Lecture Notes 10

Some applications to function theory

Recall that if $f: X \to \mathbb{R}^n$ is a vector valued function on some measure space (X,μ), then one usually defines (where sensible)

$$\int_X f(x)\,d\mu(x) = \left(\int_X f_1(x)\,d\mu(x), \cdots, \int f_n(x)\,d\mu(x)\right)$$

where $f(x) = (f_1(x), \cdots, f_n(x))$.

One way to describe this is that

$$\Lambda\left(\int_X f\,d\mu\right) = \int_X (\Lambda f)\,d\mu$$

for all linear f'ls Λ on $\mathbb{R}^n$.

This motivates the following def'n.

Def If $f: X \to V$, where (X,μ) is a measure space, V is a TVS whose dual V^* separates points, and f a fn s.t. all scalar fns $\Lambda f: X \to \mathbb{R}$ or $\mathbb{C}$ are integrable over (X,μ), then $\int f\,d\mu$ is defined as the unique vector $v \in V$ (if it exists) p2

Such that $\Lambda v = \int \Lambda f \, d\mu$ for all $\Lambda \in V^*$.

<u>Remarks</u> (1) v is unique, if it exists, because Λ separates points, & so no other vector has the same values for all Λ.

(2) Recall $V \hookrightarrow (V^*)^*$.
The def'n of integral is really saying that we take the element in the image which agrees w/ the element of $(V^*)^*$ given by
$\Lambda \longmapsto \int \Lambda f \, d\mu$.

Of course, we have to prove it is in the image, under suitable hypotheses.

We suppose X cpt Hausdorff, $\int_X d\mu = 1$ (Borel probability meas),
f closed convex hull of $f(X)$ is cpt
(automatic in Frechet spaces, by last thm of previous notes)

f also cts

$\Rightarrow$ Existence for any real or $\mathbb{C}$-valued Borel meas.

Existence

Thm If V a TVS on which V^* separates points, & μ is a Borel probability meas on a compact Hausdorff space X, & $f: X \to V$ cts, & closed convex hull H of $f(X)$ is closed, then

$\int_X f \, d\mu$ exists in the sense we defined above.

Also, it is an ~~element~~ of H.

Pf First consider an arbitrary subset of V^* $L = \{\Lambda_1, \cdots, \Lambda_n\}$.

Let

$$E_L = \{v \in H \mid \Lambda v = \int_X \Lambda f \, d\mu \text{ for all } \Lambda \in L\}$$

~~So~~ E_L = intersection over L of

$$\Lambda^{-1}\left(\int_X \Lambda f \, d\mu\right),$$

hence closed.

$E_L \subseteq H = \text{cpt} \Rightarrow E_L$ cpt also.

the E_L's are closed under finite intersection,

If we can show all E_L are nonempty, the finite intersection property will show $\exists v \in$ all E_L, which is exactly what we are trying to prove.

We might as well assume $k = \mathbb{R}$, since the thm for $\mathbb{C}$-vector spaces is equivalent to it for them when considered as $\mathbb{R}$-vector spaces.

Given $L = \{\lambda_1, \dots, \lambda_n\}$ we have a mapping

$$\Phi : V \longrightarrow \mathbb{R}^n, \quad \Phi(v) = (\lambda_1(v), \dots, \lambda_n(v))$$

Let $K = \Phi(f(X))$ = cpt, since $\Phi \circ f$ is cts.

Let $m \in \mathbb{R}^n$ equal $\int_X \Phi(f(x))\,d\mu(x)$

Claim $m \in$ convex hull of K

Pf of claim convex hull of K is cpt in $\mathbb{R}^n$ since K is (last thm).
If m is not in the hull, then $\exists$ a linear fnl separating it from K:

$$-\sum c_i u_i + \sum c_i m_i > \varepsilon = \text{fixed}$$

for some $(c_1, \dots, c_n)$ & all $u \in K$.

So

$$-\sum c_i \lambda_i f(x) + \sum c_i m_i > \varepsilon$$

integ over $(X, d\mu)$: Get 0, a contradiction (note slight gap in lecture — you should have ε here). ✓

So: $m \in$ convex hull of K,

$\Downarrow$

$m =$ for some

$= \Phi(y)$, $y \in H$ "closure of convex hull of $f(X)$.

