Intertwining Operators, L–Functions, and Representation Theory*

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Introduction

These notes are based on a series of twelve lectures delivered at the Korea Advanced Institute of Science and Technology (KAIST), Taejeon, during their biannual workshop in algebra that took place over the period of July 29–August 4, 1996.

As its main goal, I have chosen to sketch a proof of the estimate

$$q_v^{-1/5} < |\alpha_v| < q_v^{1/5}$$

for Hecke eigenvalues of a cusp form on $GL_2(\mathbb{A}_F)$, where F is an arbitrary number field, using the method initiated by Langlands and developed by myself. The first half of the notes (Sections 1–8) is devoted to developing the problem and introducing some important L–functions (symmetric power L–functions for $GL(2)$ and Rankin–Selberg product L–functions for $GL(m) \times GL(n)$). The second half (Section 9) is spent to explain the method in the split case and finally prove the estimate. The notes are concluded by describing a number of local results in harmonic analysis and representation theory of local groups that can be proved using this method (Section 10).

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§1. Notation

Throughout these lectures F will be either a local or global field of characteristic zero. More precisely, F is a global field of characteristic zero if it is a number field, i.e. a finite extension of $\mathbb{Q}$. A local field of characteristic zero is either $\mathbb{R}$, $\mathbb{C}$, or a finite extension of $\mathbb{Q}_p$, the field of p–adic numbers, i.e. the completion of $\mathbb{Q}$ with respect to p–adic absolute value $|\ \ |_p$, $|rp^m/s|_p = p^{-m}$, $p \nmid rs$. Then $|\ \ |_p$ extends to an absolute value $|\ \ | = |\ \ |_v$, on F where v is the place lying over p.

Suppose F is a p–adic local field (non–archimedean). Then

$$O = \{x \in F||x| \leq 1\}$$

is a ring, called ring of integers. The set

$$P = \{x \in O||x| < 1\}$$

is the maximal ideal of O and $O^* = O \backslash P$ is the group of units. The field $\overline{F} = O/P$ is called the residue field of F whose cardinality which is finite is denoted by $q = p^f$. If e is the ramification index of p in F, i.e. $pO = P^e$, then $[F : \mathbb{Q}_p] = ef$. The ideal P is principal and one usually fixes a uniformizing parameter $\varpi \in P$ for P. Then $|\varpi|_v = q^{-1}$.

Now let F be a number field. For each place v of F, let F_v be its completion with respect to v. A place is an equivalence class of absolute values for F. We will say $v < \infty$ if $v|p$ for some rational prime p. Otherwise $F_v \cong \mathbb{R}$ or $\mathbb{C}$ and we say $v = \infty$. In either case $F \otimes_{\mathbb{Q}} \mathbb{Q}_p = \oplus_{v|p} F_v$, $F \otimes_{\mathbb{Q}} \mathbb{R} = \oplus_{v|\infty} F_v$. If $v < \infty$, let O_v, P_v, $\overline{F}_v = O_v/P_v$, q_v, ϖ_v, and O_v^* be as before.

Let $\mathbb{A} = \mathbb{A}_F \subset \prod_v F_v$ be the subring of $\prod_v F_v$ defined by

$$\mathbb{A} = \{(x_v)_v | x_v \in O_v,\ \forall' v\},$$

i.e. the direct limit of $\prod_v F_v$ with respect to $\prod_{v<\infty} O_v$. With direct limit topology, $\mathbb{A}$ becomes a locally compact ring, called the ring of *adeles* of F. The group of units in $\mathbb{A}$, $\mathbb{A}^* = \mathbb{I}$ will be

$$\mathbb{A}^* = \{(x_v)_v | x_v \in F_v^*,\ x_v \in O_v^*,\ \forall' v\}.$$

Considered as the direct limit of $\prod_v F_v^*$ with respect to $\prod_{v<\infty} O_v^*$ and with direct limit topology, $\mathbb{A}^*$ is called the group of *ideles* of F. Then one can define

$$\|(x_v)_v\| = \prod_v |x_v|_v$$

whose kernel $I^1 = (\mathbb{A}^*)^1$ contains F^* by diagonal imbedding. Then F^* is a lattice in I^1, i.e. F^* is discrete in I^1 and I^1/F^* is compact. So is F in $\mathbb{A}$. The finiteness of class number, the number of ideals (fractional) modulo the principal ones, is equivalent to the compactness of I^1/F^*.

Fix an integer $n > 0$ and let $GL_n(R)$ be the group of invertible matrices with R–entries, where R is a commutative ring with 1.

Let $R = \mathbb{A}$. Then $GL_n(\mathbb{A})$ is called the *adelization* of GL_n over F. If $g \in GL_n(\mathbb{A})$, then $g = (g_v)_v,\ g_v \in GL_n(F_v)$ with $g_v \in GL_n(O_v),\ \forall' v < \infty$.

Let $\mathbf{G}$ be a Zariski–closed subgroup of a $GL_n(F_a)$, where F_a is an algebraic closure of F. Assume $\mathbf{G}$ is defined over F. Then it is defined over every F_v. Let $G_v = \mathbf{G}(F_v)$. Then $g \in GL_n(\mathbb{A})$ belongs to $\mathbf{G}(\mathbb{A})$, adelization of G/F, if and only if each $g_v \in G_v$. For almost all $v < \infty$, $\mathbf{G}$ is defined over O_v and therefore $\mathbf{G}(O_v)$ makes sense and $G_v \cap GL_n(O_v) = \mathbf{G}(O_v)$. Consequently $(g_v)_v \in \mathbf{G}(\mathbb{A})$ implies that $g_v \in \mathbf{G}(O_v)$ for almost all $v < \infty$.

§2. Modular forms as forms for GL_2

In this section we shall briefly review how classical modular forms and automorphic forms on $GL_2(\mathbb{A}_\mathbb{Q})$ are related.

Let Γ be a congruence subgroup for SL_2, i.e. a subgroup of $SL_2(\mathbb{Z})$, containing a principal congruence subgroup

$$\Gamma_N = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) | g \equiv I (\operatorname{mod} N) \right\}$$

for some N. The group $SL_2(\mathbb{Z})$ and therefore every congruence subgroup Γ acts on upper half plane $\mathfrak{h}$

$$\mathfrak{h} = \{ z \in \mathbb{C} | \operatorname{Im}(z) > 0 \}$$

by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

In fact $\mathrm{Im}(g \cdot z) = \mathrm{Im}(z)/|cz+d|^2$ as long as $g \in SL_2(\mathbb{R})$. Let $\mathfrak{h}^*$ be the compactification of $\mathfrak{h}$ by adding cusps of Γ to $\mathfrak{h}$. Roughly speaking, a modular form of weight $k > 0$, k an integer, is a complex function on $\mathfrak{h}^*$ which is holomorphic there and satisfies

$$f(\gamma \cdot z) = (cz+d)^k f(z) \qquad \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\right). \tag{2.1}$$

We may fix an integer $N > 0$ and consider the Hecke subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| c \equiv 0(N) \right\} \subset SL_2(\mathbb{Z}).$$

Given a character $\chi \mod N$ of $(\mathbb{Z}/N\mathbb{Z})^*$, extended to all of $\mathbb{Z}$, we may consider holomorphic forms f such that

$$f(\gamma \cdot z) = \chi(a)^{-1}(cz+d)^k f(z), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

Then given any congruence subgroup Γ and a holomorphic form with respect to Γ, one can find a $\Gamma_0(N)$ and a χ such that f becomes a χ–modular form with respect to $\Gamma_0(N)$. Thus one may only consider forms with respect to $\Gamma_0(N)$ but with respect to arbitrary characters of $(\mathbb{Z}/N\mathbb{Z})^*$. A modular form f is a *cusp form* if it vanishes on every cusp of Γ (or $\Gamma_0(N)$). The best reference for this is [Shi1].

Let us show how modular cusp forms on $\mathfrak{h}$ are in fact functions on $GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}_\mathbb{Q})$, i.e. an automorphic form. We refer to [B,G] for details.

First observe that every $g \in SL_2(\mathbb{R})$ can be written as

$$g = \begin{bmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{bmatrix} k(\theta) \qquad (y > 0)$$

with

$$k(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Thus

$$g \mapsto (z = x + iy \in \mathfrak{h},\ \theta)$$

and $g \cdot i = z$ since $k(\theta) \cdot i = i$. Given a modular cusp form f, define:

$$\begin{aligned} \phi_f(g) &= f(g \cdot i)(ci+d)^{-k} \\ &= f(g \cdot i) y^{\frac{k}{2}} e^{-i\theta k}. \end{aligned}$$

If $j(g,z) = cz + d$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, then

$$j(g_1 g_2, z) = j(g_1, g_2 \cdot z) j(g_2, z).$$

Thus

$$j(\gamma g, i) = j(\gamma, g \cdot i) j(g, i)$$

from which we get

$$\begin{aligned} \phi_f(\gamma g) &= f(\gamma g \cdot i) j(\gamma g, i)^{-k} \\ &= f(g \cdot i) j(\gamma, g \cdot i)^k j(\gamma g, i)^{-k} \\ &= \phi_f(g) \end{aligned}$$

It is then clear that ϕ_f is bounded as a function of $\Gamma \backslash G$, $G = SL_2(\mathbb{R})$, and by the finiteness of the volume of $\Gamma \backslash G$, $\phi_f \in L^2(\Gamma \backslash G)$. Thus the space of modular cusp forms with respect to Γ can be embedded into $L^2(\Gamma \backslash G)$. It can be easily seen that

$$\iint_F |f(z)|^2 y^k \frac{dxdy}{y^2} = \int_{\Gamma \backslash G} |\phi_f(g)|^2 dg < \infty$$

where F is a fundamental domain.

One likes to find a larger space so that it will cover for all Γ's. It is here that we use $GL_2(\mathbb{A}_{\mathbb{Q}})$. We may assume $\Gamma = \Gamma_0(N)$ for some N and assume f is attached to a character χ.

It can be shown, using approximation, that given $\Gamma_0(N)$

$$GL_2(\mathbb{A}_{\mathbb{Q}}) = GL_2(\mathbb{Q}) GL_2^+(\mathbb{R}) \prod_{p \nmid N} GL_2(\mathbb{Z}_p) \prod_{p \mid N} K_{p,N}$$

$K_{p,N} = GL_2(\mathbb{Z}_p)$, for $p \nmid N$, while $K_{p,N} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}_p) | c \equiv 0(N) \right\}$, $p|N$ and $GL_2^+(\mathbb{R})$ is the subgroup of elements with positive determinant, i.e. the connected component of $GL_2(\mathbb{R})$. Moreover

$$GL_2(\mathbb{Q}) \cap GL_2^+(\mathbb{R}) \prod_{p \nmid N} GL_2(\mathbb{Z}_p) \prod_{p \mid N} K_{p,N} = \Gamma_0(N).$$

For $g \in GL_2^+(\mathbb{R})$, define

$$j(g,z) = (cz + d)(\det g)^{-1/2}.$$

The character χ of $(\mathbb{Z}/N\mathbb{Z})^*$ then defines a character of $\mathbb{Q}^*\backslash\mathbb{A}_{\mathbb{Q}}^*$, write $\chi = \otimes_p\chi_p$. Each χ_p defines a character

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \chi_p(a)$$

of $K_{p,N}$ and thus we get a character of

$$K_0^N = \prod_{p\nmid N} GL_2(\mathbb{Z}_p) \prod_{p|N} K_{p,N}.$$

Then we let

$$\widetilde{\phi}_f(\gamma g_\infty k_0) = f(g_\infty \cdot i) j(g_\infty, i)^{-k}\chi(k_0).$$

It can then easily be checked that this is a well defined function and

$$\widetilde{\phi}_f \in L_0^2(Z(\mathbb{A}_{\mathbb{Q}})GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}_{\mathbb{Q}}),\ \chi),$$

the space of L^2–functions φ on $Z(\mathbb{A}_{\mathbb{Q}})GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}_{\mathbb{Q}})$ which satisfy $\varphi(zg) = \chi(z)\varphi(g)$,

$z \in Z(\mathbb{A}_{\mathbb{Q}})$

$$\int\limits_{\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx = 0,\ \forall' g.$$

The map $f \mapsto \widetilde{\phi}_f$ can now be extended to all of modular cusp forms, holomorphic or non–holomorphic.

Now let $GL_2(\mathbb{A}_{\mathbb{Q}})$ act by right regular action on the Hilbert space

$$L^2(Z(\mathbb{A}_{\mathbb{Q}})GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}_{\mathbb{Q}}),\ \omega),$$

ω a unitary character of $\mathbb{Q}^*\backslash\mathbb{A}_{\mathbb{Q}}^*$. It decomposes to a continuous part and a discrete part.

¿From now on assume all the modular forms f are eigenvalues of Hecke operators for all p. Recall that if f is of weight k, then the p–th Hecke operator $T_k(p)$ is defined by

$$T_k(p)f(z) = p^{k-1}\sum_{\substack{a>0 \\ ad=p}}\sum_{b=0}^{d-1} f\left(\frac{az+b}{d}\right)d^{-k}.$$

For a modular cusp form f, $\widetilde{\phi}_f$ will belong to the discrete part. Moreover

$$\int_{\mathbb{Q}\backslash\mathbb{A}} \widetilde{\phi}_f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) dx = 0 \qquad (\forall g \in GL_2(\mathbb{A}_{\mathbb{Q}}))$$

and in fact the space of cusp forms are exactly those L^2–functions which satisfy the above condition for $\forall' g \in GL_2(\mathbb{A}_{\mathbb{Q}})$.

In conclusion, the set of normalized cuspidal eigenfunctions of all weights and for all Hecke operators, holomorphic or non, are in one–one correspondence with irreducible constituents of

$$L_0^2(Z(\mathbb{A}_{\mathbb{Q}})GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}_{\mathbb{Q}}),\ \omega)$$

for all ω.

One can then translate problems from classical theory to representation theory and try to solve them. One important one will be discussed next.

§3. Maass forms and Ramanujan–Petersson's Conjecture

Holomorphic modular forms are defined to satisfy the property that the differential $f(z)(dz)^{\frac{k}{2}}$, $k =$ even, remains invariant under the action of congruence subgroup Γ, i.e.

$$f(\gamma \cdot z)(d(\gamma \cdot z))^{\frac{k}{2}} = f(z)(dz)^{\frac{k}{2}}.$$

The only holomorphic modular *functions* (i.e. $k = 0$) are constants and therefore no non–zero cuspidal holomorphic modular function exists. On the other hand if we allow the Laplace–Beltrami operator

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

to have a non–zero eigenfunction (a real analytic function) say

$$\Delta f = \frac{1}{4}(1 - s^2) f,$$

we can then pick up non–holomorphic modular functions i.e. forms of weight zero, which are cusp forms, the so called *Maass forms.* More precisely, one needs functions $f : \mathfrak{h}^* \longrightarrow \mathbb{C}$ for which

1. $f(\gamma \cdot z) = f(z) \qquad \gamma \in \Gamma,$
2. $\Delta f = \frac{1}{4}(1 - s^2) f,\ \ s \in i\mathbb{R}$ or $-1 < s < 1,$
3. f is bounded,

4. f is cuspidal,

5. f is an eigenfunction for all Hecke operators.

Then it is an easy exercise to show that

$$f(x+iy) = \sum_{n\neq 0} (|n|y)^{1/2} a_n K_{s/2}(2\pi|n|y)e^{2\pi inx},$$

where K_ν satisfies

$$z^2\frac{d^2K_\nu}{dz^2} + z\frac{dK_\nu}{dz} - (z^2+\nu^2)K_\nu = 0$$

with

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z}$$

as $z \longrightarrow +\infty$. The numbers a_n are called the Fourier coefficients of f. Let us say f is normalized if $a_1 = 1$.

There is an analogue of classical Ramanujan–Petersson's conjecture. More precisely:

Suppose f is a normalized Maass form which is an eigenfunction for every Hecke operator. Then

$$|a_p| \leq 2p^{-1/2}.$$

Define α_p by

$$a_p = p^{-1/2}(\alpha_p + \alpha_p^{-1}).$$

Then by R–P conjecture

$$(\alpha_p - \alpha_p^{-1})^2 \leq 0,$$

since a_p is real. This implies that $\alpha_p - \alpha_p^{-1}$ is pure imaginary or

$$\mathrm{Re}(\alpha_p) - \frac{\mathrm{Re}(\alpha_p)}{|\alpha_p|^2} = 0.$$

Since $\mathrm{Re}(\alpha_p) \neq 0$, $|\alpha_p| = 1$, and therefore

$$|a_p| \leq 2p^{-1/2} \Leftrightarrow |\alpha_p| = 1,$$

i.e. Ramanujan–Petersson's conjecture demands $|\alpha_p| = 1$.

§4. Automorphic cuspidal representations

We now assume F is an arbitrary number field and consider $L_0^2(Z(\mathbb{A})GL_2(F)\backslash GL_2(\mathbb{A}), \omega)$, where ω is a (unitary) character of $Z(F)\backslash Z(\mathbb{A}) \cong F^*\backslash \mathbb{A}^*$ and L_0^2 is the space of all the square integrable functions φ on the quotient space which satisfy:

1) $$\varphi(zg) = \omega(z)\ \varphi(g) \qquad (z \in Z(\mathbb{A}))$$

2) $$\int_{F\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx = 0 \qquad (\forall' g \in GL_2(\mathbb{A})).$$

Under the right regular action of $GL_2(\mathbb{A})$ this space decomposes to a direct sum of irreducible subrepresentations each of which appear with multiplicity one [JL, Go].

Let π be an irreducible constituent as above. Then there is a non–unique decomposition of π into an infinite restricted tensor product of irreducible representation π_v of each local group $GL_2(F_v)$ for every place v of F. Almost all of π_v's have a vector fixed by the action of $GL_2(O_v)$ and it's the choices made for these vectors which leads to the decomposition $\pi = \otimes_v \pi_v$ in a non–unique way. But the classes of representations π_v are all unique.

