Lecture Notes #2

Linear Mappings, especially linear functionals

very useful & important

Recall first some basics

Linear map $\Lambda : V \to W$, vector-spaces over k

means $\Lambda(av_1 + bv_2) = a\Lambda(v_1) + b\Lambda(v_2)$ ....

If W = base field, Λ is a linear functional.

Prop Image & Pre-image of a linear map preserve subspaces, convex sets, balanced sets.

Prop (i) Λ is cts @ $0 \Rightarrow \Lambda$ is cts everywhere
(ii) $\Rightarrow \Lambda$ is uniformly cts (given an open nbd U of 0 in W, $\Lambda^{-1}(U)$ is open, hence $\exists$ open nbd $U' \subseteq V$, so $v_1 - v_2 \in U'$ $\Rightarrow \Lambda(v_1) - \Lambda(v_2) \in U$)

Prop Let Λ be a nontrivial linear functional on V. TFAE
(i) Λ cts (ii) Nullspace is closed (iii) Nullspace not dense
(iv) Λ is bdd in some nhd of 0.

Proof (i) ⇒ (ii) since nullspace = inverse image of the closed set $\{0\}$.

(ii) ⇒ (iii) nullspace is closed, so its closure cannot be dense if Λ is nontrivial.

(iii) ⇒ (iv) We showed before that V has a local basis of balanced open nhds. So $\exists$ x & balanced nhd U of 0 s.t.

$$x + U \text{ \& } Null(\Lambda)$$

are disjoint.

ΛU is balanced, too.

If bounded, we are done.

If not, ΛU = base field, since

$$\alpha \Lambda U \subseteq \Lambda U \quad \text{for all } |\alpha| \leq 1.$$

Then $\exists$ $y \in U$ s.t. $\Lambda y = -\Lambda x$

$y + x \in Null(\Lambda)$, a contradiction.

(iv) ⇒ (i) If $|\Lambda v| < M$ for all $v \in U$ & some $M > 0$ then

$$|\Lambda v| < \varepsilon \quad \text{for all } v \in \tfrac{\varepsilon}{M} U$$

(defn of cty @ origin ...) ☐.

Next topic Results on finite dimensional spaces and subspaces, leading to showing all locally cpt TVS's are finite dimensional.

Lemma (to start with) Let $W \subseteq V$ be a subspace of a TVS, & assume it is locally compact in the subspace topology (subset).

Then W is a closed subspace of V.

Proof Locally compact $\Rightarrow \exists K \subseteq W$ cpt
$0 \in$ interior of K wrt W

That means $\exists$ a nhd U of 0 in V s.t.
$U \cap W \subseteq K$.

$\exists$ a nhd U' of 0 s.t. $U' + U' \subseteq U$ (cty)
replace U' by $U' \cap (-U') \rightarrow$ may assume $U' = -U'$
Therefore $\overline{U'} + \overline{U'} \subseteq \bar{U}$

Any member of a local base contains the closure of some member.
So replacing U by this member, we have shown $\exists$ a symmetric nhd U' of 0 s.t.

$$\overline{U'} + \overline{U'} \subseteq U.$$

Claim $W \cap (x + \overline{U'})$ is cpt $\forall x \in V$.

This follows from showing it is a subset of $y_0 + K$, where y_0 is an arb. elt of it.
(closed subsets of cpt sets are cpt!).
Indeed, $\forall y \in W \cap (x + \overline{U'})$

$$y - y_0 = (y - x) + (x - y_0) \in \overline{U'} + \overline{U'} \subseteq U$$

$\underbrace{y - y_0}_{W \text{ is a subspace}} \in W \cap U \subseteq K$, proving the Claim.

Now to finish, we must show $\overline{W} = W$.

Let $v \in \overline{W}$, & $\mathcal{B}$ = all open sets in V containing 0 & contained in U'.

Property: closed under finite intersection

Look at $E_S = W \cap (x + \overline{S})$, $S \in \mathcal{B}$.

$\overline{S} \subseteq U' \Rightarrow E_S$ is cpt, subset of cpt set. nonempty (contains x).

Therefore $\{E_S\}$ is a collection of nonempty cpt sets w/ the finite intersection property. (finite $\cap$ is nonempty)

equivalence of cptness (see next page for more) $\Downarrow$

$\bigcap_{S \in \mathcal{B}} E_S$ is nonempty.

Let $z \in \bigcap_S E_S$.

