# Lecture Notes 13

## Adjoints & Compact Operators

Let $\mathcal{B}(V,W)$ = bdd linear maps between normed spaces $V$ & $W$.

We recall if $T \in \mathcal{B}(V,W)$ then it is cts since $V$ & $W$ are metrizable.

<u>Thm</u> (existence of adjoint) In this setting, to each $T \in \mathcal{B}(V,W)$ $\exists!\, T^* \in \mathcal{B}(W^*, V^*)$ s.t.

$$\langle Tv, \lambda \rangle = \langle v, T^*\lambda \rangle$$

for all $v \in V$, $\lambda \in W^*$.

Moreover, $\|T^*\| = \|T\|$.

We call $T^*$ the adjoint of $T$, which is a generalization of the transpose of a finite dim'l matrix.

<u>Pf</u> Given $\lambda \in W^*$, the obvious definition of $T^*\lambda$ is as the linear functional which sends $v \mapsto \lambda(Tv)$

$$(T^*\lambda)(v) = (\lambda T)(v).$$

Obviously $T^*$ is a linear operator, since $T^*\lambda = \lambda \circ T$ is a composition, hence $\in V^*$!

Clearly this is unique, since it is basically a restatement of the property $\langle Tv, \lambda \rangle = \langle v, T^* \lambda \rangle$.

Now for the norm calculation:

Recall we showed before that ~~the~~ for fixed $v \in V$ the map $\lambda \mapsto \lambda v$ is a bdd linear fnl on $V^*$ of norm $\|v\|$.

So $\|T\| = \sup\{\|Tv\| \mid \|v\| \le 1\} = \sup\left\{ |\lambda Tv| \;\middle|\; \begin{matrix} \|v\| \le 1 \\ \|\lambda\| \le 1 \end{matrix} \right\}$

Thus

$$\begin{matrix} \sup\left\{ |\langle Tv, \lambda \rangle| \;\middle|\; \begin{matrix} \|v\| \le 1 \\ \|\lambda\| \le 1 \end{matrix} \right\} & = & \sup\left\{ |\langle v, T^*\lambda \rangle| \;\middle|\; \begin{matrix} \|v\| \le 1 \\ \|\lambda\| \le 1 \end{matrix} \right\} \\ \| & & \| \\ \|T\| & & \sup\{\|T^*\lambda\| \mid \|\lambda\| \le 1\} \\ & & \| \\ & & \|T^*\| \end{matrix}$$

$\square$

<u>Proposition</u> Let $V, W$ be Banach Spaces $T \in \mathcal{B}(V, W)$.

Then $Null(T^*) = Range(T)^{\perp}$

& $Null(T) = {}^{\perp}Range(T^*)$.

**Pf** An elt of $Null(T^*)$ is a linear f'nl

$$\lambda \in W^* \text{ s.t. } \langle v, T^*\lambda \rangle = \langle Tv, \lambda \rangle = 0 \text{ for all } v \in V.$$

So $\lambda \in Range(T)^\perp$

Likewise, an elt of $Null(T)$ is a vector $v$ for which $\langle Tv, \lambda\rangle = 0$ for all $\lambda \in W^*$.

Thus $\langle v, T^*\lambda \rangle$

~~So $v \in {}^\perp Range(T^*)$~~ $\square$.

Under these assumptions

**Prop** a) $Null(T^*)$ is weak* closed in $W^*$

**Pf** it is perp of a subspace, hence weak* closed as a $\cap$ of kernels of lin f'nls in $Im(V \hookrightarrow (V^*)^*)$ $\square$

see note on next page

b) $Range(T)$ is dense in $W \Leftrightarrow T^*$ is injective.

**Pf** both equiv to $Null(T^*) = \{0\}$

$Range(T)^\perp$

uses Hahn-Banach!
(if not dense then $Range(T)^\perp \neq \{0\}$)

c) $T$ is injective $\Leftrightarrow Range(T^*)$ is weak*-dense in $V^*$.

**Pf**

$Null(T) = 0 \Leftrightarrow {}^\perp Range(T^*) = 0$

$\Updownarrow$

$Range(T^*)$ is not annihilated by any $v \to \partial_v$ in $Im(V \to (V^*)^*)$

$\square$.

Th B uses Hahn–Banach



Note: we used ${}^{\perp}(M^{\perp})$ = norm closure of $M$ in $V$

$({}^{\perp}N)^{\perp}$ = weak*-closure of $N$ in $W^*$.

**Lemma** Let $V, W$ be Banach spaces w/ open unit balls $B_1, B_2$ resp. Let $T \in \mathcal{B}(V,W)$ & $c > 0$. Then

a) If $\overline{T(B_1)} \supseteq c B_2$, then $T(B_1) \supseteq c B_2$

b) If $c\|\lambda\| \le \|T^*\lambda\|$ for all $\lambda \in W^*$, then $T(B_1) \supseteq c B_2$. (i.e., open map.)