A representation π_v which has a vector fixed by $GL_2(O_v)$ is called a class one, spherical, or unramified representation. It can be realized on the space $V(\mu_{1,v}, \mu_{2,v})$ of smooth functions

$$f : GL_2(F_v) \longrightarrow \mathbb{C}$$

which satisfy

$$f\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right) = \mu_{1,v}(a_1)\mu_{2,v}(a_2)|a_1/a_2|_v^{1/2} f(g)$$

where $\mu_{v,1}$ and $\mu_{v,2}$ are a pair of unramified quasicharacters of F_v^*, i.e. $\mu_{v,i}|O_v^* = 1$, $a_1, a_2 \in F_v^*$, and $x \in F_v$. Moreover $\mu_{1,v}/\mu_{2,v} \neq |\ \ |_v^{\pm 1}$ as quasicharacters. If we denote the right regular action of $GL_2(F_v)$ on $V(\mu_{1,v}, \mu_{2,v})$ by $I(\mu_{1,v}, \mu_{2,v})$, the representation unitarily induced from $\mu_{1,v}$ and $\mu_{2,v}$, then $I(\mu_{1,v}, \mu_{2,v})$ and $I(\mu_{2,v}, \mu_{1,v})$ are equivalent and if $I(\mu_{1,v}, \mu_{2,v}) \cong I(\mu'_{1,v}, \mu'_{2,v})$, then $(\mu_{1,v}, \mu_{2,v}) = (\mu'_{1,v}, \mu'_{2,v})$ or $(\mu_{1,v}, \mu_{2,v}) = (\mu'_{2,v}, \mu'_{1,v})$.

Now assume $\pi = \otimes_p \pi_p$ is generated by one of our $\widetilde{\phi}_f$, where f is a classical modular cusp form with respect to a Hecke subgroup $\Gamma_0(N)$. Then for each $p \nmid N$, π_p

is of class one and if $\pi_p = I(\mu_{1,p}, \mu_{2,p})$, then $\mu_{1,p}(p) = \alpha_p$ and $\mu_{2,p}(p) = \alpha_p^{-1}$ or vice–versa. Thus R–P conjecture is equivalent to characters $\mu_{1,p}$ and $\mu_{2,p}$ being unitary or said in the language of representation theory $I(\mu_{1,p}, \mu_{2,p})$ must be *tempered.* One can then formulate a generalized R–P conjecture for an arbitrary number field.

If $\pi = \otimes_v \pi_v$ is an irreducible cuspidal representation of $GL_2(\mathbb{A})$, then each π_v is tempered.

Not every tempered representation of $GL_2(F_v)$ is of the form of above induced representation. The so called discrete series representations, i.e. those which appear discretely in $L^2(GL_2(F_v))$ will provide the rest of them.

This is a very difficult conjecture and due to Deligne's work we know its validity, but only for holomorphic cusp forms, i.e. those for which every π_v, v archimedean, is in the discrete series.

If π_v is of class one, then $\pi_v \cong I(\mu_{1,v}, \mu_{2,v}) \cong I(\mu_{2,v}, \mu_{1,v})$ and the choices of $\mu_{1,v}$ and $\mu_{2,v}$ are uniquely given by π_v up to permutation of the two characters. Moreover the two characters are determined uniquely by their evaluations at ϖ_v. Let A_v denote the conjugacy class of

$$\begin{pmatrix} \alpha_v & 0 \\ 0 & \beta_v \end{pmatrix}$$

in $GL_2(\mathbb{C})$, where $\alpha_v = \mu_{1,v}(\varpi_v)$ and $\beta_v = \mu_{2,v}(\varpi_v)$. Thus class of π_v is uniquely determined by A_v. Moreover, $\pi = \otimes_v \pi_v$ is uniquely determined by $\{A_v | v \notin S\}$, where S is a finite set of places of F such that π_v is of class one, whenever $v \notin S$. By strong multiplicity one theorem, the choice of S is irrelevant.

The Ramanujan–Petersson's conjecture is then equivalent to $|\alpha_v| = |\beta_v| = 1$ for $\forall\ v \notin S$.

When $[F : \mathbb{Q}] = 1$, the best estimate is due to Luo–Rudnick–Sarnak [LRSa] and Bump–Duke–Hoffstein–Iwaniec [BDHI] and is

$$p^{-5/28} \leq |\alpha_p| \text{ and } |\beta_p| \leq p^{5/28}.$$

For an arbitrary number field, the best estimate is

$$q_v^{-1/5} < |\alpha_v| \text{ and } |\beta_v| < q_v^{1/5}$$

and is due to Shahidi [Sh1,Sh2]. We refer to [Ra] for a density result on the conjecture.

Proof are all based on this theory of automorphic L–functions which we will discuss next.

§5. L–functions for GL_2

The group $GL_2(\mathbb{C})$ has already showed up in the study of automorphic forms for $GL_2(\mathbb{A})$, where $\mathbb{A} = \mathbb{A}_F$ is the ring of adeles of a number field F. More precisely, if $\pi = \otimes_v \pi_v$ is a cuspidal automorphic representation of $GL_2(\mathbb{A})$, then almost all components π_v of π are determined, up to isomorphism, by semisimple conjugacy classes A_v in $GL_2(\mathbb{C})$. Fix a complex number s. If r is a complex analytic (finite dimensional) representation of $GL_2(\mathbb{C})$, then $\det(I - r(A_v)q_v^{-s})^{-1}$ is the inverse of the characteristic polynomial of $r(A_v)$ at $\lambda = q_v^{-s}$ and is therefore independent of all the choices except classes of π_v and r. This is what one calls the local Langlands L–function attached to π_v and r and is denoted by

$$L(s, \pi_v, r) = \det(I - r(A_v)q_v^{-s})^{-1}.$$

In fact, Langlands defined his local L–functions in the generality of any quasisplit reductive group in [L1], using exactly the same kind of definition.

The derived group $SL_2(\mathbb{C})$ of $GL_2(\mathbb{C})$ has precisely one irreducible finite dimensional representation of any degree. They can be naturally extended to $GL_2(\mathbb{C})$ and they are called symmetric powers of the standard representation of $GL_2(\mathbb{C})$ which we shall now explain.

Fix a pair of positive integers m and n. The group $GL_n(\mathbb{C})$ acts on $\mathbb{C}^n$ by natural matrix multiplication which we call the standard representation ρ_n of $GL_n(\mathbb{C})$. It also acts on $(\mathbb{C}^n)^{\otimes^m}$, the tensor product of m–copies of $\mathbb{C}^n$ by acting naturally on each component of pure tensors of rank m. Let S_m be the symmetric group in m letters. Then the symmetrization operator

$$\mathrm{Sym}^m = \frac{1}{m!} \sum_{\sigma \in S_m} \sigma$$

acts on $(\mathbb{C}^n)^{\otimes^m}$. In fact, let $\{e_1, \ldots, e_n\}$ be a basis for $\mathbb{C}^n$ and define the action of S_m on the basis by $\sigma(e_i) = e_{\sigma(i)}$. Moreover $\{e_{i_1} \otimes e_{i_2} \ldots \otimes e_{i_m}\}$ makes a basis for $(\mathbb{C}^n)^{\otimes^m}$ and S_m acts by acting at each component. Then the subspace of symmetric tensors is by definition the image $\mathrm{Sym}^m(\mathbb{C}^n)$ of $(\mathbb{C}^n)^{\otimes^m}$ under Sym^m. The natural action of $GL_n(\mathbb{C})$ on $(\mathbb{C}^n)^{\otimes^m}$ commutes with Sym^m, making its image invariant under $GL_n(\mathbb{C})$. The resulting representation of $GL_n(\mathbb{C})$ on $\mathrm{Sym}^m(\mathbb{C})^n$ denoted by $\mathrm{Sym}^m(\rho_n)$ is irreducible and is called the m–th *symmetric power of* ρ_n.

In this section, we are interested in the case $n = 2$. Then for each m, $\dim \mathrm{Sym}^m(\rho_n)) = m+1$. If $t_v = \mathrm{diag}(\alpha_v, \beta_v) \in GL_2(\mathbb{C})$ is a representative for A_v, then fixing the standard basis $e_1 = (1,0)$ and $e_2 = (0,1)$ for $\mathbb{C}^2$, so that

$$\{\mathrm{Sym}^m(e_1^{\otimes^i} \otimes e_2^{\otimes^j}) | i+j=m\}$$

is a basis for $\mathrm{Sym}^m(\mathbb{C}^2)$, implies that

$$\mathrm{Sym}^m(\rho_2)(t_v) = \mathrm{diag}(\alpha_v^m, \alpha_v^{m-1}\beta_v, \dots, \alpha_v\beta_v^{m-1}, \beta_v^m),$$

an element in $GL_{m+1}(\mathbb{C})$.

The restriction of each $\mathrm{Sym}^m(\rho_2)$ to $SL_2(\mathbb{C})$ is irreducible and, up to isomorphism, this is the only irreducible representation of $SL_2(\mathbb{C})$ of dimension $m+1$. Moreover all the finite dimensional irreducible representations of $SL_2(\mathbb{C})$ are so obtained.

For each unramified place v, let

$$\begin{aligned} L(s, \pi_v, \mathrm{Sym}^m(\rho_2)) &= \det(I - \mathrm{Sym}^m(\rho_2)(A_v) q_v^{-s})^{-1} \\ &= \prod_{j=0}^{m} (1 - \alpha_v^j \beta_v^{m-j} q_v^{-s})^{-1} \end{aligned}$$

and if S is a finite set of places of F (including the archimedean ones) such that π_v is unramified for every $v \notin S$, let

$$\begin{aligned} L_S(s, \pi, \mathrm{Sym}^m(\rho_2)) &= \prod_{v \notin S} L(s, \pi_v, \mathrm{Sym}^m(\rho_2)) \\ &= \prod_{v \notin S} \prod_{j=0}^{m} (1 - \alpha_v^j \beta_v^{m-j} q_v^{-s})^{-1}. \end{aligned}$$

Langlands [L1]: *Suppose* $L_S(s, \pi, Sym^m(\rho_2))$ *is absolutely convergent for* $\mathrm{Re}(s) > 1$ *for every* m*. Then Ramanujan–Petersson's conjecture is valid for* π.

In fact, $|\alpha_v^m| < q_v$ and $|\beta_v^m| < q_v$ if the partial L–function converges for $\mathrm{Re}(s) > 1$. Then $|\alpha_v| < q_v^{1/m}$ and $|\beta_v| < q_v^{1/m}$. Letting $m \to +\infty$, then implies the conjecture since $|\alpha_v|^{-1} = |\beta_v|$.

The estimate

$$q_v^{-1/5} < |\alpha_v| \text{ and } |\beta_v| < q_v^{1/5}$$

is a consequence of the following result which is a special case of a general theorem proved about automorphic L–functions.

For each $v \notin S$, *the local* L*–function* $L(s, \pi_v, Sym^5(\rho_2))$ *is holomorphic for* $\mathrm{Re}(s) \geq 1$.

This is clearly a global result. In fact for an arbitrary unramified unitary representation π whose attached semisimple conjugacy class in $GL_2(\mathbb{C})$ is represented by $t = \mathrm{diag}(\alpha, \beta)$, $q^{-1/2} < |\alpha|$ and $|\beta| < q^{1/2}$ and there are unramified unitary representations for which $|\alpha| = q^{1/2-\varepsilon}$ for arbitrarily small $\varepsilon > 0$.

§6. Cuspidal representations for GL_n

Using the language of adeles and the theory of group representations, it is now quite easy to define cusp forms for GL_n. We rather define the corresponding representations.

Let ω be a unitary character of $F^*\backslash\mathbb{A}^*$. Identifying the center $Z(\mathbb{A})$ of $GL_n(\mathbb{A})$ with $\mathbb{A}^*$, ω is then a unitary character of $Z(\mathbb{A})$. By a parabolic subgroup of GL_n, we shall mean any conjugate (under GL_n) of a subgroup P of the form

$$\left\{\begin{pmatrix} \boxed{g_1} & * & & \\ & \boxed{g_2} & & * \\ 0 & & \ddots & \\ & & & \boxed{g_r} \end{pmatrix}\right\},$$

where $g_i \in GL_{n_i}$, $1 \leq i \leq r$, $n_1 + n_2 + \ldots + n_r = n$. Then

$$M = \left\{\begin{pmatrix} \boxed{g_1} & & & \\ & & & 0 \\ & \boxed{g_2} & & \\ 0 & & \ddots & \\ & & & \boxed{g_r} \end{pmatrix}\right\}$$

is called a Levi subgroup of P and $P = MN$, where

$$N = \begin{pmatrix} \boxed{I_{n_1}} & & * \\ & \ddots & \\ 0 & & \boxed{I_{n_r}} \end{pmatrix},$$

a subgroup of unipotent upper triangular elements, is called the unipotent radical of P.

An L^2–function φ is called cuspidal if

$$\int_{N(F)\backslash N(\mathbb{A})} \varphi(ng)dn = 0 \qquad (\forall' g \in GL_n(\mathbb{A}))$$

for every possible unipotent radical. It can be shown that it is enough to consider only those P which are maximal, i.e. $r = 2$.

Let $L_0^2(Z(\mathbb{A})GL_n(F)\backslash GL_n(\mathbb{A}),\ \omega)$ be the Hilbert space of square integrable *cuspidal* functions satisfying $\varphi(zg) = \omega(z)\varphi(g),\ z \in \mathbb{Z}(\mathbb{A}),\ g \in GL_n(\mathbb{A})$. Under right regular translation by elements in $GL_n(\mathbb{A})$, L_0^2 becomes a unitary representation which is a direct sum of irreducible unitary representation of $GL_n(\mathbb{A})$. Let π be an irreducible constituent of this space. Then again $\pi = \otimes_v \pi_v$ with almost all π_v of class one, i.e. having a vector fixed by $GL_n(O_v)$. The vector will then be unique up to scalar multiplication. Moreover, the class of a class one π_v can be uniquely determined by a unique conjugacy class A_v in $GL_n(\mathbb{C})$.

Again by strong multiplicity one, π is determined by almost all A_v's, i.e. if $\pi = \otimes_v \pi_v$ and $\pi' = \otimes_v \pi'_v$, and $A_v = A'_v$ or equivalently $\pi_v \cong \pi'_v$ for almost all v, then $\pi \cong \pi'$ and in fact $\pi = \pi'$ (due to Shalika [S]).

It is important to produce interesting cuspidal representations. When $n = 2$ the theory is equivalent to that of modular forms, holomorphic or non–holomorphic, as explained. Here is an important example for $n = 3$.

Gelbart–Jacquet lift. Let $\pi = \otimes_v \pi_v$ be a cuspidal representation of $GL_2(\mathbb{A})$. Choose a finite set S of places of F such that if $v \notin S$, then π_v is unramified. Let A_v be the corresponding conjugacy class for π_v for each $v \notin S$. The group $PGL_2(\mathbb{C})$ acts on the 3–dimensional Lie algebra $sl_2(\mathbb{C})$ of $SL_2(\mathbb{C})$ (and itself) by adjoint representation Ad, giving a 3–dimensional irreducible and faithful representation of $PGL_2(\mathbb{C})$. Let Ad^2 be the corresponding 3–dimensional representation of $GL_2(\mathbb{C})$, i.e. the one obtained by combining Ad with the natural projection of $GL_2(\mathbb{C})$ onto $PGL_2(\mathbb{C})$. It is this representation which is called the adjoint square representation of $GL_2(\mathbb{C})$. If $t_v = \mathrm{diag}(\alpha_v, \beta_v)$ represents A_v, then $\mathrm{Ad}^2(t_v) = \mathrm{diag}(\alpha_v\beta_v^{-1},\ 1,\ \alpha_v^{-1}\beta_v)$. Set $t'_v = \mathrm{Ad}^2(t_v)$ and let A'_v be its conjugacy class in $GL_3(\mathbb{C})$. Each A'_v, $v \notin S$, determines a unique (up to isomorphism) unramified representation π_v of $GL_3(F_v)$. The question is whether there exists an irreducible automorphic representation $\prod = \otimes_v \prod_v$ of $GL_3(\mathbb{A})$ such that $\prod_v = \pi'_v$ for $v \notin S$ and the answer is yes and is due to Gelbart and Jacquet ([GJ], [Shi2]). Before, we explain how this is proved, let us formulate the problem in the language of functoriality.

The map

$$\mathrm{Ad}^2 : GL_2(\mathbb{C}) \longrightarrow GL_3(\mathbb{C})$$

is a homomorphism from $GL_2(\mathbb{C})$, the L–group of GL_2, into the L–group $GL_3(\mathbb{C})$ of GL_3. In fact for every reductive group over a local or global field, Langlands [L1] defined a complex group by transposing the Cartan matrix of the original group, called its "L–group". When the group is GL_n, the L–group ${}^LGL_n = GL_n(\mathbb{C})$. Langlands functoriality conjecture then requires that every homomorphism between two L–groups must give rise to a "map" between automorphic forms of the original groups. Thus there must exist a map Ad^2_* which sends automorphic representations of $GL_2(\mathbb{A})$ to those of $GL_3(\mathbb{A})$ in such a way that if $\mathrm{Ad}^2_*(\pi) = \prod = \otimes_v \prod_v$, then $\mathrm{Ad}^2(t_v)$ represents the conjugacy class in $GL_3(\mathbb{C})$ attached to $\prod_v$ for all unramified places v.

The proof is based on the theory of L–functions. Gelbart and Jacquet showed that the partial L–function $L_S(s, \pi, \mathrm{Ad}^2)$ can be completed to an L–function which is entire (as a function of s) unless $\pi = \pi \otimes \chi$ for some non–trivial character χ of $F^*\backslash\mathbb{A}^*$, is bounded in vertical strips, and satisfies a functional equation. More generally, given any character $\rho = \otimes\rho_v$ of $F^*\backslash\mathbb{A}^*$, one can define a local L–function

$$L(s, \pi_v, \rho_v, \mathrm{Ad}^2) = \det(I - \mathrm{Ad}^2(t_v)\rho_v(\varpi_v)q_v^{-s})^{-1}$$

whenever π_v and ρ_v are unramified with $t_v \in GL_2(\mathbb{C})$ representing π_v. Similar statements are then proven in [Sh3] for the completed form of

$$L_S(s, \pi, \rho, \mathrm{Ad}^2) = \prod_{v \notin S} L(s, \pi_v, \rho_v, \mathrm{Ad}^2).$$

Converse theorem for $GL_3(\mathbb{A})$ due to Jacquet, Piatetski–Shapiro, and Shalika now applies, generalizing that of Hecke, Weil, and Jacquet–Langlands, proving the existence of an automorphic representation Π of $GL_3(\mathbb{A})$ such that

$$L(s, \pi, \rho, \mathrm{Ad}^2) = L(s, \Pi \otimes \rho, \rho_3),$$

for every ρ, where ρ_3 is the standard representation of $GL_3(\mathbb{C})$ and $(\Pi \otimes \rho)(g) = \Pi(g)\rho(\det g)$. Moreover Π is cuspidal if and only if π is cuspidal and $\pi = \pi \otimes \chi$ implies $\chi \equiv 1$.

