Toric arrangements that come from graphs

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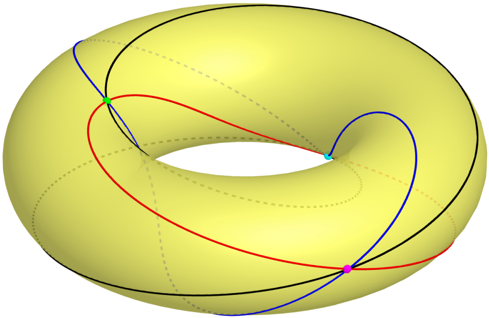
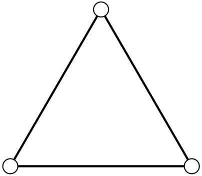
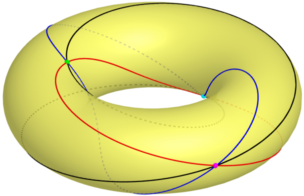


Toric arrangements

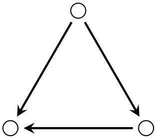


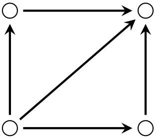


- Layman's terms: lines on a donut.
- Studied in connection to Kostant partition functions (De Concini-Procesi '05), arithmetic matroids (Moci '12), arithmetic Tutte polynomial (D'Adderio-Moci '13), etc.
- This talk is about toric arrangements that are built from graphs.

Motivation

- The current study of toric graphic arrangements is mainly focused on the case of the standard torus.
- We study graphic arrangements on two other types of tori, the coweight torus and the coroot torus.
- We will see that these two arrangements tell us new things about the acyclic orientations of the input graph.

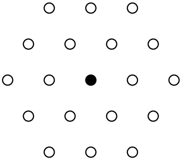




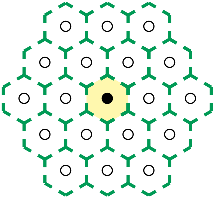
Tori

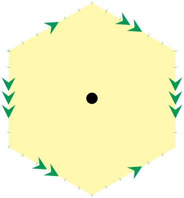
Let V be a real vector space.

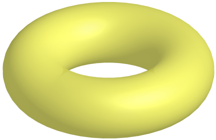
A lattice L is the integer-span of a basis of V.
The associated torus is the quotient $T := V/L$.



L



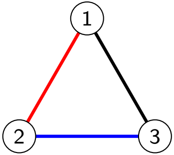




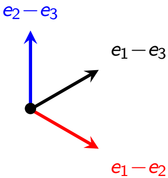
T

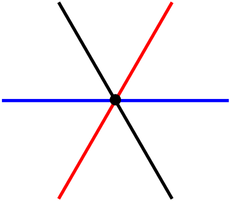
Graphic arrangements

Let G be a simple connected graph.
$\mathcal{A}(G)$ is called the linear graphic arrangement.
$\widetilde{\mathcal{A}}(G)$ is called the affine graphic arrangement.

