_n is enough)

In a metric space	In a locally compact Hausdorff space
There is some ball B_1 of small radius such that $B_1 \subseteq S \cap U_1$. If we make B_1 smaller (taking half the radius) we may assume $\overline{B_1} \subseteq S \cap U_1$. Repeat to find a ball B_2 of small radius such that $\overline{B_2} \subseteq B_1 \cap U_2$ etc. $\overline{B_3} \subseteq B_2 \cap U_3$ ⋮ $\overline{B_{n+1}} \subseteq B_n \cap U_{n+1}$ with radii $\to 0$. Then the centers of these balls form a Cauchy sequence which converges to a point in S & $\bigcap U_n$.	$S \cap U_1$ is open & thus contains an open subset called S_1. Let $x \in S_1$. Since the space is locally compact, $\exists$ an open nhd U_x of x s.t. $\overline{U_x}$ is compact. Since closed subsets of compact sets are compact, we may assume $U_x \subseteq S_1$ (continued...)

So we have an open set $U_x \subseteq S \cap U_1$ with compact closure. Let $\partial U_x = \overline{U_x} - U_x^{\circ} = \overline{U_x} - U_x$ be the boundary of U_x, which is a closed subset of $\overline{U_x}$, hence compact. For each $y \in \partial U_x$, there exists an open set U_y which does not intersect an open nhd T_y of x. Since ∂U_x is compact, this open cover has a finite subcover $y_1, \dots, y_n$ s.t.

$$\bigcup_{j=1}^{n} U_{y_j} \text{ is disjoint from}$$

$$B_1 = \left(\bigcap_{j=1}^{n} T_{y_j}\right) \cap U_x$$

The latter neighborhood B_1 of x is open. We claim

$$\overline{B_1} \subseteq S \cap U_1.$$

Indeed, we know B_1, which is a subset of U_x, so $\overline{B_1} \subseteq \overline{U_x}$.
$\overline{U_x}$ is a disjoint union of U_x & ∂U_x, but ∂U_x is contained in the open set $U_{y_1} \cup \cdots U_{y_n}$, & hence

$$B_1 \text{ is disjoint from } \partial U_x.$$

Thus

$$\overline{B_1} \subseteq U_x \subseteq S \cap U_1, \ \& \ \overline{B_1} \text{ is compact}$$

Repeating, we find an open nhd B_2 such that

$$\overline{B_2} \subseteq B_1 \cap U_2, \text{etc.... as before.}$$

WARNING - RUDIN SKIPS THIS!

Now $B_1 \supseteq \overline{B_2}$, $B_2 \supseteq \overline{B_3}$, etc...
so the $\{\overline{B_n}\}$ have nonempty finite intersections. By compactness of $\overline{B_1}$,
$\bigcap_{n=1} \overline{B_n}$ is nonempty, & is contained in all U_n, and S. Thus
$S \cap \bigcap U_n \neq \{\}$ in both cases. $\square$

<u>Remark</u> This theorem says complete metric spaces & locally compact Hausdorff spaces are second category subsets of themselves.
Why? Otherwise, they are countable unions of nowhere dense sets ("first category").
The complement of the closure of a nowhere dense set is open & dense
(since if A is nowhere dense, $\bar{A}$ has no interior).
So the theorem says the intersections of the complements is dense, & in particular not empty.
This is a contradiction.

Definition:
If V, W are TVS's & $\mathcal{L} = \{\Lambda\}$ is a collection of linear maps from $V \to W$, then $\mathcal{L}$ is an <u>equicontinuous</u> family if for each open nhd U of 0 in W, there exists a fixed open nhd S of 0 in V such that

$\Lambda(S) \subseteq U$ for all $\Lambda \in \mathcal{L}$.

Eg. If Λ is the only member of $\mathcal{L}$, then equicontinuous means the same as continuous.

<u>Proposition</u> In such an equicontinuous family, if B_1 is a bounded subset of V, then there exists a bounded subset $B_2 \subseteq W$ such that $\Lambda(B_1) \subseteq B_2$ for all $\Lambda \in \mathcal{L}$.

<u>Pf</u> We claim $B_2 = \bigcup_{\Lambda \in \mathcal{L}} \Lambda(B_1)$ satisfies this assertion – we only need to check that it is indeed bounded. Let U be any nbd of 0 in W. By equicontinuity, there exists an open nbd S of 0 in V s.t. $\Lambda(S) \subseteq U$ for all $\Lambda \in \mathcal{L}$. Since B_1 is bounded, $B_1 \subseteq t\cancel{S}$ for all large t. Thus $\Lambda(B_1) \subseteq t\cancel{S}$ for all large t, &

$B_2 = \bigcup \Lambda(B_1) \subseteq tU$, so B_2 is therefore bounded. ☐

Banach-Steinhaus Theorem

("Uniform Boundedness Principle" since boundedness is related to continuity.)

If $\mathcal{L}$ is a collection of continuous linear maps from TVS's $V \to W$, & $B = \{v \in V \mid \text{the orbit } \{\Lambda v \mid \Lambda \in \mathcal{L}\} \text{ is bounded in } W\}$, then B is either of first category in V, or instead $B=V$ & $\mathcal{L}$ is equicontinuous.

Proof There exists, as we have seen before, balanced open nhds U, U' of 0 in W s.t. $\bar{U}+\bar{U} \subseteq U'$.

Let $T = \bigcap_{\Lambda \in \mathcal{L}} \Lambda^{-1}(\bar{U})$.

Observe that if $v \in B$, then $\{\Lambda v \mid \Lambda \in \mathcal{L}\}$ is contained in tU for all large t. (this is by boundedness), so $B \subseteq \bigcup_{n \geq 1} nT$.

Assume B is second category; we need to show $B=V$ & $\mathcal{L}$ is equicontinuous.

If all nT, $n \geq 1$, are first category in V then $\bigcup nT$ is also a countable union of nowhere dense sets, hence $\bigcup nT$ is first category also, & so is B, a contradiction.
So some nT is second category, but of course in a TVS, that means T is also (because scalar multiplication is continuous, so T & nT are homeomorphic).

Also, T = intersection of closed sets (since each Λ is cts).

A second category closed set must have an interior point,
(prop on p.1, part (c)).

Let $v \in T$ be an interior point,
so that $v + S \subseteq T$, for an open nhd S of 0 in V.

Now, $\mathcal{U}'$ is an arbitrary open nhd of 0 in W. We note now that

$$\Lambda(S) \subseteq \Lambda(v) - \Lambda(T) \subseteq \overline{\mathcal{U}} - \overline{\mathcal{U}}$$
$$\subseteq \overline{\mathcal{U}} + \overline{\mathcal{U}} \subseteq \mathcal{U}',$$

so $\mathcal{L}$ is equicontinuous.

Now, let x be any vector in V, & O an open nhd of 0 in V. We proved O is always absorbing, so $x \in tO$ for all large t, hence

singletons $\{x\}$ are bdd.

Thus by the proposition
$\{\Lambda x \mid \Lambda \in \mathcal{L}\}$ is bounded.

So all $x \in B$. Thus $B = V$. $\square$