# Course Notes for Math 574: Adeles, Automorphic Forms, and Representations Spring 2002

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## 1 Introduction

### 1.1 Outline of Course

- $p$-adic numbers
- Adeles
- Tate's thesis
- Automorphic forms on $SL_2(\mathbb{R})$
- Automorphic representations of $GL_2(\mathbb{R})$
- Adelic automorphic representations
- Study of congruence subgroups, Hecke operators, Atkin-Lehner theory
- Jacquet-Langlands theory of automorphic L-functions, comparison with Atkin-Lehner

## 1.2 Some Recommended Texts

(Listed in order of the chronology above)

- Koblitz [Kob]
- Robert [Rob]
- Ramakrishnan and Valenza [Ram-Val]
- Bump [Bump]
- Gelbart [Gelb]
- Iwaniec's yellow/blue AMS book [Iwan2]
- Jacquet-Langlands (online) [Jac-Lan]

## 1.3 Goals and Emphasis of the Course

We want to become equally adept using the adelic and classical languages, and understand constructions and insights provided by each language in terms of the other; in short, to become bilingual. We assume some familiarity with the classical language, either from Iwaniec's graduate courses or his texts [Iwan1] and [Iwan2].

## 1.4 A Note on these Notes

These notes often provide some unusual and unorthodox explanations of things which are typically explained in a different, and perhaps superior, way. This is being done for two reasons: to complement existing sources, and to develop a background with automorphic forms itself in mind. Furthermore, and more importantly, many technical details (such as convergence) will be omitted, as they are explained very well in other sources. Many arguments will be purely formal. This is being done to give a conceptual explanation of the techniques.

## 2 The $p$-adic Numbers $\mathbb{Q}_p$

The $p$-adic numbers are themselves divergent series! They were first invented/discovered by Kurt Hensel about a century ago (see [Rosen] for a short biography). They are comparable to the decimal (or, more correctly base $p$) expansions of real numbers

$$1.351325\cdots,$$

except that they are written

$$\cdots 5019325.135$$

from right to left instead of left to right. Truncating these infinite expansions gives rational numbers in either case, and both $\mathbb{R}$ and $\mathbb{Q}_p$ are different completions of the rationals $\mathbb{Q}$ under different metrics. Roughly speaking, two numbers are "close" to each other in either case if a large initial stretch of digits agree. Addition and multiplication of real numbers is done from right to left (even though the expansions continue infinitely far to the right). Addition and multiplication in $\mathbb{Q}_p$ is done also from right to left; one has a well-defined starting point, but the process literally takes forever.

The $p$-adic integers $\mathbb{Z}_p$ are those $p$-adic numbers with nothing to the right of the decimal point. A more formal/algebraic definition is as the inverse limit

$$\mathbb{Z}_p = \lim_{\leftarrow} \mathbb{Z}/p^n\mathbb{Z} = \left\{ \sum_{n=0}^{\infty} a_n p^n, 0 \leq a_n < p \right\}.$$

When one adds or multiplies $p$-adic integers, one must carry as with the usual decimal addition and mulitplication. The $p$-adic numbers are

$$\mathbb{Q}_p = \cup_{k=0}^{\infty} p^{-k}\mathbb{Z}_p = \left\{ \sum_{n=-k}^{\infty} a_n p^n, 0 \leq a_n < p \right\}.$$

**Exercise 2.1.** *A. Show that it makes sense to reduce $u \in \mathbb{Q}_p$ (mod $p^k$) for any integer $k$.*

*B. Prove that if $u_1, u_2 \in \mathbb{Q}_p$, that*

$$u_1 = u_2 \quad \Leftrightarrow \quad u_1 \equiv u_2 \ (\text{mod } p^k) \text{ for all } k \in \mathbb{Z}.$$

*C. Check that the reduction is consistent with the addition and multiplication laws in $\mathbb{Z}_p$ and $\mathbb{Z}/p^k$.*

## 2.1 Division

**Lemma 2.2.** *If $x \neq 0$ is in $\mathbb{Q}_p$, then it is invertible in $\mathbb{Q}_p$.*

**Proof:** By factoring out the leading term, we may assume

$$x = p^k u \tag{2.1}$$

where $u = \sum a_n p^n \in \mathbb{Z}_p$ starts with $1 \leq a_0 \leq p-1$. Since $a_0$ is invertible in $(\mathbb{Z}/p)^*$, we may multiply $u$ by an integer and assume $a_0 = 1$. Writing $u = 1 - py, y = -\sum_{n=0}^{\infty} a_{n+1} p^n \in \mathbb{Z}_p$, we find that

$$u^{-1} = (1-py)^{-1} = 1 + py + (py)^2 + \cdots .$$

This last expression is convergent because the powers of $p$ increase, and so the $n$-th digit stabilizes after adding $(py)^n$. $\square$

This proof introduces two important concepts. First, the integer $k$ in (2.1) is called the *order*, $ord(x)$. Secondly, the order gives a topology that makes our infinite sum converge quite easily. Under the $p$-adic absolute value $|x|_p = p^{-ord(x)}$, two numbers are indeed close when a large initial stretch of their digits agree. This metric has the *ultrametric* property

$$|x+y| \leq \max(|x|, |y|) \Longleftrightarrow ord(x+y) \geq \min(ord(x), ord(y)). \tag{2.2}$$

**Exercise 2.3.** *Prove (2.2), and that every convergent sequence is a also a cauchy sequence.*

**Exercise 2.4.** *Prove that any series $\sum a_n$ converges so long as $a_n \to 0$.*

**Exercise 2.5.** *Show that $\mathbb{Z}_p$ and $\mathbb{Z}_p^*$ are compact by using the convergent subsequence property.*

This shows that any continuous image of $\mathbb{Z}_p$ is compact; in particular, continuous characters of $\mathbb{Z}_p$ and $\mathbb{Z}_p^*$ take values in the unit circle $\{z \in \mathbb{C} \mid z\bar{z} = 1\}$.

In fact, the structure of $\mathbb{Q}_p$ can be described naturally as follows:

$$\mathbb{Q}_p = \{0\} \cup \mathbb{Q}_p^*$$

(by Lemma 2.2), and

$$\mathbb{Q}_p^* = \coprod_{k\in\mathbb{Z}} p^k \mathbb{Z}_p^*,$$

where $\mathbb{Z}_p^*$ are the units in $\mathbb{Z}_p$; $\mathbb{Z}_p^* = \{|x| = 1\}$, a basic open set. All open sets are unions of the basic open neighborhoods that are translates $ax + b$ of $\mathbb{Z}_p$ itself, so long as $a \neq 0$. For example,

$$\mathbb{Z}_p^* = \coprod_{j=1}^{p-1} (j + p\mathbb{Z}_p).$$

This gives a tree-like structure to $\mathbb{Q}_p$.

