Answers to review questions for Midterm 1

Let $R$ and $H$ be positive numbers. In questions (1) and (2), the region $R$ is the triangular region determined by the vertices $(0,0)$, $(0,R)$ and $(H,R)$ in the $xy$-plane.

(1) Use the Washer Method to find the volume of the solid obtained when $R$ is rotated about the $x$ axis.

The region $R$ sits over the interval $0 \leq x \leq H$, is bounded above by $y = R$, and is bounded below by $y = \left(\frac{R}{H}\right)x$. The volume of the solid is

$$
\pi \int_0^H R^2 - \left(\frac{R}{H}x\right)^2 \, dx = \pi R^2 H - \pi \frac{R^2}{H^2} \int_0^H x^2 \, dx = \pi R^2 H - \pi \frac{R^2}{H^2} \cdot \frac{H^3}{3} = \frac{2}{3} \pi R^2 H.
$$

(2) Use the Shell Method to find the volume of the solid obtained when $R$ is rotated about the $x$ axis. Is there a way to check the accuracy of the answers to questions (1) and (2) using the formula for the volume of a cone?

The region $R$ is bounded by $x = \left(\frac{H}{R}\right)y$ and $x = 0$ for $0 \leq y \leq R$. The volume of the solid is

$$
\int_0^R 2\pi y\left(\frac{H}{R}\right)y \, dy = 2\pi \left(\frac{H}{R}\right) \int_0^R y^2 \, dy = 2\pi \left(\frac{H}{R}\right) \left(\frac{R^3}{3}\right) = \frac{2}{3} \pi R^2 H.
$$

The solid is what remains when a cone with height $H$ and base radius $R$ is removed from a cylinder with height $H$ and base radius $R$. Therefore, the volume of the remaining portion is $\pi R^2 H - \left(\frac{1}{3}\right) \pi R^2 H = \left(\frac{2}{3}\right) \pi R^2 H$.

In questions (3) and (4), the region $R$ is the region in the $xy$-plane that is bounded by $y = (x-1)^2$ and $y = 2x + 2 = 0$.

(3) Use the Washer Method to find the volume of the solid obtained when $R$ is rotated about the axis $x = -1$.

The equation $y - 2x + 2 = 0$ is equivalent to $y = 2x - 2$. The curves $y = (x-1)^2$ and $y = 2x - 2$ intersect when $(x-1)^2 = 2x - 2$, and this happens when $x = 1$ and $x = 3$. This tells us that $R$ sits over the interval $1 \leq x \leq 3$ on the $x$-axis. This implies that $R$ sits to the right of the interval $0 \leq y \leq 4$ on the $t$-axis. The region $R$ is bounded by $x = 1 + \sqrt{y}$ and $x = 1 + y/2$ for $0 \leq y \leq 4$. The volume of the solid is

$$
\pi \int_0^4 (1 + \sqrt{y} - (-1))^2 - (1 + y/2 - (-1))^2 \, dy = \pi \int_0^4 (2 + \sqrt{y})^2 - (2 + y/2)^2 \, dy
$$

$$
= \pi \int_0^4 4y^{1/2} - y - y^2/4 \, dy = 8\pi.
$$
(4) Use the Shell Method to find the volume of the solid obtained when $R$ is rotated about the axis $y = 4$.

As in problem (3), the region $R$ is bounded by $x = 1 + \sqrt{y}$ and $x = 1 + y/2$ for $0 \leq y \leq 4$. The volume of the solid is

$$
\int_0^4 2\pi(4-y)((1 + \sqrt{y}) - (1 + y/2)) \, dy = 2\pi \int_0^4 4y^{1/2} - 2y - y^{3/2} + y^2/2 \, dy
$$

$$
= 2\pi \left( \frac{8}{3}y^{3/2} - y^2 - \frac{2}{5}y^{5/2} + \frac{1}{6}y^3 \right) \bigg|_0^4 = \frac{32\pi}{5}.
$$

(5) Use the Disk Method to find the volume of a hemisphere with radius $R$.

Consider the region $R$ in the $xy$-plane that is over the interval $0 \leq x \leq R$ and under the curve $y = \sqrt{R^2 - x^2}$. If we rotate $R$ about the $x$-axis then we get a hemisphere with radius $R$. This implies that the volume of a hemisphere with radius $R$ is given by

$$
\int_0^R \pi \left( \sqrt{R^2 - x^2} \right)^2 \, dx = \int_0^R \pi(R^2 - x^2) \, dx = \frac{2}{3}\pi R^3.
$$

(6) A solid has a base on the $xy$-plane. The boundary of this base is the circle $x^2 + y^2 = 16$. You get equilateral triangles when you take cross sections of this solid perpendicular to the $y$ axis. Find the volume of this solid.

If an equilateral triangle has sides of length $L$, then its area is $(\sqrt{3}/4)L^2$. This implies that the volume of the solid is

$$
\int_{-4}^4 (\sqrt{3}/4) \left( 2\sqrt{16 - y^2} \right)^2 \, dy = \int_{-4}^4 \sqrt{3}(16 - y^2) \, dy = \frac{256}{\sqrt{3}}.
$$

(7) A rod with length $\pi/4$ meters has density $\rho(x) = 1 + \tan x$ grams per meter, where $x$ is the distance in meters to an end of the rod. Find the mass of the rod in kilograms.

