Toric arrangements that come from graphs

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Joint work with Marcelo Aguiar
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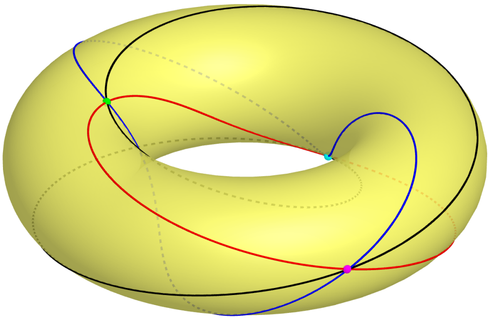
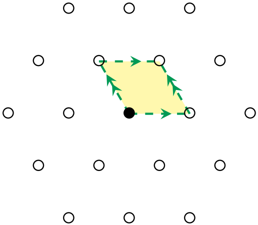


Torus

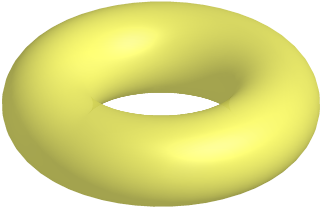
Let V be a real vector space.

A lattice L is the additive subgroup of V generated by a basis of V.

The associated torus is the quotient group $T := V/L$.



L



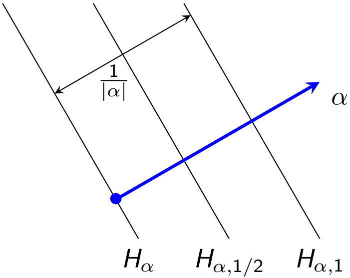
T

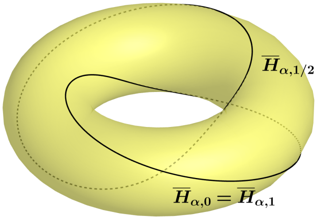
Hyperplanes and hypertori

Fix an inner product $\langle\ ,\ \rangle$ on V.

For each nonzero vector $\alpha \in V$ and $k \in \mathbb{R}$, let

$$H_\alpha := \{x \in V \mid \langle \alpha, x \rangle = 0\} \quad \text{(linear hyperplane)};$$
$$H_{\alpha,k} := \{x \in V \mid \langle \alpha, x \rangle = k\} \quad \text{(affine hyperplane)};$$
$$\overline{H}_{\alpha,k} := \text{the image of } H_{\alpha,k} \text{ in the torus } T \quad \text{(hypertorus)}.$$

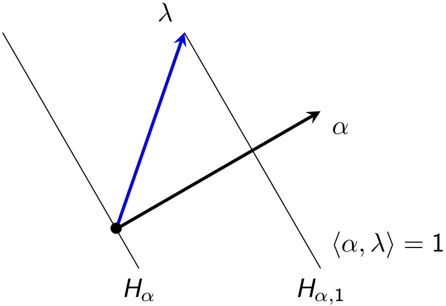


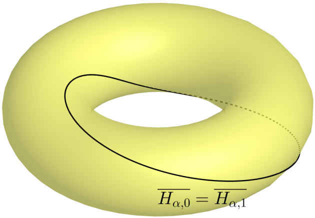


Integrality

A vector $\alpha \in V$ is L-integral if $\langle \alpha, \lambda \rangle \in \mathbb{Z}$ for all $\lambda \in L$.

If α is L-integral, then $\{\overline{H}_{\alpha,k} \mid k \in \mathbb{Z}\}$ is a finite set.