## 2.2 Haar Measure and Integration

There is an additive measure $dx$, normalized to give $\mathbb{Z}_p$ measure 1, and a multiplicative measure $d^*x$, which gives $\mathbb{Z}_p^*$ measure 1. For example, $I = p\mathbb{Z}_p$ has additive measure $1/p$, since its $p$ translates $I + j, 1 \leq j \leq p$ form $\mathbb{Z}_p$.

**Lemma 2.6.** *The map $x \mapsto yx$ changes $dx \mapsto |y|dx$, and does not change $d^*x$.*

**Proof:** This is clear if $y$ is a unit (i.e. $|y| = 1$), and the example above can be iterated for the general case. □

In fact, the multiplicative measure is

$$d^*x = \frac{dx}{|x|}\frac{p}{p-1}.$$

**Example 2.7.** *As an example of an integral, let us compute*

$$\int_{\mathbb{Z}_p} |x|^s dx = \sum_{k=0}^{\infty} \int_{p^k\mathbb{Z}_p^*} |x|^s dx$$

$$= \frac{p}{p-1} \sum p^{-ks} = \frac{p}{(p-1)(1-p^{-s})}.$$

This is a very typical example; it shows that p-adic integration is often just summation. An integrand like ours (which depends only on $|x|$) often turns out to be a geometric series.

## 2.3 Additive and Multiplicative Characters

There is a standard additive character on $\mathbb{Q}_p$ which is trivial on the integers $\mathbb{Z}_p$:

$$e_p(x) = e^{-2\pi i q}, \quad x \in q + \mathbb{Z}_p$$

which is well defined because it only depends on the fractional part of the $p$-adic number $x$. As we will show momentarily, the general continuous additive character on $\mathbb{Q}_p$ is

$$\psi_a(x) = e_p(ax), \ a \in \mathbb{Q}_p.$$

Its kernel is $|a|\mathbb{Z}_p$, the *conductor* of $\psi_a$. We will also refer to $|a|$ as the conductor here as well.

**Theorem 2.8.** *If $\psi$ is a continuous additive character of $\mathbb{Q}_p$, then $\psi(x) = e_p(ax)$ for some $a \in \mathbb{Q}_p$.*

We will use

**Lemma 2.9.** *If a sequence of complex numbers $x_0, x_1, \dots$ with $x_{n+1} = x_n^p$ converges to 1, then $x_0 = e^{2\pi i \alpha}$ for some $\alpha \in \mathbb{Q}$.*

**Proof of Lemma 2.9:** Firstly, we must clearly have that $|x_n| = 1$, and that $|\alpha_N| < p^{-2}$ for $N$ sufficiently large, where $x_n = e^{2\pi i \alpha_n}$. We may also assume that $|\alpha_n| \geq |\alpha_N|$ for $n \geq N$. But then $|\alpha_{N+1}| = |p\alpha_N| > |\alpha_N|$ unless $\alpha_N = 0$. □

**Proof of Theorem 2.8:** We use that 1 is a *topological generator* of $\mathbb{Z}_p$, i.e. the set $\mathbb{N} = \{\ 1,\ 1{+}1,\ 1{+}1{+}1,\ \dots\ \}$ is dense in $\mathbb{Z}_p$. Then $\psi$'s restriction to $\mathbb{Z}_p$ is determined purely by its values on $\mathbb{Z}$. We have that $\psi(n) = \psi(1)^n$, and in particular

$$\psi(p^k) = \psi(1)^{p^k}.$$

Since $\psi$ is continuous and $|p^k| \to 0$ as $k \to \infty$, $\psi(1)^{p^k} \to 1$ as $k \to \infty$, and Lemma 2.9 shows that $\psi(1) = e^{2\pi i j/p^k}$ for some $k \geq 0$, $(j, p) = 1$. By using the character $\psi\left(\frac{p^k}{j}x\right)$ instead, we may thus assume that $\psi$ is trivial on $\mathbb{Z}_p$. Now, $\psi(1/p)^p = \psi(1) = 1$, and so

$$\psi(1/p) = e^{2\pi i a_1/p}.$$

Also,

$$\psi(1/p^2)^{p^2} = 1,$$

and so

$$\psi(1/p^2) = e^{2\pi i a_2/p^2},$$

with the compatibility condition that $a_2 \equiv a_1 \pmod p$. Continuing we obtain a $p$-adic integer $a$ such that

$$\psi(1/p^k) = e^{2\pi i a/p^k}, \quad k \geq 0$$

and thus $\psi(x) = e_p(-ax)$. □

We now turn to the multiplicative characters (Theorem 2.11), but first give an overview. Since $\mathbb{Q}_p^*$ is isomorphic to a product of its "spine" $p^{\mathbb{Z}} = \{|x| \mid x \in \mathbb{Q}_p^*\}$ with $\mathbb{Z}_p^*$, continuous characters of $\mathbb{Q}_p^*$ will be of the form $|x|^s\chi(x)$, where $\chi(x)$ is a finite-order continuous character of $\mathbb{Z}_p^*$. By continuity, its kernel contains an open subgroup of the form $1 + p^k\mathbb{Z}_p$, which is called the *conductor* of $\chi$ for the smallest such value of $k \geq 0$. Again, the conductor is often referred to just as $p^k$. $\chi$ is essentially a dirichlet character on $\mathbb{Z}/p^k$. This is because

**Exercise 2.10.** *Show that $\mathbb{Z}_p^*/(1 + p^k\mathbb{Z}_p) \simeq (\mathbb{Z}/p^k)^*$.*

**Theorem 2.11.** *All continuous multiplicative characters of $\mathbb{Q}_p^*$ are of the form*

$$|x|^s\chi(x), \tag{2.3}$$

*where $\chi$ is a finite order character of $\mathbb{Z}_p^*$.*

**Proof:** Clearly as

$$\mathbb{Q}_p^* = \{p^k \mid k \in \mathbb{Z}\} \times \mathbb{Z}_p^*,$$

such a decomposition as in (2.3) exists. We need only show that $\chi$ is indeed a finite-order character. To simplify, we argue here only for odd primes (the case $p = 2$ can be handled similarly). Recall that for all primes $p > 2$ there is an integer $g$ which generates any $(\mathbb{Z}/p^k)^*$. Such a $g$ is a topological generator of $\mathbb{Z}_p^*$. Since

$$g^{(p-1)p^{k-1}} \equiv 1 (\bmod\ p^k),$$

$\chi\left((g^{p-1})^{p^k}\right) = \chi(g^{p-1})^{p^k} \to 1$ as $k \to \infty$. Thus Lemma 2.9 shows that $\chi(g)$ is the exponential of a rational number, and $\chi$ is of finite order. □