The substitution $u = \sec x$, $du = \tan x \sec x \, dx$ helps us to determine that the mass is

$$
\int_0^{\pi/4} 1 + \tan x \, dx = \int_0^{\pi/4} 1 \, dx + \int_0^{\pi/4} \tan x \, dx = \pi/4 + \int_0^{\pi/4} \frac{\tan x \sec x \, dx}{\sec x}
$$

$$
= \pi/4 + \int_1^{\sqrt{2}} \frac{du}{u} = \pi/4 + \ln|u| \bigg|_1^{\sqrt{2}} = \pi/4 + \ln(\sqrt{2}) = \pi/4 + (1/2) \ln 2,
$$

measured in grams.

(8) Evaluate $\int \frac{x^2 \, dx}{\sqrt{2x + 1}}$.
The substitution \( u = 2x + 1, \) \( du = 2 \, dx, \) \( (1/2) \, du = dx, \) \( x = (u - 1)/2 \) gives

\[
\int \frac{x^2 \, dx}{\sqrt{2x + 1}} = \int \frac{((u - 1)/2)^2 (1/2) \, du}{\sqrt{u}} = \frac{1}{8} \int \frac{(u - 1)^2}{\sqrt{u}} \, du = \frac{1}{8} \int \frac{u^2 - 2u + 1}{\sqrt{u}} \, du
\]

\[
= \frac{1}{8} \int u^{3/2} - 2u^{1/2} + u^{-1/2} \, du = \frac{1}{8} \left( \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + 2u^{1/2} \right) + C
\]

\[
= \frac{1}{8} \left( \frac{2}{5}(2x + 1)^{5/2} - \frac{4}{3}(2x + 1)^{3/2} + 2(2x + 1)^{1/2} \right) + C.
\]

(9) Evaluate \( \int \sin^4 x \cos^3 x \, dx. \)

The substitution \( u = \sin x, \) \( du = \cos x \, dx \) gives

\[
\int \sin^4 x \cos^3 x \, dx = \int \sin^4 x \cos^2 x \cos x \, dx = \int \sin^4 x (1 - \sin^2 x) \cos x \, dx
\]

\[
= \int u^4 (1 - u^2) \, du = \int u^4 - u^6 \, du = \frac{u^5}{5} - \frac{u^7}{7} + C
\]

\[
= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C.
\]

(10) Evaluate \( \int \tan^3 x \sec^6 x \, dx. \)

The substitution \( u = \tan x, \) \( du = \sec^2 x \, dx \) gives

\[
\int \tan^3 x \sec^6 x \, dx = \int \tan^3 x \sec^4 x \sec^2 x \, dx = \int \tan^3 x (\sec^2 x)^2 \sec^2 x \, dx
\]

\[
= \int \tan^3 x (1 + \tan^2 x)^2 \sec^2 x \, dx = \int \tan^3 x (1 + u^2)^2 \, du
\]

\[
= \int u^3 (1 + 2u^2 + u^4) \, du = \int u^3 + 2u^5 + u^7 \, du
\]

\[
= \frac{u^4}{4} + \frac{2u^6}{6} + \frac{u^8}{8} + C = \frac{\tan^4 x}{4} + \frac{2\tan^6 x}{6} + \frac{\tan^8 x}{8} + C.
\]

(11) Evaluate \( \int x^3 \cos(5x) \, dx. \)

An integration by parts, with \( u = x^3 \) and \( v = (1/5) \sin(5x), \) gives

\[
\int x^3 \cos(5x) \, dx = x^3((1/5) \sin(5x)) - \int 3x^2((1/5) \sin(5x)) \, dx.
\]
A second integration by parts, with \( u = x^2 \) and \( v = -(1/5) \cos(5x) \), gives
\[
\int x^2 \sin(5x) \, dx = x^2(-1/5 \cos(5x)) - \int 2x(-1/5 \cos(5x)) \, dx.
\]

A third integration by parts, with \( u = x \) and \( v = (1/5) \sin(5x) \), gives
\[
\int x \cos(5x) \, dx = x((1/5) \sin(5x)) - \int (1/5) \sin(5x) \, dx = (x/5) \sin(5x) + (1/5^2) \cos(5x) + C.
\]
Combining all this, we get
\[
\int x^3 \cos(5x) \, dx = \frac{x^3}{5} \sin(5x) + \frac{3x^2}{5^2} \cos(5x) - \frac{6x}{5^3} \sin(5x) - \frac{6}{5^4} \cos(5x) + C.
\]

(12) Evaluate \( \int (\ln(3x))^2 \, dx \).

