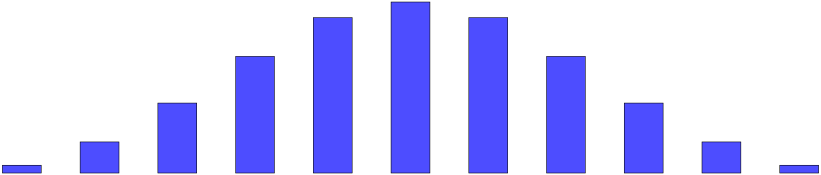
Complexity of Log-concave Inequalities for Matroids

Swee Hong Chan

joint with Igor Pak



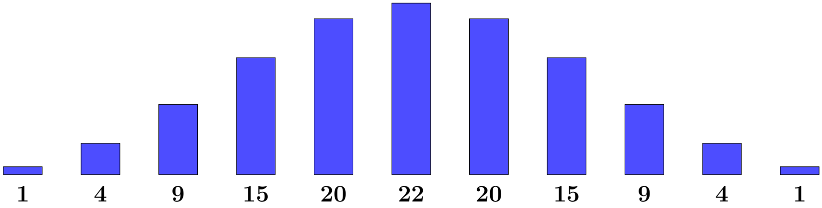
What is log-concavity?

A sequence $a_1, \ldots, a_n \in \mathbb{N}_{\geq 0}$ is log-concave if

$$a_k^2 \;\geq\; a_{k+1}\, a_{k-1} \qquad (1 < k < n).$$

Log-concavity (and positivity) implies unimodality:

$$a_1 \;\leq\; \cdots \;\leq\; a_m \;\geq\; \cdots \;\geq\; a_n \quad \text{for some } 1 \leq m \leq n.$$



Log-concave shaped objects in real life

Cheonmachong (천마총) in Gyeongju.

Example 1: Binomial coefficients

$$a_k = \binom{n}{k} \qquad k = 0, 1, \ldots, n.$$

This sequence is log-concave because

$$\frac{a_k^2}{a_{k+1}\, a_{k-1}} = \frac{\binom{n}{k}^2}{\binom{n}{k+1}\binom{n}{k-1}} = \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right),$$

which is greater than 1.

Example 2: Permutation inversion sequence

Let

$$a_k \;:=\; \text{number of } \pi \in S_n \text{ with } k \text{ inversions},$$

where inversion of π is pair $i < j$ s.t. $\pi_i > \pi_j$.

This sequence is log-concave because

$$\sum_{0 \le k \le \binom{n}{2}} a_k\, q^k \;=\; [n]_q! \;=\; \prod_{i=1}^{n-1} (1 + q + q^2 + \ldots + q^i)$$

is a product of log-concave polynomials.

Example 3: Forests of a graph

a_k = number of forests with k edges of graph G.

Forest is a subset of edges of G that has no cycles.

Log-concavity was conjectured for all matroids (Mason '72), and was proved through combinatorial Hodge theory (Huh '15).

G forest not forest spanning tree

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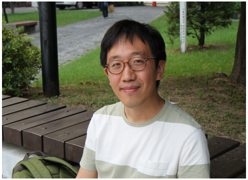
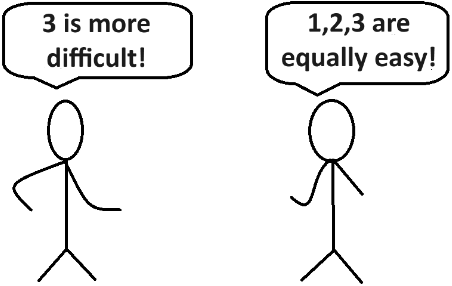


June Huh

Fields Medal

Motivation

Which log-concave inequality is more "difficult"?