## 2.4 An Important Integral

**Example 2.12.** *The "p-adic Gamma function": If $\psi$ is any additive character and $\chi$ any multiplicative character (both assumed to be continuous)*

$$G(\psi, \chi) := \int_{\mathbb{Q}_p^*} \psi(x)\chi(x)d^*x. \tag{2.4}$$

This is in analogy to the classical situation of the field of real numbers, where $\Gamma(s) = \int_0^\infty e^{-t}t^s \frac{dt}{t}$. We can equivalently study

$$\int_{\mathbb{Q}_p} \psi(x)\chi(x)dx, \tag{2.5}$$

or by changing variables, the more-explicit

$$\int_{\mathbb{Q}_p} e_p(x)\chi(x)|x|^s dx, \tag{2.6}$$

where $\chi$ is now of finite order. Then (2.6) can be broken up as

$$\sum_{k\in\mathbb{Z}} p^{-ks} \int_{\substack{x\in\mathbb{Q}_p \\ |x|=p^{-k}}} e_p(x)\chi(x)dx,$$

and we shall now compute

$$\int_{\substack{x\in\mathbb{Q}_p \\ |x|=p^{-k}}} e_p(x)\chi(x)dx \tag{2.7}$$

through some exercises:

**Exercise 2.13.** *If $\phi$ is a homomorphism from a finite group $G$ to $\mathbb{C}$, then*

$$\sum_{g\in G} \phi(g) = \begin{cases} |G| & , \text{ if } \phi \text{ is trivial,} \\ 0 & , \text{ otherwise.} \end{cases}$$

**Exercise 2.14.** *Assume that $\chi$ is trivial. Show that*

$$\int_{\substack{x\in\mathbb{Q}_p \\ |x|=p^{-k}}} e_p(x)\chi(x)dx = \begin{cases} \frac{p-1}{p}p^{-k} & , k \geq 0, \\ -1 & , k = -1, \\ 0 & , \text{otherwise.} \end{cases}$$

*Deduce that (2.6) equals*

$$\frac{1-p^s}{1-p^{-s-1}}.$$

*Hint: when $k<0$ the integral is the sum $\sum_{j\in(\mathbb{Z}/p^{-k})^*} e(-j/p^{-k})$.*

Now assume that $\chi$ is nontrivial of conductor $p^n$, $n>0$. To simplify notation, let $\nu=-k$ so that $\ell=\max(n,\nu)$.

**Exercise 2.15.** *Show that if $\nu<n$,*

$$\int_{\substack{x\in\mathbb{Q}_p\\|x|=p^\nu}} e_p(x)dx=0.$$

Hint: First express it in terms of the sum

$$\sum_{j\in(\mathbb{Z}/p^n)^*} e(-j/p^\nu)\chi(j)=\sum_{\substack{j\in(\mathbb{Z}/p^\nu)^*\\ \ell\in\mathbb{Z}/p^{n-\nu}}} e\left(-\frac{j+p^\nu\ell}{p^\nu}\right)\chi(j+p^\nu\ell),$$

where we also use $\chi$ to refer to the associated dirichlet character. Then show that the inner sum

$$\sum_{\ell\in\mathbb{Z}/p^{n-\nu}}\chi(j+p^\nu\ell)=\chi(j)\sum_{\ell\in\mathbb{Z}/p^{n-\nu}}\chi(1+p^\nu\ell)=0$$

because $\{1+p^\nu\ell\}$ forms a subgroup of order $p^{n-\nu}$, which is strictly smaller than the conductor $p^n$.

**Exercise 2.16.** *Show that if $\nu=n$, then*

$$\int_{\substack{x\in\mathbb{Q}_p\\|x|=p^\nu}} e_p(x)\chi(x)dx=\chi(-1)\tau_\chi,$$

*where the Gauss sum*

$$\tau_\chi=\sum_{j\in(\mathbb{Z}/p^n)^*} e(j/p^n)\chi(j).$$

**Exercise 2.17.** *Use Exercise 2.13 to show that*

$$\int_{\substack{x\in\mathbb{Q}_p\\|x|=p^\nu}} e_p(x)\chi(x)dx=0$$

*also if $\nu>n$.*

We conclude:

**Proposition 2.18.** *Let $s \in \mathbb{C}$ and $\chi$ be a finite-order continuous character of $\mathbb{Q}_p^*$. Then the integral*

$$\int_{\mathbb{Q}_p} e_p(x)\chi(x)|x|^s dx = \begin{cases} \frac{1-p^s}{1-p^{-s-1}} & , \ \chi \equiv 1, \\ p^{ns}\chi(-1)\tau_\chi & , \ \textit{otherwise.} \end{cases}$$

# 3 Adeles

In this section we will introduce and motivate the adeles by their use in harmonic analysis.

## 3.1 The Adelization of the Unit Interval

To start let us recall our definition of the $p$-adic exponential function

$$e_p(x) = e^{-2\pi i q} = e(-q),$$

where $q$ is a rational number such that $x \in q + \mathbb{Z}_p$. We saw that $e_p(x)$ is well-defined since the exponential function is trivial on integers. In fact, any periodic function can be extended to the $p$-adics in this fashion. By the same principle, we can also translate periodic functions by $\mathbb{Q}_p$ through the action

$$f(x) \mapsto f(x-y), \ y \in \mathbb{Q}_p. \tag{3.1}$$

This is again well-defined, because $\mathbb{Z}_p$, the closure of $\mathbb{Z}$ in $\mathbb{Q}_p$, acts trivially. Thus $\mathbb{Q}_p$ acts by shifting the argument of $f$ by a rational number whose denominator is a power of $p$:

$$\mathbb{Q}_p/\mathbb{Z}_p \simeq \{\frac{a}{p^k} \mid (a,p) = 1, \ 0 \leq a < p^k\}.$$

In fact, we may act by $\mathbb{Q}_{p_1} \times \mathbb{Q}_{p_2}$ or any finite product $\prod_{p \in S} \mathbb{Q}_p$, or even

$$\left[\prod_{p\in S} \mathbb{Q}_p\right] \times \left[\prod_{p\notin S} \mathbb{Z}_p\right], \quad S \text{ finite.}$$

