Mathematics 80600

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Text: Rudin, Functional Analysis (in library)

First main topic: Topological Vector Spaces

First we give the (somewhat dry) definition.

Recall a vector space V over a field k is a set of vectors satisfying the following axioms:

- 2 vectors can be added, and V forms a group under addition
- There is a scalar multiplication action $k \times V \longrightarrow V$ such that

$$1v = v \qquad a(bv) = (ab)v$$
$$a(v+w) = av + aw \qquad (a+b)v = av + bv$$

All our vector spaces will be over $k = \mathbb{R}$ or $\mathbb{C}$.

Obvious example: $\mathbb{R}^n$, $\mathbb{C}^n$ – have basically only one interesting topology. Recall all norms give the same topology.

Hard example: $C^\infty(\Omega)$, Ω compact subset of $\mathbb{R}$. What is the natural topology?

Recall also the definition of <u>topological space</u>: a set X with a specified collection of <u>open subsets</u> τ satisfying the following axioms:

- $X, \{\}$ are open
- $T_1, T_2 \in \tau \implies T_1 \cap T_2 \in \tau$
- $\{T_\alpha\} \in \tau \implies \bigcup_\alpha T_\alpha \in \tau$.

Topological Vector Space

A vector space with a topology! in which

- <u>all sets $\{v\}$ of single points ("singletons") are closed</u>
  - closed means its complement is open
- <u>vector addition is continuous</u> from $V \times V \to V$: given $X_1, X_2 \in V$ & an open nbhd U of $X_1 + X_2 \in V$, there exists open neighborhoods $U_1, U_2 \subseteq V$ such that $U_1 + U_2 = \{v_1 + v_2 \mid v_1 \in V_1, v_2 \in V_2\} \subseteq U \subseteq V$.
- <u>Scalar multiplication is continuous</u> (requires k have a topology).

Now, having given the formal definitions, we present a number of examples.

Examples of TVS's

1. k^n. Any finite dim'l Hausdorff topological vector space is homeomorphic to k^n, under the product topology.

Reminder, (since first day of class)

- Hausdorff: distinct points have distinct open neighborhoods
- Homeomorphic: bijection, continuous in both directions
- Product topology: topology generated by sets of the form $U_1 \times \cdots \times U_n \subseteq k^n$, $U_1, \cdots, U_n$ open in k.

2. $C([0,1])$, under L^p norm $\|f\|_p^p = \int |f|^p$, $p > 0$, in which open sets are generated by $\{f \mid \|f\|_p < \varepsilon\}$, $\varepsilon > 0$.

3. $C(\mathbb{R})$, or $C(\Omega)$, $\Omega \subseteq \mathbb{R}$ or $\mathbb{C}$ .... topology of local uniform convergence (more later).

4. $H(\Omega)$: holom. functions in $C(\Omega)$ (such a limit of holom f'ns is holom)

5. $C^\infty(\Omega)$. Now we get meaty. Let $\Omega \subseteq \mathbb{R}^n$
$\mathcal{D}_K$ = those w/ support in K.

We will say much more about these later. The point to introducing them now is that these are the main examples we need to illustrate features later on. Another nice example: Distributions

Let U be open set.

$C_c^\infty(U)$ is a vector space, w/ topology of convergence of derivatives (given later...). —complicated

Dual space = cts linear f'nls on it is the <u>space of distributions</u> on U.

Now, back to the axioms we need to prove theorems....

<u>Normed Spaces & Metric Spaces</u>

Metric: $d(x,y)$ on $X \times X \longrightarrow [0,\infty)$

(i) $d(x,y)=0 \iff x=y$

(ii) $d(x,y)=d(y,x)$

(iii) $d(x,z) \leq d(x,y)+d(y,z)$

<u>Metric Space</u>

Topology gen. by open balls $B_r(x)=\{y \mid d(x,y)<r\}$.

Norm: $V \xrightarrow{} \mathbb{R}$ w/range $[0,\infty)$ ← vector space

(i) $\|x\| = 0 \iff x = 0$

(ii) $\|\alpha x\| = |\alpha| \cdot \|x\|$, $\alpha \in k$, $x \in V$

(iii) $\|x+y\| \le \|x\| + \|y\|$

gives metric $d(x,y) = \|x-y\|$ & hence a topology in which V is a T.V.S.