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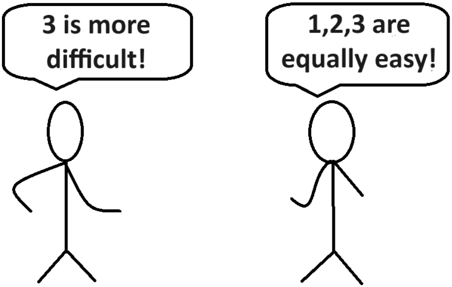


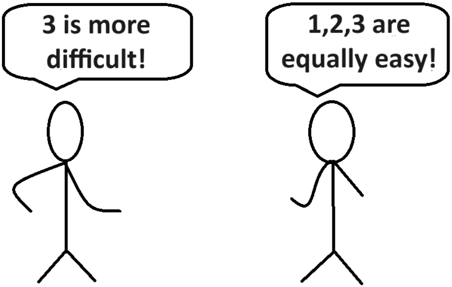
We will show that [redacted] (3)

is strictly more difficult than the rest, using

Complexity Theory.

Motivation

Which log-concave inequality is more "difficult"?



We will show that a generalization of (3) is strictly more difficult than the rest, using **Complexity Theory**.

Matroids

Object: Matroids

Matroid $\mathcal{M} = (X, \mathcal{I})$ is ground set X with collection of independent sets $\mathcal{I} \subseteq 2^X$.

Graphic matroids

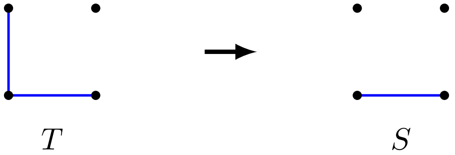
- X = edges of a graph G,
- $\mathcal{I}$ = forests in G.

Realizable matroids

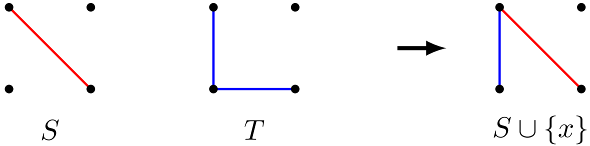
- X = finite set of vectors over field $\mathbb{F}$,
- $\mathcal{I}$ = sets of linearly independent vectors.

Matroids: Axioms

- (Hereditary) If $S \subseteq T$ and $T \in \mathcal{I}$, then $S \in \mathcal{I}$.



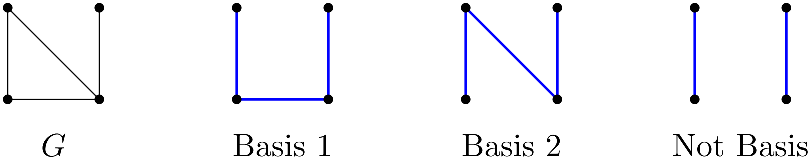
- (Exchange) If $S, T \in \mathcal{I}$ and $|S| < |T|$, then there is $x \in T \setminus S$ such that $S \cup \{x\} \in \mathcal{I}$.



Matroid: Bases and ranks

A basis of $\mathcal{M}$ is a maximal independent set.

Rank r of $\mathcal{M}$ is the size of the bases.



Matroid generalizes the notion of vector spaces.

Mason’s conjecture

Mason's conjecture

For matroid $\mathcal{M}$, let

$$\mathrm{I}(k) \;:=\; \text{no. of independents sets with } k \text{ elements.}$$

For graphic matroid, $\mathrm{I}(k)$ is no. of forest with k edges.

Conjecture (Mason '72)

The sequence $\mathrm{I}(1), \mathrm{I}(2), \ldots$ *is log-concave,*

$$\mathrm{I}(k)^2 \;\geq\; \mathrm{I}(k+1)\,\mathrm{I}(k-1) \qquad (k \in \mathbb{N}),$$

Mason's conjecture (continued)

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Conjecture was proved for graphic matroids by (Huh '15), and for all matroids by (Adiprasito–Huh–Katz '18).

Both proofs used combinatorial Hodge theory.

Mason's conjecture (continued)

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Both proofs used combinatorial Hodge theory.

We will show that Mason's conjecture is consequence of a stronger inequality.

Stanley–Yan inequality

Stanley–Yan inequality (simple case)

Let $\mathcal{M}$ be a matroid with ground set X and rank r.

Fix a subset S of X. Let

$\mathrm{B}(k)$:= no. of **bases** B such that $|B \cap S| = k$,

multiplied by $r! \times \binom{r}{k}^{-1}$.