To make this universal, we take the union of these sets over all finite sets of primes $S$. This is the *finite adeles* $\mathbb{A}_f$, a restricted direct product of the $\mathbb{Q}_p$ with respect to the $\mathbb{Z}_p$. The adeles

$$\mathbb{A} = \mathbb{R} \times \mathbb{A}_f = \{(a_\infty; a_2, a_3, a_5, \dots) \mid \text{ almost all } a_p \in \mathbb{Z}_p\}$$

are the restricted direct product of all $\mathbb{Q}_p,\ p \leq \infty$, using the convention that $\mathbb{Q}_p = \mathbb{R}$. The "prime" $p = \infty$ is called the archimedean prime, because its valuation (the usual absolute value on $\mathbb{R}$) obeys Archimedes' Axiom

$$\text{for all } |y| > |x| > 0, \text{ there exists } n \in \mathbb{Z} \text{ such that } |nx| > |y|. \tag{3.2}$$

This fails for the "non-archimedean" valuations $|\cdot|_p$, $p < \infty$, because $|nx|_p \leq |x|_p$ by the ultrametric property (2.2).

We can thus define an *adelization* any function $f$ on $\mathbb{R}/\mathbb{Z}$ by

$$f_{\mathbb{A}}(a) = f_{\mathbb{A}}(\alpha_\infty; a_f) = f(a_\infty - q) \tag{3.3}$$

where $a = (a_\infty; a_f) \in \mathbb{A}$ and

$$a_p \in q + \mathbb{Z}_p \text{ for all } p < \infty. \tag{3.4}$$

Thus $\mathbb{A}_f$ acts to shift periodic functions. One might think of $\mathbb{R}$ as the melody, but $\mathbb{A}_f$ as its key.

**Exercise 3.1.** *Prove that such a $q$ as in (3.4) always exists.*

Hint: Use the Chinese Remainder Theorem. Alternatively, first suppose that $a_f \in \mathbb{Z}_p$ except at one prime, and conclude the general case by summing the rational numbers obtained this way.

## 3.2 Diagonal Embedding of $\mathbb{Q}$

On abstract grounds, each $\mathbb{Q}_p$ is a completion of $\mathbb{Q}$, and hence $\mathbb{Q}$ embeds into each $\mathbb{Q}_p$. To be more precise, let us recall from §2.1 that

**Proposition 3.2.** *If $m, n \in \mathbb{Z}$ and $p \nmid n \neq 0$, then $\frac{m}{n} \in \mathbb{Z}_p$.*

**Proof:** This follows from the proof of Lemma 2.2. □

It thus makes sense to diagonally embed $\mathbb{Q} \hookrightarrow \mathbb{A}$ because almost all components will be in $\mathbb{Z}_p$. We will also refer to this image as $\mathbb{Q}$.

**Proposition 3.3.** *$f_{\mathbb{A}}(a)$ as defined in (3.3) is invariant under the diagonally-embedded $\mathbb{Q}$.*

It is not surprising that $\mathbb{A}/\mathbb{Q}$ is similar to $\mathbb{R}/\mathbb{Z}$. After all, in this setting $\mathbb{A}_f$, the non-archimedean component of $\mathbb{A}$, serves to act on $\mathbb{R}$ via the diagonal embedding of $\mathbb{Q}$. Let

$$\hat{\mathbb{Z}} = \prod_{p<\infty} \mathbb{Z}_p \simeq \lim_{\leftarrow N} \mathbb{Z}/N\mathbb{Z}.$$

We have the

**Theorem 3.4.** *(Strong Approximation Theorem)*

$$\mathbb{A} = \mathbb{Q} + \mathbb{R} + \hat{\mathbb{Z}}.$$

**Corollary 3.5.** *A fundamental domain for $\mathbb{A}/\mathbb{Q}$ is $(\mathbb{R}/\mathbb{Z}) + \hat{\mathbb{Z}}$.*

Since $\hat{\mathbb{Z}}$ acts trivially on periodic functions, these facts are readily visible from our viewpoint in the discussion above. Also, Corollary 3.5 shows us how to reverse the $f \mapsto f_{\mathbb{A}}$ correspondence: if a function on $\mathbb{A}/\mathbb{Q}$ is trivial on the compact subgroup $\hat{\mathbb{Z}}$, it corresponds to a periodic function on $\mathbb{R}/\mathbb{Z}$.

If we had replaced some of the $\mathbb{Z}_p$ in $\hat{\mathbb{Z}} = \prod_{p<\infty} \mathbb{Z}_p$ by their open, compact subgroups $p^{k_p}\mathbb{Z}_p$, we would obtain $N\hat{Z}$ and functions periodic on $\mathbb{R}/N\mathbb{Z}$, where $N = \prod p^{k_p}$. Indeed, the "missing" translations in $\hat{\mathbb{Z}} \smallsetminus N\hat{\mathbb{Z}}$ correspond to shifts by the integers $1, \ldots, N-1$. Equivalently, one could view periodic functions on $\mathbb{R}/N\mathbb{Z}$ as vector-valued functions on $\mathbb{R}/\mathbb{Z}$ which transform according to a shift matrix. This viewpoint is very important when considering congruence subgroups of $GL_2(\mathbb{Z})$ adelically.

# 4 Fourier Analysis on the Adeles

Suppose, completely formally, that $f$ is a function on $\mathbb{A}$ invariant under $\mathbb{Q}$ and $\hat{\mathbb{Z}}$, which is equivalently a function on $\mathbb{R}/\mathbb{Z}$. By general principles we have a Fourier series

$$f(a) = \sum_{\text{characters } \psi \text{ of } \mathbb{A} \text{ trivial on } \mathbb{Q}} \int_{\mathbb{A}/\mathbb{Q}} f(a+x)\psi(x)^{-1}dx = \sum_{\psi} \psi(a) < f, \psi > . \tag{4.1}$$

It is worthwhile to understand how this corresponds to the usual Fourier series on $\mathbb{R}/\mathbb{Z}$. First, we need to extend our classification of the characters of $\mathbb{Q}_p$ from §2.3 to $\mathbb{A}$. One character is clearly given by

$$e_{\mathbb{A}}(x) = \prod_{p\leq\infty} e_p(x), \quad e_\infty(x) = e(x) = e^{2\pi i x},$$

which is the adelization of $e(x)$, the example we started with.

