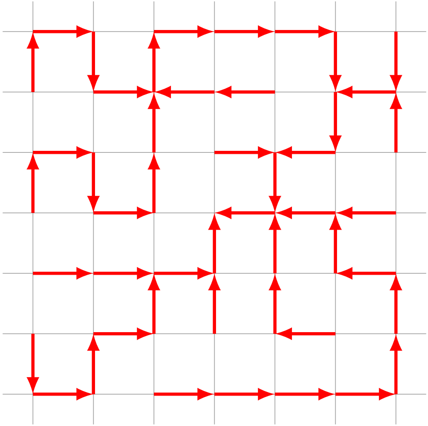
In between random walk and rotor walk

Swee Hong Chan

Rutgers University

Joint with Lila Greco, Lionel Levine, Peter Li

with very generous advice from Yuval Peres



Forbidden Palace's exploration experience (2019)

Initially, I explored the Forbidden Palace randomly without following any particular route.

Over time, I kept returning to the same gates and courtyards I have previously visited.

Everytime this happened, I would make a conscious decision to visit a new, unvisited direction.

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This talk will focus on the interplay between random and deterministic walks.

Simple random walk on $\mathbb{Z}^2$

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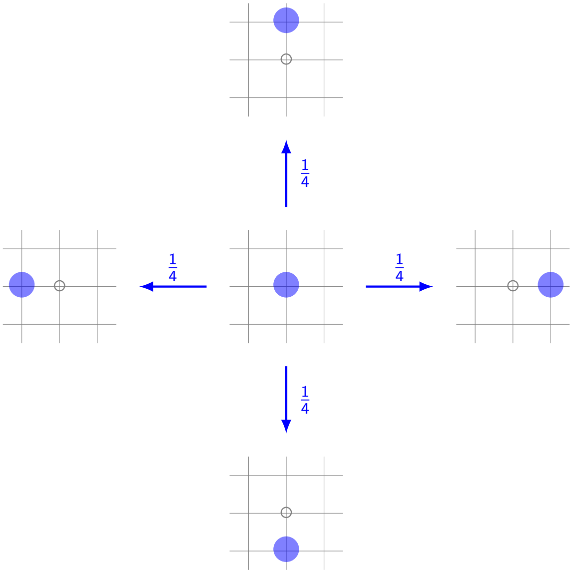


Simple random walk on $\mathbb{Z}^2$

- Visits every site infinitely often? Yes!
- Number of distinct points visited in n steps is $\asymp \frac{n}{\log n}$.
- Scaling limit? The standard 2-D Brownian motion:

$$(\frac{1}{\sqrt{n}} X([nt]))_{t \geq 0} \overset{n \to \infty}{\Longrightarrow} \frac{1}{\sqrt{2}} (B_1(t), B_2(t))_{t \geq 0},$$

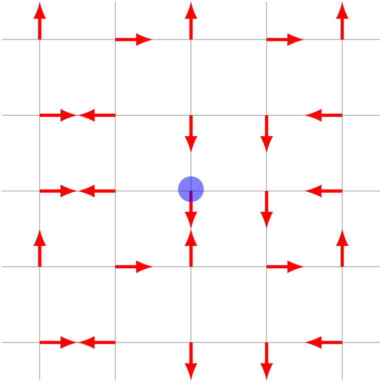
where

$$X(t) := \text{ Location of the walker at time-step } t,$$
$$B_1, B_2 := \text{Independent standard Brownian motions.}$$

Rotor walk on $\mathbb{Z}^2$

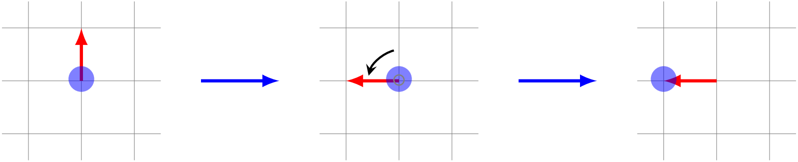
Rotor walk on $\mathbb{Z}^2$

Put a rotor (signpost) at each site.



Rotor walk on $\mathbb{Z}^2$

Turn the rotor 90° counterclockwise, then follow the rotor.



The rotor says:
"This is the way you went the last time you were here",
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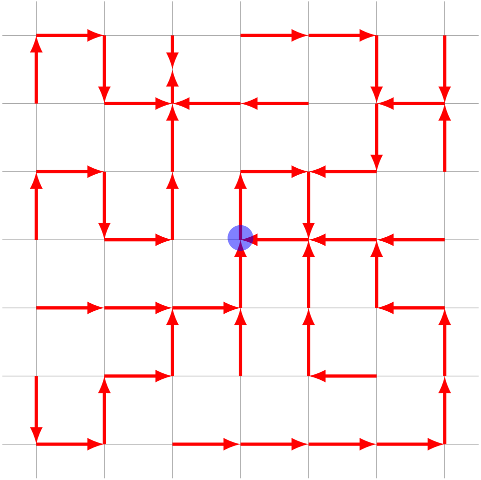

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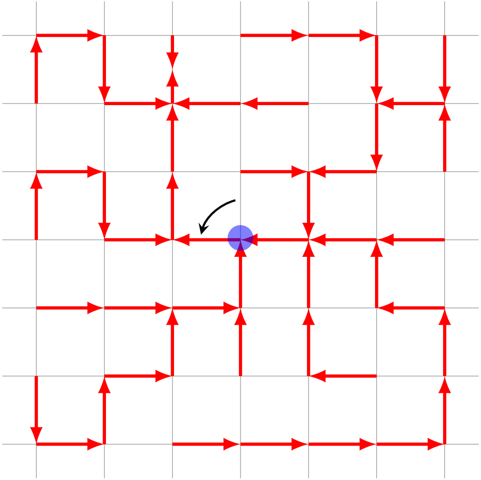

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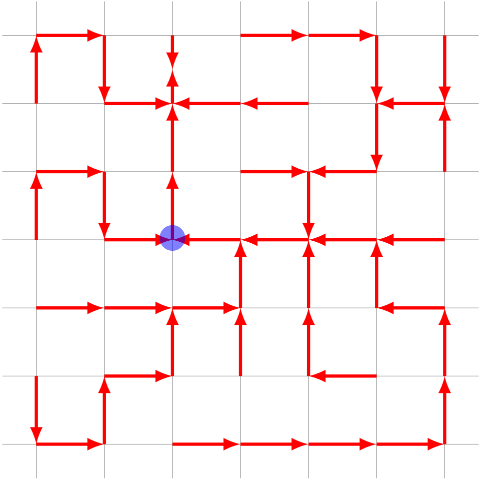