Theorem (Stanley '81, Yan '23)

The sequence $\mathrm{B}(1), \mathrm{B}(2), \ldots$ *is log-concave,*

$$\mathrm{B}(k)^2 \ \geq \ \mathrm{B}(k+1)\,\mathrm{B}(k-1) \qquad (k \in \mathbb{N}).$$

Stanley–Yan inequality (simple)

Theorem (Stanley '81, Yan '23)

$$\mathrm{B}(k)^2 \;\geq\; \mathrm{B}(k+1)\,\mathrm{B}(k-1) \qquad (k \in \mathbb{N}).$$

Proved for regular matroids by (Stanley '81) using Alexandrov–Fenchel inequality for mixed volumes.

Proved for all matroids by (Yan '23) using theory of Lorentzian polynomials.

Proof of Mason's conjecture using Stanley–Yan inequality

Direct sum of matroids

Direct sum of $\mathcal{M}_1 = (X_1, \mathcal{I}_1)$ and $\mathcal{M}_2 = (X_2, \mathcal{I}_2)$

is the matroid $\mathcal{M}' = (X', \mathcal{I}')$ given by

$$X' := X_1 \sqcup X_2 \quad \text{(disjoint union)}$$
$$\mathcal{I}' := \{ S_1 \cup S_2 : S_1 \in \mathcal{I}_1, S_2 \in \mathcal{I}_2 \}.$$

This generalizes the notion of

direct sum for vector spaces.

Proof of Mason's conjecture using SY inequality

Let

$$\mathcal{M} \;:=\; \text{original matroid in Mason's conjecture;}$$

$$\mathcal{F} \;:=\; \text{matroid with } r \text{ elements and with every subset being independent;}$$

$$\mathcal{M}' \;:=\; \text{direct sum of } \mathcal{M} \text{ and } \mathcal{F};$$

$$S \;:=\; \text{ground set of } \mathcal{M}.$$

Then

$$\mathrm{I}(k) \;\text{ for }\; \mathcal{M} \quad = \quad \frac{1}{r!} \times \mathrm{B}(k) \;\text{ for }\; \mathcal{M}'.$$

Proof of Mason's conjecture using SY inequality

Since

$$\mathrm{I}(k) \;\text{ for }\; \mathcal{M} \quad = \quad \tfrac{1}{r!} \times \mathrm{B}(k) \;\text{ for }\; \mathcal{M}',$$

we then conclude that

Stanley–Yan inequality for $\mathcal{M}'$
implies Mason's conjecture for $\mathcal{M}$.

Stanley–Yan inequality (full version)

Fix $d \geq 0$, disjoint subsets $S, S_1, \ldots, S_d$ of X, and $\ell_1, \ldots, \ell_d \in \mathbb{N}$.

$\mathrm{B}_d(k) :=$ number of bases B of $\mathcal{M}$ such that $|B \cap S| = k,\ |B \cap S_i| = \ell_i$ for $i \in [d]$,

multiplied by $r! \times \binom{r}{k, \ell_1, \ldots, \ell_d}^{-1}$.

Theorem (Stanley '81, Yan '23)

The sequence $\mathrm{B}_d(1), \mathrm{B}_d(2), \ldots$ *is log-concave,*

$$\mathrm{B}_d(k)^2 \;\geq\; \mathrm{B}_d(k+1)\,\mathrm{B}_d(k-1) \qquad (k \in \mathbb{N}).$$

What we want to do

Theorem (Stanley '81, Yan '23)
The sequence $\mathrm{B}_d(1), \mathrm{B}_d(2), \ldots$ *is log-concave,*

$$\mathrm{B}_d(k)^2 \;\geq\; \mathrm{B}_d(k+1)\,\mathrm{B}_d(k-1) \qquad (k \in \mathbb{N}).$$

Both LHS and RHS of this inequality has combinatorial interpretations.

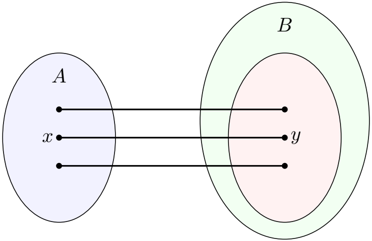
But we will show that this inequality has **no combinatorial injective proof**.