Experts familiar with classical theory must realize that the introduction of L–functions for each $\rho \in F^*\backslash A^*$ is equivalent to twisting the original Dirichlet series with primitive characters as needed by the converse theorem. It is very special of GL_3 that only twisting with characters of $GL_1(\mathbb{A})$ is enough for applying the converse theorem.

Let ω be the central character of π. Then setting $\mathrm{Sym}^2_*(\pi) = \mathrm{Ad}^2_*(\pi) \otimes \omega$ we see that Sym^2_* is the dual "map" to

$$\mathrm{Sym}^2(\rho_2) = \mathrm{Sym}^2 : GL_2(\mathbb{C}) \longrightarrow GL_3(\mathbb{C})$$

as defined before, i.e. $\mathrm{Sym}^2(\mathrm{diag}(\alpha_v, \beta_v)) = \mathrm{diag}(\alpha_v^2, \alpha_v\beta_v, \beta_v^2)$.

It is natural to ask whether Sym^m_* is defined for any other $m > 2$. At present, our best chance is $m = 3$. In fact, lots of information is available about

$$L_S(s, \pi, \mathrm{Sym}^3(\rho_2)).$$

Local L–functions are canonically defined for all v and therefore the L–function is completed [Sh3]. The completed L–function satisfies a functional equation sending s to $1 - s$. The fact that it is entire unless π is *monomial*, i.e. $\pi = \pi \otimes \chi$ for some non–trivial $\chi \in F^* \widehat{\backslash} A^*$, is still incomplete and combining the work in [BGiH] and [Sh3], one only knows that poles are real and only between $-3/4$ and $3/4$.

To apply the converse theorem one needs to twist with $\rho \in F^* \widehat{\backslash} \mathbb{A}^*$ as well as cusp forms on $GL_2(\mathbb{A})$. With respect to the first one, one needs another irreducible four dimensional representation of $GL_2(\mathbb{C})$ which we have called in [Sh3], *adjoint cube*, denoted by Ad^3. It is defined by

$$\mathrm{Ad}^3\left(\begin{pmatrix} \alpha_v & 0 \\ 0 & \beta_v \end{pmatrix}\right) = \mathrm{diag}(\alpha_v^2\beta_v^{-1},\ \alpha_v, \beta_v, \alpha_v^{-1}\beta_v^2).$$

If Ad^3_* exists, then $\mathrm{Sym}^3_* = \mathrm{Ad}^3_* \otimes \omega$, where ω is the central character of π, also exists. To twist $\mathrm{Ad}^3_*(\pi)$ with a $\rho \in \widehat{F^*\backslash A^*}$, it is enough to consider

$$L_S(s, \pi \otimes \omega,\ \mathrm{Ad}^3).$$

It is proved in [Sh3], that $L_S(s, \pi \otimes \omega, \mathrm{Ad}^3)$ can be completed, the completed one satisfies a functional equation, and has only real poles, which using [BGiH], must lie between $-3/4$ and $3/4$.

Twisting with a cusp form π' on $GL_2(\mathbb{A})$ is much harder. We know that [Sh4]

$$L_S(s, \mathrm{Ad}^3_*(\pi) \times \pi')$$

is meromorphic and satisfies a functional equation. But not much more. When full properties of these L–functions are established, we will have the existence of Ad^3_* and therefore Sym^3_*. Methods of analytic number theory will then apply, implying

$$p^{-3/22} \leq |\alpha_p| \leq p^{3/22},$$

an estimate even better than $p^{1/7}$. But until the necessary properties of L–functions are proved, these estimates remain out of our reach.

We conclude by pointing out that establishing Langlands functoriality is one of the most important goals of modern theory of automorphic forms. Besides the converse theorem which may be called the method of L–functions, the trace formula as developed by Arthur in remarkable generality [A1,A2], can be used to establish certain cases of functoriality. The most notable is that of endoscopy, a major work in progress, which when established, proves one case of functoriality, but in generality of every reductive group.

§7. Rankin–Selberg L–functions for GL_n

Fix two positive integers m and n. Let $\pi = \otimes_v \pi_v$ and $\pi' = \otimes_v \pi'_v$ be irreducible cuspidal automorphic representations of $GL_m(\mathbb{A})$ and $GL_n(\mathbb{A})$, respectively. Fix a finite set of places S of F such that for $v \notin S$, both π_v and π'_v are unramified. Given $v \notin S$, let A_v and A'_v denote the corresponding semisimple conjugacy classes of $GL_m(\mathbb{C})$ and $GL_n(\mathbb{C})$, attached to π_v and π'_v, respectively. Let

$$\begin{aligned} L(s, \pi_v \times \pi'_v) &= L(s, \pi_v \otimes \pi'_v,\ \rho_m \otimes \rho_n) \\ &= \det(I - A_v \otimes A'_v q_v^{-s})^{-1} \\ &= \prod_{\substack{1 \le i \le m \\ 1 \le j \le n}} (1 - \alpha_{v,i}\alpha'_{v,j} q_v^{-s})^{-1}, \end{aligned}$$

when $t_v = \mathrm{diag}(\alpha_{v,1}, \ldots, \alpha_{v,m}) \in A_v$ and $t'_v = \mathrm{diag}(\alpha'_{v,1}, \ldots, \alpha'_{v,n}) \in A'_v$. Finally, let

$$L_S(s, \pi \times \pi') = \prod_{v \notin S} L(s, \pi_v \times \pi'_v).$$

This is what is usually called the (partial) Rankin–Selberg product L–function for π and π'. Each $L(s, \pi_v \times \pi'_v)$ is called the corresponding local Rankin–Selberg product L–function. We have

Theorem 7.1 (Jacquet–Shalika [JS1])**.** *The partial L–function $L_S(s, \pi \times \pi')$ converges absolutely for Re $(s) > 1$.*

The fact that any partial Langlands L–function converges absolutely for $\mathrm{Re}(s)$ sufficiently large is a general fact due to Langlands [L1]. But a bound as little as 1 is usually hard to get and proof of 7.1 is fairly hard. Let us sketch it very roughly here.

Sketch of the proof of 7.1. One first represents the L–function by means of an integral representation which is holomorphic for Re $(s) > 1$, showing that it is holomorphic as a function of s for Re $(s) > 1$. This is quite involved and requires lots of work. Now, techniques of Dirichlet series with positive coefficients apply, showing that the L–function must be absolutely convergent to the right of its first pole which will happen at $s = 1$, if $m = n$ and $\pi' = \overline{\pi}$ for which $A'_v = \overline{A}_v$ for almost all v. Observe that in this case $L_S(s, \pi \times \overline{\pi})$ gives a Dirichlet series with non–negative coefficients. The general case follows from this and Schwarz Lemma.

Theorem 7.2 (Langlands [L2])**.** *The partial L–function $L_S(s, \pi \times \pi')$ extends to a meromorphic function of s on all of $\mathbb{C}$.*

Sketch. One realizes $GL_m \times GL_n$ as a Levi subgroup of GL_{m+n} and considers $\widetilde{\pi} \otimes \pi'$ as a cuspidal representation of $GL_m(\mathbb{A}) \times GL_n(\mathbb{A})$. For definition and detail see proof of Theorem 7.6 here. The ratio of partial L–functions

$$L_S(s, \pi \times \pi')/L_S(1+s, \pi \times \pi')$$

appears in the constant term of the corresponding Eisenstein series which is meromorphic on all of $\mathbb{C}$. Starting with s with Re (s) large and using induction, one concludes the meromorphy of $L_S(s, \pi \times \pi')$ for all s.

Remark. This work of Langlands [L2] is the origin of what is now called the Langlands–Shahidi method.

The theory of L–functions as predicted by Langlands [L1] requires local factors to be defined at all places and not only at unramified ones. There is one type of L–functions which are defined at all places. They are the so called Artin L–functions and Langlands philosophy is that Artin L–functions are automorphic and that is how Langlands proved their holomorphy, the Artin's conjecture on Artin L–functions, for a large number of two dimensional representations of Galois groups.

As usual, Artin L–functions are defined by a product of local L–functions. Let F be a local field. The Weil group $W_{E/F}$ of F is an extension of Gal (E/F) by E^* by means of a non–trivial 2–cocycle. Here E/F is a finite Galois extension. The Weil group of F, denoted by W_F, as $W_{\overline{F}/F}$, is defined as a projective limit for all $W_{E/F}$. There is a further thickening of these groups by a unipotent group, the Deligne–Weil group $W'_{E/F}$ or W'_F, which will be necessary when F is non–archimedean. What follows is expected in the generality of every reductive algebraic group. But let us restrict ourselves to the case of GL_n.

First, one should mention that given any continuous representation ρ of W'_F on a complex vector space V, the Artin L–function $L(s, \rho)$ is defined by

$$L(s, \rho) = \det(I - \rho(\phi)q^{-s}|V^I)^{-1},$$

where ϕ denotes an inverse Frobenius and V^I is the subspace of all vectors fixed in V by the inertia subgroup I. We refer to [T] for definitions of ϕ and I. Also given a non–trivial additive character ψ of F, one can attach an *Artin root number* $\varepsilon(s, \rho, \psi)$ which is a monomial in q^{-s}, and we again refer to [L3] for its definition.

To make matters more precise and clear, let us consider the easier case of archimedean fields. Thus let $F = \mathbb{C}$ or $\mathbb{R}$. Then $W_{\mathbb{C}} = \mathbb{C}^*$ and $W_{\mathbb{R}}$ consist of pairs $(z, \tau) \in \mathbb{C}^* \times$ Gal$(\mathbb{C}/\mathbb{R})$ with multiplication rule

$$(z_1, \tau_1)(z_2, \tau_2) = (z_1 \tau_1(z_2) a_{\tau_1,\tau_2}, \tau_1\tau_2),$$

with $a_{\tau_1,\tau_2} = 1$ unless $\tau_1 = \tau_2 = 1$ in which case $a_{\tau_1,\tau_2} = -1$ (cf. [L4]).

Irreducible representations of $W_{\mathbb{C}}$ are 1–dimensional and if ρ is one such, then $\rho(z) = |z|_{\mathbb{C}}^t (z/\overline{z})^{n/2}$, $|z|_{\mathbb{C}} = z\overline{z}$, for some $t \in \mathbb{C}$ and integer n. The corresponding Artin L–function is

$$L(s, \rho) = 2(2\pi)^{-(s+t+|n|/2)}\Gamma(s + t + |n|/2)$$

with

$$\Gamma(s) = \int_0^\infty e^{-t} t^s dt/t,$$

the ordinary Γ–function.

Now assume $F = \mathbb{R}$ and let ρ be an irreducible representation of $W_{\mathbb{R}}$. If $\dim \rho = 2$, then $\rho = \operatorname{Ind}_{\mathbb{C}^*}^{W_{\mathbb{R}}} \theta$, where θ is a character of $\mathbb{C}^*$. Then

$$L(s, \rho) = L(s, \theta)$$

with $L(s, \theta)$ defined before.

On the other hand, if $\dim \rho = 1$, then using $W_{\mathbb{R}}/[W_{\mathbb{R}}, W_{\mathbb{R}}] = \mathbb{R}^*$, ρ becomes a character of $\mathbb{R}^*$. Here $[W_{\mathbb{R}}, W_{\mathbb{R}}]$ is the commutator group of $W_{\mathbb{R}}$. Then

$$\rho(x) = (x/|x|)^\varepsilon |x|^t$$

with $t \in \mathbb{C}$ and $\varepsilon = 0$ or 1. The corresponding Artin L–function is

$$L(s, \rho) = \pi^{-1/2(\varepsilon+t+s)}\Gamma(1/2(\varepsilon + t + s)).$$

Let F be a local field. The *parametrization problem* for $GL_n(F)$ requires that every irreducible admissible representation π of $GL_n(F)$ be parametrized by a continuous representation φ of W_F', i.e. a continuous homomorphism of W_F' into

$GL_n(\mathbb{C}) = {}^L GL_n$. If r is a representation of $GL_n(\mathbb{C})$, then the corresponding Langlands L–function $L(s,\pi,r)$ will be defined to equal the Artin L–function $L(s,r\cdot\varphi)$. Similarly

$$\varepsilon(s,\pi,r,\psi) = \varepsilon(s,r\cdot\varphi,\psi)$$

with the one on the right, the Artin root number.

The parametrization problem is quite general and for any reductive group and in the generality of all reductive groups it is only solved when F is archimedean, i.e. $F = \mathbb{R}$ or $\mathbb{C}$. This is due to Langlands [L4]. For non–archimedean F, the results are quite fragmented and we will make no effort to explain them.

In practice $L(s,\pi,r)$ and $\varepsilon(s,\pi,r,\psi)$ are defined by other means and one task is to show

$$L(s,\pi,r) = L(s,r\cdot\varphi)$$

and

$$\varepsilon(s,\pi,r,\psi) = \varepsilon(s,r,\varphi\cdot\psi).$$

In the case in hand, i.e. for $\mathbf{G} = GL_m \times GL_n$ and $r = \rho_m \otimes \rho_n$, both

$$L(s,\pi_v \times \pi'_v) = L(s,\pi_v \otimes \pi'_v,\ \rho_m \otimes \rho_n)$$

and

$$\varepsilon(s,\pi_v \times \pi'_v,\ \psi_v) = \varepsilon(s,\pi_v \otimes \pi'_v,\ \rho_m \otimes \rho_n,\ \psi_v)$$

are defined using the two different methods, Rankin–Selberg in [JPSS] and Langlands–Shahidi in [Sh5] and were proved to be equal to each other in [Sh6], when F_v is non–archimedean. When $F_v = \mathbb{R},\mathbb{C}$, the factors defined from Langlands–Shahidi methods and in the full generality of the method are proved to be those of Artin in [Sh7], in particular so are $L(s,\pi_v \times \pi'_v)$ and $\varepsilon(s,\pi_v \times \pi'_v,\ \psi_v)$. The method of Rankin–Selberg for archimedean fields is addressed in [JS3]. Here ψ_v is a non–trivial additive character of F_v.

Now, let $\pi = \otimes_v \pi_v$ and $\pi' = \otimes_v \pi'_v$ be cuspidal representations of $GL_m(\mathbb{A})$ and $GL_n(\mathbb{A})$ as in the beginning of the section. Fix a non–trivial additive character $\psi = \otimes_v \psi_v$ of $F\backslash\mathbb{A}$. Let

$$L(s,\pi \times \pi') = \prod_v L(s,\pi_v \times \pi'_v)$$

and

$$\varepsilon(s, \pi \times \pi') = \prod_v \varepsilon(s, \pi_v \times \pi_v, \psi_v),$$

where the factors are as in the previous paragraph. Let $\widetilde{\pi}$ and $\widetilde{\pi}'$ be contragredients of π and π', i.e. their $\prod\limits_v GL_m(O_v)$–finite (respectively $\prod\limits_v GL_n(O_v)$–finite) duals.

Theorem 7.3 [Sh5]. *The functional equation*

$$L(s,\ \pi \times \pi') = \varepsilon(s,\ \pi \times \pi')L(1-s,\ \widetilde{\pi} \times \widetilde{\pi}')$$

is valid.

The fact that a non–trivial additive character is necessary to define local root numbers which will have no appearance in the global root number is one of the mysteries of the subject.

Sketch of the proof. In the setting of the proof of Theorem 7.2, one considers a non–constant Fourier coefficient of the Eisenstein series built on $\widetilde{\pi} \otimes \pi'$. Up to a finite number of factors, in fact attached to those in S, the Fourier coefficient is equal to $L_S(1+s,\ \pi \times \pi')^{-1}$. One then applies the functional equation satisfied by the Eisenstein series and combines with lots of local work. We refer to [Sh1,Sh8,CS] for details.

The method of the proof is quite general and applies to a large number of L–functions. We will discuss this later.

Just as it is the case with classical Dirichlet series, one can also prove a non–vanishing for these L–functions on the line $\mathrm{Re}(s) = 1$ and, as its celebrated classical application to the infiniteness of number of primes in an arithmetic progression has shown, many applications are expected. Some has already been established and are truly significant such as classification of automorphic forms for $GL_n(\mathbb{A})$ proved in [JS2]. Many more will be proved in future, especially since the results are true in a much more general setting than $L_S(s, \pi \times \pi')$.

Theorem 7.4. $L(s,\ \pi \times \pi') \neq 0$ *for* $Re(s) = 1$.

Sketch of the proof. The Fourier expansion discussed in the proof of Theorem 7.3 is equal (up to a finite number of local Whittaker functions) to $L_S(1+it,\ \pi \times \pi')^{-1}$.

It must also be holomorphic for all $t \in \mathbb{R}$, i.e. on the unitary axis by the general theory of Eisenstein series [L5,MW1]. One must then apply some deep local theory at archimedean places, due to Casselman–Wallach, to show that local Whittaker functions are not identically zero. (This was a serious problem about the time when the theory was being developed). Details are given in [Sh8]. The theorem follows.

Next, one must address the question of poles. The first result is due to Jacquet–Shalika [JS2].

Theorem 7.5 (Jacquet–Shalika [JS2]). *The only poles for $L_S(s,\ \pi \times \pi')$ on the half plane $Re(s) \geq 1$ are on the line $Re(s) = 1$. They happen if and only if $m = n$ and $\pi' \cong \widetilde{\pi} \otimes |\det(\quad)|^t,\ t \in \mathbb{C}$, i.e. when π' is an unramified twist of $\widetilde{\pi}$. The partial L–function $L_S(s,\ \pi \times \widetilde{\pi})$ has a simple pole at $s = 1$.*

Sketch of the proof. This is accomplished using an integral representation which itself involves an Eisenstein series if $m = n$. Again good amount of local result for $\mathrm{Re}(s) > 1$ is necessary. For more details see [JS1,JS2].