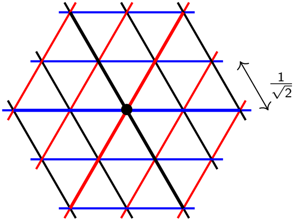


K_3



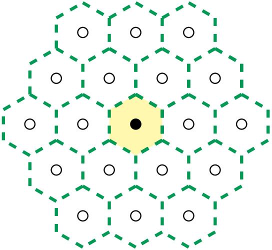


$\mathcal{A}(K_3)$

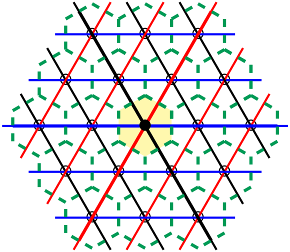


$\widetilde{\mathcal{A}}(K_3)$

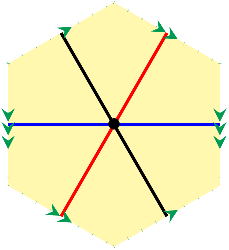
Toric graphic arrangements, example 1

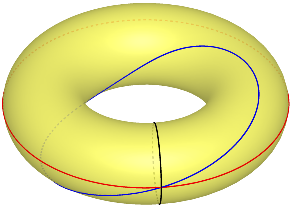


L_1



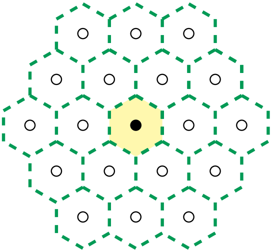
$\widetilde{\mathcal{A}}(K_3)$





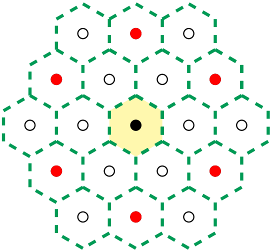
$\overline{\mathcal{A}}(K_3, L_1)$

Toric graphic arrangements, example 2

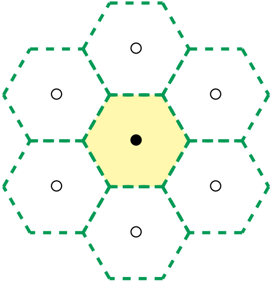


L_1

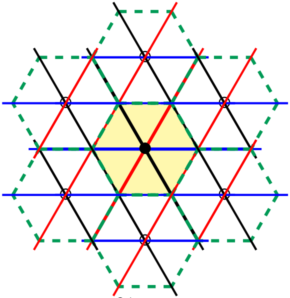
Toric graphic arrangements, example 2



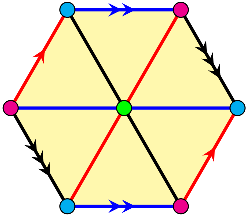
Toric graphic arrangements, example 2

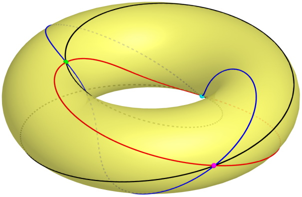


L_2



$\widetilde{\mathcal{A}}(K_3)$

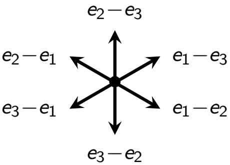




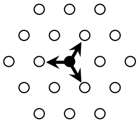
$\overline{\mathcal{A}}(K_3, L_2)$

Root system of type A

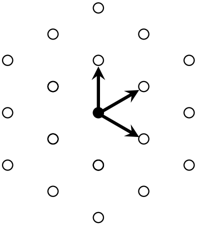
$$V_n := \{x \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0\} \quad \text{(ambient space)};$$
$$A_{n-1} := \{e_i - e_j \mid 1 \le i < j \le n\} \quad \text{(root system of type A)};$$
$$\widehat{\mathbb{Z}A_{n-1}} := \mathbb{Z}\{1/n(e_1 + \cdots + e_n) - e_i \mid 1 \le i \le n\} \quad \text{((co)weight lattice)};$$
$$\mathbb{Z}A_{n-1} := \mathbb{Z}\{e_i - e_j \mid 1 \le i < j \le n\} \quad \text{((co)root lattice)}.$$



V_3



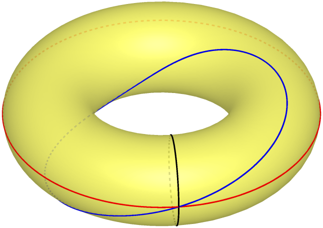
$\widehat{\mathbb{Z}A_2}$



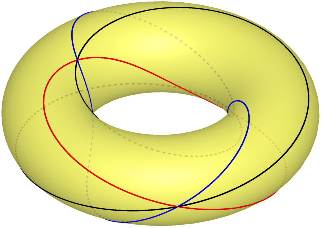
$\mathbb{Z}A_2$

One graph, three toric arrangements

- The standard arrangement: $V = \mathbb{R}^n$, $L = \mathbb{Z}^n$.
- The coweight arrangement: $V = V_n$, $L = \widehat{\mathbb{Z}A_{n-1}}$.
- The coroot arrangement: $V = V_n$, $L = \mathbb{Z}A_{n-1}$.