Pf The statements scale (if you instead rescale the norm) & allow only $c = 1$.

a): Assume $\overline{T(B_1)} \supseteq B_2$
closed

so $\overline{T(B_1)} \supseteq \overline{B_2}$.

We must show each elt of $B_2$ is in $T(U)$.

Let $w \in B_2$, & choose a sequence $\{\varepsilon_n > 0\}$, s.t. $\sum \varepsilon_n < 1 - \|w\|$.

Note: given any vector $y \in W$ & $\varepsilon > 0$

by this $\rightarrow \exists\, v \in V$ s.t. $\|v\| \le \|y\|$ & $\|y - Tv\| < \varepsilon$.

Let $v_1$ be such a $v$ for $y_1 = w$, $\varepsilon = \varepsilon_1$.

& choose

Now set $y_2 = y_1 - Tv_1$

Pick $\cancel{v}_2$ s.t. $\|v_2\| \le \|y_2\|$

& $\|y_2 - Tv_2\| < \varepsilon_2$

Keep doing this, getting $\|v_n\| \le \|y_n\|$

$$y_{n+1} - y_n = -Tv_n$$

$$\|y_n - Tv_n\| < \varepsilon_n.$$

Thus $\|v_{n+1}\| \le \|y_{n+1}\| \le \|y_n - Tv_n\| < \varepsilon_n$.

So

$$\sum_{n \ge 1} \|v_n\| \le \|v_1\| + \sum \varepsilon_n$$

$$< \|y_1\| + (1 - \|y_1\|) = 1.$$

So

$\sum_{n \le m} v_n$ is a Cauchy sequence which converges

to $v := \sum_{n \ge 1} v_n \in B_1$.

also, $Tv = \lim_{m\to\infty} \sum_{n\le m} Tv_n$ by cty

$= \lim_{m\to\infty} \sum_{n\le m} (y_n - y_{n+1})$

$= y_1$ since $\|y_{m+1}\| < \varepsilon_m \to 0$.

Thus $w = y_1 = Tv$, for $v \in B_1$
& $w \in T(B_1)$. ✓

b) Let $E = \overline{T(B_1)}$, = closed, convex, [illegible]

Choose $w_0 \in W \setminus E$.

Then by Hahn-Banach $\{w_0\}$ cpt, $E$ closed, disjoint in a Banach space

$\Rightarrow \exists\, \lambda \in W^*$ s.t.

$|\langle w, \lambda\rangle| \le 1 < |\langle w_0, \lambda\rangle|$

↑ for all $w \in E$.

Let $v \in B_1$, so that $Tv \in E$:

$|\langle Tv, \lambda\rangle| = |\langle v, T^*\lambda\rangle| \le 1$, for any $v \in B_1$

Hence $\|T^*\lambda\| \le 1$,
so $\|\lambda\| \le \|T^*\lambda\|$ (by assumption in (b)) is also $\le 1$.

so
$1 < |\langle w_0, \lambda\rangle| \le \|w_0\| \cdot \|\lambda\| \le \|w_0\|$.

Conclusion each $w_0$ not in $E = \overline{T(B_1)}$ is not in $B_2$,

so each elt in $B_2$ is in $\overline{T(B_1)}$,

i.e., the assumptions of (a) hold;

(a) & (b) have the same conclusion,

so we are done. $\square$

Theorem Let $V, W$ be Banach spaces, $T \in \mathcal{B}(V,W)$,

TFAE

(a) Range$(T)$ is closed in $W$

(b) Range$(T^*)$ weak closed in $V^*$

(c) Range$(T^*)$ norm ~~closed~~ closed in $V^*$.

(Recall weak closure of convex set in locally convex space coincides w/ strong closure. But this isn't enough!)

(b) $\Rightarrow$ (c) is obvious

Pf We show (a) $\Rightarrow$ (b) & (c) $\Rightarrow$ (a).

First, assume (a). Since (p. 2, bottom)

$$^{\perp}\text{Range}(T^*) = \text{Null}(T)$$

$$\text{Null}(T)^{\perp} = \left({}^{\perp}\text{Range}(T^*)\right)^{\perp}$$

We are in a Banach space, so $({}^{\perp}N)^{\perp}$ = weak$^*$-closure of $N$.

So $Null(T)^\perp$ = weak$^*$-closure of $Range(T^*)$ 

Thus (b) $\Leftarrow\!\!\Rightarrow$ $Null(T)^\perp \subseteq Range(T^*)$.

Recall: open mapping theorem says any ~~surjective~~ linear cts map between Banach spaces is onto.

This applies to $T: X \to Range(T)$, since assumption (a) says $Range(T) \subseteq W$ is closed, hence a Banach subspace.