The continuation to the whole complex plane is due to Moeglin and Waldspurger [MW2]. It is an ingenious application of results of Jacquet–Piatetski–Shapiro–Shalika and Shahidi.

Theorem 7.6 (Moeglin–Waldspurger [MW2]). *The complete L–function $L(s,\ \pi \times \pi')$ is entire unless $m = n$ and $\pi' \cong \widetilde{\pi} \otimes |\det(\quad)|^t,\ t \in \mathbb{C}$. The only poles of $L(s,\ \pi \times \widetilde{\pi})$ are at $s = 0$ and 1 and are both simple.*

Sketch of the proof. Using Theorems 7.1–7.5, it is enough to prove $L(s, \widetilde{\pi} \times \pi')$ has no poles for $0 < \mathrm{Re}(s) \leq 1/2$. Assume s with $0 < \mathrm{Re}(s) \leq 1/2$ is a pole of $L(\cdot, \widetilde{\pi} \times \pi')$ of order r.

Let us first say a few words about Eisenstein series in this case which is the main tool in this whole program. Let Q and Q' denote standard parabolic subgroups of $G = GL_{m+n}$ whose Levi subgroups are $M = GL_m \times GL_n$ and $M' = GL_n \times GL_m$, respectively. Let U and U' be their unipotent radicals. Set $\underline{\pi} = \pi \otimes \pi'$ and $\underline{s} = (s, s'),\ s, s' \in \mathbb{C}$ and denote by $I(\underline{\pi})$ and $I(\underline{\pi}, \underline{s})$, representations induced from

$\underline{\pi}$ and $\underline{\pi}[\underline{s}] = (\pi \otimes |\det(\ \ )|^{s}) \otimes (\pi' \otimes |\det(\ \ )|^{s'})$. If $\varphi \in I(\underline{\pi})$, define $\varphi(\underline{s}) \in I(\underline{\pi}, \underline{s})$ by

$$\varphi(\underline{s})(mug) = |\det m|^{\underline{s}} \varphi(mg).$$

Let

$$G_Q = \{f : Q(F)U(\mathbb{A})\backslash G(\mathbb{A}) \longrightarrow \mathbb{C}\}$$

and define $i : I(\underline{\pi}, \underline{s}) \longrightarrow G_Q$ by

$$(if)(g) = f(g)(1,1).$$

The Eisenstein series attached to φ and $\underline{s}$ is defined by

$$E(\varphi, \underline{s}, g) = \sum_{\gamma \in Q(F)\backslash G(F)} [i\varphi(\underline{s})](\gamma g).$$

Consider

$$E(\varphi, g) = \lim_{s' \longrightarrow s} (s' - s)^r E(\varphi, \underline{s}', g)$$

with $\underline{s}' = (s'/2,\ -s'/2)$, where $E(\varphi, \underline{s}', g)$ is the Eisenstein series built on representation $(\pi \otimes |\det(\ \ )|^{s'/2}) \otimes (\pi' \otimes |\det(\ \ )|^{-s'/2})$ of the Levi subgroup $M(\mathbb{A}) = GL_m(\mathbb{A}) \times GL_n(\mathbb{A})$ of $GL_{m+n}(\mathbb{A})$ discussed above. Since the poles of $E(\varphi, \underline{s}', g)$ in this interval are those of $L(s', \widetilde{\pi} \times \pi')$, $E(\varphi, g)$ is not identically zero. Let U be the unipotent radical of the parabolic subgroup $Q = MU$ of GL_{m+n} on which the Eisenstein series is built. Next they show that if $E^U(\varphi, g)$ is the limit

$$E^U(\varphi, g) = \lim_{s' \longrightarrow s} (s' - s)^r E^U(\varphi, \underline{s}', g),$$

where $E^U(\varphi, \underline{s}', g)$ is the constant term of $E(\varphi, \underline{s}', g)$ along Q, then

$$E(\varphi, g) = \sum_{\gamma \in (P \cap Q)(F)\backslash P(F)} E^U(\varphi, \gamma g), \tag{7.6.1}$$

where P is the standard parabolic subgroup of GL_{m+n} whose Levi is isomorphic to $GL_{m+n-1} \times GL_1$.

Using density of $GL_{m+n}(F)B(\mathbb{A})$ in $GL_{m+n}(\mathbb{A})$, where B is the Borel subgroup of upper triangular elements in GL_{m+n}, one concludes that if $E^U(\varphi,\ \ ) \neq 0$, then there exist $\gamma \in P(F)$ and $b \in B(\mathbb{A})$ such that $E^U(\varphi, \gamma b) \neq 0$.

Now, up to a constant, $E^U(\varphi, g)$ is given by a normalized intertwining operator $N(\underline{s})\varphi(\underline{s})(g)$. Moreover

$$\{N(\underline{s})\varphi(\underline{s})|\varphi \in I(\underline{\pi})\} = \otimes_v J_v,$$

where each J_v is a space of functions on $GL_{m+n}(F_v)$ with values in the space of $\pi'_v \otimes \pi_v$. More precisely, J_v is a subspace of the space of representation induced from a twist of $\pi'_v \otimes \pi_v$. They then show that if $v \notin S$, then there exists a function $j_v \in J_v$, $j_v \neq 0$, such that $j_v|Q(F_v)P(F_v) \equiv 0$.

Choose φ such that $N(\underline{s})\varphi(\underline{s})$ is not identically zero and its v–th component is j_v for some $v \notin S$. Then $E^U(\varphi, g) \neq 0$ and consequently $E(\varphi, \quad)$ is not identically zero. But then by (7.6.1) and the density argument below it $E^U(\varphi, \gamma b) \neq 0$ for some $\gamma \in P(F)$ and $b \in B(\mathbb{A})$. On the other hand $N(\underline{s})\varphi(\underline{s})(\gamma b) = 0$ since $N(\underline{s})\varphi(\underline{s})$ has a j_v–component. This contradicts $E^U(\varphi, \gamma b) \neq 0$.

The proof is quite remarkable and mixes the results of the two different methods to prove the full holomorphy which is usually hardest step. It has the potential of being applied to other cases which have remained unresolved.

§8. Applications of properties of $L(s, \pi \times \pi')$

8.1 Classification of automorphic forms for GL_n. Let $G = GL_r(\mathbb{A})$. Fix a cusp form $\sigma = \sigma_1 \otimes \ldots \otimes \sigma_u$ of $M = GL_{r_1}(\mathbb{A}) \times \ldots \times GL_{r_u}(\mathbb{A})$, not necessarily unitary, $r_1 + \ldots + r_u = r$. Here M is considered as the standard Levi subgroup of the standard parabolic subgroup $P = MN$ of G. Let $\xi = \otimes_v \xi_v$ be the representation induced from $\sigma \otimes \mathbf{1}$ of $MN = P$. Similarly let Q be another standard parabolic subgroup of G with a cuspidal representation τ of the standard Levi subgroup of Q. Denote by η the representation of G induced from $\tau \otimes \mathbf{1}$. Write $\eta = \otimes_v \eta_v$. The following theorem is proved in [JS2].

Theorem 8.1 (Jacquet–Shalika [JS2]). *Let S be a finite set of places of F such that for $v \notin S$, σ_v and τ_v are both unramified. Then ξ_v and η_v have the same unramified components if and only if (σ, P) and (τ, Q) are conjugate, i.e. up to a permutation they are equivalent.*

When $M = G$, this is the *strong Multiplicity one Theorem* alluded to before.

The connection with classification is explained in [JS2]. Its significance and connection with global parametrization problem which requires introduction of certain Tanakian categories is well explained in an important article of Langlands [L6].

Proof of 8.1 is a clever application of Theorems 7.1, 7.4, and 7.5.

8.2 Base change for GL_n. Let E/F be a cyclic extension of F. Set $r = [E : F]$ and let $\mathrm{Gal}(E/F) = \langle\tau\rangle$. There is a group denoted by $\mathrm{Res}_{E/F}GL_n$. It is defined by

$$\mathrm{Res}_{E/F}GL_n = \overbrace{GL_n(E) \times \ldots \times GL_n(E)}^{r} \rtimes \mathrm{Gal}(E/F)$$

in which $\tau(g_1, \ldots, g_r) = (\tau(g_r),\ \tau(g_1), \tau(g_2), \ldots, \tau(g_{r-1}))$. Thus $\tau(g_1, \ldots, g_r) = (g_1, \ldots, g_r)$ implies $g_1 = \tau(g_r),\ g_2 = \tau(g_1),\ g_3 = \tau(g_2), \ldots, g_r = \tau(g_{r-1})$, and therefore $(g_1, \ldots, g_r) = (g_1, \tau(g_1), \tau^2(g_1), \ldots, \tau^{r-2}(g_1), \tau^{r-1}(g_1))$. Consequently $(\mathrm{Res}_{E/F}GL_n)(F) = GL_n(E)$. Its L–group is $(GL_n(\mathbb{C}) \times \ldots \times GL_n(\mathbb{C})) \rtimes \mathrm{Gal}(E/F)$ in which

$$\tau(g) = \begin{pmatrix} & & 1 \\ & & \\ 1 & & \end{pmatrix} {}^t g^{-1} \begin{pmatrix} & & 1 \\ & & \\ 1 & & \end{pmatrix}$$

for every $g \in GL_n(\mathbb{C})$.

Let

$$\theta : GL_n(\mathbb{C}) \times \mathrm{Gal}(E/F) \longrightarrow \overbrace{GL_n(\mathbb{C}) \times \ldots \times GL_n(\mathbb{C})}^{r} \rtimes \mathrm{Gal}(E/F)$$

be defined by

$$\theta(g, \tau) = (g, \ldots, g) \rtimes \tau.$$

Then it is a major work of Arthur and Clozel [ACl] that θ_* from the space of automorphic representations of $GL_n(\mathbb{A}_F)$ into those of $GL_n(\mathbb{A}_E)$ exists, generalizing the fundamental work of Langlands [L7] from $n = 2$, in which he proves Artin's conjecture for a large number of irreducible two dimensional continuous representations of $\mathrm{Gal}(\overline{F}/F)$. In fact for all those whose image in $PGL_2(\mathbb{C})$ are solvable.

Proof makes a fundamental use of trace formula and its twisted version. Again L–functions $L(s, \pi \times \pi')$ play a role both through the normalizing factors for intertwining operators, as well as by means of their analytic properties and their defining role in the theory of local base change.

8.3 Residual spectrum for GL_n. Langlands theory of Eisenstein series [L5] gives a recipe to how to determine non–cuspidal discrete spectrum of reductive groups. One must determine residues of Eisenstein series coming from different conjugacy classes of parabolic subgroups. This is a hard and difficult process. One must determine residues of constant terms which are given, up to a finite number of factors, as ratios of partial L–functions. For $GL_r(\mathbb{A})$ and the parabolic subgroup $Q = MU$ in which $M \cong GL_m \times GL_n,\ r = m + n$, with inducing cuspidal representations $\pi \otimes \pi'$ on $M(\mathbb{A})$, the ratio is

$$L_S(s, \widetilde{\pi} \times \pi')/L_S(1 + s, \widetilde{\pi} \times \pi'),$$

and clearly analytic properties of $L_S(s, \widetilde{\pi} \times \pi')$ play an important role in determination of these residues. This whole project has been masterfully executed by Moeglin and Waldspurger in [MW2], completely classifying the residual spectrum for $GL_n(\mathbb{A})$.

8.4 Ramanujan–Petersson's and Selberg's Conjectures. Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$. The Laplace operator

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

acts on $L^2(\Gamma \backslash \mathfrak{h})$. Let $\lambda_1(\Gamma)$ be the smallest non–zero eigenvalue of Δ. The corresponding eigenfunctions are called Maass forms.

Selberg's Conjecture [Se]. $\lambda_1(\Gamma) \geq 1/4 = 0.25$**.**

Recall that

$$\lambda_1(\Gamma) = \frac{1}{4}(1 - s^2)$$

in which s must be either pure imaginary or $-1 \leq s \leq 1$. If the Maass form $\pi = \otimes_p \pi_p$ has π_∞ as its archimedean component, then $\pi_\infty = I(\mu_{1,\infty}, \mu_{2,\infty})$, where $\mu_{1,\infty} = |\ \ |^{s/2}$ and $\mu_{2,\infty} = |\ \ |^{-s/2}$ with $|\ \ | = |\ \ |_{\mathbb{R}}$. Thus

$$\frac{1}{4}(1 - s^2) = \frac{1}{4}$$

implies $s = 0$ and consequently π_∞ is tempered and therefore Selberg's conjecture may be considered as the archimedean version of Ramanujan–Petersson's conjecture.

Selberg himself proved $\lambda_1(\Gamma) \geq 3/16$ or $-1/4 \leq s/2 \leq 1/4$ which was sharpened to $-1/4 < s/2 < 1/4$ by Gelbart and Jacquet [GJ]. We again refer to [I], Section 11.3, for further discussion of this subject and its history.

While $R-P$ conjecture was making progress at all the finite places, its archimedean version, the Selberg's conjecture remained resilient to any progress. Except for some partial results by Iwaniec, no real progress took place until the recent paper of Luo, Rudnick, and Sarnak [LRSa] in which they proved

Theorem 8.2 (Luo–Rudnick–Sarnak) [LRSa]**.** $\lambda_1(\Gamma) \geq 21/100$ *or equivalently*

$$-1/5 \leq s/2 \leq 1/5.$$

The proof relies in a fundamental way on the theory of Rankin–Selberg L–functions for $GL_m \times GL_n$ when $m = n = 3$. Their proof also gives the estimate

$$p^{-1/5} \leq |\alpha_p| \ \& \ |\beta_p| \leq p^{1/5}$$

discussed earlier. Using GL_2–theory alone Iwaniec has also obtained the slightly weaker bound $\lambda_1(\Gamma) \geq 10/49$.

Sketch of the proof. Let $\Pi = \otimes_p \prod_p$ be the Gelbart–Jacquet lift (§6) of $\pi = \otimes_p \pi_p$. Fix a place $p = p_0$, finite or infinite. We need the following theorem whose proof is in [LRSa] and becomes unavailable as we replace $\mathbb{Q}$ by a number field.

Theorem 8.3 [LRSa]**.** *Given s with $Re(s) > 4/5$, there exists a character $\chi = \otimes_p \chi_p$ of $\mathbb{Q}^* \backslash \mathbb{A}_{\mathbb{Q}}^*$ with $\chi_{p_0} = 1$, such that*

$$L_{\{p_0\}}(s,\ (\Pi \otimes \chi) \times \Pi) = \prod_{p \neq p_0} L(s,\ (\Pi_p \otimes \chi_p) \times \Pi_p)$$

is non–zero.

Proof. See [LRSa].

Fix a $s = s_0$ with $4/5 <\ \mathrm{Re}(s_0) < 1$ and choose χ as in Theorem 8.3. By Theorem 7.6

$$L(s_0,\ (\Pi \otimes \chi) \times \Pi)$$

is holomorphic. But

$$L(s,\ (\Pi\otimes\chi)\times\Pi)=L(s,\ \Pi_{p_0}\times\Pi_{p_0})L_{\{p_0\}}(s,\ (\Pi\otimes\chi)\times\Pi).$$

Since $L_{\{p_0\}}(s,\ (\Pi\otimes\chi)\times\Pi)$ is non–zero at $s=s_0$, $L(s,\ \Pi_{p_0}\times\Pi_{p_0})$ must be holomorphic at $s=s_0$. Let us first assume $p_0<\infty$ and π_{p_0} (and thus Π_{p_0}) is unramified. Recall that if

$$t_{p_0}=\begin{pmatrix}\alpha_{p_0} & 0\\ 0 & \beta_{p_0}\end{pmatrix}$$

is attached to π_{p_0}, then

$$\mathrm{Ad}^2(t_{p_0})=\begin{pmatrix}\alpha_{p_0}\beta_{p_0}^{-1} & & 0\\ & 1 & \\ 0 & & \alpha_{\rho_0}^{-1}\beta_{p_0}\end{pmatrix}$$

represents the conjugacy class of $GL_3(\mathbb{C})$ attached to Π_{p_0} (cf. §6). For a Maass form $\beta_{p_0}^{-1}=\alpha_{p_0}$ and

$$(1-p^{-s}\alpha_{p_0}^4)$$

divides $L(s,\ \Pi_{p_0}\times\Pi_{p_0})^{-1}$. Consequently $(1-p^{-s}\alpha_{p_0}^4)$ must have no zeros for any s with $4/5<\mathrm{Re}(s)<1$. Thus

$$p^{-(4/5)+\varepsilon}|\alpha_{p_0}|^4<1$$

for every $\varepsilon>0$. Taking the limit as $\varepsilon\longrightarrow 0$ one gets

$$|\alpha_{p_0}|\le p_0^{1/5}.$$

Now suppose $p_0=\infty$ and $\pi_\infty=I(|\ \ |^{s'/2},\ |\ \ |^{-s'/2})$, $s'\in\mathbb{R}$. In fact this is always the case for a Maass form. If

$$\varphi: W_{\mathbb{R}}\longrightarrow GL_2(\mathbb{C})$$

parametrizes π_∞, then

$$\varphi(x)=\begin{pmatrix}|x|^{s'/2} & 0\\ 0 & |x|^{-s'/2}\end{pmatrix}$$

for $\forall\ x\in\mathbb{R}^*\subset W_{\mathbb{R}}$. Moreover $\Pi_\infty=I(|\ \ |^{s'},\ \mathbf{1},|\ \ |^{-s'})$ and $L(s,\Pi_\infty\times\Pi_\infty)$ will have $\Gamma(s-2s')$ as a factor. By the same argument $\Gamma(s-2s')$ must be holomorphic for $\mathrm{Re}(s)>4/5$, or

$$4/5+\varepsilon-2s'>0$$

for every $\varepsilon > 0$. Then $s' \leq 2/5$ and $s'/2 \leq 1/5$.

The estimate 5/28 which for a classical Maass form $\pi = \otimes_p \pi_p$ is the best presently proved, uses a different L–function. Again one takes $\Pi = \mathrm{Ad}^2_*(\pi)$ which is cuspidal unless π is monomial in which case $R - P$ is valid anyway, π being parametrized by a representation of the global Weil group. One then uses the following theorem of Bump and Ginzburg.