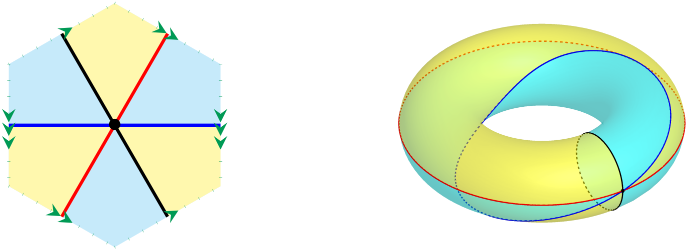


$\overline{\mathcal{A}}(K_3, \widehat{\mathbb{Z}A_2})$

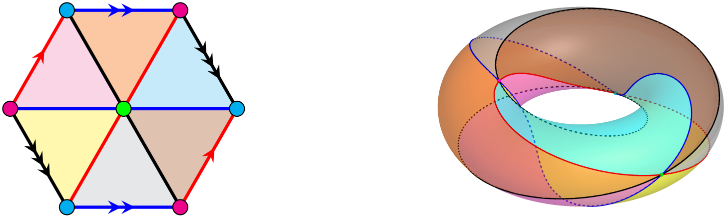


$\overline{\mathcal{A}}(K_3, \mathbb{Z}A_2)$

Toric chambers



$\overline{\mathcal{A}}(K_3, \widehat{\mathbb{Z}A_2})$

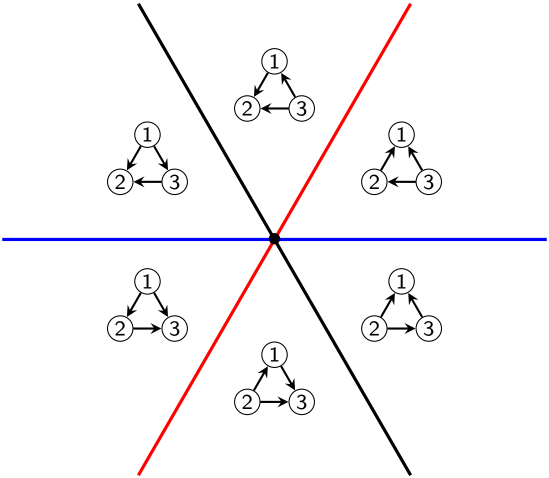


$\overline{\mathcal{A}}(K_3, \mathbb{Z}A_2)$

Acyclic orientations

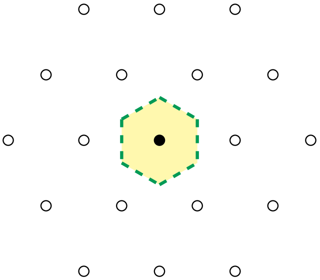
Recall the bijection of Greene and Zaslavsky ('83):

$$\text{Chambers of } \mathcal{A}(G) \leftrightarrow \text{Acyclic orientations of } G$$