Hence E_L is nonempty. $\square$

<u>Theorem</u> Let V be a TVS such that V^* separates points, $K =$ cpt subset of V
$H =$ convex hull of K
$\bar{H}$ is cpt,

then $y \in \bar{H} \iff \exists$ a regular Borel probability measure μ on K s.t.

$$y = \int_K x \, d\mu$$

as interpreted in previous thm.

"regular" means for all Borel E
$\mu(E) = \sup\{\mu(A) \mid A \subseteq E \text{ cpt}\}$
$= \inf\{\mu(B) \mid B \supseteq E \text{ open}\}$

<u>Note</u>: If V Frechet then we need only assume K is cpt (the rest follows).

<u>Pf</u> Again, we can regard V as a real vector space (as in the pf of the previous thm).

Let $C(K)$ = Banach space of cts $f: K \to \mathbb{R}$, under sup norm.

Riesz-Rep'n thm: says $C(K)^* =$ all real Borel measures $\mu = \mu_1 - \mu_2$, where μ_1 & μ_2 are regular positive Borel measures.

this is not a trivial result!

Construct a map $\varphi: C(K)^* \to V$

$$\mu \longmapsto \int_K x \, d\mu$$

Let $P = \left\{ \begin{array}{l} \text{all regular, positive, prob} \\ \text{measures on } K \end{array} \right\} \subseteq C(K)^*$

convex

So we need to show $\varphi(P) = \overline{H}$.

Example of element of P: δ-mass @ $v \in K$

δ_v.

$K \subseteq \varphi(P)$ since $\delta_v \in P$

& $\int_K x \, d\delta_v = v = \varphi(\delta_v)$.

φ linear, P convex $\Rightarrow H \subseteq \varphi(P)$.

Last thm: $\varphi(P) \subseteq \overline{H}$. So $H \subseteq \varphi(P) \subseteq \overline{H}$

w/ f = Identity

Thm follows if $\varphi(P)$ is closed

Each elmt $\mu \in P$ has the property that

$$\|h\| < 1 \Rightarrow \left|\int_K h\, d\mu\right| \le 1.$$

↑ sup norm

The set of such measures

$$\left\{ \mu \;\middle|\; \|h\| \le 1 \Rightarrow \left|\int_K h\, d\mu\right| \le 1 \right\}$$

If U = open nbhd of zero $= \{\|h\| \le 1\}$
Then Banach-Alaoglu (w/ this choice of U) shows precisely that this set is weak*-cpt.

Let E_h, for $h \ge 0$ in $C(K)$, equal

$$E_h = \left\{ \mu \;\middle|\; \int_K h\, d\mu \ge 0 \right\}$$

↑ cts linear fnal on μ

E_h = weak*-closed

Set $E = \{\mu \mid \int 1\, d\mu = 1\}$ ~ Mass 1
is also weak*-closed.

uses a bit of measure theory e.g. "Pos meas $\Leftrightarrow$ integs vs. $h \ge 0$ is ≥ 0."

Thus $P = \left[\bigcap_{\substack{h \in C(K) \\ h \ge 0}} E_h\right] \cap E$ is weak* closed, hence cpt. (since it is a subset of this)

Now, we claim $\varphi: C(K)^* \to V$ is a cts map

$\uparrow$ weak* topology $\uparrow$ weak topology,

It follows $\varphi(P)$ is weakly cpt, convex

$\Downarrow$ — in Hausdorff spaces

weakly closed.

$\Downarrow$ — defn of weak

Strongly closed, what we need to finish,

~~Pf of claim:~~

CTY of φ means we check given a weak nhd U of 0 in X, $\varphi^{-1}(U)$ contains a weak*-open nhd in $C(K)^*$.

U may be assumed to be of the form

$$U = \{ v \in V \mid |\lambda_i v| < r_i \} \quad \text{for } \lambda_1, \dots, \lambda_n \in V^*, \; r_1, \dots, r_n > 0$$

Let $S = \{ \mu \in C(K)^* \mid |\int_K \lambda_i(v) d\mu(v)| < r_i \}$.

