Lecture Notes 4
Seminorms and local convexity

Definition $p: V \to \mathbb{R}$ is a *seminorm* if it is:

- subadditive: $p(v+w) \leq p(v) + p(w)$
- $p(\alpha v) = |\alpha| \cdot p(v)$ for all $\alpha \in k$

Proposition For any seminorm p on a vector space V, we have

(i) $p(0) = 0$

(ii) $|p(v) - p(w)| \leq p(v-w)$

(iii) $p(v) \geq 0$

(iv) $p^{-1}(\{0\})$ is a vector subspace of V.

Proof (i) $p(0v) = p(0) = 0 \cdot p(v) = 0$

(ii) $p(v) = p(v-w+w) \leq p(v-w) + p(w)$

$\Rightarrow$ $p(v) - p(w) \leq p(v-w)$. Reverse v & w:

$p(w) - p(v) \leq p((-1)(v-w))$

$= p(v-w)$.

(iii) If $w=0$ in (ii)

$p(v) \geq |p(v)| \geq 0$.

(iv) Given $v, w \in p^{-1}(\{0\})$,

$0 \leq p(\alpha v + \beta w) \leq |\alpha| \cdot p(v) + |\beta| \cdot p(w) \leq 0$

so $p^{-1}(\{0\})$ is a subspace. $\square$

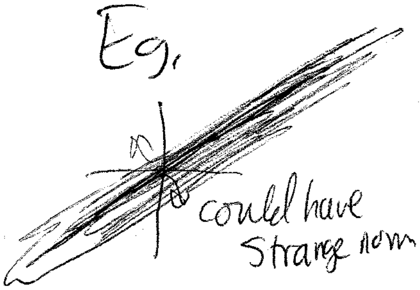
We say a subset $A \subseteq V$ is <u>absorbing</u> if $\bigcup_{t>0} tA = V$.

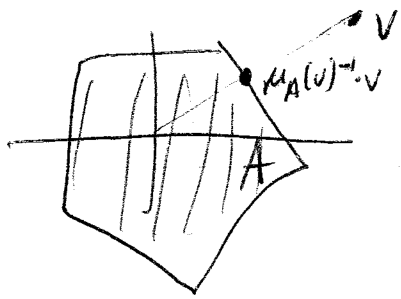
Open Neighborhoods of 0 are

If A is a convex, absorbing set, there is a very important notion of <u>Minkowski Functional</u>

$$\mu_A(v) = \inf\{t>0 \mid t^{-1}v \in A\}$$

Eg.





measure is the norm if A is unit ball.

<u>Proposition</u> If p is any seminorm,

$$p^{-1}([0,1)) = \{v \text{ s.t. } p(v) < 1\}$$

is a convex, balanced, absorbing set, & in fact its Minkowski functional is <u>exactly</u> p !

<u>Proof</u> Let $B = p^{-1}([0,1))$. It is clearly balanced. If $v,w \in B$ & $0 \le t \le 1$

$$p(tv + (1-t)w) \le tp(v) + (1-t)p(w) < 1$$

so it is convex.

It is absorbing since for any $v \in V$ & $s > p(v)$, $p(s^{-1}v) = s^{-1}p(v) < 1$,

so $v \in s \cdot B$

Furthermore, this shows that $p(v) \geq \mu_B(v)$.

To complete the proof we need to show $\mu_B(v) \geq p(v)$.
Indeed, this is obvious if $p(v)=0$.
If $p(v)>0$, let t be an arbitrary real number $0<t\leq p(v)$.
Then $p(t^{-1}v) \geq 1$

$$\Downarrow$$

$$t^{-1}v \notin B.$$

So $\mu_B(v) \geq p(v)$. $\square$

<u>Proposition</u> If A is a convex absorbing set in V

(i) $\mu_A(v+w) \leq \mu_A(v)+\mu_A(w)$

(ii) $\mu_A(tv) = t\mu(v)$ for $t\geq 0$

(iii) A balanced $\Longrightarrow$ μ_A a seminorm.