Banach Space – complete, normed TVS (every Cauchy sequence converges...)

Major Family includes L^p Lebesgue spaces, Hilbert Spaces, ...

Does NOT include the examples of $C(\Omega)$, $H(\Omega)$, $C_c^\infty(U)$, distribution spaces...

This is why we need to study TVS's separately.

Another axiom: **Balanced set**: $B \subseteq V$ s.t. $\alpha B \subseteq B$ $\forall \alpha$ w/ $|\alpha| \le 1$.

It is nice to have a topology generated by balanced open neighborhoods (like normed vector spaces do).

Base a subset of a topology which generates it through unions of open sets

<u>Local Base</u> Base of the form

$$\bigcup_{v\in V} \bigcup_{S\in B_0} (v+S)$$

where B_0 = collection of open nbhds of the origin.

<u>Convex</u> subset of V: $tC+(1-t)C \subseteq C$ $\forall 0\le t\le 1$.

<u>Locally Convex TVS</u> (important notion) - has a local base of convex sets.

<u>Locally Bounded TVS</u>. origin of V is ~~co~~ contained in a bdd open set,

$A\subseteq V$ is bounded if $\forall$ nhd U of 0 $\exists$ a pos # t w/ $A\subseteq t'U$ for all $t'\ge t$.

<u>Locally Compact TVS</u> $0\in A$, open, & $\bar{A}$ cpt.

<u>Fréchet Space</u> (very important)

locally convex TVS w/ topology induced from a complete, invariant metric: $d(x,y)=d(x+z,y+z)$

Some amazing theorems that tell you if a topology comes from a norm or a metric or not.

Thm locally bdd TVS $\Rightarrow$ TVS w/ a countable local base $\Leftrightarrow$ Metrizable

Thm TVS topology comes from a norm

$\Updownarrow$

locally convex + locally bdd

Thm locally compact TVS $\Longleftrightarrow$ finite dim'l

$\Uparrow$

locally bdd + Heine Borel (all closed + bdd subsets are cpt)

E.g. $H(\Omega)$, $C^\infty(\Omega)$ are Frechet, Heine-Borel, obviously not finite dim'l.

$\Downarrow$

not locally bdd

$\Downarrow$

not normable.

$\Downarrow$

not Banach Spaces

Let's finally prove a result.

<u>Thm</u> TVS (which we defined so that $\{0\}$ is closed) $\Rightarrow$ Hausdorff

<u>Proof</u> We shall show

(*) If K cpt, C closed are disjoint subsets of a TVS, then $\exists$ a nhd U of 0 s.t. $K+U$ & $C+U$ are disjoint.

If K & U are singletons, this implies they reside in ~~the~~ disjoint open sets (which is the def'n of Hausdorff). It remains to prove (*).

Let W be a nhd of 0.
Add'n is cts, so $\exists$ open sets W_1, W_2 about 0 for which $W_1 + W_2 \subseteq W$.
Let $W' = W_1 \cap W_2 \cap (-W_1) \cap (-W_2)$.
Then $W' = -W'$
& $W' + W' \subseteq W$.
Repeat: get $W'' + W'' + W'' + W'' \subseteq W$, $W'' = -W''$ ....

We may assume $K \neq \emptyset$, else obvious.
Let $k \in K$. $k \notin C$ by assumption
Let $W = V - C + (-k)$ above
$\exists W''$ s.t. $W'' + W'' + W'' + W'' \subseteq (-k) + V - C$

That means

$$\underbrace{k + W'' + W'' + W''}_{\text{disjoint from } C.} \subseteq k + W' + W' + W' + W' \subseteq V - C$$

Moreover $(C + W'') \cap (k + W'' + W'') = \emptyset$

(else $x \in C + W''$
$\in k + W'' + W''$, but $W'' = -W''$
so get contradiction),

The sets $k + W_k''$ formed this way
are an open cover of K

K cpt $\Rightarrow \exists\, k_1, \dots, k_n$ s.t. $\bigcup_{j \leq n} (k_j + W_{k_j}'') \supseteq K$.

Let $U = \bigcap W_{k_j}''$.