Then λ_i, restricted to K, is an element of $C(K)$, so S is a weak* nhd of 0 in $C(K)^*$.

Now Given $\mu \in S$

$$\varphi(\mu) = \int_K v \, d\mu.$$

&

The context of this def'n is that

$$\lambda_i(\varphi(\mu)) = \lambda_i\left(\int_K v\,d\mu\right) = \int_K \lambda_i(v)\,d\mu$$
$$= \int_K \lambda_i\,d\mu$$

This means $\varphi(S) \subseteq U$ ✓ $\square$.

Thm Quantitative version of the existence theorem

Let K = cpt Hausdorff space
V = Banach space
$f: K \to V$ cts
μ = pos Borel measure on K.

Then $\left\| \int_K f\,d\mu \right\| \le \int_K \|f\|\,d\mu$

(cts version of Δ-inequality).

Pf Under these hypotheses V^* separates pts (V is Frechet)

$f(K)$ is cpt, & earlier result says the closure of its convex hull is too.

So $\int f\,d\mu$ exists, call it v.

We saw early on if V is a normed TVS, then $\exists \Lambda \in V^*$ w/

$\Lambda v = \|v\|$,

& $|\Lambda(x)| \le \|x\|$ for all $x \in V$.

so

$|\Lambda f(k)| \le \|f(k)\|$ for all $k \in K$.

so

$$\|v\| = \Lambda v = \int_K \Lambda f \, d\mu \le \int_K |\Lambda f| \, d\mu$$
$$\le \int \|f\| \, d\mu. \quad \square$$

Holomorphic Functions

Now study only complex TVS.

Consider $\Omega \subseteq \mathbb{C}$ open set, V a TVS,

weakly holomorphic $f: \Omega \to X$ s.t.

Λf holom for all $\Lambda \in V^*$

Strongly holomorphic: $\lim\limits_{w \to z} \dfrac{f(w) - f(z)}{w - z}$ exists in V for all $z \in \Omega$.

Strong $\Rightarrow$ weak by cty of $\Lambda \in V^*$.

Recall $\mathrm{Ind}_\Gamma(z)$ = index of a pt $z \in \mathbb{C}$ wrt a closed path Γ s.t. $z \notin \Gamma$

$$= \frac{1}{2\pi i} \int_\Gamma \frac{d\zeta}{\zeta - z}.$$

Thm If $\Omega \subseteq \mathbb{C}$ open, V $\mathbb{C}$-Frechet Then weakly holom $\Leftrightarrow$ strongly holom.

In more detail

f weakly holom $\Rightarrow$ (1) f strongly cts in Ω

(2) Cauchy: $\forall$ closed paths $\Gamma \subseteq \Omega$ s.t. $\mathrm{Ind}_\Gamma(w)$ for $w \notin \Omega$

$$\int_\Gamma f(\zeta)\,d\zeta = 0$$

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

for all $z \in \Omega$ w/ $\mathrm{Ind}_\Gamma(z) = 1$

Value of $\int_{\Gamma_1 \text{ or } \Gamma_2} f(\zeta)\,d\zeta$ is indep of Γ_1 or Γ_2 so long as $\mathrm{Ind}_\Gamma(w)$ is indep of Γ_1 or Γ_2 for $w \notin \Omega$.

Pf We translate Ω so that $0 \in \Omega$.

We First prove strong cty @ zero of a weakly holom. $f : \Omega \to \mathbb{C}$.

Let D_r = closed disc of radius r in $\mathbb{C}$.
$\exists\, r$ s.t. $D_{2r} \subseteq \Omega$.
For any $\Lambda \in V^*$ we know (by our assumption of weak holomorphy) that

$$\frac{(\Lambda f)(z) - (\Lambda f)(0)}{z} = \frac{1}{2\pi i} \int_{|\zeta|=2r} \frac{(\Lambda f)(\zeta)}{(\zeta - z)\zeta}\, dz$$

for all $|z| \leq r$.

We deduce

$$|\Lambda[f(z) - f(0)]| \leq \frac{|z|}{r} \cdot C \, , \quad C \text{ depends on max } \Lambda f(\zeta) \text{ over } |\zeta| \leq 2r.$$

So $\left\{ \frac{f(z)-f(0)}{z} \right\}_{|z| \leq r} \subseteq V$ is a weakly bdd set.