Thus $T$(unit ball on $V$)
$= T$(unit ball on $V$ + $Null(T)$) is open in $W$
$\supseteq \varepsilon *$unit ball in $W$

So, letting $K = \varepsilon^{-1}$, for each $w \in Range(T)$ $\exists v \in V$ s.t. $Tv = w$ & $\|v\| \le K \cdot \|w\|$.

Let $\Lambda \in Null(T)^\perp$

Now define a linear f'nl on $Range(T)$ by

$\lambda : Tv \mapsto \Lambda v$, well defined since $\Lambda \equiv 0$ on $Null(T)$.

It is cts because

$$|\lambda w| = |\lambda Tv| = |\Lambda v| \le \|\Lambda\| \cdot \|v\| \le K \cdot \|\Lambda\| \cdot \|w\|,$$

($w, v$ as above

Thus $\lambda \in \text{Range}(T)^*$,

By Hahn-Banach $\lambda$ extends to some $\mu \in W^*$.

Now for any $x \in V$

$$T^*\mu(x) = \mu(Tx) \overset{\text{since agree on Range}(T)}{=} \lambda(Tx) = \Lambda x$$

so $\Lambda = T^*\mu$,

so each $\Lambda \in \text{Null}(T)^\perp$ is also in $\text{Range}(T^*)$. ✓

Now we assume ⓑ & try to prove ⓐ.

Let $S \in \mathcal{B}(V, \overline{\text{Range}(T)})$ be defined by $T$.

$\text{Range}(S) \subseteq \overline{\text{Range}(T)}$, of course, so $S^*$ is 1-1 between $\left(\overline{\text{Range}(T)}\right)^*$ & $V^*$. (p.3 part 6))

Since each elt $\lambda$ of $\left(\overline{\text{Range}(T)}\right)^*$ extends to an elt $\mu$ of $W^*$ (Hahn-Banach)

we see for each $v \in V$

$$(S^*\lambda)(v) = \lambda(Sv) = \mu(Tv) = (T^*\mu)(v),$$

i.e., $S^*\lambda = T^*\mu$.

Therefore $\text{Range}(S^*) = \text{Range}(T^*) \subseteq V^*$

ⓒ says both are norm-closed,

Thus $\text{Range}(S^*)$ is complete & a Banach space.

[remember: $V^*$ is a Banach space under norm topology!]

We apply the open mapping thm to the cts bijection between Banach spaces:

$$S^*: (\overline{\text{Range}\,T})^* \to \text{Range}(S^*)\;:$$

open map $\Rightarrow$ $\exists\, c > 0$ s.t. $\|S^*\lambda\| \geq c\|\lambda\|$ for each $\lambda \in (\overline{\text{Range}\,T})^*$

By the last lemma (p.4, part b) $S: V \to \overline{\text{Range}(T)}$ is open also.

Open sets absorb, so $S$ is onto.

so $\text{Range}(S) = \text{Range}(T)$ is infact equal to $\overline{\text{Range}(T)}$, i.e. closed $\square$.

<u>Corollary</u> Let $V, W$ be Banach Spaces, $T \in \mathcal{B}(V, W)$, then

$T$ is onto

$\Updownarrow$

$\exists\, c > 0$ s.t. $\|T^*\lambda\| \geq c\|\lambda\| \;\forall \lambda \in W^*$.

(A way to show $T$ is onto, if $T^*$ resembles an open map ---)

Proof $T$ is onto $\Leftrightarrow$ Range$(T)$ is closed + dense.

(*) Range$(T)$ dense $\Leftrightarrow$ $T^*$ is injective (p.3, part (b)).

(**) Range$(T)$ closed $\Leftrightarrow$ Range$(T)^*$ norm-closed in $V^*$ (last thm, ⓐ⇔ⓒ).

We need to show (*) + (**)

$\Updownarrow$

(***) $\exists c>0$ s.t. $\|T^*\lambda\| \geq c\|\lambda\|$ $\forall \lambda \in W^*$

Assume (*) + (**). Then $T^*$ is an open map on its range, & (***) follows as it did in the previous proof.

Assume (***). Then $T^*$ is injective. (*)

If $\{T^* v_n\}$ is Cauchy, so is $v_n$.

so $v_n \to$ some $v \in W^*$.

$T^* v_n \to T^* v$ since $T^*$ cts.

$\Rightarrow$ Range$(T^*)$ is complete, hence norm closed (**).

Thus (*) + (**) $\Leftrightarrow$ (***). $\square$

Compact Operators $\boxed{\text{very}}$ rich & important subject in analysis

**Def'n** If $V, W$ Banach spaces, $T \in \mathcal{B}(V,W)$ is a compact operator if

$T$(unit ball) is cpt in $W$.