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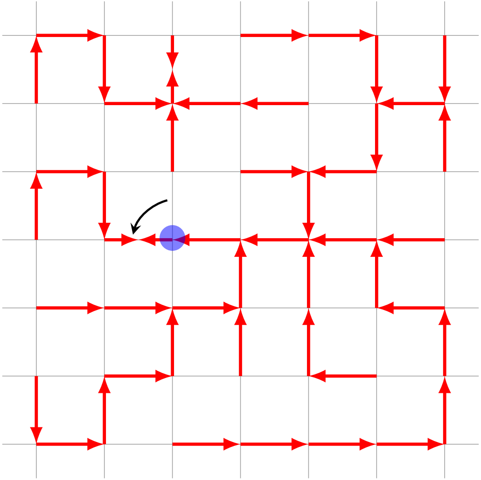

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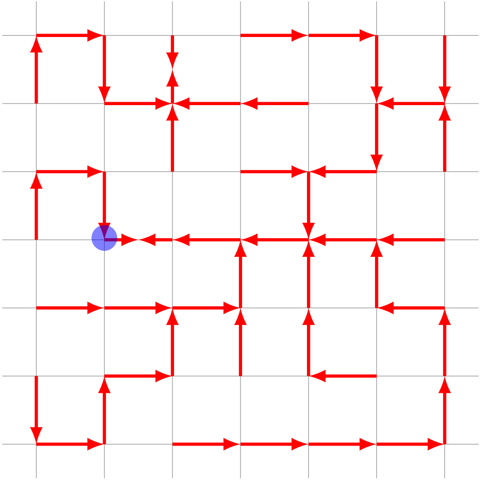

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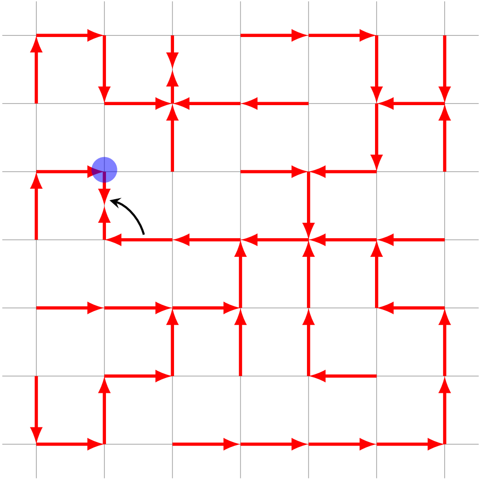

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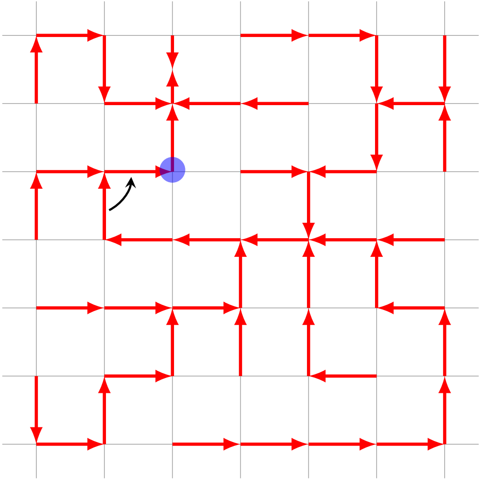

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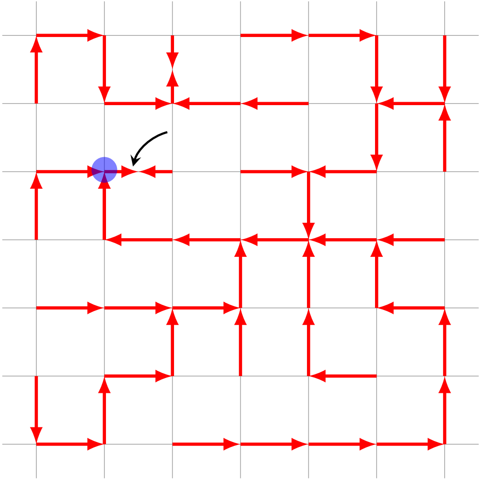

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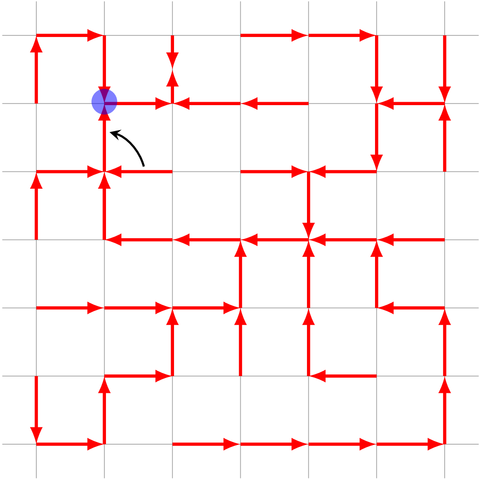

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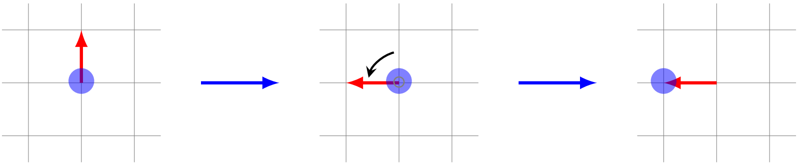


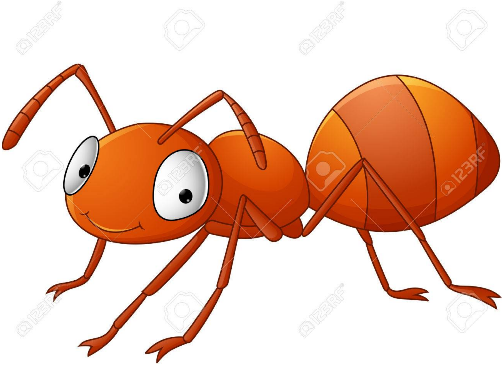
The rotor says:
"This is the way you went the last time you were here",
(assuming you ever were!)

Why rotor walk?

Randomness can be (was) expensive to simulate!