Theorem 8.4 (Bump–Ginzburg [BGi]). *Let $\Pi = \otimes_v \Pi_v$ be a cuspidal automorphic representation of $GL_3(\mathbb{A})$ and choose a finite set S of places of F such that Π_v is unramified for every $v \notin S$. Then*

$$L_S(s, \Pi, Sym^2(\rho_3)) = \prod_{v \notin S} L(s, \Pi_v, Sym^2(\rho_3))$$

is entire with possibly only a pole at $s = 1$.

The 5/28 estimate of Bump–Duke–Hoffstein–Iwaniec [BDHI]

$$p^{-5/28} \leq |\alpha_p| \leq p^{5/28}$$

follows from Theorem 8.4, the functional equation satisfied by this L–function, and an ingenious idea of Duke and Iwaniec [DI].

Although it may be possible to extend the 5/28 estimate to quadratic extensions of $\mathbb{Q}$ (private communications with Sarnak), still the best estimate for an arbitrary number field is (cf. [Sh1, Sh2])

$$q_v^{-1/5} < |\alpha_v| \ \& \ |\beta_v| < q_v^{1/5}.$$

This follows from the theory developed in [L2, Sh1, Sh8] which is now being called the Langlands–Shahidi method which we will discuss next.

§9. The Method

In 1967 Langlands gave a series of lectures at Yale which were published in a book titled "Euler Products" by Yale University Press [L2]. The book is the origin for his notion of an L-group. This was later pursued by the author and is now being called the Langlands–Shahidi method and is one of the two methods (the Rankin–Selberg being the other one) to study L–functions directly. Efforts are now being made to generalize the method to non–generic representations [FGol]. For the purpose of these lectures, let us consider only the case of split groups.

9.1 ***L*–groups.** Let F be a number field and denote by $\mathbf{G}$ a Zariski–closed subgroup of $GL_n(F_a)$, for some n, where F_a denotes an algebraic closure of F. This means that $\mathbf{G}$ is given as common zeros of a finite number of polynomials (in several variables). If the polynomials are with coefficients in F, then $\mathbf{G}$ will be said to be *defined over* F. We refer to §1 for adelization of $\mathbf{G}$. *Radical* of $\mathbf{G}$ is the largest connected solvable normal subgroup of $\mathbf{G}$. $\mathbf{G}$ will be called *reductive*, if its radical consists of only semisimple (diagonalizable) elements. The radical is then equal to the connected component of the center of $\mathbf{G}$. A maximal torus of $\mathbf{G}$ is one which is maximal among its closed connected abelian subgroups whose elements are semisimple. Let $\mathbf{T}$ be one. Denote by $X^*(\mathbf{T})$ the group of rational characters of $\mathbf{T}$ into GL_1. Let $X_*(\mathbf{T})$ be the group of all rational homomorphism of GL_1 into $\mathbf{T}$. Let $X^*(\mathbf{T})_F$ be the group of F–rational character of $\mathbf{T}$ into GL_1, i.e. those defined over F which means that they send $\mathbf{T}(F)$ into $GL_1(F)$. We will say $\mathbf{T}$ is split if

$$X^*(\mathbf{T}) = X^*(\mathbf{T})_F.$$

We shall say $\mathbf{G}$ is split over F if it has maximal tori which are split over F.

For simplicity from now on we shall assume $\mathbf{G}$ is split. Let $\mathbf{T}$ be a maximal torus and fix a Borel subgroup $\mathbf{B}$, i.e. a maximal connected solvable subgroup, containing $\mathbf{T}$. Let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$, i.e. the subgroup of all its unipotent elements. Then $\mathbf{B} = \mathbf{T} \cdot \mathbf{U}$. The torus $\mathbf{T}$ acts by adjoint action on $\mathbf{U}$, defining root characters of $\mathbf{T}$ in $\mathbf{U}$. Let $\Sigma^* \subset X^*(\mathbf{T})$ be their subset. One can identify the coroots Σ_* of $\mathbf{T}$ with a subset of $X_*(\mathbf{T})$. Observe that the choice of $\mathbf{B}$, already determines a set of positive roots.

Theorem 9.1. *The root datum* $(X^*(\mathbf{T}), \Sigma^*, X_*(\mathbf{T}), \Sigma_*)$ *determines the class of* $\mathbf{G}$ *uniquely.*

Definition 9.2. The complex group whose root datum is dual of that of $\mathbf{G}$, i.e. is $(X_*(\mathbf{T}), \Sigma_*, X^*(\mathbf{T}), \Sigma^*)$ is called the L–group of $\mathbf{G}$ and is denoted by ${}^LG$.

Remark. All of these are valid if F is local.

Examples. One can quickly notice that the roots of $\mathbf{G}$ are coroots of ${}^LG$. If α and β are roots of $\mathbf{G}$ and $\alpha^\vee$ and $\beta^\vee$, their corresponding coroots, i.e. so that $\langle \alpha, \alpha^\vee \rangle = \langle \beta, \beta^\vee \rangle = 2$. Then

$$\langle \alpha, \beta \rangle = \langle \beta^\vee, \alpha^\vee \rangle.$$

The Cartan matrix of $\mathbf{G}$ is

$$C = (\langle \alpha_i, \alpha_j \rangle),$$

where α_i and α_j denote simple roots of $\mathbf{T}$ in $\mathbf{U}$. Let ${}^LC$ be the Cartan matrix of ${}^LG$, then

$$\begin{aligned} {}^LC &= (\langle \alpha_i^\vee, \alpha_j^\vee \rangle) \\ &= (\langle \alpha_j, \alpha_i \rangle) \\ &= {}^tC, \end{aligned}$$

from which one concludes that $\mathbf{G}$ and ${}^LG$ have transposed Cartan matrices.

Suppose $\mathbf{G}$ is semisimple, i.e. it is reductive and its center is finite, thus its radical is trivial, then $\mathbf{G}$ is called simply connected, if

$$X_*(\mathbf{T}) = \mathbb{Z}\text{–span of } \Sigma_*$$

and adjoint, if

$$X^*(\mathbf{T}) = \mathbb{Z}\text{–span of } \Sigma^*.$$

Thus $\mathbf{G}$ is simply connected if and only if ${}^LG$ is adjoint and conversely.

A simple way of characterizing an adjoint group is that it has no center. Examples are PGL_n, PSp_{2n}, PSO_{2n}, SO_{2n+1}, exceptional groups of type G_2, F_4, and E_8. On the other hand SL_n, Sp_{2n}, $Spin_n$, G_2, F_4, and E_8 are all simply connected. The groups SO_{2n} are neither simply connected nor adjoint. The following table for

Chevalley groups is then clear:

$\mathbf{G}$	${}^LG$
SL_n	$PGL_n(\mathbb{C})$
Sp_{2n}	$SO_{2n+1}(\mathbb{C})$
SO_{2n+1}	$Sp_{2n}(\mathbb{C})$
SO_{2n}	$SO_{2n}(\mathbb{C})$
$Spin_{2n}$	$PSO_{2n}(\mathbb{C})$
$Spin_{2n+1}$	$PSp_{2n}(\mathbb{C})$
G_2	$G_2(\mathbb{C})$
F_4	$F_4(\mathbb{C})$
simply connected E_6 & E_7	adjoint $E_6(\mathbb{C})$ and $E_7(\mathbb{C})$, resp.
E_8	$E_8(\mathbb{C})$

Examples of non–semisimple reductive groups are GL_n and similitude groups, such as GSp_{2n}. Their L–groups require a little more work to figure out. We refer to [L1] and [Bo] for details. The result is ${}^LGL_n = GL_n(\mathbb{C})$, as mentioned before, and ${}^LGSp_4 = GSp_4(\mathbb{C})$. The L–group of $GSp_{2n},\ n > 2$, is again a reductive group whose derived group is $Sp_{2n}(\mathbb{C})$. But one needs to build in the extra dimension in the maximal torus of GSp_{2n} as opposed to Sp_{2n} (cf. [Bo]).

9.2 Parabolic subgroups. Fix a maximal split torus $\mathbf{T}$ for $\mathbf{G}$ which lies in our fixed Borel subgroup $\mathbf{B} = \mathbf{TU}$. Let Δ be the subset of simple roots for roots of $\mathbf{T}$ in $\mathbf{U}$, spanning $\Sigma_+ \subset \Sigma^*$ the set of (positive) roots in $\mathbf{U}$. Given a set $\theta \subset \Delta$, let $\mathbf{A} = \mathbf{A}_\theta = (\bigcap_{\alpha\in\theta} \ker\alpha)^0$, the connected component of $\bigcap_{\alpha\in\theta} \ker\alpha$. Let $\mathbf{M} = \mathbf{M}_\theta$ be the centralizer of $\mathbf{A}$ in $\mathbf{G}$. Clearly $\mathbf{T} \subset \mathbf{M}$. Let $\mathbf{N} = \mathbf{N}_\theta$ be the subgroup of $\mathbf{U}$ spanned by roots in $\Sigma_+ \backslash \langle\theta\rangle$, where $\langle\theta\rangle$ is the subset of Σ_+ spanned by the roots in θ. Then $\mathbf{P} = \mathbf{P}_\theta = \mathbf{MN}$ is called the *standard parabolic* subgroup of $\mathbf{G}$ attached to θ with a Levi subgroup $\mathbf{M}$ and unipotent radical $\mathbf{N}$. Being standard means that $\mathbf{N} \subset \mathbf{U}$ for a fixed $\mathbf{B}$. The decomposition $\mathbf{P} = \mathbf{MN}$ is called a *Levi decomposition.* A *parabolic subgroup* of $\mathbf{G}$ is then a conjugate of a $\mathbf{P}_\theta$ for some θ. Every parabolic subgroup is its own normalizer. $\mathbf{P}$ is called maximal if $\mathbf{A}/Z(\mathbf{G})$ is one–dimensional.

If F is a local non–archimedean field, then

$$G = \mathbf{G}(F) = \mathbf{P}(F)\mathbf{G}(O),$$

where O is the ring of integers of F. The subgroup $K = \mathbf{G}(O)$ is a maximal compact subgroup of G and $G = PK$ is called an *Iwasawa decomposition* of G.

Let $\mathbf{P} = \mathbf{MN}$ be a parabolic subgroup of $\mathbf{G}$ standard with respect to a Borel subgroup $\mathbf{B} = \mathbf{TU}$ of $\mathbf{G}$. Let ${}^LG$ be the L–group of $\mathbf{G}$, and denote by ${}^LM$ the Levi subgroup of ${}^LG$ defined by $\theta^\vee$ the coroots of $\mathbf{T}$ defined by roots in θ. It should now be clear that ${}^LG$ has a maximal torus ${}^LT$ such that

$$X^*({}^LT) = X_*(\mathbf{T})$$

and

$$X_*({}^LT) = X^*(\mathbf{T}).$$

The subset of roots of ${}^LT$ is Σ_*, the coroots of $\mathbf{T}$. Fix a Borel subgroup ${}^LB$ of ${}^LG$ such that ${}^LT \subset {}^LB$. Let ${}^LU$ be its unipotent radical. Then the set of coroots of $\mathbf{T}$ defined by positive roots of $\mathbf{T}$ can be identified with the roots of ${}^LT$ in ${}^LU$. Let ${}^LN$ be the subgroup of ${}^LU$ spanned by positive roots in ${}^LU$ which are not spanned by $\theta^\vee$. Then the subgroup ${}^LP = {}^LM{}^LN$ of ${}^LG$ is called the L–group of $\mathbf{P}$. It is standard with respect to ${}^LB$.

Example. The group Sp_4 has two maximal parabolic subgroups. To explain, let α and β be the short and the long simple roots of Sp_4. Then $\mathbf{M}_1 = \mathbf{M}_{\langle\alpha\rangle} \cong GL_2$ and $\mathbf{M}_2 = \mathbf{M}_{\langle\beta\rangle} \cong SL_2 \times GL_1$. Moreover ${}^LSp_4 = SO_5(\mathbb{C})$. Since

$$\langle\alpha,\beta\rangle = \langle\beta^\vee,\alpha^\vee\rangle,$$

then $\alpha^\vee$ and $\beta^\vee$ are the long and the short simple roots of $SO_5(\mathbb{C})$, respectively. It can be easily checked that $M_{\langle\alpha^\vee\rangle} \subseteq SO_5(\mathbb{C})$ is in fact $GL_2(\mathbb{C})$ which is ${}^LM_1$, the L–group of the reductive group $\mathbf{M}_1$. On the other hand

$$\begin{aligned} {}^LM_2 &= PGL_2(\mathbb{C}) \times GL_1(\mathbb{C}) \\ &\cong SO_3(\mathbb{C}) \times GL_1(\mathbb{C}) \end{aligned}$$

for the reductive group $\mathbf{M}_2$. But this is exactly $\mathbf{M}_{\langle\beta^\vee\rangle} \subset SO_5(\mathbb{C})$.

Now assume F is either local or global. Let $\mathbf{G}$ be as before a split reductive group over F. Fix $\mathbf{B}, \mathbf{T}, \mathbf{U}, \ldots$ as before. Let $\mathbf{P} = \mathbf{MN}$ be a standard parabolic subgroup of $\mathbf{G}, \mathbf{N} \subseteq \mathbf{U}$, and let $\mathbf{A}$ be its split component, i.e. the connected component of the center of $\mathbf{M}$. Let $X(\mathbf{M})_F$ and $X(\mathbf{A})_F$ be the groups of F–rational character of $\mathbf{M}$ and $\mathbf{A}$, respectively. Set

$$\begin{aligned}\mathfrak{a} &= \mathrm{Hom}(X(\mathbf{M})_F, \mathbb{R}) \\ &= \mathrm{Hom}(X(\mathbf{A})_F, \mathbb{R}),\end{aligned}$$

since $X(\mathbf{M})_F$ is of finite index in $X(\mathbf{A})_F$. The $\mathbb{R}$–vector space $\mathfrak{a}$ is called the real Lie algebra of $\mathbf{A}$. Then its $\mathbb{R}$–dual is

$$\begin{aligned}\mathfrak{a}^* &= X(\mathbf{M})_F \otimes_{\mathbb{Z}} \mathbb{R} \\ &= X(\mathbf{A})_F \otimes_{\mathbb{Z}} \mathbb{R}.\end{aligned}$$

In fact, if $\chi \otimes r \in \mathfrak{a}^*$ and $\lambda \in \mathfrak{a}$, then

$$\langle \lambda, \chi \otimes r \rangle = \lambda(\chi) r.$$

Let $\mathfrak{a}^*_{\mathbb{C}} = \mathfrak{a}^* \otimes_{\mathbb{R}} \mathbb{C}$.

If F is global, then for every place v of F, $X(\mathbf{M})_F \hookrightarrow X(\mathbf{M})_{F_v}$ will then induce an embedding $\mathfrak{a}_v = \mathrm{Hom}(X(\mathbf{M})_{F_v}, \mathbb{R}) \hookrightarrow \mathfrak{a} = \mathrm{Hom}(X(\mathbf{M})_F, \mathbb{R})$.

Suppose first that F is local. Define the homomorphism

$$H_M : M = \mathbf{M}(F) \longrightarrow \mathfrak{a}$$

by

$$\exp\langle \chi, H_M(m) \rangle = |\chi(m)|$$

where $\chi \in X(\mathbf{M})_F$ and $|\ \ |$ is that of F. Extend H_M to H_P on all of $G = MNK$ by extending it trivially on NK.

Next suppose F is global. Let v be a place of F with F_v the completion of F at v. The embedding $X(\mathbf{M})_F \hookrightarrow X(\mathbf{M})_{F_v}$ induces a map $\mathfrak{a}_v \to \mathfrak{a}$, where $\mathfrak{a}_v = \mathrm{Hom}(X(\mathbf{M})_{F_v}, \mathbb{R})$. We again define a homomorphism

$$H_M : M = \mathbf{M}(\mathbb{A}) \longrightarrow \mathfrak{a}$$

by

$$\exp\langle \chi, H_M(m)\rangle = \prod_v |\chi(m_v)|_v,$$

where $\chi \in X(\mathbf{M})_F$ and $m = (m_v) \in \mathbf{M}(\mathbb{A})$. We extend H_M to H_P on all of $G = \mathbf{G}(\mathbb{A})$ in the same manner. Observe that

$$\exp\langle \chi, H_M(m)\rangle = \prod_v \exp\langle \chi, H_{M_v}(m_v)\rangle.$$

Since $m_v \in \mathbf{M}(O_v)$ for almost all $v < \infty$, the product is a finite product.

When F is non–archimedean, it is natural and is usual to use q rather than exp to define H_M locally, i.e.

$$q^{\langle \chi, H_M(m)\rangle} = |\chi(m)| \quad (\forall \chi \in X(\mathbf{M})_F).$$

Now assume F is global and define

$$H_M : M = \mathbf{M}(\mathbb{A}) \longrightarrow \mathfrak{a}$$

as before by

$$\exp\langle \chi, H_M(m)\rangle = \prod_v |\chi(m_v)|_v \quad (\forall \chi \in X(\mathbf{M})_F)$$

and therefore

$$\exp\langle \chi, H_M(m_v)\rangle = |\chi(m_v)|_v.$$

If H'_{M_v} is the one defined by

$$q_v^{\langle \chi, H'_{M_v}(m)\rangle} = |\chi(m)|_v \quad (m \in M_v),$$

then

$$H_M(m_v) = (\log q_v) H'_{M_v}(m_v),$$

where $q_v = \exp$ if F is archimedean. Therefore if we define

$$H'_M : M \longrightarrow \mathfrak{a}$$

by

$$H'_M | M_v = (\log q_v)^{-1} H_M | M_v,$$

then

$$\exp\langle \chi, H_m(m)\rangle = \prod_{v=\infty} \exp\langle \chi, H'_M(m_v)\rangle \cdot \prod_{v<\infty} q_v^{\langle \chi, H'_M(m_v)\rangle}.$$

Suppose $\chi = 2\rho_{\mathbf{P}}$, sum of positive roots in $\mathbf{N}$, and $t \in \mathbb{C}$. Then

$$\begin{aligned}\exp\langle t\rho_{\mathbf{P}}, H_M(m)\rangle &= \exp\frac{t}{2}\langle 2\rho_{\mathbf{P}}, H_M(m)\rangle \\ &= \prod_v |(2\rho_{\mathbf{P}})(m_v)|_v^{t/2} \\ &= \prod_v \delta_{P_v}(m_v)^t,\end{aligned}$$

where δ_{P_v} is the modular character of P_v. In fact $2\rho_{\mathbf{P}}$ is the determinant of the adjoint action of $\mathbf{M}$ on the Lie algebra of $\mathbf{N}$, a rational character of $\mathbf{M}$. The modulus character δ_{P_v}, the ratio of the right and the left invariant measures on N_v, when evaluated at $m_v \in M_v$ is just $|(2\rho_{\mathbf{P}})(m_v)|_v^{1/2}$.