**Proposition 4.1.** *The continuous additive characters of $\mathbb{A}$ which are trivial on $\mathbb{Q}$ are precisely those given by $e(qx)$ for some $q \in \mathbb{Q}$.*

**Proof:** Clearly these characters $e(qx)$ are all trivial on $\mathbb{Q}$. Conversely, any additive character can be decomposed as a product $\psi = \prod_{p\leq\infty}\psi_p$ of continuous additive characters $\psi_p$ of $\mathbb{Q}_p$. When $p = \infty$, all characters are of the form $e(c_\infty x)$, and we have already seen in §2.3 that for $p < \infty$ all continuous additive characters are of the form $e_p(c_p x)$. Let $U = \{|z-1| < 1/10\}$, so that the only subgroup of $\mathbb{C}^*$ contained in $U$ is the trivial subgroup $\{1\}$. Since $\psi$ is continuous, the inverse image $\psi^{-1}(U)$ is an open subset of $\mathbb{A}$, and hence contains a basic open neighborhood of the form $\mathbb{R}\times N\hat{\mathbb{Z}}$ for some integer $N$. Then the image $\psi(N\hat{\mathbb{Z}})$ is a subgroup contained in $U$, and hence trivial. We conclude that $\psi_p = e(c_p x_p)$ is trivial on $\mathbb{Z}_p$ for almost all $p$, i.e. those not dividing $N$. Writing $c_p = p^k u$, where $|u| = 1$, we see that $e(c_p u^{-1}) = e^{-2\pi i p^k} = 1$, and and thus these $k \geq 0$, i.e. $c_p \in \mathbb{Z}_p$ for all $p \nmid N$. Thus, we conclude that globally $\psi(x) = e_{\mathbb{A}}(cx)$, for $c \in \mathbb{A}$.

Because $e(c+q) = e(c\cdot 1)e(q) = 1$, we may use the strong approximation theorem to reduce to the case that $c \in \mathbb{R}\times\hat{\mathbb{Z}}$. Thus $e(c) = e^{2\pi i c_\infty} = 1$, and $c_\infty \in \mathbb{Z}$. The condition $e(cq) = 1$ for any rational $q$, in particular $q = p^{-k}$, forces

$$e^{2\pi i c_\infty/p^k} = e^{2\pi i c_p/p^k}.$$

Thus $c_p \equiv c_\infty \pmod{p^k}$ for all $k$, and $c_p$ is the image of the integer $c_\infty$ for each $p$. We conclude that $c$ is in the diagonal embedding of $\mathbb{Q}$. □

With this description of the additive characters, we can understand the meaning of $< f, \psi >$, where $\psi = e(qx)$.

$$< f, \psi >= \int_{\mathbb{A}/\mathbb{Q}} f(x)\psi(x)^{-1}dx$$

$$= \int_{\mathbb{A}/\mathbb{Q}} f(x)e(-qx)dx$$

$$= \int_{\mathbb{R}/\mathbb{Z}} \int_{\hat{\mathbb{Z}}} f(x_\infty + x_f)e(-qx_\infty)e(-qx_f)dx_f dx_\infty,$$

using the fundamental domain in Corollary 3.5. Since $f$ is periodic on $\mathbb{R}/\mathbb{Z}$, this is

$$= \int_{\mathbb{R}/\mathbb{Z}} f(x_\infty)e(-qx_\infty)dx_\infty \int_{\hat{\mathbb{Z}}} e(-qx_f)dx_f.$$

Now, this inner integral factors as a product

$$\prod_{p<\infty} \int_{\mathbb{Z}_p} e_p(-q_p x_p)dx_p = \begin{cases} 1 & , \ q_p \in \mathbb{Z}_p \text{ for all } p \\ 0 & , \ \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & , \ q \in \mathbb{Z}, \\ 0 & , \ \text{otherwise.} \end{cases}$$

So $< f, \psi >$ is the usual Fourier coefficient if $q \in \mathbb{Z}$ and is zero otherwise. Hence if the classical Fourier expansion of a periodic function $f(x) = \sum_{n\in\mathbb{Z}} a_n e(nx)$, then the adelic Fourier expansion of its adelization is

$$\sum_{n\in\mathbb{Z}} a_n e_{\mathbb{A}}(nx).$$

In other words, our original adelization (that of $e^{2\pi ix}$) is in some sense the only example!

## 4.1 The Poisson Summation Formula

The classical Poisson Summation Formula

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n) \tag{4.2}$$

can be proven by periodizing $f(x) \mapsto \sum_{n\in\mathbb{Z}} f(x+n)$, and then expanding its Fourier series at $x = 0$. One needs, of course, to assume $f$ decays rapidly enough so that the average converges, and that it is regular enough so that the right hand side of (4.2) does also. The space of Schwartz functions

$$\mathcal{S}(\mathbb{R}) = \left\{ f \in C^\infty(\mathbb{R}) \mid |x^m f^{(n)}(x)| \to 0 \text{ as } |x| \to \infty \text{ for all } m, n \geq 0 \right\}$$

is certainly sufficient for this purpose.

**Exercise 4.2.** *Prove that the Fourier transform maps $\mathcal{S}(\mathbb{R})$ to itself.*

There is a $p$-adic analogue, the Schwartz-Bruhat space

$$\mathcal{S}(\mathbb{Q}_p) = \{f \in C(\mathbb{Q}_p) \mid f \text{ has compact support and is locally constant}\}.$$

Locally constant means that $f(x)$ is constant on small open neighborhoods of $x$.

**Proposition 4.3.** *If $f \in \mathcal{S}(\mathbb{Q}_p)$, then $f$ is a finite sum of the form*

$$\sum c_j \cdot \chi_{a_j+b_j\mathbb{Z}_p},$$

*with $c_j \in \mathbb{C}$ and $a_j \in \mathbb{Q}_p, b_j \in \mathbb{Q}_p^*$.*

**Proof:** For each $a \in \mathbb{Z}_p$ there is some $N$ such that $f(a) = f(x)$ whenever $|x-a| < p^{-N}$. The collection of such open sets covers the support of $f$, which is assumed compact. Thus there is a finite collection of $a_j \in \mathbb{Q}_p$ such that $f$ is constant of the set $a_j + p^{N+1}\mathbb{Z}_p$, and zero otherwise. We may assume that these sets are disjoint by the ultrametric property (2.2). $\square$

One can define the global $\mathcal{S}(\mathbb{A})$, the adelic Schwartz space, by first creating "pure tensors," functions $f = \prod_{p\le\infty} f_p$ on $\mathbb{A}$, with each $f_p \in \mathcal{S}(\mathbb{Q}_p)$ and all but finitely many $f_p = \chi_{\mathbb{Z}_p}$, the characteristic function of $\mathbb{Z}_p$. $\mathcal{S}(\mathbb{A})$ is the span of these pure tensors.