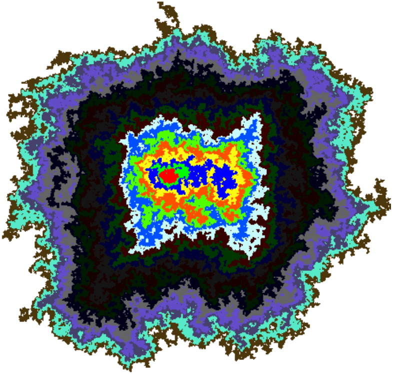
Why rotor walk?

As a model for ants' foraging strategy.



Why rotor walk?

As a model of self-organized criticality for statistical mechanics.



Visited sites after 80 returns to the origin (by Laura Florescu).

Conjectures for rotor walk on $\mathbb{Z}^2$

For initial rotors i.i.d. uniform among the four directions,

- (PDDK '96) Visits every site infinitely often?
- (PDDK '96) $\#\{X_1, \ldots, X_n\}$ is $\asymp n^{2/3}$?
  (compare with $n/\log n$ for the simple random walk.)
- (Kapri-Dhar '09) The asymptotic shape of $\{X_1, \ldots, X_n\}$ is a disc?

The bottleneck: lack of randomness

Well studied

Many open problems

Random

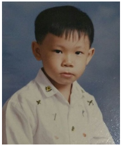
Deterministic

The bottleneck: lack of randomness

Well studied

Random

Let's study this!!!

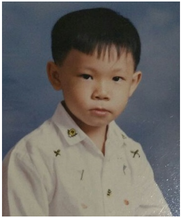


Something in between

Many open problems

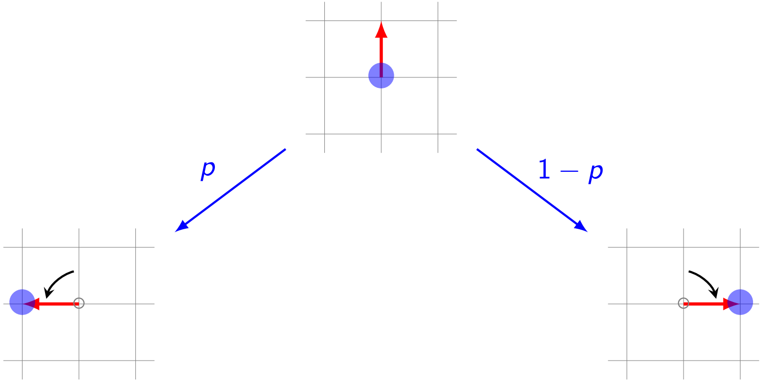
Deterministic

p-rotor walk on $\mathbb{Z}^2$



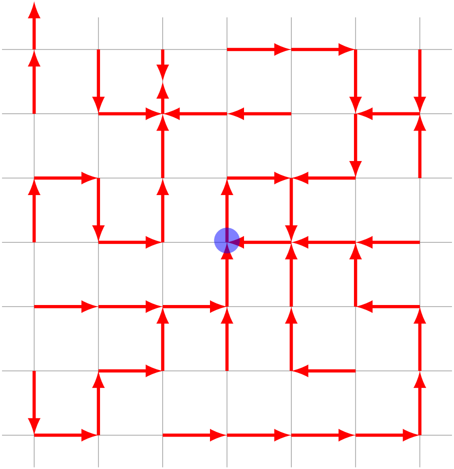
p-rotor walk on $\mathbb{Z}^2$

With probability p, turn the signpost 90° counter-clockwise.
With probability $1-p$, turn the signpost 90° clockwise.



p-rotor walk on $\mathbb{Z}^2$

Follow rotor walk rule with prob. p, do the opposite with prob. $1-p$



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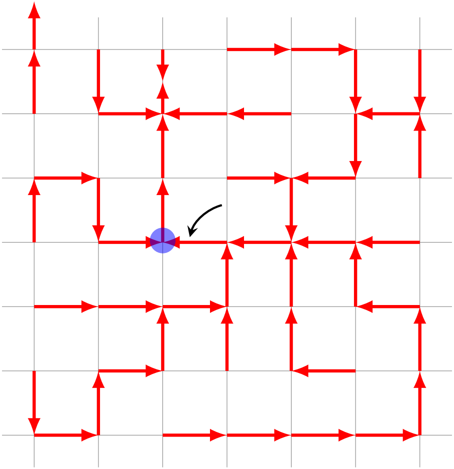
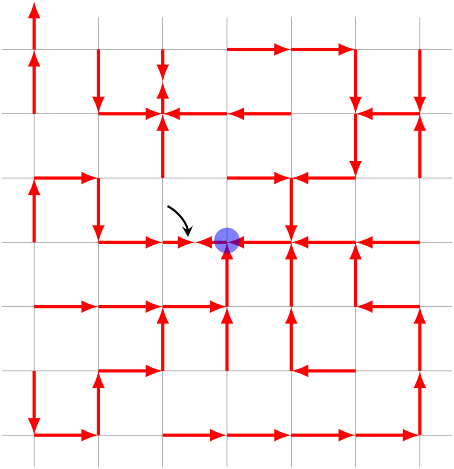


Follow the rule.

p-rotor walk on $\mathbb{Z}^2$

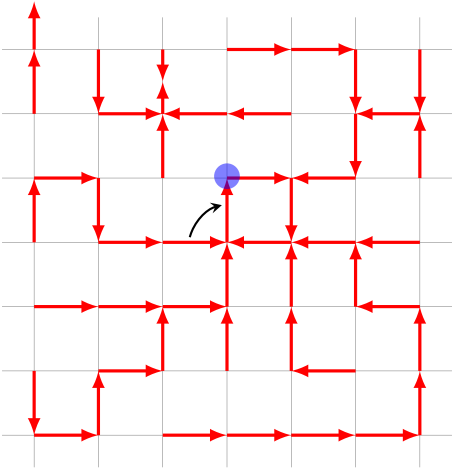
Follow rotor walk rule with prob. p, do the opposite with prob. $1 - p$



Do the opposite.

p-rotor walk on $\mathbb{Z}^2$

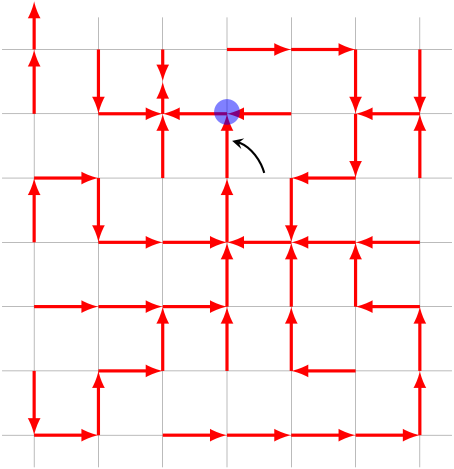
Follow rotor walk rule with prob. p, do the opposite with prob. $1 - p$



