Lecture Notes 5 - Some Examples

Example 1 — $C(\Omega)$

Let Ω be a nonempty subset of $\mathbb{R}^n$ (or of $\mathbb{C}^n$, viewed as the Euclidean space $\mathbb{R}^{2n}$)

$C(\Omega)$ = vector space of continuous functions from Ω to $\mathbb{C}$.

Let $K_n \subseteq K_{n+1}^{\circ}$ (← interior) be an increasing family of subsets of Ω which exhausts Ω, i.e. $\Omega = \bigcup_{n\geq 1} K_n$.

The topology on $C(\Omega)$ is given by seminorms

$$p_n(f) = \max\{|f(x)| \mid x \in K_n\}$$

(this is finite, since f is a continuous function on the compact set K_n).

Proposition The sets $V(p_n, n) = \{f \mid p_n(f) < \frac{1}{n}\}$ form a convex local basis for $C(\Omega)$.

Notation: $V(p_k, n) = \{f \mid p_k(f) < 1/n\}$ is convex by previous thm.

Proof: By our earlier theorem, a convex local base is given by finite intersections

$$V(p_{j_1}, n_1) \cap \cdots \cap V(p_{j_k}, n_k)$$

$j_1, \dots, j_k$ indices

Since $K_n \subseteq K_{n+1}$, $p_1(f) \le p_2(f) \le \cdots$,

$$V(p_j, n) = \{f \mid p_j(f) < \tfrac{1}{n}\} \supseteq \{f \mid p_k(f) < \tfrac{1}{n}\} = V(p_k, n)$$
if $j \le k$

Also, $V(p_j, n) \supseteq V(p_j, m)$ if $m \ge n$.

So $V(p_{j_1}, n_1) \cap \cdots \cap V(p_{j_k}, n_k) \supseteq$

$$\supseteq V(p_M, M),$$

$$M = \max\{j_1, \cdots, j_k, n_1, \cdots, n_k\}. \square$$

$\{V(p_n, n)\}$ is countable, so $C(\Omega)$ is metrizable.

EXERCISE (HW) Prove $d(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(f-g)}{1+p_n(f-g)}$ is a metric which induces the same topology on $C(\Omega)$.

The family $\{p_n\}$ is clearly a separating family of seminorms, since a nonzero function cannot vanish on all the subsets K_n.

Proposition $C(\Omega)$ is a Frechet Space (means complete, locally convex, with an invariant metric).

Proof:
The last proposition showed it is locally convex, & the exercise gives an invariant metric. We only need to show completeness.

Suppose $\{f_i\}$ is a Cauchy sequence. That means for each $k \in \mathbb{N}$

$$p_k(f_i - f_j) \to 0 \quad \text{as } i,j \to \infty$$

$$\Downarrow$$

f_i converges uniformly on compacta, to a continuous function f. (Uniform limit of cts functions is cts)

"on compacta" means on all cpt sets.

Then $p_k(f_i - f) \to 0$ as $i \to \infty$ for all k. ◻

Proposition $C(\Omega)$ is not locally bounded

(This implies $C(\Omega)$ is not normable).

Proof Suppose, to the contrary that $0 \in U$ & U is a bounded open set.
We may assume $U = \{p_n < \frac{1}{n}\}$, since these sets are a local base.
We showed earlier that when a TVS topology is given by a separating family of seminorms, boundedness is equivalent to insisting

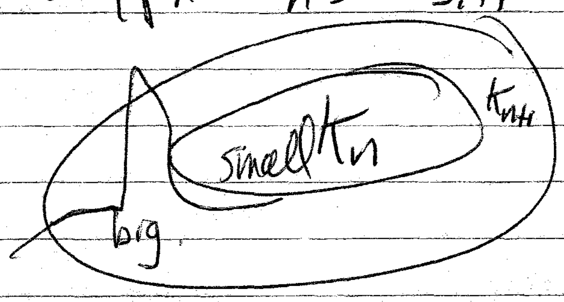
$p_n(U)$ is bounded for each seminorm p_n.

$$\Updownarrow$$

all ~~$f \in U$~~ are uniformly bounded on each K_n.

We can construct, however,

$f \in \{p_n < \frac{1}{n}\}$ s.t. $p_{n+1}(f)$ is arbitrarily large!



— $K_{n+1} \setminus K_n$ is open

so on K_n+1 there is an open ball, and you can make a continuous function supported in an open ball which is arbitrarilylarge

Example 2 $H(\Omega)$

Now $\Omega \subseteq \mathbb{C} \cong \mathbb{R}^2$, & $H(\Omega) \subseteq C(\Omega)$ consists of holomorphic ("analytic") functions.

Limits in $H(\Omega)$ which converge uniformly on compacta (which is the $C(\Omega)$ topology) lie in $H(\Omega)$. Hence $H(\Omega)$ is a closed subspace of a Frechet space
$\Rightarrow$ it is itself a Frechet space (complete, locally convex, invariant metric).

