# Homework in complex analysis

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Assignment 1

Due Wednesday March 23rd, 2005 at 17:00

**You only need to do 2 out of the 3 problems**. Be sure to **indicate which 2 you want me to grade**. My advice is that you try all 3 first, because they will help you to see if you understand the material. You may try to solve the problems in working groups together; however, you should submit your own solutions that are written *without* looking at any notes or papers from work you did with other people. Questions? E-mail me at `miller@math.huji.ac.il`.

**1**. The purpose of this exercise is to give another proof of the Poisson summation formula. Recall that $\mathcal{F}_a = \{\text{functions which are holomorphic in the strip } |\text{Im } z| < a\}$ and that $\mathcal{F} = \cup_{a>0}\mathcal{F}_a$. The Poisson summation formula asserts that $\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)$ for $f \in \mathcal{F}$.

a. Prove that $\mathcal{F}$ is a vector space over the field of complex numbers $\mathbb{C}$.

b. Prove that, as a vector space, $\mathcal{F}$ is the direct sum of the subspaces

$$\mathcal{F}^+ = \{f \in \mathcal{F} \mid f(z) = f(-z)\} \quad \text{and} \quad \mathcal{F}^- = \{f \in \mathcal{F} \mid f(z) = -f(-z)\}\,.$$

c. Explain why the Poisson Summation Formula for $f \in \mathcal{F}^-$ is trivial. Be sure to explain the convergence.

d. Consider the contour integral

$$\mathcal{I}_\varepsilon \quad := \quad \frac{1}{2\pi i}\int_{Im\, z\, =\, \varepsilon} f(z)\, \pi\, \cot \pi z\, dz$$

for $\varepsilon \in \mathbb{R}$. Explain why for $f \in \mathcal{F}^+$ and $\varepsilon > 0$ sufficiently small that

$$-2\,\mathcal{I}_\varepsilon \quad = \quad \sum_{n\in\mathbb{Z}} f(n)\,.$$

This will use the fact that the residue of $\pi \cot \pi z$ at any integer $z$ is equal to 1.

e. Prove that

$$\pi \cot \pi z = -i\pi \left( 1 + 2 \sum_{n \geq 1} e^{2\pi i n z} \right)$$

for Im $z > 0$.

f. Use parts c, d, and e to give a new proof the Poisson summation formula for functions in $\mathcal{F}$.

**2.** Use a contour shift to prove that the function $\operatorname{sech}(\pi x) = \frac{2}{e^{\pi x} + e^{-\pi x}}$ is equal to its own Fourier transform. Hint: Examine the rectangular contour $C_R$ whose corners are at the points $\{-R, R, R + 2i, -R + 2i\}$.

**3.** For this exercise we will use the space of Schwartz functions

$$\mathcal{S}(\mathbb{R}) := \{ f \in C^{\infty}(\mathbb{R}) \quad | \quad \lim_{|x| \to \infty} |x|^k \frac{d^n}{dx^n} f(x) = 0 \;\text{ for each }\; k, n \geq 0 \} .$$

Suppose $f \in \mathcal{S}(\mathbb{R})$ is an arbitrary Schwartz function in each part of this exercise.

a. Let $g(x) := f(xt)$. Prove that for any $t \neq 0$ that $\widehat{g}(r) = \frac{1}{t} \widehat{f}(r/t)$.

b. Let $g(x) := f(x + u)$, $u \in \mathbb{R}$. Prove that $\widehat{g}(r) = e^{2\pi i r u} \widehat{f}(r)$.

c. Let $g(x) = f'(x)$. Prove that $\widehat{g}(r) = 2\pi i \, r \, \widehat{f}(r)$.

d. Let $g(x) = x f(x)$. Prove that $\widehat{g}(r) = -\frac{1}{2\pi i} \frac{d}{dr} \widehat{f}(r)$.