Do the opposite again.

p-rotor walk on $\mathbb{Z}^2$

Follow rotor walk rule with prob. p, do the opposite with prob. $1 - p$



Follow the rule.

p-rotor walk on $\mathbb{Z}^2$

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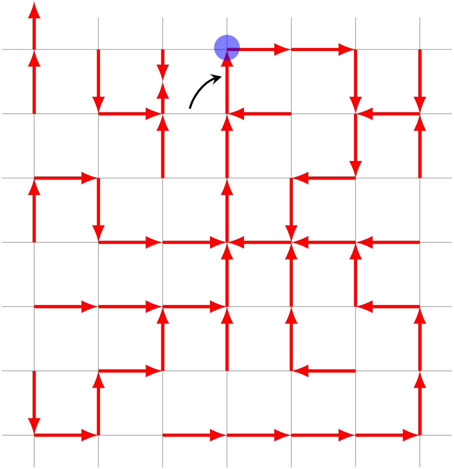

Do the opposite.

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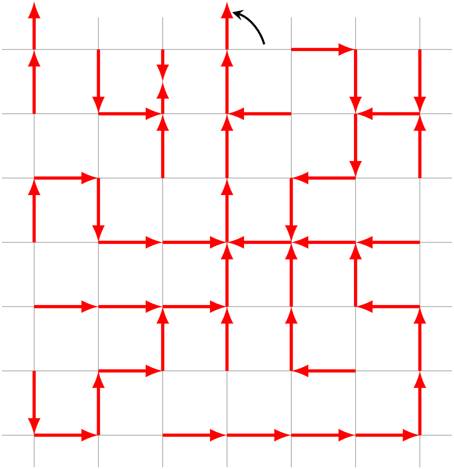
Follow rotor walk rule with prob. p, do the opposite with prob. $1-p$



Ops...

p-rotor walk on $\mathbb{Z}^2$

With probability p, turn the rotor 90° counter-clockwise.
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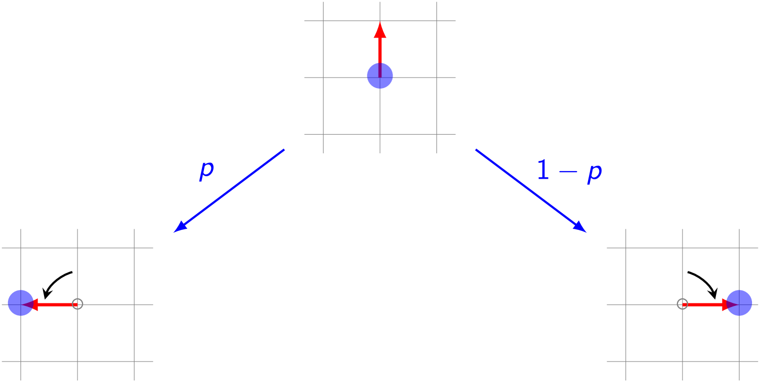


Recover the rotor walk if $p = 1$.

Recurrence result for p-rotor walk

Recurrence for p-rotor walk on $\mathbb{Z}^2$

Theorem (C., '23)

Let $p = \frac{1}{2}$ and let the i.i.d uniform among four directions be the initial rotor configuration. Then the p-rotor walk visits every vertex infinitely often almost surely.

Contrast this with rotor walk, where recurrence is still an open problem (PDDK '96).

Proof of recurrence for the simple random walk

Consider the following martingale:

$$M(t) := a(X(t)) \; - \; N(t),$$

where

$$\begin{aligned} X(t) &:= \text{ Location of walker at time } t; \\ a(x) &:= \text{ Potential kernel of } x \text{ at } \mathbb{Z}^2; \\ N(t) &:= \text{ Number of times walker leaving } o \text{ by time } t. \end{aligned}$$

Proof of recurrence for the simple random walk

Use the optional stopping theorem:

$$0 = \mathbb{E}[M(\tau(r))] \;\approx\; \frac{2}{\pi} \ln r \,(1 - p_{\mathrm{ret}}(r)) - 1,$$

where

$$\begin{aligned} \partial B_r &:= \text{Boundary of ball of radius } r; \\ \tau(r) &:= \text{First time walker reaching } \partial B_r \text{ or returning to } o; \\ p_{\mathrm{ret}}(r) &:= \text{Probability of reaching } \partial B_r \text{ before returning to } o. \end{aligned}$$

Proof of recurrence for the simple random walk

We rewrite the equation to

$$p_{\text{ret}}(r) \;\approx\; 1 - \frac{\pi}{2\ln r}.$$

Taking the limit $r \to \infty$, we conclude

$$p_{\text{rec}} \;=\; 1 - \lim_{r\to\infty} \frac{\pi}{2\ln r} = 1,$$

where

$$p_{\text{rec}} \;=\; \text{Probability of walker returning to } o,$$

is the recurrence probability.

Proof of recurrence for p-rotor walk

Consider the following martingale:

$$M(t) := a(X(t)) \; - \; N(t) \; + \underbrace{\sum_{x \in \{X_0, \ldots, X_t\}} \mathtt{w}(x; \rho_t)}_{\text{compensator}},$$

where $\mathtt{w}$ is a local compensator function.

By the same argument as before,

$$p_{\text{rec}} \; = \; 1 - \lim_{r \to \infty} \frac{\pi}{2 \ln r} \left(\sum_{|x| \leq r} \mathbb{E}[\mathtt{w}(x; \rho_{\tau(r)})] \right).$$

Proof of recurrence for p-rotor walk

We can estimate the terms in the compensator locally by

$$\left|\mathbb{E}[\mathtt{w}(x;\rho_{\tau(r)})]\right| \;\leq\; \left(1-\frac{1}{2^{70}}\right)\frac{2}{\pi|x|^2}.$$

Plugging this estimate into previous equation,

$$p_{\mathrm{rec}} \geq 1 - \lim_{r\to\infty}\frac{\pi}{2\ln r}\left(\sum_{|x|\leq r}\left(1-\frac{1}{2^{70}}\right)\frac{2}{\pi|x|^2}\right) \;=\; \frac{1}{2^{70}} > 0.$$

By Kolmogorov zero-one law, the recurrence probability is 1.

So we have proved ...