9.3. Cusp forms and Eisenstein series. We will be interested in cusp forms on $M = \mathbf{M}(\mathbb{A})$. Let us fix a character χ of $A = \mathbf{A}(\mathbb{A})$, where $\mathbf{A}$ is the split component of $\mathbf{M}$. Let $L_0^2(A\mathbf{M}(F)\backslash M, \chi)$ be the Hilbert space of all L^2–functions φ on $A\mathbf{M}(F)\backslash M$ which transform under A according to χ and moreover

$$\int_{\mathbf{N_M}(F)\backslash\mathbf{N_M}(\mathbb{A})} \varphi(nm)dn = 0 \quad (\forall' m \in M),$$

where $\mathbf{N_M}$ is the unipotent radical of a parabolic subgroup $\mathbf{P_M}$ of $\mathbf{M}$ for all such parabolic subgroups. The space $L_0^2(A\mathbf{M}(F)\backslash M, \chi)$ is called the space of *χ–cusp forms* on M and M acts on it by right translations. It decomposes discretely to a direct sum of irreducible subspaces each called a *cuspidal representation* of M. Let π be one such representation. Then again $\pi = \otimes_v \pi_v$ with almost all π_v, $v < \infty$, having a vector fixed by $\mathbf{M}_v(O_v)$, where $\mathbf{M}_v$ denotes $\mathbf{M}$ considered as a group over F_v. The representation π_v is then uniquely determined by a unique semisimple conjugacy class in ${}^L M_v$ where ${}^L M_v = {}^L M$ since $\mathbf{M}$ is split over F.

Choose a function $\varphi \in L_0^2(A\mathbf{M}(F)\backslash M, \chi)$, belonging to π such that

$$V_\varphi = \langle \pi(k)\varphi | k \in K_M = \prod_v K_{M_v}\rangle$$

is finite dimensional. Such functions exist by density arguments and are dense. There are natural ways of extending φ to a function Φ on $G = \mathbf{G}(\mathbb{A})$, for example via a finite dimensional representation of K, a maximal compact subgroup of G with $K_M = K \cap \mathbf{M}(\mathbb{A})$, which contains the representation of K_M on V_φ upon restriction [L5, MW1, HC1].

Given $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, let

$$\Phi_\nu(g) = \Phi(g) \exp\langle \nu + \rho_{\mathbf{P}}, H_P(g) \rangle$$

where $\rho_{\mathbf{P}}$ is half the sum of roots in $\mathbf{N}$, an element in $\mathfrak{a}^*$. Let

$$E(\nu, \varphi, g) = \sum_{\gamma \in \mathbf{P}(F) \backslash \mathbf{G}(F)} \Phi_\nu(\gamma g),$$

where $g \in G = \mathbf{G}(\mathbb{A})$, the *Eisenstein series* attached to ν and ϕ. It converges for ν in a cone inside the positive cone, i.e. the one defined by

$$\langle \nu, H_\alpha \rangle > 0$$

for every simple root α in $\mathbf{N}$. Here

$$H_\alpha = 2\alpha / (\alpha, \alpha)$$

is the corresponding coroot. It extends to a meromorphic function of ν on all of $\mathfrak{a}_{\mathbb{C}}^*$ (cf. [L5]).

Now assume $\mathbf{P}$ is maximal. Let s be a complex number. Denote by α the unique simple root in $\mathbf{N}$. Set

$$\widetilde{\alpha} = \langle \rho_{\mathbf{P}}, \alpha \rangle^{-1} \rho_{\mathbf{P}} \in \mathfrak{a}^*.$$

Then $s\widetilde{\alpha} \in \mathfrak{a}_{\mathbb{C}}^*$.

9.4 Constant term and intertwining operators. Recall that $\mathbf{A} \subset \mathbf{T}$, a maximal split torus of $\mathbf{G}$. Let W be the Weyl group of $\mathbf{T}$ in $\mathbf{G}$, i.e. the quotient of its normalizer by its centralizer. Assume $\mathbf{M} = \mathbf{M}_\theta,\ \theta \subset \Delta$ and $\theta \cup \{\alpha\} = \Delta$. There exists a unique element $\widetilde{w}_0 \in W$ such that $\widetilde{w}_0(\theta) \subset \Delta$ while $\widetilde{w}_0(\alpha) \in \Sigma_- = -\Sigma_+$. Fix a representative $w_0 \in K \cap \mathbf{G}(F)$ for $\widetilde{w}_0$.

For every place v of F, let

$$I(s\widetilde{\alpha}, \pi_v) = \operatorname*{Ind}_{M_v N_v \uparrow G_v} \pi_v \otimes q_v^{\langle s\widetilde{\alpha}, H_{P_v}(\quad) \rangle} \otimes \mathbf{1}.$$

More precisely, the space $V(s\widetilde{\alpha}, \pi_v)$ of $I(s\widetilde{\alpha}, \pi_v)$ is the vector space of all smooth functions f_v from $G_v = \mathbf{G}(F_v)$ into the space $\mathcal{H}(\pi_v)$ of π_v satisfying

$$f_v(mng) = \pi_v(m) q_v^{\langle s\widetilde{\alpha} + \rho_{\mathbf{P}}, H_{P_v}(m) \rangle} f_v(g),$$

for all $m \in M_v, n \in N_v$, and $g \in G_v$. We use exp instead of q_v if $v = \infty$. Finally given $f_v \in V(s\widetilde{\alpha}, \pi_v)$, define the corresponding *intertwining operator* by

$$A(s\widetilde{\alpha}, \pi_v, w_0) f_v(g) = \int_{N'_v} f_v(w_0^{-1} n g) dn,$$

for all $g \in G_v$. Here $\mathbf{N}'$ is the unipotent radical of the standard parabolic subgroup $\mathbf{P}'$ of $\mathbf{G}$ which has $\mathbf{M}' = \mathbf{M}_{\widetilde{w_0}(\theta)}$ as its Levi subgroup.

Let

$$M(s\widetilde{\alpha}, \pi) = \otimes_v A(s, \pi_v, w_0).$$

It acts on

$$I(s\widetilde{\alpha}, \pi) = \otimes_v I(s\widetilde{\alpha}, \pi_v),$$

which itself is a restricted tensor product, to be explained below, as follows. Given a function $f \in V(s\widetilde{\alpha}, \pi)$, there exists a finite set of places, including the archimedean ones, such that

$$f \in \otimes_{v \in S} V(s\widetilde{\alpha}, \pi_v) \otimes \otimes_{v \notin S} \{f_v^0\},$$

where each f_v^0, $v \notin S$, satisfies, $f_v^0(k_v) = 1$, $\forall k_v \in \mathbf{G}(O_v)$. Clearly for such $v \notin S$, π_v would have to be unramified. The operator $M(s\widetilde{\alpha}, \pi)$ then acts by linearity on each component of f. It is therefore important to explain how the resulting infinite product behaves. The operator $M(s\widetilde{\alpha}, \pi)$ is called, by abuse of notation in the maximal case, the constant term of $E(s\widetilde{\alpha}, -, -)$, attached to π. The poles of $M(s\widetilde{\alpha}, \pi)$ are exactly those of $E(s\widetilde{\alpha}, -, -)$.

9.5 ***L*–functions in the constant term.** The L–group ${}^L M$ of $\mathbf{M}$ acts on the Lie algebra ${}^L\mathfrak{n}$ of the complex Lie group ${}^L N$ by adjoint action r (conjugation). Given a positive integer i, let V_i be the subspace of ${}^L\mathfrak{n}$ generated by those dual roots $\beta^\vee$ for which $\langle \widetilde{\alpha}, \beta \rangle = i$. There is a positive integer m, such that $V_i = \phi$ for every $i > m$, while $V_i \neq \phi$ for every $1 \leq i \leq m$. Moreover each V_i is invariant under ${}^L M$. Let $r_i = r | V_i$.

Examples.

1) Let $\mathbf{G} = GL_{m+n}, \mathbf{M} = GL_m \times GL_n$, and $\mathbf{N} = M_{m \times n}$. Then ${}^L M = GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$ acts on $M_{m \times n}(\mathbb{C})$ by

$$r(g_1, g_2) X = g_1 X g_2^{-1},$$

$g_1 \in GL_m(\mathbb{C})$, $g_2 \in GL_n(\mathbb{C})$, and $X \in M_{m\times n}(\mathbb{C})$. The representation r is irreducible and $m = 1$. This is the case of Rankin–Selberg L–functions discussed in Section 7.

2) Let $\mathbf{G}$ be the exceptional group of type G_2. It is a rank 2 split group. Let α and β be its simple roots. We assume α is short and β is long. That simply means $\|\alpha\|^2 = 1$ and $\|\beta\|^2 = 3$. Moreover $(\alpha,\beta) = (\beta,\alpha) = -\dfrac{3}{2}$. Here $\|\alpha\|^2 = (\alpha,\alpha)$, where $(\ ,\ )$ is the standard inner product on $\mathbb{R}^2$. The Cartan matrix is

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

Assume $\theta = \{\beta\}$. Then $\mathbf{M}_\theta \cong GL_2$. (So is $\mathbf{M}_\theta$ if $\theta = \{\alpha\}$.) The Lie algebra ${}^L\mathfrak{n}$ is five dimensional. The L–group ${}^LM = {}^LM_\theta \cong GL_2(\mathbb{C})$ is the Levi subgroup generated by $\beta^\vee$ which is the short root of $G_2(\mathbb{C})$. The integer $m = 2, \dim V_1 = 4$, and dim $V_2 = 1$. Moreover $r_1 = Ad^3$, the adjoint cube discussed in Section 6, and $r_2 = \det$. Recall that

$$Ad^3(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}) = \mathrm{diag}(a^2b^{-1}, a, b, a^{-1}b^2),$$

$a, b \in \mathbb{C}^*$. This example was called by Langlands in [L2] "extremely striking."

One can quickly determine the subspaces V_1 and V_2. We will use α and β to also denote the short and the long simple roots of $G_2(\mathbb{C})$. Then ${}^L\mathfrak{n}$ is generated by root vectors of $\beta,\ \alpha+\beta,\ 2\alpha+\beta,\ 3\alpha+\beta$, and $3\alpha+2\beta$, since ${}^LM$ is generated by the short simple root α. The simple root in ${}^LN$ is β and the numbers i are coefficients of β in the above roots. Then V_1 is spanned by $\beta,\ \alpha+\beta,\ 2\alpha+\beta,\ 3\alpha+\beta$, while V_2 is spanned by $3\alpha+2\beta$. Their dimensions are obviously 4 and 1.

On the other hand if $\theta = \{\alpha\}$, then $m = 3$, dim $V_2 = 1$, while dim $V_1 = \mathrm{dim} V_3 = 2$.

A good part of progress on the symmetric cube L–functions for $GL_2(\mathbb{A})$ has come from this example (cf. [Sh3, Sh4, Sh9]).

Choose a finite set S of places of F such that π_v is unramified. Then π_v is uniquely determined by a semisimple conjugacy class A_v in ${}^LM_v = {}^LM$, $\mathbf{M}$ being split. For each complex analytic (finite dimensional) representation r of ${}^LM$, there exists a Langlands L–function $L(s,\pi_v,r)$ defined by

$$L(s,\pi_v,r)=\det(I-r(A_v)q_v^{-s})^{-1}.$$

The following theorem is due to Langlands [L2].

Theorem 9.1. *Fix $v\notin S$. Then there exists a choice of w such that:*

$$A(s\widetilde{\alpha},\pi_v,w)f_v^0=\prod_{i=1}^{m}L(is,\pi_v,\widetilde{r}_i)/L(1+is,\pi_v,\widetilde{r}_i)\widetilde{f}_v^0$$

where $\widetilde{f}_v^0\in V(w(s\widetilde{\alpha}),w(\pi_v))$ satisfies $\widetilde{f}_v^0(k)=1,\ \forall k\in\mathbf{G}(O_v)$.

We should remark that the functions f_v^0 and $\widetilde{f}_v^0$ are realized by complex valued functions that one obtains by identifying π_v on a constituent of an unramified principal series, i.e. one induced from an unramified character of $T\subset M$ and trivial on $U\cap M$ to M.

Corollary (Langlands)**.** *There exists a $s_0\in\mathbb{R}$ such that $M(s\widetilde{\alpha},\pi)$ is defined by an absolutely convergent product for $Re(s)>s_0$.*

We refer to [GPSR] for another important application of Theorem 9.1.

Finally let us point out that intertwining operators can be defined for any parabolic subgroup $\mathbf{P}_\theta=\mathbf{M}_\theta\mathbf{N}_\theta$ of $\mathbf{G}$, where $\mathbf{G}$ is any (split) group over a local field F. Again let $\widetilde{w}\in W$ be such that $\widetilde{w}(\theta)\subset\Delta$. Let $\mathbf{N}^-=\mathbf{N}_{-\theta}$ and

$$\mathbf{N}_{\widetilde{w}}=\mathbf{U}\cap w\mathbf{N}^-w^{-1}.$$

Given $\nu\in\mathfrak{a}_{\mathbb{C}}^*$, where $\mathfrak{a}$ is the real Lie algebra of $\mathbf{A}_\theta$, a representative w of $\widetilde{w}$, and an irreducible admissible representation σ of $M=\mathbf{M}_\theta(F)$, let

$$A(\nu,\sigma,w)f(g)=\int_{N_{\widetilde{w}}}f(w^{-1}ng)dn \qquad (g\in G),$$

for every $f\in V(\nu,\sigma)$, the space of the representation $I(\nu,\sigma)$ induced from

$$\sigma\otimes q^{\langle\nu,H_M(\quad)\rangle}\otimes\mathbf{1}.$$

It converges absolutely for ν in a cone inside the positive Weyl chamber.

There is a product formula giving $A(\nu,\sigma,w)$ as a product of operators for maximal parabolics as we defined using elements $\widetilde{w}_0$ before ([L5, Sh8]) and therefore one needs to study those defined earlier more closely as they are building blocks of the general ones.

9.6 Generic representations.

We will first assume F is a local field whose ring of integer is O. Let $\mathbf{G}$ be a split reductive group over F and fix $\mathbf{B}, \mathbf{T}$, and $\mathbf{U}$ as before. Let Δ be the set of simple roots in $\mathbf{U}$. Then

$$U/[U,U] \cong \prod_{\alpha\in\Delta} \mathbf{U}_\alpha(F),$$

where each $U_\alpha = \mathbf{U}_\alpha(F)$ is the one dimensional subgroup generated by

$$\exp(tX_\alpha) = \sum_{n=0}^{\infty} \frac{t^n X_\alpha^n}{n!}, \quad (t \in F)$$

where X_α is the corresponding root vector. Let ψ be a character of U. Then

$$\psi = \prod_\alpha \psi_\alpha$$

with each ψ_α a character of U_α. The character ψ is called *generic* or *non–degenerate* if each ψ_α is non–trivial. Having fixed $X_\alpha, \alpha \in \Delta$, then each ψ_α is a character of F which we still denote by ψ_α. A character of F is called *unramified* if O is the largest ideal on which the character is trivial. The character ψ is called *unramified* if and only if each ψ_α is unramified.

An irreducible admissible representation $(\sigma, \mathcal{H}(\sigma))$ of $G = \mathbf{G}(F)$ is called ψ–generic, where ψ is a non–degenerate character of U, if there exists a functional $\lambda \in \mathcal{H}(\sigma)'$, the full dual of $\mathcal{H}(\sigma)$, such that

$$\lambda(\sigma(u)v) = \psi(u)\lambda(v)$$

for all $u \in U$ and $v \in \mathcal{H}(\sigma)$. When F is archimedean one would require λ to be continuous with respect to semi–norm topology defined by elements of universal enveloping algebra of the Lie algebra of G, or more simply by right invariant differential operators on $\mathfrak{g}$. A non–zero such λ is called a *Whittaker functional.*

Theorem 9.2 (Shalika [S], Gelfand–Kazhdan)**.** *The dimension of the space of Whittaker functionals for σ is at most 1.*

Now suppose F is global. A character $\chi = \otimes_v \chi_v$ of $\mathbf{U}(F)\backslash\mathbf{U}(\mathbb{A})$ is generic if and only if each χ_v is generic. Let φ be a cusp form. Set:

$$W_\varphi(g) = \int_{\mathbf{U}(F)\backslash\mathbf{U}(\mathbb{A})} \varphi(ug)\overline{\chi(u)}du,$$

$g \in G = \mathbf{G}(\mathbb{A})$. φ is called globally χ–generic if there exists a $g \in G$ such that $W_\varphi(g) \neq 0$. The function W_φ is called the *Whittaker function* attached to φ. Finally if φ belongs to the space of a cuspidal representation $\pi = \otimes_v \pi_v$, then the vector space of all W_φ is called a *Whittaker model* for π.

Let F be local and assume σ is an irreducible admissible ψ–generic representation of G. Fix a Whittaker functional λ and define a function W_ν for each $v \in \mathcal{H}(\sigma)$ by

$$W_v(g) = \lambda(\sigma(g)v).$$

The space of all W_v's is called a *Whittaker model* for σ.

9.7. Local coefficients. Let F be local and σ be an irreducible admissible ψ_M generic representation of $M = \mathbf{M}(F)$, F–points of a Levi subgroup for the parabolic subgroup $\mathbf{P} = \mathbf{MN}$ with $\mathbf{N} \subset \mathbf{U}$. As usual $\mathbf{B} = \mathbf{TU}$ is a Borel subgroup of $\mathbf{G}$. The character ψ_M is defined by means of restriction from a character ψ of $U = \mathbf{U}(F)$ which is assumed to be generic. For the sake of simplicity from now on we shall assume all the ψ_α's appearing in ψ, $\alpha \in \Delta$, are equal.