Combinatorial injective proof

Combinatorial injection

An injection $f : A \to B$ is combinatorial if

- Given $x \in A$, the image $f(x)$ is computable in poly($|x|$) steps;
- Given $y \in B$, it takes poly($|y|$) steps to decide if y is in image of f; and if so, the pre-image $f^{-1}(y)$ is computable in poly($|y|$) steps.



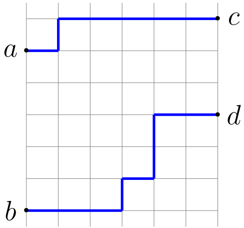
Example: Injective proof of binomial inequality

$$\binom{n}{k}^2 \geq \binom{n}{k+1}\binom{n}{k-1} \qquad (1 < k < n).$$

This inequality has a lattice path interpretation:

$K(a \to c, b \to d) :=$ no. of pairs of north-east lattice paths from a to c and b to d,

for $a, b, c, d \in \mathbb{Z}^2$.



Example: Injective proof of binomial inequality

Let

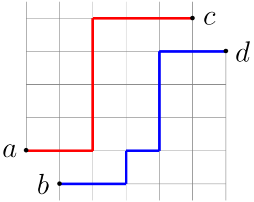
$$a = (0,1), \qquad c = (k, n-k+1),$$

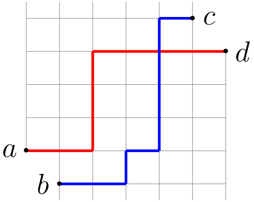
$$b = (1,0), \qquad d = (k+1, n-k).$$

Then

$$K(a \to c, b \to d) \;=\; \binom{n}{k}^2,$$

$$K(a \to d, b \to c) \;=\; \binom{n}{k-1}\binom{n}{k+1}.$$

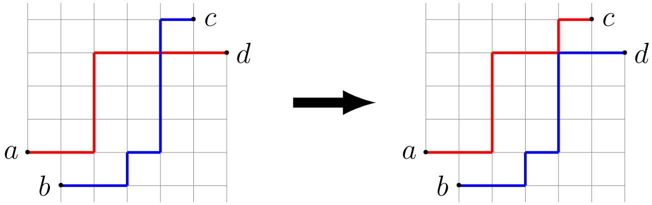




Example: Injective proof of binomial inequality

$$f : K(a \to d, b \to c) \to K(a \to c, b \to d)$$

is defined by path-swapping injections.



Images of f are pairs of lattice paths that intersects.

First main result

Theorem 1 (C.–Pak '24+)
There is no combinatorial injective proof for Stanley–Yan inequality, assuming $\mathsf{NP}^{\mathsf{NP}} \neq \mathsf{coNP}^{\mathsf{NP}}$.

The assumption above is slightly stronger than $\mathsf{P} \neq \mathsf{NP}$, and is widely used in Complexity Theory.

First main result

Theorem 1 (C.–Pak '24+)

There is no combinatorial injective proof for Stanley–Yan inequality, assuming $\mathrm{NP}^{\mathrm{NP}} \neq \mathrm{coNP}^{\mathrm{NP}}$.

This result is a consequence of Stanley–Yan inequality being not in $\#P$ (explained next slide).

Complexity class $\#P$

Complexity class $\#P$

Informal definition for intuition:

$\#$P := Problems of counting the number of objects satisfying some **property**; this **property** is simple to verify.

Example (Problem in $\#$P)

Count number of **proper** 3-*colorings of graph* G.

Complexity class NP

NP := Problems asking about **existence** of a solution S for input x, where validity of S can be verified in polynomial time.

Example (Problem in NP)

Does graph G have a proper 3-coloring?

Complexity class #P: Formal definition

$\#\mathrm{P}$:= Problems asking for **number** of solutions S for input x, where validity of S can be verified in polynomial time.

Example (Problem in #P)

Count the number of proper 3-colorings of graph G.

It might take exponential time to solve a problem in $\#\mathrm{P}$.