Theorem (C., '23)

Let $p = \frac{1}{2}$ and let the i.i.d uniform among four directions be the initial rotor configuration. Then the p-rotor walk visits every vertex infinitely often almost surely.

We need to assume $p = \frac{1}{2}$, as otherwise the bound for the compensator decays linearly instead of quadratically.

Open problem

Conjecture

Let $p \neq \frac{1}{2}$. Prove that p-rotor walk with i.i.d. uniform rotor configuration is recurrent.

Obstacle: Need a good estimate for the compensator.

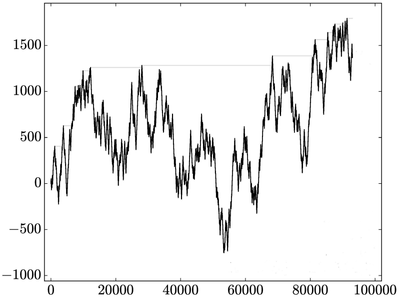
$$\underbrace{M(t)}_{\text{martingale}} := a(X(t)) - N(t) + \underbrace{\sum_{x \in \{X_0, \ldots, X_t\}} \mathtt{w}(x; \rho_t)}_{\text{compensator}}.$$

Scaling limit result for p-rotor walk

Scaling limit for p-rotor walk on $\mathbb{Z}$

(Huss, Levine, Sava-Huss 18) The scaling limit for p-rotor walk on $\mathbb{Z}$ is a perturbed Brownian motion $(Y(t))_{t\geq 0}$,

$$Y(t) = \underbrace{B(t)}_{\substack{\text{standard}\\\text{Brownian}\\\text{motion}}} + \underbrace{a \sup_{0\leq s\leq t} Y(s)}_{\substack{\text{perturbation at}\\\text{maximum}}} + \underbrace{b \inf_{0\leq s\leq t} Y(s)}_{\substack{\text{perturbation at}\\\text{minimum}}}, \qquad t \geq 0.$$



$Y(t)$ for $a = -0.998$, and $b = 0$ (by Wilfried Huss).

Scaling limit for p-rotor walk on $\mathbb{Z}^2$

Question: Is the scaling limit for p-rotor walk on $\mathbb{Z}^2$ a "2-D perturbed Brownian motion"?

Problem: How to define "2-D perturbed Brownian motion"?.

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Conjecture: The scaling limit for p-rotor walk on $\mathbb{Z}^2$ when $p = \frac{1}{2}$ is the standard 2-D Brownian motion.

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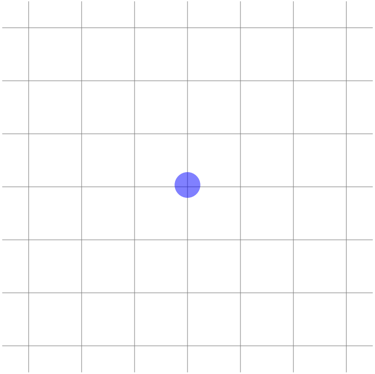
Conjecture: The scaling limit for p-rotor walk on $\mathbb{Z}^2$ when $p = \frac{1}{2}$ is the standard 2-D Brownian motion.

Obstacle: Some technical issues persist with i.i.d. rotor configurations, so we need different initial rotor configurations.

Uniform spanning forest plus one edge (USF$^+$)



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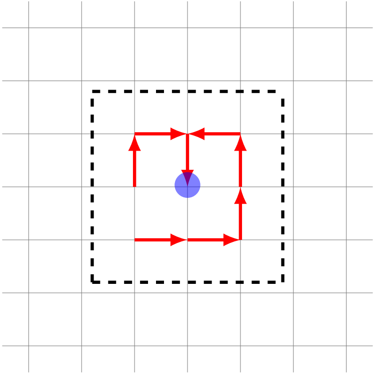
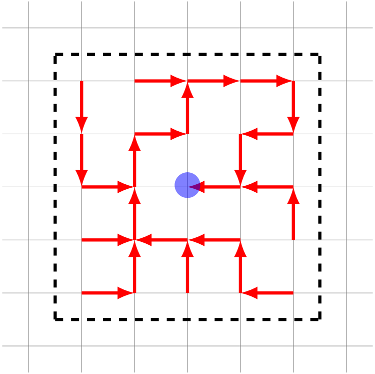


Pick a spanning tree of the black box directed to the origin (uniformly at random).

Uniform spanning forest plus one edge (USF$^+$)



Take the limit as the black box grows until it covers $\mathbb{Z}^2$.

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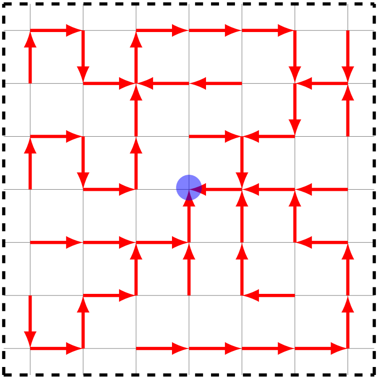

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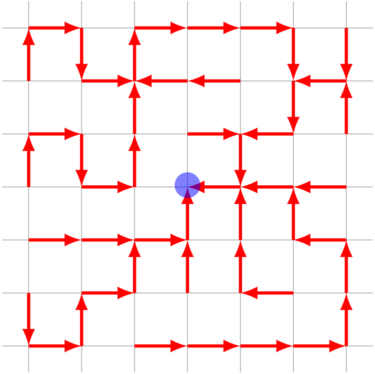

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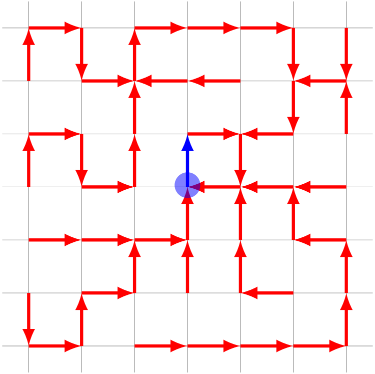


Add a rotor to the origin, uniform among the four directions.