**Proposition 4.4.** *If $f \in \mathcal{S}(\mathbb{A})$, then*

$$\sum_{q\in\mathbb{Q}} f(q) = \sum_{q\in\mathbb{Q}} \hat{f}(q), \tag{4.3}$$

*where*

$$\hat{f}(a) = \int_{\mathbb{A}} f(x)e_{\mathbb{A}}(-ax)dx. \tag{4.4}$$

**Remark 4.5.** *In (4.4) we could actually take any additive character $\psi$ which is trivial on $\mathbb{Q}$ instead of the standard character $e_{\mathbb{A}}(x)$.*

**Proof of Proposition 4.4:** We follow the usual argument. Let

$$F(x) = \sum_{q\in\mathbb{Q}} f(x+q) = \sum_{q\in\mathbb{Q}} \int_{\mathbb{A}/\mathbb{Q}} F(x)e(-qx)dx.$$

By "unfolding"

$$\int_{\mathbb{A}/\mathbb{Q}} F(x)e(-qx)dx = \int_{\mathbb{A}/\mathbb{Q}} \sum_{r\in\mathbb{Q}} f(x+r)e(-qx)dx$$

$$= \sum_{r\in\mathbb{Q}} \int_{r+\mathbb{A}/\mathbb{Q}} f(x)e(-qx)dx = \int_{\mathbb{A}} f(x)e(-qx)dx = \hat{f}(q).$$

(strictly speaking, we should replace be speaking of a translates of a fundamental domain rather than just $r + \mathbb{A}/\mathbb{Q}$). Now set $x = 0$ to obtain (4.3). $\square$

## 4.2 Classical Interpretations

Let use see what the adelic Poisson Summation Formula (4.3) means classically in the crucial case when $f$ is a pure tensor. In this situation, the Fourier transform factors as a product of local Fourier transforms

$$\hat{f}(a) = \prod_{p\le\infty} \int_{\mathbb{Q}_p} f_p(x)e_p(-a_p x_p)dx_p = \prod_{p\le\infty} \hat{f}_p(a_p).$$

**Exercise 4.6.** *Show that if $f_p = \chi_{\mathbb{Z}_p}$, then $\hat{f}_p = \chi_{\mathbb{Z}_p}$ also. Show that the adelic Fourier transform of a function $f \in \mathcal{S}(\mathbb{A})$ remains in $\mathcal{S}(\mathbb{A})$. (Hint: Use the fact that if $f_t(x) = f(tx)$, then $\hat{f}_t(x) = \frac{1}{|t|}\hat{f}\left(\frac{x}{t}\right)$.)*

To simplify matters, let us first assume that all $f_p = \chi_{\mathbb{Z}_p},\ p < \infty$. Then

$$\hat{f}(a) = \hat{f}_\infty(a_\infty)\chi_{\hat{Z}}(a_f) = \begin{cases} \hat{f}_\infty(a_\infty) \text{ the usual F.T.} & ,\ a_f \in \hat{\mathbb{Z}} \\ 0 & ,\ \text{otherwise.} \end{cases}$$

Thus because $q_f \in \hat{\mathbb{Z}} \Leftrightarrow q \in \mathbb{Z}$, the adelic Poisson Summation Formula recovers the classical one.

**Exercise 4.7.** *Show that if $f_p = \sum_{j=1}^{p} a_j\ \chi_{j+p\mathbb{Z}_p}$, then*

$$\hat{f}_p(a) = \begin{cases} 0 & ,\ \text{if } a \notin p^{-1}\mathbb{Z}_p, \\ \frac{1}{p}\sum_{\ell=1}^{p} \hat{a}_\ell & ,\ \text{if } a \in \frac{\ell}{p} + \mathbb{Z}_p \end{cases}$$

*where $\hat{a}_\ell$ is the Discrete Fourier transform $\sum_{j=1}^{p} a_j e(j\ell/p)$.*

In the finite case, Poisson Summation reduces to the inversion formula $a_j = \frac{1}{p}\sum_{\ell=1}^{p} \hat{a}_\ell e(-j\ell/p)$.

## 4.3 The Riemann $\zeta$-Function

First, we give a little background about distributions. They will not be important here *per se*, but will be later on in the course. All that is needed here is the Poisson Summation Formula itself.

In terms of distributions (=linear functions on spaces of functions), the classical Poisson Summation Formula shows that the delta function of the integers

$$\delta_{\mathbb{Z}}(x) = \sum_{n \in \mathbb{Z}} \delta_n(x)$$

is self-dual under the Fourier transform. Adelically, $\delta_{\mathbb{Q}} = \hat{\delta}_{\mathbb{Q}}$ also. There are many examples of self-dual functions, e.g. $e^{-\pi x^2}, \operatorname{sech}(\pi x)$, and Hermite polynomials. The functional equations of dirichlet $L$-functions can be derived from applying these distributional identities to multiplicative characters. We will develop the multiplicative theory of the adeles in the next section. For now we use the classical viewpoint to illustrate how the identity

$$\delta_{\mathbb{Z}} = \hat{\delta}_{\mathbb{Z}} \tag{4.5}$$

implies the functional equation of the Riemann $\zeta$-function. Indeed, it can be shown that the two are equivalent.

We will ingore the polar terms coming from $\delta_0$ in these sums; they can be dealt with by applying (4.5) to Schwartz functions e.g. $e^{-\pi x^2}$ ala Riemann, or by careful distributional arguments. We apply (4.5) to $f(x) = |x|^{-s}$

$$< \delta_{\mathbb{Z}}, f > = < \hat{\delta}_{\mathbb{Z}}, f > = < \delta_{\mathbb{Z}}, \hat{f} > \text{ by Parseval's Theorem.} \tag{4.6}$$

We compute that

$$\hat{f}(\xi) = \int_{\mathbb{R}} e(-\xi x)|x|^{-s} dx = |\xi|^{s-1} \int_{\mathbb{R}} e(x)|x|^{1-s} d^*x.$$

This last integral is very important in distributional derivations of the functional equations of L-functions.