Second main result

Theorem 2 (C.–Pak '24+)

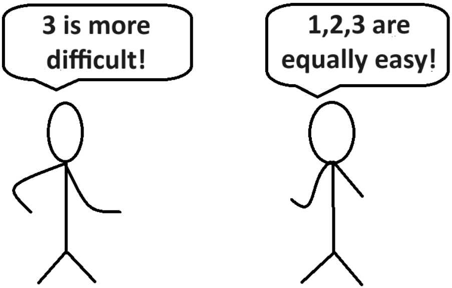
Let $\mathcal{M}$ be a binary matroid. Then the defect of Stanley–Yan inequality

$$\mathrm{B}_d(k)^2 \ - \ \mathrm{B}_d(k+1)\,\mathrm{B}_d(k-1)$$

is not in $\#\mathrm{P}$, *assuming* $\mathsf{NP}^{\mathsf{NP}} \neq \mathsf{coNP}^{\mathsf{NP}}$.

This means LHS and RHS of Stanley–Yan inequality belongs to $\#\mathrm{P}$, but their difference does not.

Recall our goal



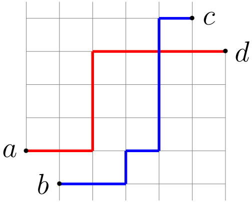
We will now show that Stanley–Yan inequality is strictly more difficult than the binomial inequality and permutation inversion inequality.

Example 1: Binomial inequality

It follows from path-swapping injections that

$$\binom{n}{k}^2 - \binom{n}{k+1}\binom{n}{k-1} \;=\; \text{number of non-intersecting}$$

lattice paths from a to c and b to d.



Thus the defect of this inequality belongs to #P.

Example 2: Permutation inversion inequality

Let a_k = number of $\pi \in S_n$ with k inversions.

Then $$\sum_{0 \le k \le \binom{n}{2}} a_k \, q^k \; = \; \prod_{i=1}^{n-1} (1 + q + \ldots + q^i)$$

is computable in poly(n) time.

Thus $a_k^2 - a_{k+1} a_{k-1}$ is computable in poly(n) time;

and thus belongs to #P.

Conclusion

We compare three log-concave inequalities:

Binomial inequality: in #P;

Permutation inversion inequality: in #P;

Stanley–Yan inequality: not in #P.

This differentiates Stanley–Yan inequality from binomial inequality and permutation inversion inequality.

Open Problem

Conjecture

Defect of Mason's conjecture

$$\mathrm{I}(k)^2 \; - \; \mathrm{I}(k+1)\,\mathrm{I}(k-1) \quad \notin \; \#\mathrm{P}.$$

We have shown Stanley–Yan inequality is not in $\#P$, but not Mason's conjecture.

감사합니다!

THANK YOU!

Preprint: www.arxiv.org/abs/2407.19608

Webpage: www.math.rutgers.edu/~sc2518/

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Complexity class NP

NP := Problems asking about **existence** of a solution S for input x, where validity of S can be verified in polynomial time.

Example (Problem in NP)

Is given graph G 3-colorable?

Complexity class coNP

coNP := Problems asking about **non-existence** of a solution S for input x, where validity of S can be verified in polynomial time.

Example (Problem in coNP)

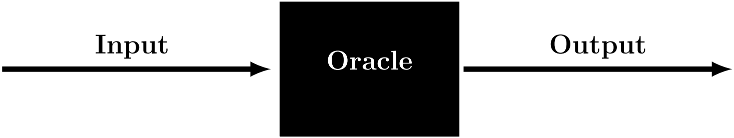
Is given graph G not 3-colorable?

It is known that

$$\text{NP} \neq \text{coNP} \implies \text{P} \neq \text{NP}.$$

NP-oracle

An NP-oracle is a black box that is able to solve any problem in NP in a single operation.



Complexity class $\mathsf{coNP}^{\mathsf{NP}}$

$\mathsf{NP}^{\mathsf{NP}}$:= Problems asking about existence of a solution S for input x, where validity of S can be verified in polynomial time, **with an** NP-**oracle**.

Complexity class $\mathsf{NP}^{\mathsf{NP}}$

$\mathsf{coNP}^{\mathsf{NP}}$:= Problems asking about non-existence of a solution S for input x, where validity of S can be verified in polynomial time, **with an** NP-**oracle.**

It is known that

$$\mathsf{NP}^{\mathsf{NP}} \neq \mathsf{coNP}^{\mathsf{NP}} \implies \mathsf{NP} \neq \mathsf{coNP} \implies \mathsf{P} \neq \mathsf{NP}\,.$$