Scaling limit for p-rotor walk on $\mathbb{Z}^2$

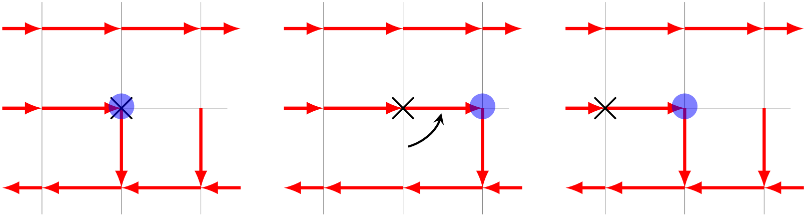
Theorem (C., Greco, Levine, Li '21)

Let $p = \frac{1}{2}$ and let the uniform spanning forest plus one edge be the initial rotor configuration. Then, with probability 1, the p-rotor walk on $\mathbb{Z}^2$ scales to the standard 2-D Brownian motion:

$$\frac{1}{\sqrt{n}}(\underbrace{X_{[nt]}}_{\substack{\text{location of the} \\ \text{walker at time } [nt]}})_{t\geq 0} \overset{n\to\infty}{\Longrightarrow} \frac{1}{\sqrt{2}}(\underbrace{B_1(t), B_2(t)}_{\substack{\text{independent} \\ \text{Brownian motions}}})_{t\geq 0}.$$

Stationarity from the walker's point of view

A random rotor configuration is stationary from the walker's POV if its statistical distribution looks identical before and after a step, as long as we shift the grid to keep the walker at the center.

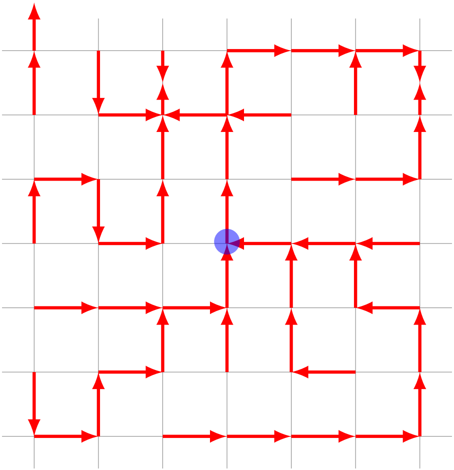


Initial state

Walker moves one step to the right

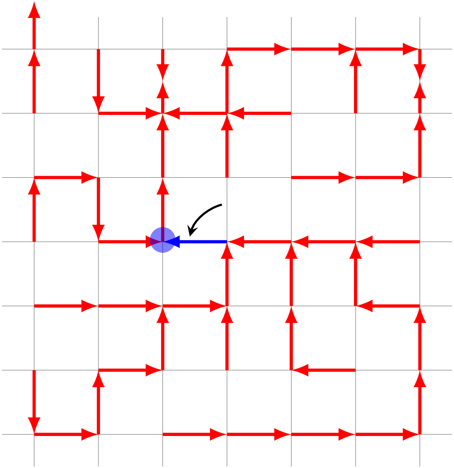
Shift grid one unit to the left

Why is USF$^+$ stationary from walker's POV?



The rotors at previously visited sites form a tree oriented toward the walker.

Why is USF$^+$ stationary from walker's POV?



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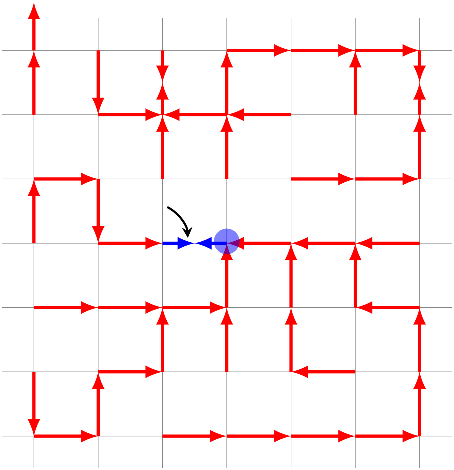

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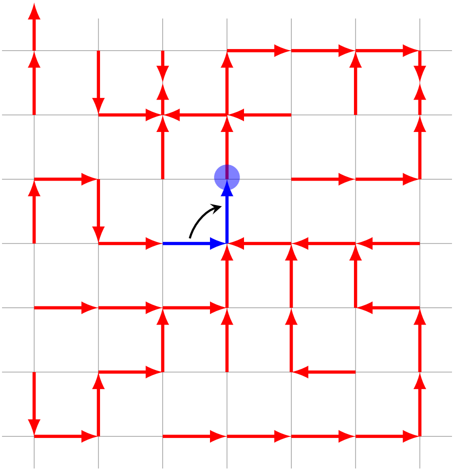
Why is USF^{+} stationary from walker's POV?



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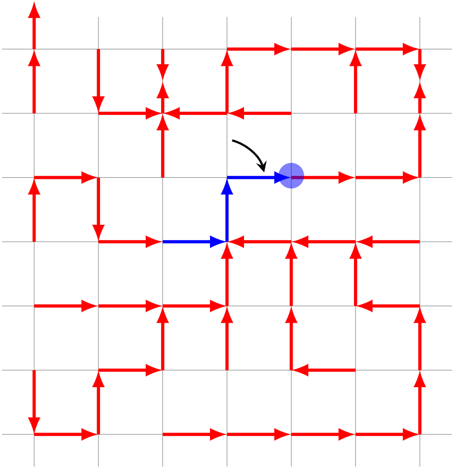
Why is USF$^+$ stationary from walker's POV?



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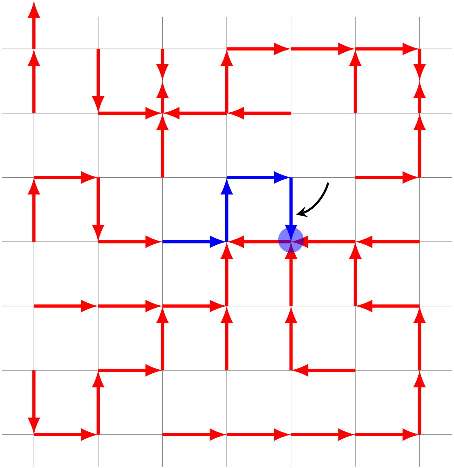

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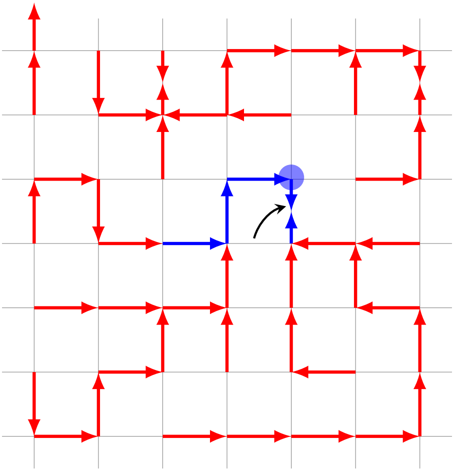
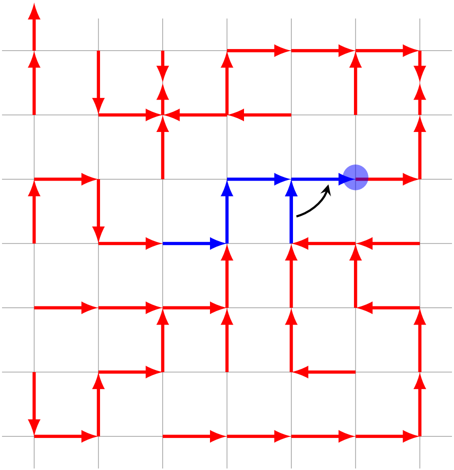
Why is USF$^+$ stationary from walker's POV?