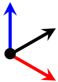
Arrangements

Let Φ be a finite set of nonzero L-integral vectors in V.
To Φ and L we associate three arrangements:

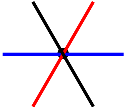
$$\mathcal{A}(\Phi) := \{H_\alpha \mid \alpha \in \Phi\} \text{ (linear)};$$

$$\widetilde{\mathcal{A}}(\Phi) := \{H_{\alpha,k} \mid \alpha \in \Phi,\ k \in \mathbb{Z}\} \text{ (affine)};$$

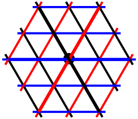
$$\overline{\mathcal{A}}(\Phi, L) := \{\overline{H}_{\alpha,k} \mid \alpha \in \Phi,\ k \in \mathbb{Z}\} \text{ (toric)}.$$



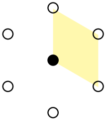
Φ



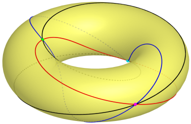
$\mathcal{A}(\Phi)$



$\widetilde{\mathcal{A}}(\Phi)$



L



$\overline{\mathcal{A}}(\Phi, L)$

Root systems

Let Φ be a crystallographic root system in V.

The coroot lattice $\mathbb{Z}\Phi^\vee$ is

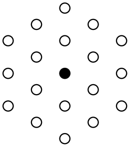
$$\mathbb{Z}\Phi^\vee := \mathbb{Z}\{2\beta/\langle\beta,\beta\rangle \mid \beta \in \Phi\}.$$

The coweight lattice is

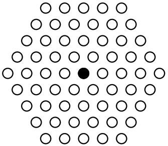
$$\widehat{\mathbb{Z}\Phi} := \{\alpha \in V \mid \langle\alpha,\lambda\rangle \in \mathbb{Z} \text{ for all } \lambda \in \Phi\}.$$

Then Φ is integral with respect to either lattice.

A_2



$\mathbb{Z}A_2^\vee$



$\widehat{\mathbb{Z}A_2}$

One graph, two toric arrangements

Let A_{n-1} denote the root system of type A,

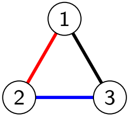
$$A_{n-1} := \{e_i - e_j \mid 1 \leq i \neq j \leq n\}.$$

Let G be a simple connected graph with vertex set $[n]$.
View G as this finite subset of A_{n-1}:

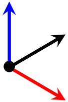
$$\{e_i - e_j \mid \{i, j\} \text{ is an edge of } G\}.$$

There are two kinds of toric graphic arrangements:

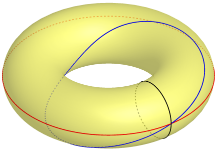
- $\overline{\mathcal{A}}(G, \widehat{\mathbb{Z}A_{n-1}})$, the coweight graphic arrangement.
- $\overline{\mathcal{A}}(G, \mathbb{Z}A_{n-1}^{\vee})$, the **coroot graphic arrangement**.



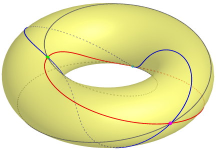
K_3



"K_3"

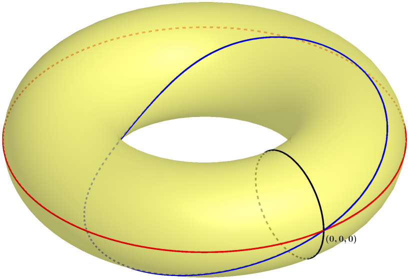


$\overline{\mathcal{A}}(K_3, \widehat{\mathbb{Z}A_2})$

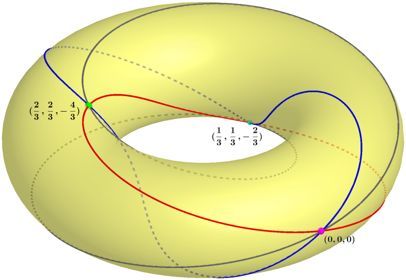


$\overline{\mathcal{A}}(K_3, \mathbb{Z}A_2{}^{\vee})$

The coweight and coroot arrangement for the graph K_3



Coweight arrangement



Coroot arrangement

Note that:

$$V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\};$$

$$L = \begin{cases} \langle (1,-1,0), (0,1,-1), (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}) \rangle_{\mathbb{Z}} & \text{(coweight lattice)}; \\ \langle (1,-1,0), (0,1,-1) \rangle_{\mathbb{Z}} & \text{(coroot lattice)}. \end{cases}$$

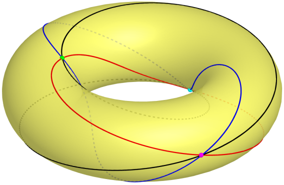
Characteristic polynomial

A flat of a toric arrangement $\overline{\mathcal{A}}$ is a connected component of an intersection of hypertori in $\overline{\mathcal{A}}$.