<u>Proposition</u> $H(\Omega)$ has the Heine-Borel property (all closed and bounded sets are compact).

This implies $H(\Omega)$ is not locally bounded, since it is infinite dimensional.

Also, it implies $H(\Omega)$ is not normable.

Proof Recall if C is a subset of a metric space with the convergent subsequence property (all sequences have a convergent subsequence), then C is compact. P.5/1

Let C be a closed and bounded subset of $H(\Omega)$.

Earlier we saw bounded means all f in C are uniformly bounded on any fixed compact subset $K \subseteq \Omega$.

Montel's Theorem on Normal Families implies that all subsequences in C have a subsequence which converges uniformly on all compact sets to a function f, $f \in C$, since C is closed. So C has the convergent subsequence property, & hence is compact. $\square$

Example 3 $C^\infty(\Omega)$

First, we need notation.

$\Omega \subseteq \mathbb{R}^n$ open set

$\alpha =$ "multi-index" $= (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}_{\geq 0}$

α describes partial derivatives:

Let $D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$

be a differential operator of order $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$.

By convention $D^{(0,\dots,0)} = D^{(0)} =$ identity operator & $D^{\alpha+\beta} = D^\alpha D^\beta = D^\beta D^\alpha$.

<u>Define</u> $C^k(\Omega) = \{ f : \Omega \to \mathbb{C} \mid D^\alpha f \in C(\Omega) \text{ for all } |\alpha| \le k \}$

$$C^\infty(\Omega) = \bigcup_{k \ge 1} C^k(\Omega) = \text{smooth functions on } \Omega.$$

For any function f, the <u>support</u> of f

$$\mathrm{supp}(f) = \text{closure of } f^{-1}(\mathbb{C} \setminus \{0\}).$$

More <u>notation</u>: if $K \subseteq \mathbb{R}^n$ is compact

$$\mathcal{D}_K = \{ \text{smooth } f \in C^\infty(\mathbb{R}^n) \text{ w/ } \mathrm{supp}(f) \subseteq K \}$$

= "space of test functions".

Naturally, we may regard $\mathcal{D}_K$ as a subspace of $C^\infty(\Omega)$, if $K \subseteq \Omega$.

<u>Topology on $C^\infty(\Omega)$</u> We use the increasing family $K_n \subseteq \mathring{K}_{n+1}$ as before, which has $\bigcup K_n = \Omega$.

Seminorms: $p_N(f) = \max\{ |D^\alpha f(x)| \mid x \in K_N, |\alpha| \le N \}$

These are like the ones on $C(\Omega)$, but taking derivatives into account.

$\{p_N(f)\}$ is a separating family of seminorms (since if $p_N(f)=0$ for all N, $f\equiv 0$ on all K_N & thus $f\equiv 0$ on Ω. Thus we can use earlier theorems.

Proposition For any fixed $x\in\Omega$, the map

$$\delta_x : f \longmapsto f(x)$$

is a continuous linear functional on $C^\infty(\Omega)$.

(δ_x is the "Dirac δ-function" at x).

Proof It is obviously a linear functional. To prove continuity let $\varepsilon>0$. The inverse image of $(-\varepsilon,\varepsilon)$ contains $\{p_1(f)<\varepsilon\}$, & hence is open. Thus δ_x is continuous at zero, & thus on all of $C^\infty(\Omega)$. $\square$

Remark The proof shows δ_x extends to a continuous linear functional on $C(\Omega)$.

Proposition For any compact subset $K\subseteq\Omega$, $\mathcal{D}_K$ is closed in $C^\infty(\Omega)$.

Pf $\mathcal{D}_K$ is the intersection of ~~the~~ $\operatorname{Ker}(\delta_x)$ — a closed subspace — over all $x\in\Omega\setminus K$. $\square$

Proposition The sets $\{f \in C^\infty(\Omega) \mid p_n(f) < \frac{1}{n}\}$ form a local open base of convex sets.

Proof these sets are obviously open & convex since $\{p_n\}$ are seminorms. We must show every open nbd of 0 contains one of them. Our earlier theorem says a local base is generated by sets of the form

$$(*) \quad V(p_{j_1}, n_1) \cap \cdots \cap V(p_{j_k}, n_k),$$

where $V(p_j, n) = \{f \mid p_j(f) < \frac{1}{n}\}$.

As w/ $C(\Omega)$, we have the relations

$$V(p_j, n) \supseteq V(p_{j'}, n) \quad \text{if } j \leq j'$$

$$V(p_j, n) \supseteq V(p_m, m) \quad \text{if } m = \max\{j, n\},$$

so $(*) \supseteq V(p_m, m)$, where $m = \max\{j_1, \dots, j_k, n_1, \dots, n_k\}$. $\square$

Theorem $C^\infty(\Omega)$ & its subspaces $\mathcal{D}_K$, K compact, are Frechet spaces.