The rotors at previously visited sites form a tree oriented toward the walker.

Why is USF$^+$ stationary from walker's POV?



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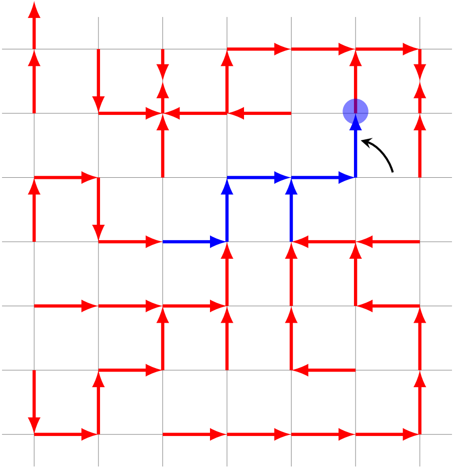

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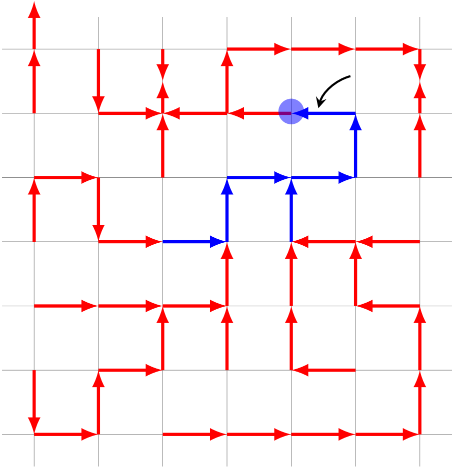

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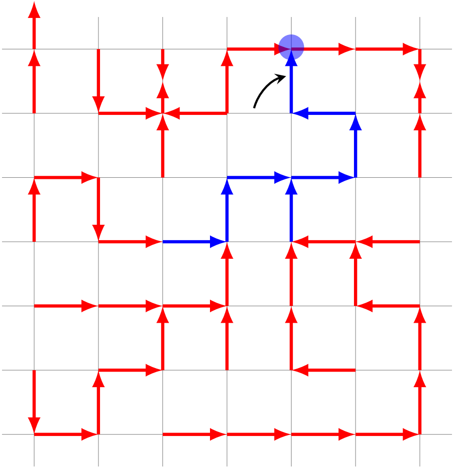

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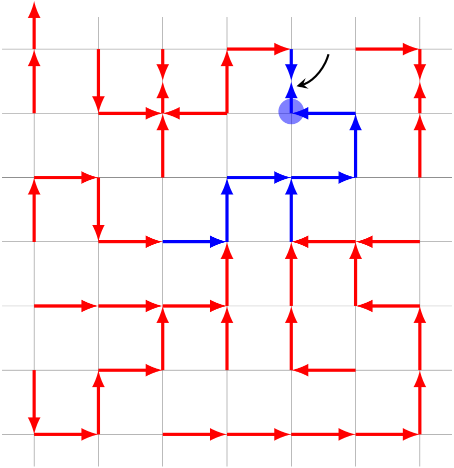
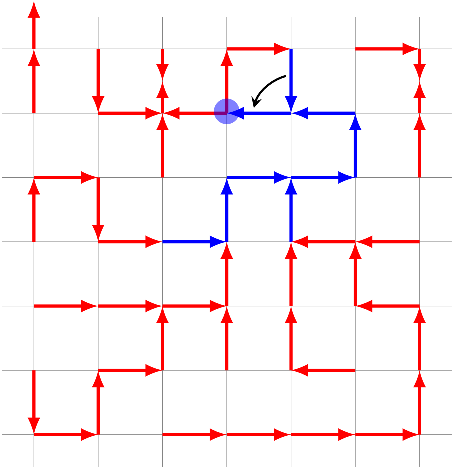


The rotors at previously visited sites form a tree oriented toward the walker.

Why is USF$^+$ stationary from walker's POV?



The rotors at previously visited sites form a tree oriented toward the walker.

Sketch of the scaling limit proof

Scaling limit

Martingale CLT

Encounters vertical rotors half the time a.s.

Ergodic theorem

Stationarity and ergodicity of USF$^+$ from walker's POV

So we have proved...

Theorem (C., Greco, Levine, Li '21)

Let $p = \frac{1}{2}$ and let the uniform spanning forest plus one edge be the initial rotor configuration. Then, with probability 1, the p-rotor walk on $\mathbb{Z}^2$ scales to the standard 2-D Brownian motion:

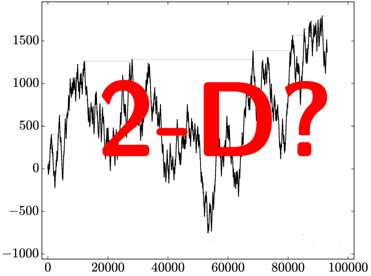
$$\frac{1}{\sqrt{n}}(\underbrace{X_{[nt]}}_{\substack{\textit{location of the}\\ \textit{walker at time } [nt]}})_{t\geq 0} \overset{n\to\infty}{\Longrightarrow} \frac{1}{\sqrt{2}}(\underbrace{B_1(t), B_2(t)}_{\substack{\textit{independent}\\ \textit{Brownian motions}}})_{t\geq 0}.$$

Open Problem

Problem

Find the *scaling limit* *for the p-rotor walk with i.i.d. uniform rotor configuration.*

Obstacle: Need to define "2-D perturbed Brownian motion (?)".

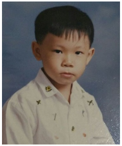


Back to our motivation

Well studied

Know a little bit now

Many open problems



Simple random walk

p-rotor walk

Rotor walk

Let's apply what we have learnt to rotor walk.