**Exercise 4.8.** *1. Show that for* $0 < Re\ s < 1$*,*

$$\int_{\mathbb{R}} e(x)|x|^s d^*x = \Gamma_{\mathbb{C}}(s) \cos\left(\frac{\pi s}{2}\right),$$

*where*

$$\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s).$$

*2. Use the duplication formula*

$$\Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1),$$

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$$

*and the functional equation*

$$\Gamma(s)\Gamma(1-s) = \pi \csc(\pi s)$$

*to prove*

$$\int_{\mathbb{R}} e(x)|x|^s d^*x = \Gamma_{\mathbb{C}}(s)\cos(\pi s/2) = \frac{\Gamma_{\mathbb{R}}(s)}{\Gamma_{\mathbb{R}}(1-s)}.$$

Thus the Fourier transform of $f(x) = |x|^{-s}$ is $\hat{f}(\xi) = \frac{\Gamma_{\mathbb{R}}(1-s)}{\Gamma_{\mathbb{R}}(s)}|\xi|^{1-s}$, and Poisson Summation gives

$$\sum f(n) = \sum \hat{f}(n)$$

$$2\zeta(s) = \frac{\Gamma_{\mathbb{R}}(1-s)}{\Gamma_{\mathbb{R}}(s)} 2\zeta(1-s),$$

the functional equation $\xi(s) = \Gamma_{\mathbb{R}}(s)\zeta(s) = \xi(1-s)$.

# 5 Ideles

In this section we describe the multiplicative nature of the adeles and its invertible elements, the *ideles*

$$\mathbb{A} = \{a \in \mathbb{A} \mid a_p \in \mathbb{Q}_p^*, \text{ for all } p \leq \infty\}.$$

Actually, the ideles predate the adeles; their name is derived from "ideals," and the additive "adeles" are a backformation. We have already described the Haar measures for $p < \infty$; for $p = \infty$ it is the customary $d^*x_\infty = \frac{dx}{|x|}$.

**Exercise 5.1.** *Prove that the ideles have measure zero under the* additive *Haar measure. (Hint: use the fact that $\zeta(1) = \infty$.)*

The topology on the ideles $\mathbb{A}^*$ comes from the restricted direct product. There is an absolute value on the adeles, $|x|_{\mathbb{A}} = \prod_{p \leq \infty} |x_p|_p$. The change of coordinates $x \mapsto ax$ changes $dx \mapsto |a|dx$. For example, multiplying by rationals does not change the additive Haar measure! There is a version of the Strong Approximation Theorem for the ideles, involving

$$\hat{\mathbb{Z}}^* = \prod_{p<\infty} \mathbb{Z}_p^* = \lim_{\leftarrow N} (\mathbb{Z}/N)^*.$$

**Theorem 5.2.** *(Strong Approximation for Ideles)*

$$\mathbb{Q}^* \times \hat{\mathbb{Z}} \times \mathbb{R}^* = \mathbb{A}^*,$$

*and a fundamental domain for*

$$\mathbb{Q}^* \backslash \mathbb{A}^*$$

*is given by* $(0, \infty) \times \hat{\mathbb{Z}}^*$.

**Proof:** Given any idele $a$, all $a_p \in \mathbb{Z}_p^*$ except for those $p$ in a finite set $S$. We may factor

$$a = a_\infty \cdot \left[ \prod_{p \in S} a_p \right] \cdot a_r,$$

where $a_\infty \in \mathbb{R}$, $a_r = \prod_{p \notin S} a_p \in \prod_{p \notin S} \mathbb{Z}_p^*$. For each $p \in S$, multiply $a$ by the rational number $|a_p|_p$. Since $p_1 \in \mathbb{Z}_{p_2}^*$ for $p_1 \neq p_2$, this process will yield a rational multiple of $a$ which lies in $\hat{\mathbb{Z}}^*$. The description of the fundamental domain follows from the fact that $\mathbb{Q}^* \cap (\mathbb{R}^* \hat{\mathbb{Z}}^*) = \{\pm 1\}$. $\square$

## 5.1 Character Theory

All continuous multiplicative characters of $\mathbb{R}$ are of the form

$$|x|^s sgn(x)^\epsilon, \quad s \in \mathbb{C}, \quad \epsilon = 0 \text{ or } 1.$$

Combining the multiplicative characters at the non-archimedean places (i.e. the ones from §2.3 related to dirichlet characters), we have a complete description of the multiplicative characters of $\mathbb{A}^*$. We need to know, in analogy with Proposition 4.1, the continuous characters of $\mathbb{A}^*$ which are trivial on $\mathbb{Q}^*$. In view of the Strong Approximation Theorem, these characters are on $(0, \infty) \times \hat{\mathbb{Z}}^*$, and hence are products of $|x|^s$ times continuous characters of finite order. As with the additive characters (Proposition 4.1), the continuity assumption ensures triviality on a subgroup of the form

$$\left[\prod_{p\in S}(1+p^{k_p}\mathbb{Z}_p)\right]\prod_{p\notin S}\mathbb{Z}_p^*$$

for some finite set of primes $S$.

**Exercise 5.3.** *Prove that*

$$\hat{\mathbb{Z}}^*/\left(\prod_{p\in S}(1+p^{k_p}\mathbb{Z}_p)\right)\simeq(\mathbb{Z}/N)^*,\quad N=\prod p^{k_p}.$$

*(Hint: see Exercise 2.10.)*

In view of this exercise, finite order continuous characters are equivalent to characters of $(\mathbb{Z}/N)^*$, and hence to dirichlet characters. Let us now be more explicit in describing the correspondance between dirichlet characaters and finite order, continuous characters of $\mathbb{Q}^*\backslash\mathbb{A}^*$. Because the Chinese Remainder Theorem allows us to factor

$$(\mathbb{Z}/N)^*=\prod(\mathbb{Z}/p^{k_p})^*,$$

any character $\chi$ of $(\mathbb{Z}/N)^*$ factors into a product of characters of $(\mathbb{Z}/p^{k_p})^*$, so we will focus on the case that $N=p^k$ is a prime power.

We will now describe the corresponding adelic character $\omega=\prod_{p\le\infty}\omega_p$. First,

$$\omega_\infty(x)=\begin{cases}1 & ,\ \chi(-1)=1,\\ sgn(x) & ,\ \chi(-1)=-1,\end{cases}$$

i.e. $\omega_\infty$ is even or odd depending on whether not $\chi$ is.