(iv) If $B = \mu_A^{-1}([0,1))$
$C = \mu_A^{-1}([0,1])$ Then $B \subseteq A \subseteq C$
& $\mu_A = \mu_B = \mu_C$.

<u>Proof</u> $\mu_A(v) = \inf H_A(v)$,

First some remarks

where $H_A(v)$ is the set $\{t>0 \mid t^{-1}v \in A\}$.
If $t\in H_A(v)$ & $s>t$, $s\in H_A(v)$ since A is convex.
Thus $H_A(v) = [\mu_A(v),\infty)$ or $(\mu_A(v),\infty)$.

(i) Assume $\mu_A(v)<s$ & $\mu_A(w)<t$, so $s^{-1}v$ & $t^{-1}w \in A$.
By convexity
$$\frac{1}{s+t}(v+w) = \frac{s}{s+t}(s^{-1}v) + \frac{t}{s+t}(t^{-1}v) \in A,$$
so $\mu_A(v+w) \leq s+t$. Taking infimum gives (i).

(ii) $H_A(tv) = \{s>0 \mid s^{-1}tv \in A\}$
$tH_A(v) = t\{s>0 \mid s^{-1}v\in A\} = \{ts>0 \mid s^{-1}v \in A\} = \{s>0 \mid s^{-1}tv \in A\}$
So $\mu_A(tv) = t\mu_A(v)$.

(iii) Need to show $\mu_A(-v) = \mu_A(v)$, obvious, since $-A \subseteq A \subseteq -A$, $A=-A$.

(iv): $v \in B \Rightarrow 1 \in H_A(v)$.

$\Updownarrow$

$v \in A$ so $B \subseteq A$.

$v \in A \Rightarrow 1 \in H_A(v) \Rightarrow \mu_A(v) \leq 1$,

so $A \subseteq C$.

This shows $H_B \subseteq H_A \subseteq H_C$,

therefore $\mu_C \leq \mu_A \leq \mu_B$,

To finish, we need to show $\mu_B(v) \leq \mu_C(v)$

for all v.

Suppose $\mu_C(v) < s < t$ for reals s, t,

then $s^{-1}v \in C$

$\Updownarrow$

$\mu_A(s^{-1}v) \leq 1$

$\parallel$

$s^{-1} \cdot \mu_A(v)$

so $\mu_A(v) \leq s$

$\mu_A(t^{-1}v) \leq \frac{s}{t} < 1$

so $t^{-1}v \in B$

$\Rightarrow \mu_B(v) \leq t$,

so taking $t \to \mu_C(v)$

we see

$\mu_C(v) \geq \mu_B(v)$

& $\mu_A = \mu_B = \mu_C$. $\square$

__Definition__ a family of seminorms on V is called __separating__ if for each nonzero vector $v \in V$, $\exists$ a seminorm p in the family s.t. $p(v) \neq 0$.

__Proposition__ Let $\mathcal{B}$ be a balanced, convex local base in a TVS V. (Every locally convex TVS has one, as we proved earlier).
Then the set of all Minkowski functionals of elements of $\mathcal{B}$ is a separating family of continuous seminorms on V.

__Proof__ These assumptions show μ_A is a seminorm for all $A \in \mathcal{B}$ by the previous prop (since A is open it is absorbing)

__Separating__: let $v \in V$, $v \neq 0$. V Hausdorff $\Longrightarrow \exists A \in \mathcal{B}$ such that $v \notin A$.
$1 \notin H_A(v)$ so $\mu_A(v) \geq 1$.

__Continuity__: Let $A \in \mathcal{B}$. For each $v \in A$, cty of scalar mult shows that $\exists \varepsilon > 0$ s.t. $tv \in A$ for all $|t-1| < \varepsilon$ (recall A is open).

So $\mu_A(v) < 1$ for all $v \in A$.