Proof Since the topology is defined by seminorms, they are locally convex. Since they have a countable local basis (given in the proposition), they are metrizable with an invariant metric.

We need to only show $C^\infty(\Omega)$ is complete, because then the closed subspaces $\mathcal{D}_K$ will automatically be complete.

Suppose $\{f_i\}$ is a Cauchy sequence. Then for each fixed N, $p_N(f_i - f_j) < \frac{1}{N}$ for i, j large.

So on K_N, $|D^\alpha f_i - D^\alpha f_j| < \frac{1}{N}$ if $|\alpha| \le N$,

$\Downarrow$

$D^\alpha f_i \in C(\Omega)$ converges uniformly on compacta to a function $g_\alpha \in C(\Omega)$.

$f_i \to g_0$ uniformly

There is a bit of a gap in Rudin towards showing $D^\alpha g_0 = g_\alpha$ & that $f_i \to g_0$ in the C^∞ topology.

For one thing, it is not obvious g_0 is differentiable!

Actually, recall that if a sequence of cts functions $h_n \to h$ uniformly on an interval (or more generally, a compact set), then

$\int h_n \to \int h$ over the compact set.

More generally,

$$\int h_n \varphi \longrightarrow \int h \varphi \quad \text{where}$$

φ is any cts function.

Thus if

$D^\alpha f_i \to g_\alpha$, we have (using the fundamental theorem of calculus)

that

$$\begin{array}{ccc} f_i & = \text{some integral of} & D^\alpha f_i \\ \downarrow & & \downarrow \\ g_0 & & \text{the same integral of } g_\alpha \end{array}$$

$\uparrow$ therefore, $g_\alpha = D^\alpha g_0$

(Example: in one variable, if

$f_i' \to g_1$

$$f_i(x) = \int_a^x f_i'(t)\,dt \longrightarrow \int_a^x g_1(t)\,dt$$

$\downarrow$

$g_0(x)$

so $g_0'(x) = g_1(x)$.)

So $D^\alpha g_0 = g_\alpha$.

Finally, we need to observe

$$|D^\alpha f_i - D^\alpha g_0| \le |D^\alpha f_i - g_\alpha| + |g_\alpha - D^\alpha g_0|$$

$$= |D^\alpha f_i - g_\alpha| \to 0$$

as $i \to \infty$, uniformly on any cpt set.

So $f_i \to g_0$ in the $C^\infty(\Omega)$ topology, & $C^\infty(\Omega)$ is complete. $\square$

Remark: Using Ascoli's Theorem, one can show $C^\infty(\Omega)$ has the Heine-Borel property. Thus it is not locally bounded & not normable, like $C(\Omega)$.

Example 4 $L^p([0,1])$, $0<p<1$

(this is an example of a quotient space,

We already noted this is a metric space under the metric

$$d(f,g) = \int_0^1 |f(x)-g(x)|^p dx,$$

which is not a norm. In fact, this metric is not normable, as follows from this:

Prop $L^p([0,1])$ has only 2 convex open sets: $\{0\}$ & $L^p([0,1])$ itself.

It follow also that $L^p([0,1])$ is not a Frechet space.

Pf Suppose $f \neq 0$ & U is an open convex nbd of 0 (if U does not contain 0,

we may translate it).
Thus $\mathcal{U}$ contains an open ball

$$B = \{ g \in L^p([0,1]) \mid \int_0^1 |g|^p < \varepsilon \}$$

for some $\varepsilon > 0$.

Let $f \in L^p$. We will show $f \in \mathcal{U}$, & hence conclude $\mathcal{U} = L^p([0,1])$.

There exist n points
$0 < x_1 < x_2 < \cdots < x_n = 1$ s.t.

$$\int_{x_{i-1}}^{x_i} |f|^p = \frac{1}{n} \int_0^1 |f|^p = \frac{1}{n} d(f,0)$$

(this follows from the fact that the definite integrals of L^1 functions are continuous).

Let $g_i = n \cdot f \cdot \chi_{[x_{i-1}, x_i]}$.

Then
$$d(g_i, 0) = \int_{x_{i-1}}^{x_i} n^p f^p = n^{p-1} d(f,0).$$

$p < 1$, so for n large this is $< \varepsilon$.
Thus $g_i \in B \subseteq \mathcal{U}$.

But $f = \frac{1}{n} g_1 + \frac{1}{n} g_2 + \cdots + \frac{1}{n} g_n$

is thus a convex linear combination of elements of U, so $f \in U$. $\square$

<u>Remark</u> Any cts linear functional on $L^p([0,1])$ must be trivial, because the inverse image of $(-\varepsilon, \varepsilon)$ is a convex set, hence all of $L^p([0,1])$.
(A linear functional is trivial if its range is bounded