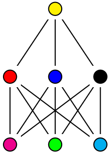
The intersection poset $\overline{\Pi}(\overline{\mathcal{A}})$ is the (po)set of flats of $\overline{\mathcal{A}}$, ordered by inclusion.

The characteristic polynomial of $\overline{\mathcal{A}}$ is

$$\overline{\chi}(\overline{\mathcal{A}};\, t) := \sum_{X \in \overline{\Pi}(\overline{\mathcal{A}})} \underbrace{\mu(X, T)}_{\text{Möbius function}} \; t^{\dim X}.$$



$\overline{\mathcal{A}}$



$\overline{\Pi}$

$$2 + 2 + 2 - t - t - t + t^2$$
$$= t^2 - 3t + 6$$

$\overline{\chi}(\overline{\mathcal{A}};\, t)$

Examples of coroot characteristic polynomials

- For the path graph P_n,

$$\overline{\chi}(P_n, \mathbb{Z}A_{n-1}^{\vee};\, t) = (-1)^{n-1} \sum_{d|n} \varphi(d)(1-t)^{\frac{n}{d}-1},$$

  where φ is Euler's totient function.

- For the star graph $K_{1,n-1}$,

$$\overline{\chi}(K_{1,n-1}, \mathbb{Z}A_{n-1}^{\vee};\, t) = (t-1)^{n-1} + (-1)^{n-1}(n-1).$$

- For the complete graph K_n,

$$\overline{\chi}(A_{n-1}, \mathbb{Z}A_{n-1}^{\vee};\, t) = (-1)^{n-1}(n-1)! \sum_{d|n} (-1)^{\frac{n}{d}-1} \varphi(d) \binom{\frac{t}{d}-1}{\frac{n}{d}-1}$$

(Ardila Castillo Henley '15)

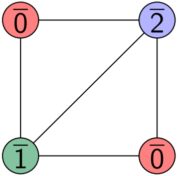
Divisible colorings

A proper divisible m-coloring of G is a function $f : V \to \mathbb{Z}_m$ with

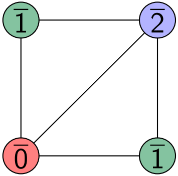
- $f(i) \neq f(j)$ if i and j are adjacent in G; and
- $\sum_{i \in V} f(i) \equiv 0 \pmod{m}$.

Theorem

For any positive multiple m of n, the number of proper divisible m-colorings of G is equal to $|\overline{\chi}(G, \mathbb{Z}A_{n-1}^{\vee}; m)|$.



Divisible 3-coloring

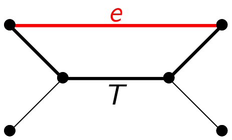


Non-divisible 3-coloring

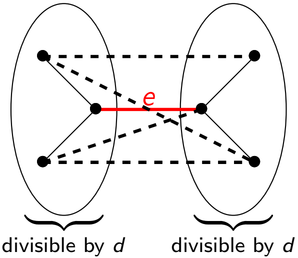
Divisible activities

Fix a total order on $E(G)$. For any $d \in \mathbb{N}$ and any $T \in \text{Tree}(G)$,

- An edge e is externally active w.r.t T if $e \notin T$ and e is the minimum edge in $\text{Cycle}(T, e)$.



- An edge e is d-internally active w.r.t T if $e \in T$ and e is the minimum edge in $\text{d-Cut}(T, e)$.