Fix $\nu \in \mathfrak{a}^*_{\mathbb{C}}$ and consider the representation $I(\nu, \sigma)$ of G induced from

$$\sigma \otimes q^{\langle \nu, H_M(\ \ ) \rangle} \otimes \mathbf{1}.$$

Let $\widetilde{w}_0 \in W$ be such that $\widetilde{w}_0(\theta) \subset \Delta$, $\mathbf{P} = \mathbf{P}_\theta$, while $\widetilde{w}_0(\alpha)$ is negative for every α in $\mathbf{N}$. Let $\mathbf{M}' = \mathbf{M}_{\widetilde{w}_0(\theta)}$ and $\mathbf{P}' = \mathbf{M}'\mathbf{N}'$, $\mathbf{P}' = \mathbf{P}_{\widetilde{w}_0(\theta)}$, $\mathbf{N}' = \mathbf{N}_{\widetilde{w}_0(\theta)} \subset \mathbf{U}$. Let λ_M be a ψ_M–Whittaker functional for the space $\mathcal{H}(\sigma)$ of σ. For each $f \in V(\nu, \sigma)$, define

$$\lambda_\psi(\nu, \sigma)(f) = \int_{N' = \mathbf{N}'(F)} \overline{\psi(n')} \lambda_M(f(w_0^{-1} n'))dn',$$

where w_0 is a representative for $\widetilde{w}_0$. Clearly

$$\lambda(I(u)f) = \psi(u)\lambda(f),$$

where $\lambda = \lambda_\psi(\nu, \sigma)$ and $I = I(\nu, \sigma)$, i.e. λ is a ψ–generic Whittaker functional for $I(\nu, \sigma)$. It is easy to see $\lambda \neq 0$. Moreover, as a function of ν, $\lambda_\psi(\nu, \sigma)$ is holomorphic for all ν (cf. [CS]).

Now fix a $\widetilde{w} \in W$ and choose a representative w for $\widetilde{w}$. Let $A(\nu, \sigma, w)$ be the intertwining operator from $I(\nu, \sigma)$ into $I(w(\nu), w(\sigma))$. Denote by $\lambda_\psi(w(\nu), w(\sigma))$

the Whittaker functional defined as above for $I(w(\nu), w(\sigma))$. Then

$$\lambda_\psi(w(\nu), w(\sigma))A(\nu, \sigma, w)$$

is another non–zero (generically) Whittaker functional for $I(\nu, \sigma)$. There is a theorem of Rodier [Ro] which states that such functionals for the induced representation $I(\nu, \sigma)$ are unique up to scalars. Thus there exists a scalar $C_\psi(\nu, \sigma, w)$, called the *local coefficient* [Sh8] attached to ν, σ, and w such that

$$\lambda_\psi(\nu, \sigma) = C_\psi(\nu, \sigma, w)\lambda_\psi(w(\nu), w(\sigma))A(\nu, \sigma, w).$$

We shall now explain the most important property of local coefficients.

Assume F is global with $\mathbb{A}$ as its ring of adeles. Let $G = \mathbf{G}(A)$, $M = \mathbf{M}(\mathbb{A}),\ N, P,\ B, T, U$ be as before. We will assume P is maximal. Let ψ_M be a generic character of $\mathbf{U_M}(F)\backslash\mathbf{U_M}(\mathbb{A})$, where $\mathbf{U_M} = \mathbf{U} \cap \mathbf{M}$. Let φ be a globally ψ_M–generic cusp form on $M = \mathbf{M}(\mathbb{A})$, belonging to the space of the irreducible cuspidal representation $\pi = \otimes_v \pi_v$ of M. Write $\psi_M = \otimes_v \psi_{M,v}$. Let $s \in \mathbb{C}$ and if α is the unique simple root in $\mathbf{N}$, let $\widetilde{\alpha} = \langle \rho, \alpha \rangle^{-1}\rho$ be as in 9.3. Finally let $r_i,\ 1 \le i \le m$, be representations of ${}^L M$ as defined in 9.5.

Theorem 9.3 (Crude functional equation [Sh8]). *Let S be a finite set of places such that π_v and ψ_v are both unramified for every $v \notin S$. Then*

$$\prod_{i=1}^{m} L_S(is, \pi, r_i) = \prod_{v \in S} C_{\overline{\psi}_v}(s\widetilde{\alpha}, \widetilde{\pi}_v, w_0) \prod_{i=1}^{m} L_S(1 - is, \pi, \widetilde{r}_i).$$

Here $\psi = \otimes_v \psi_v$ is any generic character of $\mathbf{U}(F)\backslash U$ which restricts to ψ_M and L_S is as usual the product of corresponding local L–functions at all $v \notin S$.

Sketch of the proof. Let $E(s\widetilde{\alpha}, \Phi, g)$ be the Eisenstein series defined by extension Φ of φ as in 9.3. Consider the non–constant Fourier coefficient

$$E_\psi(s\widetilde{\alpha}, \Phi, g) = \int_{\mathbf{U}(F)\backslash U} E(s\widetilde{\alpha}, \Phi, ug)\overline{\psi(u)}du$$

of $E(s\widetilde{\alpha}, \Phi, -)$. Then it can be shown [Sh1, Sh8] that

$$E_\psi(s\widetilde{\alpha}, \Phi, e) = \prod_{v \in S} W_v(e_v) \prod_{i=1}^{m} L_S(1 + is, \pi, \widetilde{r}_i)^{-1},$$

where for $v \in S$

$$W_v(e_v) = \lambda_{\psi_v}(s\widetilde{\alpha}, \pi_v)(f_v)$$

with $f_v \in V(s\widetilde{\alpha}, \pi_v)$. Here we assume that the extension Φ of φ to $G = \mathbf{G}(\mathbb{A})$ corresponds to $f = \otimes_v f_v \in I(s\widetilde{\alpha}, \pi)$. The first important ingredient in the proof is the formula

$$\lambda_{\psi_v}(s\widetilde{\alpha}, \pi_v)(f_v^0) = \prod_{i=1}^{m} L(1 + is, \pi_v, \widetilde{r}_i)^{-1}$$

for every $v \notin S$ due to Casselman and Shalika [CS]. Here $f_v^0 \in I(s\widetilde{\alpha}, \pi_v)$ is as in Theorem 9.1, i.e. $f_v^0(k) = 1$ for all $k \in \mathbf{G}(O_v)$.

Next one needs to apply (cf. [Sh7, Sh8]) some deep technical results at the archimedean places due to Casselman [C1] and Wallach [Wa] to show that W_v's can be chosen so that $W_v(e_v) \neq 0$ if $v = \infty$. The crude functional equation is now a consequence of that of Eisenstein series $E(s\widetilde{\alpha}, \Phi, g)$, namely

$$E(s\widetilde{\alpha}, \Phi, g) = E(-s\widetilde{\alpha}, M(s)\Phi, g),$$

and the definition of local coefficients, using

$$M(s\widetilde{\alpha}, \pi) = \otimes_v A(s\widetilde{\alpha}, \pi_v, w_0).$$

In the next paragraph we shall state the main global results of the method.

9.8. Main global results. F is now a global field and $\mathbf{G}, \mathbf{P}, \mathbf{M}, \mathbf{N}, \mathbf{U}, \mathbf{T}, \mathbf{B}, \psi_M, \psi$, $\widetilde{w}_0, w_0, \ldots$ are as before, and $\mathbf{P}$ is assumed maximal. We fix a ψ–generic cuspidal representation $\pi = \otimes_v \pi_v$ of $M = \mathbf{M}(\mathbb{A})$ and we choose representations r_i as before. The set S is always a finite set of places of F such that π_v and ψ_v are both unramified for $v \notin S$.

Theorem 9.4 [Sh8]. *Fix $t \in \mathbb{R}$. Then*

$$\prod_{i=1}^{m} L_S(1 + it\sqrt{-1}, \pi, r_i) \neq 0.$$

Sketch of the Proof. We have

$$E_\psi(s\widetilde{\alpha}, \Phi, e) = \prod_{v \in S} W_v(e_v) \prod_{i=1}^{m} L_S(1 + is, \pi, \widetilde{r}_i)^{-1}. \tag{9.4.1}$$

By the general theory of Eisenstein series, $E(s\widetilde{\alpha}, -, -)$ is holomorphic if s is pure imaginary, i.e. $s = \sqrt{-1}t,\ t \in \mathbb{R}$. But E_ψ is obtained by integrating E over the compact set $\mathbf{U}(F)\backslash U$ which remains holomorphic on the imaginary axis. Using the archimedean results [C1, Wa, Sh7, Sh8] discussed in the proof of Theorem 9.3, each $W_v(e_v)$ can be made non–zero for a given $s \in \mathbb{C},\ v = \infty$. Their non–vanishing for $v < \infty$ is easily verified. The theorem now follows from (9.4.1).

We refer to Theorem 7.4 and [Sh4, Sh8] for important special cases of Theorem 9.4 and the application of Theorem 7.4 to the important classification theorem of Jacquet and Shalika [JS2] (Theorem 8.1 here).

Theorem 9.5 a) (cf. [L2, Sh1]). *Each $L_S(s, \pi, r_i)$ extends to a meromorphic function of s on $\mathbb{C}$.*

b) [Sh1] Assume $m = 1$ or $m = 2$ and $\dim r_2 = 1$. Then $L_S(s, \pi, r_1)$ extends to a meromorphic function of s on $\mathbb{C}$ with only a finite number of poles, if S is large enough to include all the conjugates of places over which π_v and ψ_v ramify.

We refer to [Sh1] for the proofs and the induction involved, as well as a list of examples where b) is valid.

Theorem 9.6. *At each place $v \notin S$ and for each i, the local L–function*

$$L(s, \pi_v, r_i) = \det(I - r_i(A_v)q_v^{-s})^{-1}$$

is holomorphic for $Re(s) \geq 1$. Here $A_v \in {}^L M$ is a semisimple element in the conjugacy class of ${}^L M$ attached to π_v.

This is proved in [Sh1] using an induction on m whose full version was later proved in [Sh5], together with;

Lemma 9.7. *For $Re(s) \geq 1,\ L_S(s, \pi, \widetilde{r}_i)$ and the quotient*

$$L_S(s, \pi, \widetilde{r}_i)/L_S(1 + s, \pi, \widetilde{r}_i)$$

both have only a finite number of poles and zeros, $1 \leq i \leq m$.

Sketch of the proof. Assume that for $Re(s) \geq 1$ and each $i,\ 2 \leq i \leq m$, the corresponding quotient has a finite number of poles and zeros. Then by finiteness of poles of $M(s\widetilde{\alpha},\pi)$ for $Re(s) \geq 0$, the relation

$$M(s\widetilde{\alpha},\pi) = \otimes_v A(s\widetilde{\alpha},\pi_v,w_0),$$

and Theorem 9.1, the quotient for $i=1$ has a finite number of poles for $Re(s) \geq 1$. Now using this line of absolute convergence for each $L_S(s,\pi,\widetilde{r}_i)$, one concludes that each $L_S(s,\pi,\widetilde{r}_i),\ 1 \leq i \leq m$, has a finite number of poles for $Re(s) \geq 1$.

On the other hand by (9.4.1)

$$\prod_{i=1}^{m} L_S(1+is,\pi,\widetilde{r}_i)$$

has a finite number of zeros for $Re(s) \geq 0$. Applying the finiteness of poles for each $L_S(s,\pi,\widetilde{r}_i)$ for $Re(s) \geq 1$ and $2 \leq i \leq m$, we conclude that $L_S(s,\pi,\widetilde{r}_1)$ has also a finite number of zeros for $Re(s) \geq 1$. Consequently the quotient for $i=1$ has a finite number of poles and zeros for $Re(s) \geq 1$, completing the induction.

Sketch of the proof of Theorem 9.6. We enlarge S to include v and use

$$M(s\widetilde{\alpha},\pi)(\otimes_u f_u) = \otimes_{u\in S} A(s\widetilde{\alpha},\pi_u,w_0)f_u \otimes \prod_{i=1}^{m} L_S(is,\pi,\widetilde{r}_i)/L_S(1+is,\pi,\widetilde{r}_i)\widetilde{f}_S^0$$

for $f = \otimes_u f_u$ for which $f_u = f_u^0$ for $u \notin S$ and

$$\widetilde{f}_S^0 = \otimes_{u \notin S} \widetilde{f}_u^0$$

with f_u^0 and $\widetilde{f}_u^0$ as in Theorem 9.1.

Now suppose $A(s,\pi_v,w_0)$ has a pole for some s with $Re(s) \geq 1$. For each $u \in S,\ u \neq v$, choose f_u such that

$$A(s\widetilde{\alpha},\pi_u,w_0)f_u(w_0) \neq 0.$$

The operator $A(s,\pi_v,w_0)$ having values which are rational functions of q_v^{-s} will then have infinitely many poles parallel to the imaginary axis. By Lemma 9.7 so must be $M(s\widetilde{\alpha},\pi)$, a contradiction. Consequently $A(s,\pi_v,w_0)$ is holomorphic for $Re(s) \geq 1$. This then implies

$$\prod_{i=1}^{m} L(is,\pi_v,\widetilde{r}_i)/L(1+is,\pi_v,\widetilde{r}_i)$$

is holomorphic for $Re(s) \geq 1$.

Assume each local quotient is holomorphic for $Re(s) \geq 1$ when $2 \leq i \leq m$. Then using $Re(s)$ large enough so that local L–functions are holomorphic there, one concludes that each $L(s, \pi_v, \widetilde{r}_i)$, $2 \leq i \leq m$, is holomorphic for $Re(s) \geq 1$. Consequently each local quotient for $2 \leq i \leq m$ is also non–zero for $Re(s) \geq 1$, implying the holomorphy of the quotient for $i = 1$. We again use induction and the holomorphy of $L(s, \pi_v, \widetilde{r}_1)$ for $Re(s)$ large, to conclude it for $Re(s) \geq 1$. The induction is now complete and the Theorem 9.6 follows.

One important corollary to this which gives the best estimate presently proved for Hecke eigenvalues of cusp forms over an arbitrary number field can be formulated as:

Corollary 1 [Sh1, Sh2]**.** *Let $\pi = \otimes_v \pi_v$ be an irreducible cuspidal representation of $GL_2(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of a number field F. For each v where π_v is unramified, let the corresponding conjugacy class in $GL_2(\mathbb{C})$ be given by*

$$t_v = \begin{pmatrix} \alpha_v & 0 \\ 0 & \beta_v \end{pmatrix}.$$

Then

$$q_v^{-1/5} < |\alpha_v| \text{ and } |\beta_v| < q_v^{1/5}.$$

Proof in the adjoint case. Let us give a proof when π has trivial central character. The general case requires a little more knowledge of exceptional groups. We refer to [Sh2] for detail of the general case. The basic idea is the same.

Let $\mathbf{G}$ be a split group of type F_4. Its Dynkin diagram is

$$\begin{array}{ccccccc} \circ & \text{———} & \circ & \text{———} & \circ & \text{———} & \circ \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 \end{array}$$

with α_1 and α_2 long and α_3 and α_4 short. Let Π be the Gelbart–Jacquet lift of π. Since Ramanujan–Petersson is valid for monomial cusp forms, we may assume Π is a *cuspidal* representation of $PGL_3(\mathbb{A})$. Let $\mathbf{M} = \mathbf{M}_\theta$, where $\theta = \{\alpha_1, \alpha_2, \alpha_4\}$. There is an exact sequence

$$0 \to \mathbf{A} \to \mathbf{M} \to PGL_3 \times PGL_2 \to 0,$$

leading to a surjection

$$M \to PGL_3(\mathbb{A}) \times PGL_2(\mathbb{A}) \longrightarrow 0$$

by the lemma in page 36 of [L2]. The cuspidal representation $\Pi \times \pi$ of $PGL_3(\mathbb{A}) \times PGL_2(\mathbb{A})$ then defines a cuspidal representation ρ of M. Let

$$t_v = \begin{pmatrix} \alpha_v & 0 \\ 0 & \alpha_v^{-1} \end{pmatrix} \in SL_2(\mathbb{C}) = {}^L PGL_2$$

represent the conjugacy class attached to π_v. Then among the factors dividing $L(s,\rho,r_1)^{-1}$ is

$$(1-\alpha_v^5 q_v^{-s})(1-\alpha_v^{-5} q_v^{-s}).$$

By Theorem 9.6 this must be non–zero for $Re(s) \geq 1$. This implies

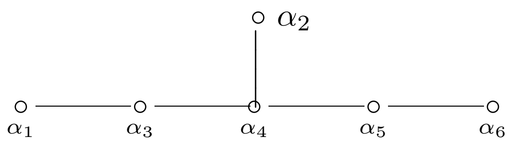
$$|\alpha_v^5| q_v^{-1} < 1$$

as well as

$$|\alpha_v^{-5}| q_v^{-1} < 1,$$

proving the Corollary.

The corollary can also be proved by letting $\mathbf{G}$ be a group of type E_6



and $\mathbf{M} = \mathbf{M}_\theta$ with $\theta = \{\alpha_1, \alpha_3, \alpha_2, \alpha_5, \alpha_6\}$, and the cuspidal representation $\Pi \times \pi \times \Pi$ of $PGL_3(\mathbb{A}) \times PGL_2(\mathbb{A}) \times PGL_3(\mathbb{A})$.

Corollary 2 [Sh1]**.** *Every L–function $L_S(s,\pi,r_i)$ is absolutely convergent for $Re(s) > 2$.*

The fact that π is globally generic is crucial. In fact one expects that such forms satisfy Ramanujan–Petersson's conjecture to the effect that each local component π_v of π is tempered, i.e. their matrix coefficients are in $L^{2+\varepsilon}(M_v)$ for each $\varepsilon > 0$.