Then $K + U \subseteq \bigcup_{j \leq n} (k_j + W_{k_j} + U)$

$$\subseteq \bigcup_{j \leq n} (k_j + \underbrace{W_{k_j}'' + W_{k_j}''}_{})$$

disjoint from $C + U$.

$\square$

Furthermore, $\overline{K + U}$ is disjoint from $C + U$

$\Rightarrow \overline{K + U}$ — — C

Consequence in a local base, each nbd contains the closure of another nbd in the base.

Thm
Our next goal is the $\boxed{\text{thm}}$ that locally bdd $\Rightarrow$ countable local base.

We'll prove the stronger result that if $\mathcal{U}$ is a bdd open nhd of 0, then $\{\delta_n \mathcal{U} : n \geq 1\}$ forms a local base for V, where $\delta_n > \delta_{n+1} > 0$ is any sequence $\to 0$ as $n \to \infty$.

For this we need to show that given any nhd S of 0, some $\delta_n \mathcal{U} \subseteq S$.
$\mathcal{U}$ is bdd, so $\exists\, s > 0$ s.t. $\mathcal{U} \subseteq tS$ for all $t \geq s$. If $\delta_n < 1/s$ (which happens for n large enough), then $S \supseteq \delta_n \mathcal{U}$. $\square$

<u>Prop.</u> (i) In any TVS, every nhd of 0 contains a balanced nhd of 0.
(ii) Every convex nhd of 0 contains a balanced convex nhd of zero.

(recall: Balanced set $B \Rightarrow \alpha B \subseteq B \;\; \forall |\alpha| \leq 1$)

In (i) we say the TVS has a <u>balanced</u> local base,
(ii) ... <u>balanced convex</u> local base.

Proof (i) Let U be such an nhd.

mult by scalars is cts, so $\exists \delta > 0$

nhd S of 0 such that $\alpha S \subseteq U$ for all $|\alpha| < \delta$.

Let $W = \bigcup_{|\alpha| \le \delta} \alpha S$.

Then $\forall |\beta| \le 1$ $\beta W \subseteq W$, hence is balanced.

(ii) Let U be the convex nhd given.

Choose W as in part (i),

Let $A = \bigcap_{|\alpha|=1} \alpha U$.

Then $\alpha^{-1} W = W$ for $|\alpha| = 1$

so $W \subseteq U \Rightarrow W \subseteq \alpha U$ for all $|\alpha| = 1$

$\Rightarrow W \subseteq A$.

Now $B =$ interior of A is convex:

– it is a nhd of 0, since W is & $W \subseteq A$.

it's convex because

$\underline{tA^o + (1-t)A^o} \subseteq A$

open, so lies in A^o.

Why is B balanced?

$\Uparrow$

A balanced

open, so $\subseteq A^o$ also.

$(\alpha A) = \alpha A^o \subseteq \alpha A \subseteq A$

But why is A balanced?

If $|\beta| = 1$, $0 \le r \le 1$

since convex & contains 0.

$r\beta A = \bigcap_{|\alpha|=1} r\beta\alpha U = \bigcap_{\alpha'} r\alpha U \subseteq \alpha U$

If we take the intersection over all α!

$r\beta A \subseteq A$.

All $|\gamma| \leq 1$ may be written as $r\beta$ this way, which completes the proof $\square$.

Prop Let $0 < r_1 < r_2 < \cdots \to \infty$ as $n \to \infty$ & U be an nhd of 0. Then $V = \bigcup_{n \geq 1} r_n U$.

Pf by cty of scalar mult., for any fixed $v \in V$, we know

$\{\alpha \mid \alpha v \in U\}$ is open & contains 0.

$\Rightarrow$ contains $\frac{1}{r_n}$ for n large.

So $\frac{1}{r_n} v \in U \Leftrightarrow v \in r_n U$ for n large. $\square$

Prop In a TVS, all cpt subsets are bdd.

Pf $\exists$ a balanced nhd of 0 (see p.10), W, in any given open nhd U of 0.

$K \subseteq \bigcup_{n \geq 1} nW$. (by previous).

K cpt, W balanced $\Rightarrow K \subseteq tW$ for some large t.

Therefore $K \subseteq tU$ for all large t. So, we have shown for all open nbhds U one has

$$K \subseteq tU$$

for all sufficiently large t, which is the definition of bounded.