An integration by parts, with \( u = x \) and \( v = (\ln(3x))^2 \), gives us
\[
\int (\ln(3x))^2 \, dx = x(\ln(3x))^2 - \int x \cdot 2 \ln(3x) (1/x) \, dx = x(\ln(3x))^2 - \int 2 \ln(3x) \, dx.
\]
Another integration by parts, with \( u = 2x \) and \( v = \ln(3x) \), gives us
\[
\int 2 \ln(3x) \, dx = 2x \ln(3x) - \int 2x (1/x) \, dx = 2x \ln(3x) - \int 2 \, dx = 2x \ln(3x) - 2x + C.
\]
We combine this to get
\[
\int (\ln(3x))^2 \, dx = x(\ln(3x))^2 - (2x \ln(3x) - 2x) + C.
\]

(13) Find the average value of the function \( f(x) = x \sec^{-1} x \) over the interval \([\sqrt{2}, 2]\).

Integration by parts, with \( u = x^2/2 \) and \( v = \sec^{-1} x \), gives us
\[
\int x \sec^{-1} x \, dx = (x^2/2) \sec^{-1} x - \int (x^2/2) \cdot \frac{1}{|x|\sqrt{x^2 - 1}} \, dx
\]
\[
= (x^2/2) \sec^{-1} x - \int (|x|^2/2) \cdot \frac{1}{|x|\sqrt{x^2 - 1}} \, dx
\]
\[
= (x^2/2) \sec^{-1} x - \int \frac{|x|^2/2}{\sqrt{x^2 - 1}} \, dx.
\]
Now the substitution $u = x^2 - 1$ gives us $(1/2)\, du = x \, dx$ and

\[
\int_{\sqrt{2}}^{2} x \sec^{-1} x \, dx = \left( x^2/2 \right) \sec^{-1} x \bigg|_{\sqrt{2}}^{2} - \int_{\sqrt{2}}^{2} \frac{|x|/2}{\sqrt{x^2 - 1}} \, dx
\]

\[
= 2(\pi/3) - \pi/4 - \int_{\sqrt{2}}^{2} \frac{x/2}{\sqrt{x^2 - 1}} \, dx
\]

\[
= 2(\pi/3) - \pi/4 - \int_{1}^{3} \frac{1}{4} \frac{1}{\sqrt{u}} \, du
\]

\[
= 2(\pi/3) - \pi/4 - (1/2)(\sqrt{3} - 1)
\]

The average value is this definite integral divided by $2 - \sqrt{2}$.

(14) Evaluate $\int \csc^3 x \, dx$.

We start with

\[
\int \csc^3 x \, dx = \int \csc x \csc^2 x \, dx = \csc x (-\cot x) - \int (-\csc x \cot x)(-\cot x) \, dx
\]

\[
= -\csc x \cot x - \int \csc x \cot^2 x \, dx
\]

\[
= -\csc x \cot x - \int \csc x (\csc^2 x - 1) \, dx
\]

\[
= -\csc x \cot x - \int \csc^3 x \, dx + \int \csc x \, dx,
\]

where we used integration by parts (with $u = \csc x$ and $v = -\cot x$) and a trigonometric identity. This gives us

\[
2 \int \csc^3 x \, dx = -\csc x \cot x + \int \csc x \, dx,
\]

which implies

\[
\int \csc^3 x \, dx = (1/2) \left( -\csc x \cot x + \int \csc x \, dx \right).
\]

Finally, the substitution $u = \csc x - \cot x$ gives us

\[
\int \csc x \, dx = \int \csc x \cdot \frac{\csc x - \cot x}{\csc x - \cot x} = \int \frac{\csc^2 x - \csc x \cot x}{\csc x - \cot x} \, dx
\]

\[
= \int \frac{-\csc x \cot x + \csc^2 x}{\csc x - \cot x} \, dx = \int \frac{du}{u}
\]

\[
= \ln |u| + C = \ln |\csc x - \cot x| + C.
\]
This and the previous result imply

\[ \int \csc^3 x \, dx = (1/2)(-\csc x \cot x + \ln |\csc x - \cot x|) + C. \]

(15) Evaluate \( \int \cos(\ln x) \, dx \) using the substitution \( u = \ln x \) and two integrations by parts.

From \( u = \ln x \) we get \( du = (1/x) \, dx \), \( x \, du = dx \), \( e^u \, du = dx \) and

\[ \int \cos(\ln x) \, dx = \int e^u \cos u \, du. \]

Now we will evaluate \( \int e^u \cos u \, du \) using two integrations by parts. The first integration by parts involves the functions \( e^u \) and \( \sin u \). The second integration by parts involves the functions \( e^u \) and \( -\cos u \). We get

\[ \int e^u \cos u \, du = e^u \sin u - \int e^u \sin u \, du \]
\[ = e^u \sin u - \left( e^u(-\cos u) - \int e^u(-\cos u) \, du \right) \]
\[ = e^u \sin u + e^u \cos u - \int e^u \cos u \, du, \]

which leads to

\[ 2 \int e^u \cos u \, du = e^u \sin u + e^u \cos u + C, \]

which gives us

\[ \int e^u \cos u \, du = (1/2)(e^u \sin u + e^u \cos u) + C. \]

Finally,

\[ \int \cos(\ln x) \, dx = \int e^u \cos u \, du = (1/2)(e^u \sin u + e^u \cos u) + C \]
\[ = (1/2)(x \sin(\ln x) + x \cos(\ln x)) + C. \]