Local factors and functional equations. Let F be local and $\mathbf{G}$ a split group over F. We fix $\mathbf{B}, \mathbf{T}, \mathbf{U}, \mathbf{P}, \mathbf{M}, \mathbf{N}$, ..., as before. We always assume $\mathbf{N} \subset \mathbf{U}$ and $\mathbf{T} \subset \mathbf{M}$. Moreover assume $\mathbf{P}$ is maximal and let α be the unique simple root in

N. Define $\widetilde{\alpha} = \langle \rho, \alpha \rangle^{-1}\rho$ as before. Let ψ be a generic character of $U = \mathbf{U}(F)$ extending ψ_M, one of $U_{\mathbf{M}} = (\mathbf{U} \cap \mathbf{M})(F)$. We assume ψ is defined by a character of F, denoted the same way. Let $r_1, , \ldots , r_m$ be representations of ${}^LM$ on ${}^L\mathfrak{n}$ as before. Let $s \in \mathbb{C}$.

Theorem 9.8. *Let σ be an irreducible admissible ψ_M–generic representation of M. Then for each i, $1 \leq i \leq m$, there exists a root number $\varepsilon(s, \sigma, r_i, \psi)$ which is a monomial in q^{-s} if F is non–archimedean, and an L–function $L(s, \sigma, r_i)$, whose inverse is a polynomial in q^{-s} with value 1 at zero, again if F is non–archimedean, with the following properties.*

1) If F is archimedean or σ has an Iwahori fixed vector (any component of an unramified principal series), let $\varphi' : W'_F \longrightarrow {}^LM$ be the homomorphism of the Deligne–Weil group W'_F parametrizing σ. Then

$$\varepsilon(s, \sigma, r_i, \psi) = \varepsilon(s, r_i \cdot \varphi', \psi)$$

and

$$L(s, \sigma, r_i) = L(s, r_i \cdot \varphi'),$$

where the factors on the right are the corresponding Artin root numbers and L–functions.

2) For each i, $1 \leq i \leq m$,

$$C_\psi(s\widetilde{\alpha}, \sigma, w_0) = \prod_{i=1}^{m} \varepsilon(is, \sigma, \widetilde{r}_i, \overline{\psi}) L(1 - is, \sigma, r_i)/L(is, \sigma, \widetilde{r}_i).$$

3) Inductive property (cf. part 3, Theorem 3.5 and equations (7.10) and (7.11) of [Sh5]).

4) Functional equations. Let K be a number field and $\widetilde{\mathbf{G}}$ a split group over K. Let $\widetilde{\mathbf{P}} = \widetilde{\mathbf{M}}\widetilde{\mathbf{N}}$, $\widetilde{\mathbf{N}} \subset \widetilde{\mathbf{U}}$, be a maximal parabolic subgroup of $\widetilde{\mathbf{G}}$. Fix a generic character $\widetilde{\psi} = \otimes_v \psi_v$ of $\widetilde{U} = \widetilde{\mathbf{U}}(\mathbb{A}_K)$, trivial on $\widetilde{\mathbf{U}}(K)$, defined by one of $K \backslash \mathbb{A}_K$, still denoted the same way. Let $\widetilde{\psi}_{\widetilde{M}} = \widetilde{\psi} | \widetilde{M} \cap \widetilde{U}$. Let $\pi = \otimes_v \pi_v$ be a $\widetilde{\psi}_{\widetilde{M}}$–generic cuspidal representation of $\widetilde{M} = \widetilde{\mathbf{M}}(\mathbb{A}_K)$. Finally let $r_1, \ldots , r_m$ be representations of ${}^L\widetilde{M}$ or ${}^L\widetilde{\mathfrak{n}}$ as before. For each i, $1 \leq i \leq m$, set

$$L(s, \pi, r_i) = \prod_v L(s, \pi_v, r_i)$$

and

$$\varepsilon(s,\pi,r_i) = \prod_v \varepsilon(s,\pi_v,r_i,\psi_v),$$

where the first product is a finite product and the second one converges absolutely for $Re(s)$ large (say bigger than 2). Then:

$$L(s,\pi,r_i) = \varepsilon(s,\pi,r_i)L(1-s,\pi,\widetilde{r}_i)$$

for each i, $1 \leq i \leq m$.

Moreover, let

$$\gamma(s,\sigma,r_i,\psi) = \varepsilon(s,\sigma,r_i,\varphi)L(1-s,\sigma,\widetilde{r}_i)/L(s,\sigma,r_i). \tag{9.8.1}$$

Then properties 1, 3, and 4, determine complex functions $\gamma(s,\sigma,r_i,\psi)$, $1 \leq i \leq m$, uniquely.

Several comments are in order:

1. When F is archimedean one may take W_F instead of W'_F since W_F suffices to parametrize all the irreducible admissible representations of real groups [L4]. We refer to [T] and pages 286–288 of [Sh5] for W'_F.

2. The γ–functions $\gamma(s,\sigma,r_i,\psi)$ are defined first using local coefficients inductively. When σ is tempered and F is non–archimedean, the L–function $L(s,\sigma,r_i)$ is defined as the inverse of the normalized numerator of $\gamma(s,\sigma,r_i,\psi)$, a polynomial in q^{-s} whose constant term is 1. The root number is then defined by means of (9.8.1). To define root numbers and L–functions for any irreducible admissible generic σ, one uses analytic continuation of the ones defined by tempered σ into the positive Weyl chamber and Langlands classification, exactly as in the archimedean one. It is then clear that uniqueness of $\gamma(s,\sigma,r_i,\varphi)$ leads to the same thing for root number and L–functions.

3. Finally, there is one important property of these L–functions that can be formulated as following conjecture, since it is not yet proved in general.

Conjecture 9.9. *Assume σ is tempered. Then each $L(s,\sigma,r_i)$ is holomorphic for $Re(s) > 0$.*

This is conjecture 7.1 of [Sh5] which is true if $F = \mathbb{R}$ or $\mathbb{C}$. Its validity is proved in following cases when F is p–adic:

1) $m = 1,\ \sigma$ is generic tempered;

2) $m = 2$ and $\dim r_2 = 1,\ \sigma$ is generic tempered;

3) σ is generic supercuspidal, i.e. its matrix coefficients are compact modulo the center of the group, and finally

4) $\mathbf{G}$ is of classical type, but σ is any generic tempered representation of M. This includes all the classical groups.

Cases 1), 2), and 3) are proved in [Sh5], while 4) which is more involved was just recently settled [CSh]. In the next section we will study certain applications of Conjecture 9.9.

§10. Some local applications

Throughout this section F is local (of characteristic zero), i.e. either $F = \mathbb{R}$ or $\mathbb{C}$ or a finite extension of $\mathbb{Q}_p$ for some prime p. $\mathbf{G}$ will be a split reductive group over F and $\mathbf{B}, \mathbf{T}, \mathbf{U}, \mathbf{P}, \mathbf{M}, \mathbf{N}, \psi, \psi_M$, ... are all as before.

10.1 Normalization of intertwining operators. Let $\mathbf{P} = \mathbf{P}_\theta,\ \theta \subset \Delta,\ \widetilde{w} \in W$. For simplicity of formulation assume $\widetilde{w}(\theta) = \theta$. Let $\mathbf{N}_{\widetilde{w}} = \mathbf{U} \cap \widetilde{w}\mathbf{N}^- \widetilde{w}^{-1}$, where $\mathbf{N}^- = \mathbf{N}_{-\theta}$. Let ${}^L\mathfrak{n}_{\widetilde{w}}$ be the Lie algebra of ${}^LN_{\widetilde{w}}$. The L–group ${}^LM$ of $\mathbf{M}$ acts on ${}^L\mathfrak{n}_{\widetilde{w}}$ by adjoint action. Let $r_{\widetilde{w}}$ be the representation of ${}^LM$ on ${}^L\mathfrak{n}_{\widetilde{w}}$. Use $\widetilde{r}_{\widetilde{w}}$ to denote its contragredient.

Let σ be an irreducible unitary ψ–generic representation of M. Fix $\nu \in \mathfrak{a}^*_{\mathbb{C}}$ and define $A(\nu, \sigma, w)$, the intertwining operator attached to ν, σ, and w as in §9.5, where w is a representative for $\widetilde{w}$. We recall for completeness that

$$A(\nu, \sigma, w) f(g) = \int_{N_{\widetilde{w}}} f(w^{-1} n g) dn \quad (f \in I(\nu, \sigma)).$$

We set

$$A(\sigma, w) = A(0, \sigma, w)$$

if the right hand side is well defined.

Using inductive property 3) of Theorem 9.8, one can define a root number $\varepsilon(s,\sigma,r_{\widetilde{w}},\psi)$ and an L–function $L(s,\sigma,r_{\widetilde{w}})$ for $r_{\widetilde{w}}$ as in Page 312 of [Sh5]. The following normalization of intertwining operators was first conjectured by Langlands in [L5]. Let

$$\mathcal{A}(\sigma,w)=\varepsilon(0,\sigma,\widetilde{r}_{\widetilde{w}},\psi)L(1,\sigma,\widetilde{r}_{\widetilde{w}})L(0,\sigma,\widetilde{r}_{\widetilde{w}})^{-1}A(\sigma,w),$$

where the right hand side is determined as a limit.

The normalized operator $\mathcal{A}(\sigma,w)$ is supposed to satisfy a list of properties set forth in Theorem 2.1 of [A3], a larger list than what was originally demanded by Langlands in [L5]. When $F=\mathbb{R}$ on $\mathbb{C}$, they are all verified by Arthur in [A3]. Theorem 7.9 of [Sh5] verifies them when F is p–adic and σ is generic, except for condition R_7 of Theorem 2.1 of [A3]. Condition R_7, which was not originally demanded in [L5], follows immediately from Conjecture 9.9. For the sake of completeness, let us state Theorem 7.9 of [Sh5] here as:

Theorem 10.1 (Langlands' Conjecture)**.** *The normalized operator $\mathcal{A}(\sigma,w)$ satisfies:*

a) $\mathcal{A}(\sigma,w_1w_2)=\mathcal{A}(w_2(\sigma),w_1)\mathcal{A}(\sigma,w_2)$ and

b) $\mathcal{A}(\sigma,w)^*=\mathcal{A}(w(\sigma),w^{-1})$, where $\mathcal{A}(\sigma,w)^*$ is the adjoint of $\mathcal{A}(\sigma,w)$, i.e. $\mathcal{A}(\sigma,w)$ is unitary.

In general, i.e. when $\widetilde{w}(\theta)$ is not necessarily equal to θ, the representation $r_{\widetilde{w}}$ is the adjoint action of ${}^LM$ on the Lie algebra of the L–group of $\overline{N}_{\widetilde{w}}^{-}=(w^{-1}\mathbf{N}_{\widetilde{w}}w)^{-}$, the unipotent group opposed to $w^{-1}\mathbf{N}_{\widetilde{w}}w$. Observe that if $\widetilde{w}(\theta)=\theta$, then $\overline{\mathbf{N}}_{\widetilde{w}}^{-}=\mathbf{N}_{\widetilde{w}}$.

We refer to Section 9 of [Sh5] for how one expects to extend the normalization to representations which are not necessarily generic.

10.2. Irreducibility of standard modules for generic representations. Let $\mathfrak{a}$ be the real Lie algebra of the split component of $\mathbf{M}$. Take $\nu\in\mathfrak{a}_{\mathbb{C}}^*$. For each simple root α in $\mathbf{N}$, realized as a root of $\mathbf{T}$, let $H_\alpha\in\mathfrak{t}$ be the corresponding coroot. Consider ν as as element of $\mathfrak{t}_{\mathbb{C}}^*$. We will say ν is in positive Weyl chamber of $\mathbf{A}$ if

$$Re\langle\nu,H_\alpha\rangle>0$$

for every simple root α in $\mathbf{N}$.

Let σ be a tempered representation of M and consider $I(\nu,\sigma)$. If ν is in the positive Weyl chamber, then $I(\nu,\sigma)$ is called a *standard module.* It has a unique quotient $J(\nu,\sigma)$, called the Langlands quotient of $I(\nu,\sigma)$ (cf. [L4, BoWa, Si2]). Every irreducible admissible representation of G is of the form $J(\nu,\sigma)$ for some ν and σ. The choice of ν and σ are unique up to conjugation.

When $F=\mathbb{R}$ or $\mathbb{C}$, it was proved by Vogan [V], that if $J(\nu,\sigma)$ is generic, then $I(\nu,\sigma)=J(\nu,\sigma)$, i.e. *the standard module of a generic representation is irreducible.* In this section we will exploit this to deduce results on reducibility of induced representations as well as study the non–archimedean case. The following is the subject matter of a joint work with Casselman [CSh].

Proposition 10.2. *Let $I(\nu_0,\sigma)$ be the standard module for a ψ–generic representation. Assume*

$$\lim_{\nu\longrightarrow\nu_0} C_\psi(w_0(\nu),w_0(\sigma))A(w_0(\nu),w_0(\sigma),w_0^{-1})$$

is well defined on $I(w_0(\nu_0),w_0(\sigma))$. Then $I(\nu_0,\sigma)$ is irreducible.

Proof. Choose $f\in I(w(\nu_0),w_0(\sigma))$ such that

$$\lambda_\psi(w_0(\nu_0),w_0(\sigma))(f)\neq 0.$$

Since

$$C_\psi(w_0(\nu),w_0(\sigma))A(w_0(\nu),w_0(\sigma),w_0^{-1})$$

is well defined at $\nu=\nu_0$, the image of f, and consequently the image of $I(w_0(\nu_0)w_0(\sigma))$, under it will be generic. By Rodier's theorem, it must equal $J(\nu_0,\sigma)$. Thus $I(\nu_0,\sigma)=J(\nu_0,\sigma)$. Converse is clear.

Corollary 1. *Suppose F is p–adic. Assume σ is supercuspidal. Then Vogan's theorem is valid, i.e. $J(\nu,\sigma)$ is generic if and only if $I(\nu,\sigma)=J(\nu,\sigma)$.*

Proof. This follows immediately from the validity of Conjecture 9.9 for σ supercuspidal which implies the holomorphy of $C_\psi(w_0(\nu),w_0(\sigma))$ for all ν in the positive

Weyl chamber, as well as the fact that $A(w_0(\nu), w_0(\sigma), w_0^{-1})$ has no poles as long as ν is regular.

More can be proved.

Theorem 10.2. *Let F be any local field (of characteristic zero). Fix ν in the positive Weyl chamber and take σ to be tempered. Assume*

$$C_\psi(w_0(\nu), w_0(\sigma))A(w_0(\nu), w_0(\sigma), w_0^{-1})$$

is well defined on all of $I(w_0(\nu), w_0(\sigma))$. Then

a) Suppose $J(\nu,\sigma)$ is ψ–generic. Then $I(\nu,\sigma) = J(\nu,\sigma)$, i.e. $I(\nu,\sigma)$ is irreducible.

b) Assume F is non–archimedean. Suppose $J(\nu,\sigma)$ is not ψ–generic. Let W be the direct sum of the irreducible subspaces of $I(\nu,\sigma)$, counted with their multiplicities. Let

$$\sigma_\nu = \sigma \otimes q^{\langle \nu, H_M(\ )\rangle}.$$

Denote by $\delta = \delta_P$ the modulus character of $P = MN$. Assume $\sigma_\nu \delta^{1/2}$ does not appear as a subquotient anywhere in the Jacquet module $(V/W)_N = V_N/W_N$ of V/W, where $V = V(\nu,\sigma)$. Then the ψ–generic subquotient of $V(\nu,\sigma)$ is a subrepresentation.

We should only remind the reader of what Jacquet modules are. Let $\mathbf{G}$ be a (split) reductive group over an archimedean local field F. Choose a parabolic subgroup $\mathbf{P} = \mathbf{MN}$ of $\mathbf{G}$. Let (π, V) be a smooth representation of G, i.e. a representation for which every $v \in V$ is fixed by an open compact subgroup of G. Let

$$V(N) = \{\pi(n)v - v | n \in N,\ v \in V\}.$$

Since M normalizes N, it acts on $V(N)$ as well as on $V/V(N) = V_N$, giving a representation π_N of M on V_N. One can then prove [C2]:

1. Let $U \longrightarrow V \longrightarrow W$ be an exact sequence of smooth representations of G. Then $U_N \longrightarrow V_N \longrightarrow W_N$ is exact.

2. (Frobenius reciprocity) Let (π, V) be a smooth representation of G and let σ be a smooth representation of M on a space U. Then the P–morphism

$$\Lambda : (I(\sigma), V(\sigma)) \longrightarrow (\sigma\delta_P^{1/2}, U)$$

defined by

$$f \mapsto f(e)$$

induces an isomorphism:

$$Hom_G(V, V(\sigma)) \cong Hom_M(V_N, U),$$

where U is given the M–structure $\sigma\delta_P^{1/2}$.

Then by 1) and 2) it is clear that $\sigma\delta_P^{1/2}$ appears as a quotient of $(V_0)_N$ exactly as many times as the number of subrepresentations of V_0 with multiplicities. Beside those, $(V_0)_N$ might also have $\sigma\delta_P^{1/2}$ as subquotients which are not quotients. The condition in Theorem 10.3.b then means that there will be no more appearance of $\sigma\delta_P^{1/2}$ in V_N, except possibly in $(\ker\lambda_\psi(\nu, \sigma))_N$.

The following result can also be proved.

Proposition 10.4. *Let $\mathbf{P} = \mathbf{MN}$ be a maximal parabolic subgroup of $\mathbf{G}$, a split reductive group over a local field F (of characteristic zero). Fix $s \in \mathbb{C}$ with $Re(s) > 0$. Let σ be an irreducible generic tempered representation of M. Assume that whenever $J(s\widetilde{\alpha}, \sigma)$ is generic, $I(s\widetilde{\alpha}, \sigma) = J(s\widetilde{\alpha}, \sigma)$. Then $I(s\widetilde{\alpha}, \sigma)$ is irreducible if and only if*

$$\prod_{i=1}^{m} L(1 - is, \sigma, r_i)^{-1} \neq 0,$$

if conjecture 9.9 is valid, when F is non–archimedean.

Corollary. *Suppose $F = \mathbb{R}$ or $\mathbb{C}$. Let*

$$\varphi : W_F \longrightarrow {}^L M$$

be the parametrization of σ. For each i, $1 \leq i \leq m$, let $L(s, r_i \cdot \varphi)$ be the corresponding Artin L–function. Then $I(s\widetilde{\alpha}, \sigma)$ is irreducible if and only if

$$\prod_{i=1}^{m} L(1 - is, r_i \cdot \varphi)^{-1} \neq 0.$$

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