Math 311 - Section 4 - Workshop 1

In this course, we will assume the basic rules of algebra, such as $1/(ab) = (1/a)(1/b)$. However, it is important to realize that these rules are consequences of a small set of axioms. The problems below are an illustration of this fact, and they provide us with a nice introduction to this course on real analysis. As a matter of fact, courses on real analysis often begin with the proofs that you will discover in this workshop.

For the purposes of this workshop, assume that we are given a set $\mathbf{R}$ with two operations: Addition and multiplication. Assume also that these operations satisfy the field axioms, called (A1), (A2), (A3), (A4), (M1), (M2), (M3), (M4) and (D):

(A1) $x + y = y + x$ for all elements x, y of $\mathbf{R}$.
(A2) $(x + y) + z = x + (y + z)$ for all elements x, y, z of $\mathbf{R}$.
(A3) There is a special element of $\mathbf{R}$, called 0, which has the property $x + 0 = x$ for all $x \in \mathbf{R}$.
(A4) For every $x \in \mathbf{R}$ there exists an element $-x \in \mathbf{R}$ such that $-x + x = 0$.
(M1) $xy = yx$ for all elements x, y of $\mathbf{R}$.
(M2) $(xy)z = x(yz)$ for all elements x, y, z of $\mathbf{R}$.
(M3) There is a special element of $\mathbf{R}$, called 1, which has the property $x \cdot 1 = x$ for all $x \in \mathbf{R}$. This element 1 is not equal to 0.
(M4) If $x \in \mathbf{R}$ and $x \neq 0$ then there exists an element $1/x \in \mathbf{R}$ such that $(1/x)x = 1$.
(D) $(x + y)z = xz + yz$ for all elements x, y, z of $\mathbf{R}$.

Prove the following assertions using only the field axioms:

(1) If $z \in \mathbf{R}$ has the property

$$x + z = x \text{ for all } x \in \mathbf{R}$$

then $z = 0$.

(2) If $u \in \mathbf{R}$ has the property

$$xu = x \text{ for every } x \in \mathbf{R}$$

then $u = 1$.

(3) $-(-x) = x$ for every $x \in \mathbf{R}$.

(4) $-(x + y) = (-x) + (-y)$ for all elements x, y of $\mathbf{R}$.

(5) $0 \cdot x = 0$ for every $x \in \mathbf{R}$.
Comment: You will have to use (D) in order to prove (5).

(6) If $x \in \mathbf{R}$, $y \in \mathbf{R}$, $x \neq 0$ and $y \neq 0$ then $xy \neq 0$.
Comment: You can use the result in (5) when you prove (6).