A formula for the coroot characteristic polynomial

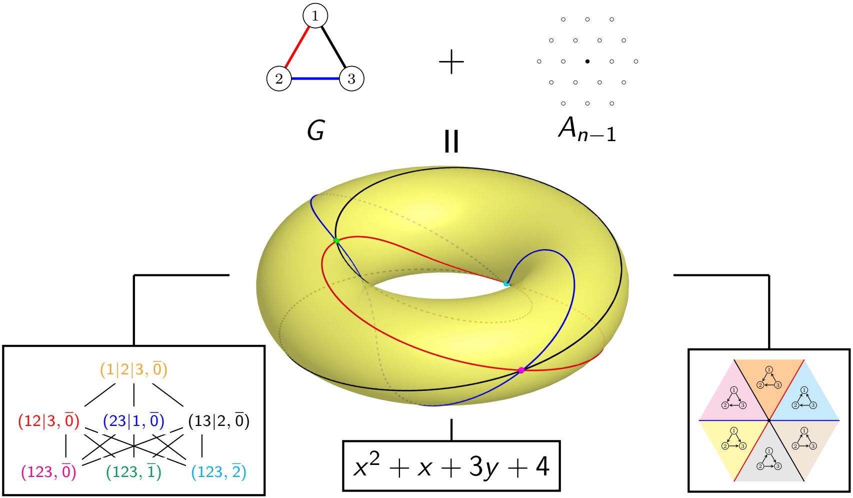
Theorem

The coroot characteristic polynomial of G is equal to

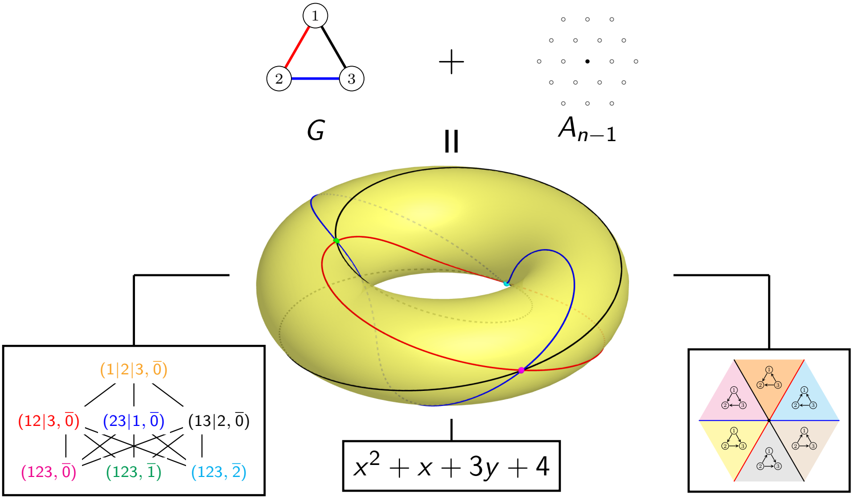
$$\overline{\chi}(G, \mathbb{Z}A_{n-1}^{\vee};\, t) = (-1)^{n-1} \sum_{d|n} \varphi(d)\, \overline{\chi}_d(G; t),$$

where φ is the Euler's totient function and $\chi_d(G; t)$ is

$$\overline{\chi}_d(G; t) := \sum_{\substack{T \text{ spanning tree of } G \\ \text{with ext. activity } 0}} (1-t)^{int_d(T)}.$$



1
2
3
G
+
A_{n-1}
=
$(1|2|3, \bar{0})$
$(12|3, \bar{0})$
$(23|1, \bar{0})$
$(13|2, \bar{0})$
$(123, \bar{0})$
$(123, \bar{1})$
$(123, \bar{2})$
$x^2 + x + 3y + 4$



THANK YOU!

Extended abstract : http://www.mat.univie.ac.at/ slc/wpapers/ FPSAC2017/84%20Aguiar%20Chan.html