Let now $\varepsilon > 0$ (different than above) be arbitrary, & $v \in V$.

for all $w \in v+\varepsilon A$,

$$|\mu_A(v)-\mu_A(w)| \leq \mu_A(v-w)$$

$$=\mu_A(\varepsilon\cdot x) \quad \text{for some } x\in A$$

$$=\varepsilon\mu_A(x) < \varepsilon,$$

so μ_A is cts

($v+\varepsilon A$ is the open set we had to produce). $\square$

<u>Theorem</u> Let $\mathcal{P}$ be a separating family of seminorms on V. For $n\in \mathbb{N}=\{1,2,\dots\}$ & $p\in\mathcal{P}$, let

$$B(p,n)=\{v \mid p(v)<\tfrac{1}{n}\}$$

(like a "ball of radius $\frac{1}{n}$").

Then $\mathcal{B}=\{\text{finite intersections of } B(p,n)\}$ is a convex, balanced local base for a topology on V under which a) V is locally convex, b) each p is cts, & c) a subset is bounded if & only if each p is bdd on it.

Proof We have seen $p^{-1}([0,1))$ is convex, hence is $p^{-1}([0,\frac{1}{n})) = \frac{1}{n}p^{-1}([0,1))$. Also these sets are balanced.

Thus all finite intersections are also convex and balanced.

So let τ = the topology generated by all translates of $\mathbb{B}$.

This, by def'n, makes V a locally convex space, which is a TVS iff $\{0\}$ is closed & operations are cts.

Since $\mathcal{P}$ separates, every $v \in V$ has $p(v) > \frac{1}{n}$ for some $p \in \mathcal{P}$ & $n \in \mathbb{N}$,

so $0 \notin V - B(p,n)$. ← open set

Thus v is not in the closure of $\{0\}$, $\{0\}$ is therefore closed. ✓

<u>Continuity of addition</u>:

If U is an open nbhd of 0, by def'n $U \supseteq$ some $B(p_1,n_1) \cap B(p_2,n_2) \cap \cdots \cap B(p_k,n_k)$.

If $U' = B(p_1,2n_1) \cap B(p_2,2n_2) \cap \cdots \cap B(p_k,2n_k)$ then for $v_1, v_2 \in U'$

$p_j(v_1+v_2) \le p_j(v_1) + p_j(v_2) < \frac{1}{2n_j} + \frac{1}{2n_j} = \frac{1}{n_j}$

$\Rightarrow v_1 + v_2 \in U$

$\Rightarrow U' + U' \subseteq U$.

U' is open, so addition is cts. ✓

Scalar Multiplication is cts

Suppose
$v \in V$, $\alpha \in K$ (Scalar)
& $\alpha v + \mathcal{U}$ is an open nhd of αv,
where $\mathcal{U}$ is an open nhd of the origin,

We will show for some nhd $v + \mathcal{U}'$ of v
& $|\beta - \alpha| < \varepsilon$ that $\beta w \in \alpha v + \mathcal{U}$,
whenever $w \in v + \mathcal{U}''$ (This is the definition of

Indeed, take $\mathcal{U}''$ as $\mathcal{U}'$ before,
but multiply it by $\frac{s}{1+|\alpha|s}$, where
$v \in s\mathcal{U}'$ for some s,
Let $\varepsilon = \frac{1}{s}$.

Then indeed If $w - v \in \frac{s}{1+|\alpha|s}\mathcal{U}'$

$$\beta w - \alpha v = \beta(w-v) + (\beta-\alpha)v$$

$$\in \beta \frac{s}{1+|\alpha|s}\mathcal{U}' + \varepsilon \cdot s\mathcal{U}'$$

$$= \frac{\beta s}{1+|\alpha|s}\mathcal{U}' + \mathcal{U}'$$

Now $\frac{\beta s}{1+|\alpha|s} = \frac{\beta}{\frac{1}{s}+|\alpha|} \leq \frac{|\alpha|+\varepsilon}{|\alpha|+\frac{1}{s}} = \frac{|\alpha|+\frac{1}{s}}{|\alpha|+\frac{1}{s}} = 1$

so $\beta w - \alpha v \in \mathcal{U}' + \mathcal{U}' \subseteq \mathcal{U}$, ✓

↑balanced

We have shown V is a locally convex TVS w.r.t. this topology

We need to prove the other 2 claims:

b) <u>Each $p \in \mathcal{P}$ is continuous</u>

For each $v \in V$ & $\varepsilon > 0$, let $\frac{1}{n} < \varepsilon$.
Then for $w \in v + B(p, \frac{1}{n})$

$$|p(w) - p(v)| \leq p(w - v) < \tfrac{1}{n} < \varepsilon, \text{ so } p \text{ is cts,}$$

c) <u>A subset $S \subseteq V$ is bdd $\Leftrightarrow$ each p is bdd on S</u>

$\Rightarrow$ S lies in $t \cdot B(p, 1)$ for some large t, on which p is bdd by t.