Fact about cpt metric spaces, such as Banach spaces: a set has cpt closure $\Leftrightarrow$ it is totally bounded.

totally bdd means the set is contained in a finite union of open $\varepsilon$-balls for any $\varepsilon > 0$.

So,

$T$ a Compact Op. $\Longleftrightarrow$ $T(U)$ totally bdd

$\Updownarrow$

$\forall \varepsilon > 0 \;\; \exists\, w_1, \dots, w_n \in W$
s.t. for all $v \in V$, $\|v\| \le 1$
$\exists\, 1 \le j \le n$ s.t.
$\|v - w_j\| < \varepsilon$.

each bdd sequence $\{v_n\} \subseteq V$ has a subsequence s.t. $\{T(v_{n_i})\}$ converges in $W$

$\Downarrow$ easy

$\Uparrow$ an earlier result about totally bdd.

The compact operators have a spectral theory.

For finite dim'l VS, spectrum = {eigenvalues} = $\{\lambda \mid A-\lambda I$ is not invertible$\}$.

But not invertible does not imply the existence of eigenvalues in the $\infty$-dim'l setting.

So consider $V$ = Banach space
$B(V) = B(V,V)$ an algebra of operators

$\|ST\| \le \|S\| \cdot \|T\|$, ~~for example~~ (also a Banach space w/ a norm...)

This is an example of a "Banach Algebra" (A Banach space w/ associative multiplication satisfying)

$T$ **Invertible** means $\exists S \in B(V)$ s.t. $ST = TS = I$. (Need both left & right inverses)

Open mapping theorem can be restated to say
$T$ is invertible iff it is 1-1 & onto
(in this a linear inverse exists, & open mapping guarantees it is cts also!).

Spectrum of $T = \sigma(T) = \{\lambda \in \mathbb{C}$ (or $\mathbb{R}$) $\mid T-\lambda I$ is not invertible$\}$.

$\lambda \in \sigma(T)$ means

T has an eigenvalue OR

$\exists v \in V$ s.t. $Tw - \lambda w \neq v$ $\forall w \in V$

(OR Both!).

Proposition (a) If $T \in \mathcal{B}(V,W)$ & range is finite dim'l, $T$ is cpt

$V, W$ Banach

(b) If $T \in \mathcal{B}(V,W)$ & range closed & $T$ cpt, then range is finite dim'l

(c) the cpt operators are a norm-closed subspace of $\mathcal{B}(V,W)$

Zero operator — is good example when $\lambda \neq 0$ here.

(d) Given $T \in \mathcal{B}(V)$, cpt, $\lambda \neq 0$, $\dim\{v \mid Tv = \lambda v\} < \infty$,

(e) If $\dim V = \infty$, $T \in \mathcal{B}(V)$ cpt, then $0 \in \sigma(T)$.

(f) cpt operators form a 2-sided ideal in $B(V)$.

Pf (a) Image of unit ball is bdd, closure is bdd & closed, hence cpt.

(b) $T: V \to \text{Range}(T)$ is a surjective map of Banach spaces. So $T$ is open. Range$(T)$ thus has an open nbd of $0$ (image of unit ball) which has cpt closure.

Thus Range$(T)$ locally cpt $\Rightarrow$ finite dim'l.

(e) follows from (b), since Range $T$ cannot be onto.

(c) postponed.

(d). Restrict. T to $Null(T-\lambda I)$, closed by cty $\subseteq V$
then it is still cpt, of course,
T maps $Null(T-\lambda I)$ to itself,

T is onto if $\lambda \neq 0$.
So (d) follows from (b).

(f) We assume $T \in B(V)$ cpt, $S \in B(V)$

Let $\{V_n\}$ be a bdd sequence in $V$
it has a subsequence s.t. $\{TV_n\}$ converges
$\Rightarrow$ $STV_n$ converges, since $S$ cts
thus $ST$ is cpt.

Also $\{SV_n\}$ bdd
So has subsequence s.t.
$\{TSV_n\}$ converges.

Back to (c)
Obviously scalar mults of cpt operators are cpt.
If $S, T \in B(V)$ cpt,
$U$ = unit ball
$S(U)+T(U)$ is still totally bdd since $S(U)$ & $T(U)$ are.
Thus $S+T$ cpt & Cpt operators form a subspace.

Now we show closure:

Let $\varepsilon > 0$ & $T$ an operator in the closure of the set of cpt operators. $\exists$ a cpt operator $S$ s.t.

$$\|S - T\| < \varepsilon/2.$$

$S(\mathcal{U})$ is totally bounded

$\exists$ points $v_1, \dots, v_n \in V$

s.t. $\bigcup_{j \le n} B_{v_j}(\varepsilon/2)$

covers $S(\mathcal{U})$.