V Frechet $\Rightarrow$ weak & strong bddness are equivalent.

Given any (ordinary) open nbd U of 0,
$f(z) - f(0) \subseteq tzU$ for some $t > 0$, & all $|z| < r$

... so $f(z) \to f(0)$ as $z \to 0$ in the usual topology. Hence f is continuous in the usual topology (we are in a metric space, so "sequential continuity" $\Leftrightarrow$ continuity).

Now since f is cts, Γ in statement of thm is cpt, then since we are in a Frechet space

$$\int_\Gamma f(\zeta)\,d\zeta\,, \quad \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

all make sense.

The identities we asserted (Cauchy ...) hold for Λf instead of f, $\forall \Lambda \in V^*$, by weak holomorphy assumption, i.e. thus, by def'n of vector valued integration, they hold for f.

Finally, we come to strong holomorphy. We use Cauchy formula:

$$\frac{f(z) - f(0)}{z} = \frac{1}{2\pi i}\int_{|\zeta|=2r} \zeta^{-2} f(\zeta)\,d\zeta$$

$$+ \frac{z}{2\pi}\int_{-\pi}^{\pi} \frac{f(2re^{i\theta})\,d\theta}{2re^{i\theta}(2re^{i\theta} - z)}$$

Why? RHS is

$$\frac{1}{2\pi i}\int_{\substack{|\rho|=2r}} d\rho \left[\frac{f(\rho)}{\rho^2} + \frac{zf(\rho)}{\rho^2(\rho-z)}\right]$$

$\rho = 2re^{i\theta}$

$d\rho = 2ire^{i\theta}$

$$= \frac{f(\rho)}{\rho^2}\left[\frac{z}{\rho-z}+1\right]$$

$$= \frac{f(\rho)}{\rho^2}\left(\frac{\rho-z+z}{\rho-z}\right)$$

$$= \frac{f(\rho)}{\rho(\rho-z)}$$

Residue @ $\rho=0$: $\frac{f(0)}{-z}$

@ $\rho = z$ $\frac{f(z)}{z}$ ✓.

Back to formula. Let U' be an arbitrary open nhd of 0. U' contains a balanced convex open nhd U of 0.

Let K = image of $f(\rho)$ on circle $|\rho|=2r$.
$=$ cpt.

$\Rightarrow$ for some t we have $K \subseteq tU \subseteq tU'$
(& hence all larger t)

Let $s = \frac{t}{r^2}$. Then if $|z|\le r$.

$$\frac{f(2re^{i\theta})}{2re^{i\theta}(2re^{i\theta}-z)} \subseteq \frac{K}{2re^{i\theta}(2re^{i\theta}-z)}$$

$$\subseteq \frac{tU}{2re^{i\theta}(2re^{i\theta}-z)} \subseteq sU.$$

Thus second integral

$$\frac{z}{2\pi}\int_{-\pi}^{\pi}\frac{f(2re^{i\theta})\,d\theta}{2re^{i\theta}(2re^{i\theta}-z)} \in \epsilon s|z|U.$$

So as $z \to 0$

$$\frac{f(z)-f(0)}{z} - \frac{1}{2\pi i}\int_{|\zeta|=2r}\zeta^{-2}f(\zeta)\,d\zeta$$

$$\in s|z|U.$$

As $z \to 0$ $\in U \subseteq U'$.

So $\frac{f(z)-f(0)}{z} \longrightarrow \frac{1}{2\pi i}\int_{|\zeta|=2r}\zeta^{-2}f(\zeta)\,d\zeta$

as $z \to 0$ in strong topology. $\square$.

<u>Theorem</u> Let V = TVS on which V^* separates points. If $f: \mathbb{C} \to V$ is weakly holomorphic & $f(\mathbb{C})$ is weakly bdd on V, then f is constant

(Generalizes Liouville).

<u>Pf</u> $\forall \Lambda \in V^*$ Λf bdd, hence constant by Liouville.
$\Rightarrow \Lambda(f(z)-f(0)) = 0$ $\forall \Lambda$. V^* separates points
$\Rightarrow f(z) = f(0)$ $\forall z$. $\square$.