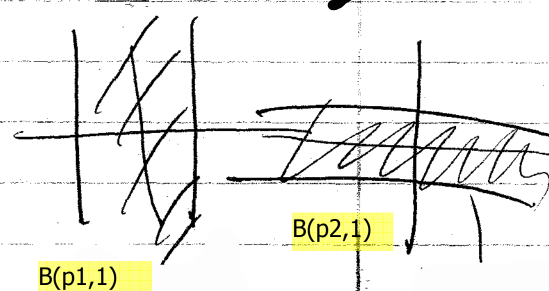
$\Leftarrow$ Let U be an arbitrary open nhd of 0.
U contains some
$$U' = B(p_1, n_1) \cap \cdots \cap B(p_k, n_k).$$
$\exists\, N$ s.t. each $p_1, \ldots, p_k \leq N$ on S.
$\exists\, M$ s.t. $\quad n_1 N, \ldots, n_k N \leq M$ on S.
Therefore $S \subseteq M \cdot U' \subseteq M \cdot U$,
& S is bdd.

(this explains why S is contained in MU')

because on $\frac{1}{M} S$ we have

$$p_j\left(\tfrac{1}{M} S\right) \leq \tfrac{1}{M} p_j(S) \leq \tfrac{N}{M} \leq \tfrac{1}{n_j} \checkmark$$

Example $\mathbb{R}^2$, (x, y) $\quad p_1 = |x|$, $p_2 = |y|$



These don't generate topology but intersections do.

Finally, we come to a very attractive theorem, which I promised before to show you:

Thm A TVS is normable $\Leftrightarrow$ locally convex + locally bdd
(means the topology comes from a

Recall: locally convex means $\exists$ local base of convex sets
locally bdd: $\exists$ a bdd nbd of 0.

Proof: $\Rightarrow$ unit ball $\{\|v\|<1\}$ is convex.

Here we use the previous theorem, with $P = \{\|\cdot\|\}$

& bdd because it is a $p^{-1}([0,1))$ for a seminorm p, the only seminorm which defines the topology

$\Leftarrow$ Harder, obviously.

Let V be a bdd nbd. It contains a balanced convex nbd U, which is also bdd. Let μ be its Minkowski functional.
We claim this is a norm:
Only need to check $\mu(v)=0 \Rightarrow v=0$
and that the topology coincides

We proved before the sets $\{\frac{1}{n}U\}$ are a local basis (this is true for any nbhd of 0 in any TVS).

So if $v \neq 0$, $v \notin \frac{1}{n}U$ for some large n. (Hausdorff)

$\Rightarrow nv \notin U$

$\Rightarrow \mu(nv) \geq 1$

$\Rightarrow \mu(v) \geq \frac{1}{n} > 0$. So μ is nonzero on nonzero vectors, and therefore indeed a norm.

Now we check the topologies coincide.

In fact, $\{\|v\| < r\} = \{\mu(v) < r\}$
$\subseteq rU$ (we showed before page 3),

so given any open nbhd S of 0, $S \supseteq$ some $\frac{1}{n}U$ since these sets form
which contains norm ball of radius $\frac{1}{n}$.

Conversely, the norm ball of radius r

$$= \{\mu(v) < r\} \supseteq \{\mu(v) \leq \tfrac{r}{2}\}$$
$$\supseteq \tfrac{r}{2}U,$$

So topologies are equivalent $\square$

(every open neighborhood of 0 in one of them contains an open neighborhood of 0 in the other)