Now let $x \in V$, $\|x\| \le 1$

$$\|T(x) - S(x)\| < \varepsilon/2.$$

So $\exists$ $v_j$ s.t. $\|T(x) - v_j\| < \varepsilon$. totally

Thus $T(\mathcal{U})$ is bdd. $\square$

<u>Proposition</u> Let $V, W$ be Banach spaces. Then $T \in \mathcal{B}(V, W)$ is cpt $\Leftrightarrow$ its adjoint $T^* \in \mathcal{B}(W^*, V^*)$ is cpt.

<u>Pf</u> Assume $T$ is cpt, & $\{\lambda_n\}$ a sequence in the unit ball of $W^*$.

The linear f'nls $\lambda_n$ are equicts on $Y$ since their norms are $\le 1$.

Let $U$ = unit ball of $V$

$T(U)$ has cpt closure.

Ascoli's thm — $\{\lambda_n\}$ has a subsequence $\{\lambda_{n_i}\}$ which converge uniformly on $\overline{T(U)}$.

So $\{T^*\lambda_{n_i}\}$ is Cauchy:

$$\|T^*\lambda_{n_i} - T^*\lambda_{n_j}\| = \sup_{v\in U} |(\lambda_{n_i}-\lambda_{n_j})(Tv)|$$

$$\to 0 \text{ as } n_i, n_j \to \infty$$

on $T(U)$,

& thus $\{T^*\lambda_{n_i}\}$ converges. So $T^*$ is cpt.

Now assume $T^*$ cpt. We will relate $T$ to $(T^*)^*$, which we know is cpt by the part of the theorem we just proved.

$$(T^*)^* : (V^*)^* \to (W^*)^*$$

Recall $I_V : V \hookrightarrow (V^*)^*$
$I_W : W \hookrightarrow (W^*)^*$ are isometric embeddings.

We claim $I_W \circ T = (T^*)^* \circ I_V \; : V \to (W^*)^*$

Indeed for any $v \in V$ & $\lambda \in W^*$

$$[I_W(T)\lambda](v) \qquad\qquad [((T^*)^* \circ I_V)\lambda](v)$$

$$= \lambda(Tv) \;=\!=\!=\; \lambda(Tv)$$

Now $(I_W \circ T)(U) = (T^*)^* \circ I_V(U)$

$\subseteq (T^*)^*((U^*)^*)$

because $I_V(U) \subseteq (U^*)^*$

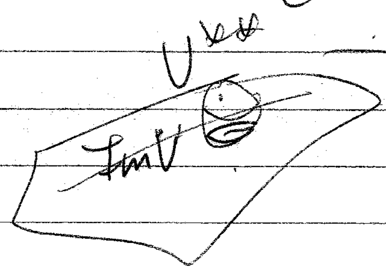
(isometry!).

Thus $T^*$ cpt $\Rightarrow (T^*)^*$ cpt $\Rightarrow (T^*)^*((U^*)^*)$ totally bdd

$\Rightarrow (I_W \circ T)(U)$ is totally bdd

So $T(U)$, isometric to $(I_W \circ T)(U)$ is also totally bdd. $\square$

Note: Rudin seems to frequently change his def'n of totally bounded.




we need also here to ~~use the fact that if~~ the ε-balls covering $I_W(T(U))$ are centered outside of $I_W(W)$,

then the intersection of the ε-ball w/ the subspace $I_W(W)$ is a δ-ball, $\delta < \varepsilon$, centered in $I_W(W)$.

Recall Direct sum: If $M, N \subseteq V$ subspaces we say

$$V = M \oplus N$$

if $M + N = V$
$M \cap N = \{0\}$.

A closed subspace $M$ of a TVS $V$ for which $\exists$ another closed subspace $N$ s.t. $M \oplus N = V$ is called complemented.

This is a useful structural property.

Lemma (Conditions for a closed subspace to be complemented)
Let $M \subset V$ be a closed subspace of a TVS.
1) If $V$ is locally convex & $M$ finite-dim'l, then $M$ is complemented.

2) If codimension of $M = \dim(V/M) < \infty$, then $M$ is complemented.

Pf 1) Let $m_1, \dots, m_n$ be a basis for $M$.
We saw before the map

$$m \in M \longmapsto \sum c_j(m) m_j$$

has coefficient functions $c_j(m)$ which are cts linear f'nls on $M$.
Hahn-Banach: since $V$ is locally convex, each $c_j$

extends to an element of $V^*$.

Let $N$ = kernel of these extns.

Then
- $N$ is closed.
- $M \cap N = \{0\}$ (since $m_j$ are a basis)
- $M + N$ = all of $V$, since
  $M$ = part where some $c_j$ is nontrivial
  $N$ = part where all $c_j$ are trivial.