For prime $\ell\neq p$, $\omega_\ell$ will be trivial on $\mathbb{Z}_\ell^*$ and $\omega_\ell(\ell)=\chi(\ell)$.[1] Thus $\omega_\ell(\ell^k u)=\chi(\ell)^k$ for all $u\in\mathbb{Z}_\ell^*,\ k\in\mathbb{Z}$. This is analogous to extending $\chi$ to the subset of the rationals

$$\{\frac{r}{s}\in\mathbb{Q}\mid(p,rs)=1\}$$

[1]We could also define $\omega_\ell$ instead using the inverse; this is only a matter of convention.

by setting

$$\chi(\frac{r}{s}) = \chi(r)\chi(s)^{-1}.$$

We now define $\omega_p$ in order to make $\omega$ globally invariant under $\mathbb{Q}^*$. In contrast to the other primes, we will have that $\omega_p$ is trivial on powers of $p$ and on $1 + p^k\mathbb{Z}_p$. If $u \in \mathbb{Z}_p$, we define

$$\omega_p(u) = \chi^{-1}(j),$$

where

$$j = u \pmod{p^k}.$$

**Exercise 5.4.** *Show that $\omega$ as defined here is indeed trivial on primes, and therefore invariant under multiplication by $\mathbb{Q}^*$.*

Recall a dirichlet character $\chi_1 \pmod{q_1}$ induces a character $\chi_2 \pmod{q_2}$ if $q_1 \mid q_2$ and

$$\chi_2(n) = \begin{cases} \chi_1(n) & , \ (n, q_2) = 1 \\ 0 & , \text{otherwise,} \end{cases}$$

and is called *imprimitive* if it is not induced from any other character.

**Exercise 5.5.** *Show that if $\chi_1$ and $\chi_2$ are dirichlet characters induced from the same primitive character, then they both correspond to the same adelic character $\omega$, and moreover this is the only ambiguity.*

This latter aspect (which also naturally removes the distinction of newforms in the setting of adelic modular forms) is very useful.

**Exercise 5.6.** *Prove that if $\chi_1$ and $\chi_2$ are primitive dirichlet characters modulo $q_1$ and $q_2$, respectively, and $(q_1, q_2) = 1$, then*

$$\tau_{\chi_1\chi_2} = \chi_1(q_2)\chi_2(q_1)\tau_{\chi_1}\tau_{\chi_2}.$$

# 6 Tate's Thesis via Distributions

To obtain L-functions of primitive dirichlet characters $\chi$, or equivalently finite order, continuous characters $\omega_\chi$ of $\mathbb{Q}^*\backslash\mathbb{A}^*$, we integrate the identity

$$\delta_{\mathbb{Q}} = \hat{\delta}_{\mathbb{Q}},$$

$$\int_{\mathbb{Q}^*\backslash\mathbb{A}^*} \delta_{\mathbb{Q}}(a)\omega(a)|a|^{s-1}da = \int_{\mathbb{Q}^*\backslash\mathbb{A}^*} \hat{\delta}_{\mathbb{Q}}(a)\omega(a)|a|^{s-1}da. \tag{6.1}$$

We are essentially taking spectral expansion of the self-dual $\delta_{\mathbb{Q}}$ over $\mathbb{Q}^*\backslash\mathbb{A}^*$, and will find L-functions as coefficients. As usual, our argument is very formal; we will assume for simplicity that the conductor $q$ of $\chi$ is prime, but this is easily removed using Exercise 5.6.

The left-hand side of (6.1) is just 1, since a fundamental domain for $\mathbb{Q}^*\backslash\mathbb{A}^*$ is $(0,\infty)\times\hat{\mathbb{Z}}^*$ (Theorem 5.2). Note that we need to use the additive Haar measure $da$ instead of the seemingly more-natural multiplicative Haar measure $d^*a$ to use the delta functions here. The computation of the right-hand side of (6.1) is more involved. Using the definitions of the Fourier transform and $\delta_{\mathbb{Q}} = \sum_{q\in\mathbb{Q}} \delta_q$, we find it is

$$\begin{aligned}\int_{\mathbb{Q}^*\backslash\mathbb{A}^*} \sum_{q\in\mathbb{Q}} \left[\int_{\mathbb{A}} \delta_q(x)e(-ax)\right] |a|^{s-1}\omega(a)da &= \sum_{q\in\mathbb{Q}} \int_{\mathbb{Q}^*\backslash\mathbb{A}^*} e(-qa)|a|^{s-1}\omega(a)da \\ &= \int_{\mathbb{Q}^*\backslash\mathbb{A}^*} |a|^{s-1}\omega(a)da + \int_{\mathbb{A}^*} e(-a)|a|^{s-1}\omega(a)da\end{aligned}$$

We essentially ignored this first term before, when $\omega$ was trivial. However, using the fundamental domain in Theorem 5.2 for $\mathbb{Q}^*\backslash\mathbb{A}^*$, it is

$$\int_0^\infty \int_{\hat{\mathbb{Z}}^*} \omega_f(x_f)dx_f, \quad \omega = \omega_\infty \cdot \omega_f.$$

However, the inner integral is $\frac{1}{N}\sum_{j=1}^N \chi^{-1}(j) = 0$ if $\chi$ is a nontrivial character of order $N$. So, (6.1) reads

$$1 = \int_{\mathbb{A}^*} e(-a)|a|^{s-1}\omega(a)da = \prod_{p\leq\infty} \int_{\mathbb{Q}_p^*} e_p(a)|a|_p^{s-1}\omega(a_p)da_p.$$

**Exercise 6.1.** *Use Exercise 4.8 to show that*

$$\int_{\mathbb{R}} e(x)|x|^s sgn(x)d^*x = i\Gamma_{\mathbb{C}}(s+1)\sin(\pi s/2) = i\frac{\Gamma_{\mathbb{R}}(s+1)}{\Gamma_{\mathbb{R}}(2-s)}.$$

The nonarchimdean factors of the integral are the ones we considered in Proposition 2.18. Combining these together with the identities $|\tau_\chi|^2 = q$, $\tau_{\bar{\chi}} = \chi(-1)\overline{\tau_\chi}$ we get the correct functional equation of $L(s,\chi) = \prod_{p<\infty}\left(1-\frac{\chi(p)}{p^s}\right)^{-1}$.

**Theorem 6.2.** *Let* $\Lambda(s,\chi) = \Gamma_{\mathbb{R}}(s+\epsilon)L(s,\chi)$*, where* $\chi(-1) = (-1)^\epsilon$*. Then*

$$\Lambda(s,\chi) = \frac{\tau_\chi}{i^\epsilon} q^{-s}\Lambda(1-s,\chi^{-1}). \tag{6.2}$$

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