Then $z \in W$

$\in x + \overline{S}$ for all $S \in \mathcal{B}$

Hausdorff $\Rightarrow z = x$

$\Rightarrow x \in W$

$\Rightarrow \overline{W} = W$, closed $\square$

Refresher Remark on how the finite $\cap$ prop. was used here:

The complement of each E_s in $W \cap (x+U') = X$ is an open set.

If $\cap E_s = \emptyset$, complements form an open cover w/o a finite subcover, which contradicts compactness.

Theorem Every finite dim'l subspace of a TVS is closed $W \subseteq V$

Pf Follows from showing every finite dim'l subspace is locally cpt.

This follows from showing every Iso of k^n & W is a homeomorphism, since k^n is locally cpt.

If $n=1$, Let $\Lambda: k \to W$ be the Iso

let $w = \Lambda(1)$

so $\alpha w = \Lambda(\alpha)$, $\alpha \in k$.

cty of TVS $\Rightarrow$ Λ is cts

Λ^{-1} is linear fnl of W w/ kernel $\{0\}$.

which is closed, & we saw before this is equivalent to continuity.

Now, we do an inductive argument & assume true for dim < n.

Let $\Lambda: k^n \to W$ be an isomorphism

$w_1, \ldots, w_n$ = image of std basis $e_1, \ldots, e_n$

$\Lambda(\alpha_1, \ldots, \alpha_n) = \alpha_1 w_1 + \ldots + \alpha_n w_n$

cty of TVS $\Rightarrow$ Λ is cts.

$w_1, \ldots, w_n$ is a basis of W since Λ is an isomorphism

Map $w \in W$ to $c_1(w) w_1 + \ldots + c_n(w) w_n$

each $c_j(w)$ is a linear fnl of w.

Null space $\cong k^{n-1}$ is closed in k^n

so each cts.

So Λ^{-1} is cts, Homeomorphic. $\square$.

Main Thm of this discussion:

Every locally cpt TVS is finite diml.

Pf Let U be a nbd of 0 w/ cpt closure

$\Rightarrow$ bdd, since $\subseteq \bar{U}$ & $\bar{U}$ cpt is bdd (all cpt subsets of a TVS are bdd).

Also, we showed before

$\{2^{-n} U\}$ form a local base for V.

$$\overline{U} \subseteq \bigcup_{v \in V} \left(v + \tfrac{1}{2} U\right) \quad \text{open cover}$$

$\Rightarrow \exists\, v_1, \dots, v_n \in V$ s.t.

$$\overline{U} \subseteq \left(v_1 + \tfrac{1}{2}U\right) \cup \dots \cup \left(v_n + \tfrac{1}{2}U\right).$$

Let $W = \operatorname{span}\{v_1, \dots, v_n\}$, a finite dim'l closed subspace of V.

$$U \subseteq W + \tfrac{1}{2}U$$

$$\Rightarrow \tfrac{1}{2}U \subseteq \tfrac{1}{2}W + \tfrac{1}{4}U = W + \tfrac{1}{4}U$$

$$\Rightarrow U \subseteq W + \tfrac{1}{4}U$$

etc. $U \subseteq \bigcap_{n=1}^{\infty} \left(W + 2^{-n} U\right)$

$\uparrow$ local base.

We claim $\overline{W} = \bigcap_{n=1}^{\infty} \left(W + 2^{-n} U\right)$

This implies $U \subseteq \overline{W} = W$ (W is closed)

We know $\bigcup_{k \geq 1} kU = V$

so $V = W$, modulo the claim.

Anyhow, let $x \in \overline{W}$. Then

$(x+S) \cap W \neq \emptyset$ for any $S = 2^{-n}U$.

That proves the claim. $\square$.

<u>Corollary</u> If V is a locally bounded TVS with the Heine-Borel Property, then V is finite dimensional.

closed + bdd ⇒ cpt

<u>Pf</u> Let U be a bdd nhd of the origin. Let S be any open nhd S contains the closure of another nhd U'.

$\overline{U'} \subseteq S$

U bdd $\Rightarrow$ $U \subseteq tU'$ for all large t

$\Downarrow$

$\overline{U} \subseteq \overline{tU'} = t\overline{U'}$

$\subseteq tS$,

Hence U bdd $\Rightarrow$ $\overline{U}$ bdd.

$\Rightarrow \overline{U}$ cpt (Heine-Borel)

$\Rightarrow V$ locally cpt

$\Rightarrow$ finite dim'l. $\square$