2) Use the quotient map

$$\pi : V \to V/M$$

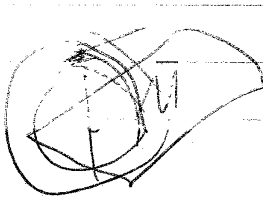
Let $b_1, \dots, b_n$ be basis for $V/M$.

Choose $v_1, \dots, v_n$ s.t. $\pi v_i = b_i$.

Let $N = \mathrm{span}\{v_j\}$.

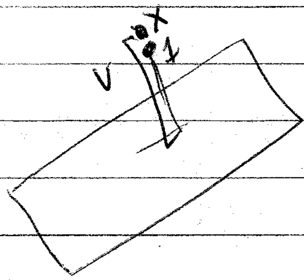
Now
- $N$ is finite dim'l, hence closed
- $M \cap N = \{0\}$, since $M$ is ker of $\pi$
  $N$ cannot be [illegible] by basis
- $M + N = \pi^{-1}(\pi(V)) = V$. $\square$.

[illegible] Suppose $M$ is a subspace of $V$ which is not dense. Then for each $r > 1$ $\exists$ $v \in V$ s.t. $\|v\| < r$ & $\|v - m\| \geq 1$ for all $m \in M$.



Pf (easy) $M$ is not dense, so $\exists v \in V$ s.t. $d(v, M) = \inf_{m \in M} d(v, m) > 0$.

Thus, by scaling, $\exists\, x$ s.t.
$$d(x, M) = 1.$$



Choose $m' \in M$ s.t. $d(x, m') < r$

let $v = x - m'$.

Then $\|v\| = d(x, m') < r$

& $\|v - m\| = \|x - (\underbrace{m' - m}_{\text{elt of } M})\| \geq 1$ for all $m \in M$.

$\square$.

Now 3 thms about spectrum.

Thm If $\underset{\text{cpt}}{T} \in B(V)$, $V$ a Banach space, then $T - \lambda I$ has closed range for all $\lambda \neq 0$.

Pf We proved before (p.14) that
$$\dim(\underbrace{Null(T - \lambda I)}) < \infty.$$

So this is a complemented space by the lemma on p.19.

Thus $\exists$ closed subspace $M \subseteq V$ s.t.

$$V = M \oplus \text{Null}(T-\lambda I),$$

Now we map $S: M \to V$
by $Sv = (T-\lambda I)v$

Obviously $S \in \mathcal{B}(M,V)$

$M$ is a Banach space itself, since it is closed.

Since $M \cap \text{Null}(T-\lambda I) = \{0\}$
$Sv \neq 0$ if $v \neq 0$,

So $S$ is 1-1 & Range$(S)$ = Range$(T-\lambda I)$,

So we need to show $S$ has closed range.

Suppose
$(*_r)$ $\|Sm\| \geq r\|m\|$ $\forall m \in M$,

fails for all $r > 0$.

Then $\exists$ $m_1, m_2, \ldots$
w/ $\|m_j\| = 1$ & $Sm_j \to 0$.

$T$ is compact, so a subsequence of
$\{Tm_j\}$ converges to some point

$x$ in $V$.

so $\|\lambda m_j - x\|$

$\leq \|\lambda m_j - Tm_j\| + \|Tm_j - x\|$

$\to 0$ on this subsequence.

So $x$ is a limit of points of the form $\lambda m_j \in M$, which is closed.

Thus $x \in M$. Also by cty of $S$

$Sx = \lim_{\text{on subsequence}} (S\lambda m_j) = \lambda \lim Sm_j = 0$.

So $x = 0$ because $S: M \to V$ is 1-1.

However $Tm_j \to x = 0$ (subsequence) $\Rightarrow \lambda m_j \to 0$

$Sm_j \to 0$

$\Rightarrow$ impossible unless $\lambda = 0$ (since $\|m_j\| = 1$).

Conclusion $(*_r)$ holds for some $r>0$.

Now assume $\{Sv_n\}$ is a Cauchy sequence in Range$(S)$. Then $(*_r)$ shows $\{v_n\}$ is Cauchy in $M$.
$M$ is closed, hence complete (as $M \subseteq V$ are Banach spaces)
$\Rightarrow v_n \to$ some $v \in M$.

Then by cty of $S$, $Sv_n \to Sv \in$ Range$(S)$.

Hence Range$(S)$ = Range$(T-\lambda I)$ is closed. $\square$.

Nice.

**Thm** A cpt operator $T \in \mathcal{B}(V)$, $V$ a Banach space has only a finite set of eigenvalues $\lambda$ s.t. $|\lambda| >$ any fixed $\varepsilon > 0$.

Furthermore, if $\lambda \neq 0$ is an eigenvalue then $T-\lambda I$ is not onto, i.e. both invertability conditions fail!

Pf Suppose first $\lambda \neq 0$ (eigenvalue) & $T - \lambda I$ is onto.
Let $S = T - \lambda I$
& $M_n = \ker(S^n)$.