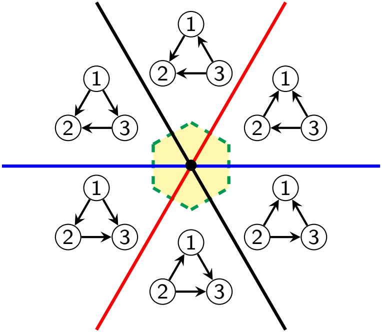


$\mathcal{A}(K_3)$

Coweight Voronoi cells

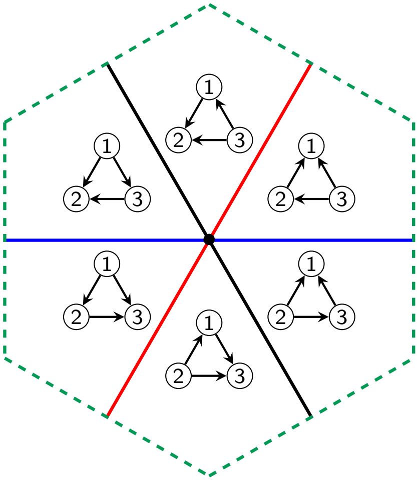


$\widehat{\mathbb{Z}A_2}$

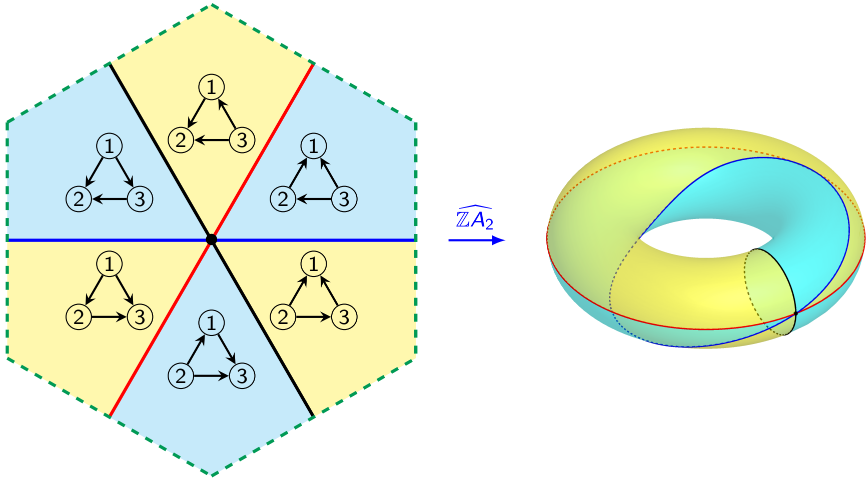


$\mathcal{A}(K_3)$

Coweight Voronoi relation

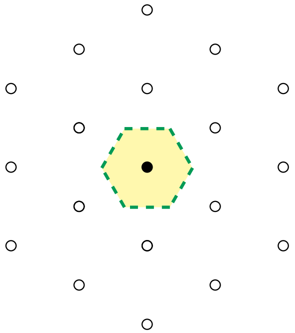


Coweight Voronoi relation

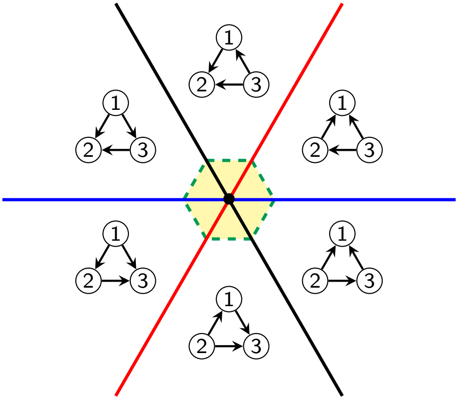


Two orientations that are projected to the same toric chamber are Voronoi equivalent.

Coroot Voronoi cells

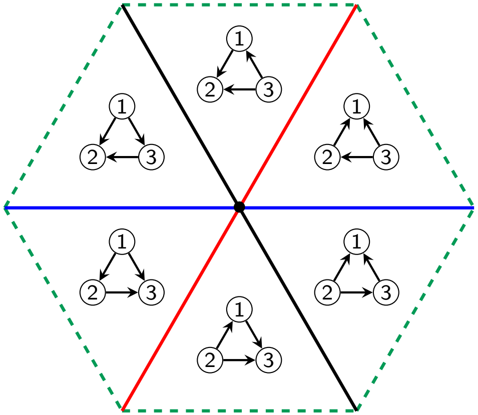


$\mathbb{Z}A_2$

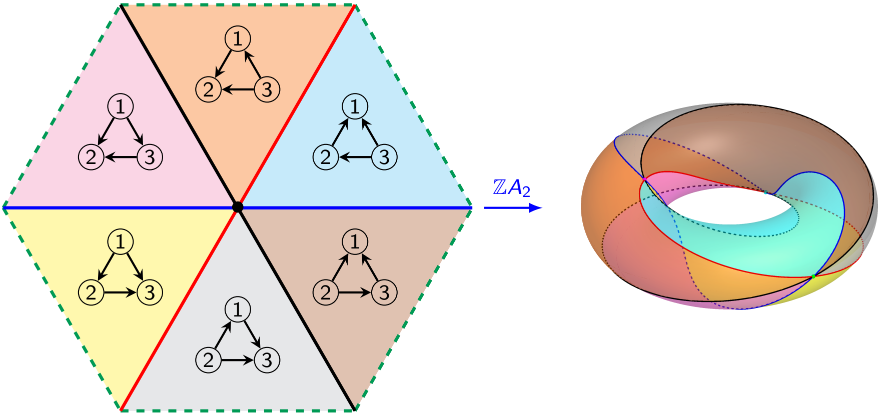


$\mathcal{A}(K_3)$

Coroot Voronoi relation



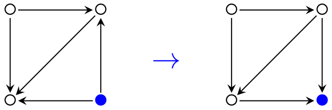
Coroot Voronoi relation



No two distinct acyclic orientations are Voronoi equivalent.

Combinatorial description for coweight Voronoi equivalence

The relation is source-to-sink flip.

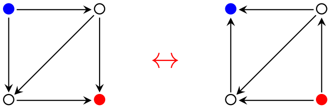


- Studied by
  - Mosesjan ('72) and Pretzel ('86) in combinatorics;
  - Eriksson and Eriksson ('09), and Speyer ('09) in connection to conjugacy of Coxeter elements;
  - Develin, Macauley and Reiner ('16) in the context of toric arrangements.
- It also arises in connection to sandpile groups and chip-firing.

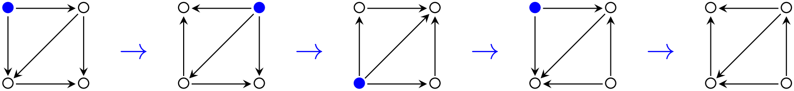
Combinatorial description for coroot Voronoi equivalence

The relation has several equivalent descriptions.

One is source-sink exchange.



Another one is n-step source-to-sink flip.



THANK YOU!