$M_1$ is nonempty since $\lambda$ is an eigenvalue.

Let $v_1$ be an eigenvector.
Since $S$ is onto, let $Sv_2 = v_1$.

Then $S^2 v_2 = 0$
continue to get $v_1, v_2, \ldots$
$S v_{n+1} = v_n$
$S^{n+1} v_{n+1} = S^n v_n = 0$.

Thus
- $M_1 \subseteq M_2 \subseteq M_3 \subseteq \ldots$
- $T$ fixes each $M_n$ ($T$ commutes w/ $S$)
- $S M_n \subseteq M_{n-1}$

each $M_n$ closed in $M_{n+1}$
(kernel condition)
& in fact proper.

Similarly, if for some $r > 0$ there are $\infty$-many eigenvalues $\lambda$ w/ $|\lambda| > r$,
pick an $\infty$ sequence of eigenvalues & eigenvectors $Av_n = \lambda v_n$.

If $M_n = \mathrm{Span}(v_1, \ldots, v_n)$
Finite dim'l, hence closed

we again have
($*_1$) • $M_1 \subseteq M_2 \subseteq \ldots$ proper
($*_2$) • $T$ fixes each $M_n$
($*_3$) • $(T - \lambda_n I) M_n \subseteq M_{n-1}$,
since $S$ kills $\langle v_n \rangle$
but not $\langle v_1, \ldots, v_{n-1} \rangle$.

Now from this filtered configuration, we will use the compactness of $T$ to get a contradiction. By the lemma on p. 20,

$\exists$ vectors $y_{n+1} \in M_{n+1}$ s.t.

$\|y_{n+1}\| \le 2$

$\|y_{n+1} - x\| \ge 1$ if $x \in M_n$.

Define $z := Ty_m - (T - \lambda_n I)y_n$ for $m < n$

$\in M_{n-1}$ by ($*_2$) & ($*_3$).

Thus $\|T(y_n - y_m)\|$

$= \|Ty_n - Ty_m - \lambda_n y_n + \lambda_n y_n\|$

$= \|z - \lambda_n y_n\|$

$= |\lambda_n| \cdot \|y_n - \underbrace{\tfrac{1}{\lambda_n} \cdot z}_{\in M_{n-1}}\| \ge |\lambda_n| > r.$

But $\{y_n\}$ is bdd

yet $\|Ty_n - Ty_m\| > r$ cannot converge!

So $T$ is not compact, a contradiction. $\square$

<u>Lemma</u> Let $V$ = locally convex TVS, $M$ closed subspace. Then $\dim V/M \le \dim \Sigma$, where $\Sigma = \{\Lambda \in V^* \mid \Lambda \text{ trivial on } M\}$

<u>pf</u> Let $v_1, \dots, v_k$ be linearly indep vectors in $V$ s.t. $\pi v_j \ne 0$. Then $\exists$ $k$ lin indep cts linear fnls on span$\{v_1, \dots, v_k\}$, trivial on $M$. They extend to elts on $V^*$, & must be linearly indep there.

# Big Final Thm of Chapter!

Let $T \in \mathcal{B}(V)$ be a cpt operator, $V$ = Banach space.

Then (a) if $\lambda \neq 0$,

$$\underbrace{\dim\left[\mathrm{Null}(T-\lambda I)\right]}_{d_1} = \underbrace{\dim\left[V/\mathrm{Range}(T-\lambda I)\right]}_{d_2}$$

$$= \underbrace{\dim\left[\mathrm{Null}(T^*-\lambda I)\right]}_{d_3} = \underbrace{\dim\left[V^*/\mathrm{Range}(T^*-\lambda I)\right]}_{d_4}$$

$< \infty$, see p14d), where this was shown.

(b) $\lambda \neq 0, \lambda \in \sigma(T) \Rightarrow \lambda$ an eigenvalue of both $T$ & $T^*$

(c) $\sigma(T)$ is cpt & countable (could be finite, of course), & has only one possible limit point: $0$. ← these follow from previous thm

Pf (a) Apply last Lemma to $V$, $M = \mathrm{Range}(S)$, $S = T - \lambda I$ again.

[Thm (p21): Range(S) is closed.]

p2

$\Sigma$ in lemma $= \mathrm{Range}(S)^\perp = \mathrm{Null}(S^*)$

So $d_2 \leq d_3$ using this lemma.

Cannot simply use duality for $d_4 \le d_1$, because $d_1$ uses $T$, not $T^{**}$,

Instead: observe $V^*$ locally convex in weak topology (basic open nbds are convex...)

$M = \text{Range}(S^*)$ is weak* closed in $V^*$ ($\Leftrightarrow R(S)$ in $V$ closed), (p,7)

Now $\Sigma$ in lemma is all weak* cts lner f'ns which annihilate $\text{Range}(S^*) = M$,

$\Sigma = \quad = {}^\perp\text{Range}(S^*)$ exactly
$\quad = \text{Null}(S)$ (p,2),

So Lemma: $\dim\left(\text{Null}(T - \lambda I)\right) \ge \dim\left(V/\text{Range}(T^* - \lambda I)\right)$, i.e. $d_4 \le d_1$,

Hard part: to show $d_1 \le d_2$ ($\Leftrightarrow$ to $d_3 \le d_4$),

Assume $d_1 > d_2$,

Actually $d_1 < \infty$ by Prop part d, p,14

So then $d_2$ also $< \infty$,

Use complementation lemma (p,19): a) $\text{Null}(S)$ is complemented & b) $\text{Range}(S)$ is complemented,
i.e. $\exists$ closed subspaces $E$ & $F$ s.t.

$V = \text{Null}(S) \oplus E$
$V = \text{Range}(S) \oplus F$, $\dim F = d_2$.

Define projection $P: V \to Null(S)$

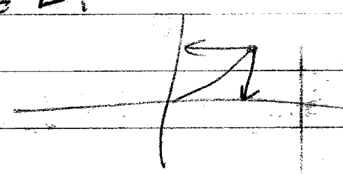
Claim: $P$ is cts

proof

if $V_n \to$ some $V$

$V_n = x_n + y_n$, $x_n \in Null(S)$

$V = x + y$ $\quad y_n \in E$.



$\underset{\in \text{ closed space}}{x_n - x} \to \underset{\in \text{ closed space}}{-(y_n - y)}$

which have $\{0\}$ intersection, so each $\to 0$, ✓

By assumption $d_1 > d_2$

$\dim Null(S) > \dim F$

Algebra: $\exists$ linear mapping $\varphi : Null(S) \twoheadrightarrow F$
w/ nontrivial kernel,
cts, since spaces are finite dim'l

Let $\Phi(v) = T(v) + \varphi(\pi(v))$
$\in \mathcal{B}(V)$

$\varphi \circ \pi : V \to V$ is cpt since range is finite dim'l
(p. 14, a))

so $\Phi$, sum, is also cpt. (p. 14, f))

$\Phi - \lambda I = S + \varphi\pi$

We saw a few lines ago $\exists\, x_0 \in Null(S)$ s.t. $\varphi(x_0) = 0$

$\Rightarrow$ $x_0$ is an eigenvector for $\Phi$ w/ eig $\lambda$,

so $Range(\Phi - \lambda I) \neq V$ (Thm p.24)

Now, $(\Phi - \lambda I)(E) = S(E)$ since $\pi$ kills $E$,

$= S(V)$ since $S$ kills $Null(S)$

$= Range(S)$. & $V = Null(S) \oplus E$.

Likewise

$$(\Phi - \lambda I)(Null(S)) = \varphi\pi(Null(S))$$
$$= \varphi(Null\, S) = F$$

($\varphi$ is onto).

So $Range(\Phi - \lambda I)$, which we just said is not all of $V$, contains $Range(S) + F = V$, contradiction!

Thus $d_1 \leq d_2$

$\Updownarrow$

$d_3 \leq d_4$

& all $d_1 = d_2 = d_3 = d_4$ are finite ✓

(b) $\lambda$ not an eig of $T$

then $d_1 = 0$.

Thus $d_2 = 0$, so $S$ is 1-1, onto, invertible

$\overset{\shortparallel}{T - \lambda I}$, so $\lambda \notin \sigma(T)$.

Also: $\lambda$ not eig of $T^*$ (using $d_3$ & $d_4$ ...).

(c) Suffices to show $\sigma(T)$ cpt, as other 2 assertions came on p.24.
That result shows any open cover of $\sigma(T) \cup \{0\}$ has a finite subcover.
Result obvious if $\dim V < \infty$ (linear algebra)

If $\dim V = \infty$, use p.14 e): $0 \in \sigma(T)$! $\sigma(T)$ is cpt $\square$.

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Finally, good examples of cpt operator

1) $\infty$-dim'l diag matrix $\begin{pmatrix} a_1 & & \\ & a_2 & \\ & & \ddots \end{pmatrix}$ w/ $\sum |a_j|^2 < \infty$. on $\ell^2(\mathbb{Z})$,

2) let $k : [0,1]^2 \to \mathbb{C}$ be cts
$I : C([0,1]) \to C([0,1])$
given by $(If)(x) = \int f(y)\, k(x,y)\, dy$,
Fredholm

3) any $T \in \mathcal{B}(V)$ s.t. $\exists$ operators $T_1, T_2, \ldots$ w/ finite rank s.t. $T_n \to T$ in the operator norm.

4) In a Hilbert space $V$, all cpt operators have this form