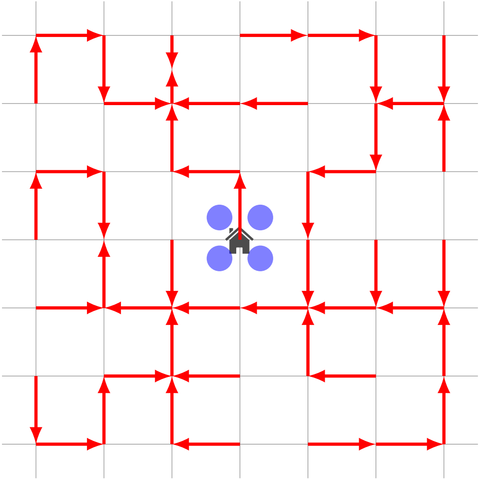
Prison break using rotor walk

Prison break using rotor walks

- Starts with n walkers at the origin (prison), with initial rotor configuration ρ.
- Walkers attempt to escape by performing rotor walks. Each walker completes their entire path before the next one starts.
- If a walker ever returns to the origin, it is removed.
- Rotors are never reset between escape attempts.

Prison break using rotor walk

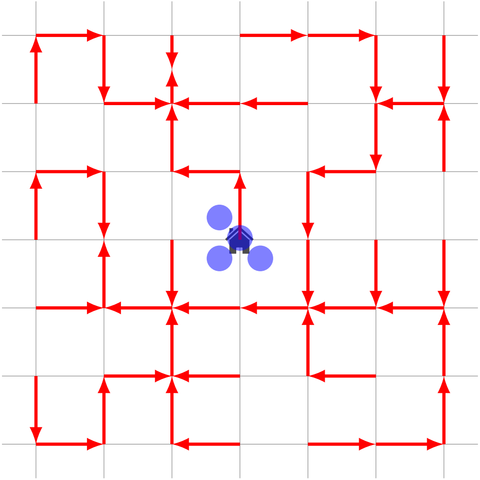
Put n walkers at the origin (the prison).



Note: Rotors are never reset between escape attempts.

Prison break using rotor walk

First walker performs rotor walk to escape the prison.



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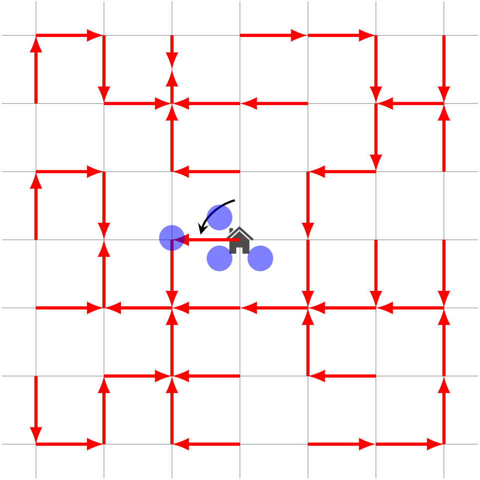

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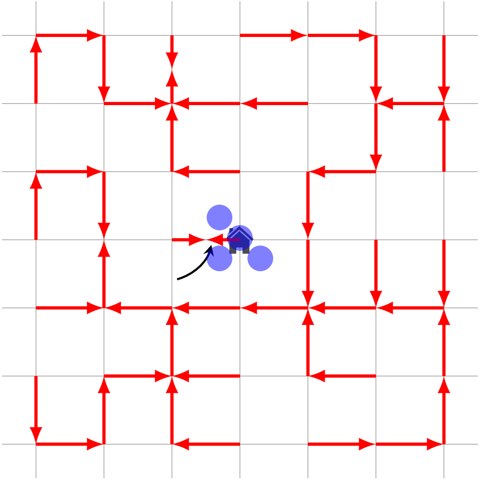
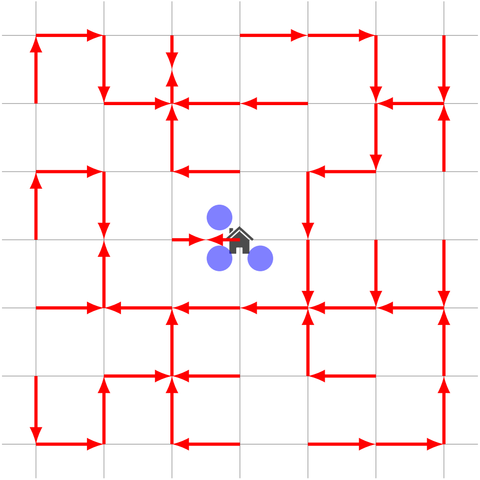


Note: Rotors are never reset between escape attempts.

Prison break using rotor walk

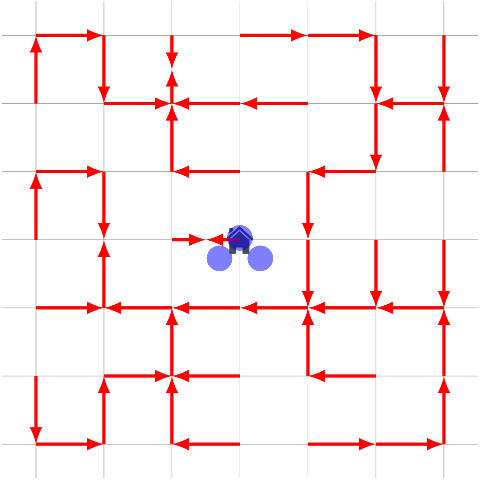
First walker returns to prison, and is removed.



Note: Rotors are never reset between escape attempts.

Prison break using rotor walk

Second walker performs rotor walk.



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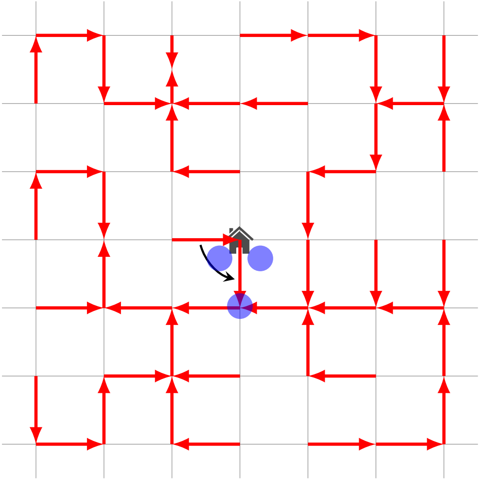

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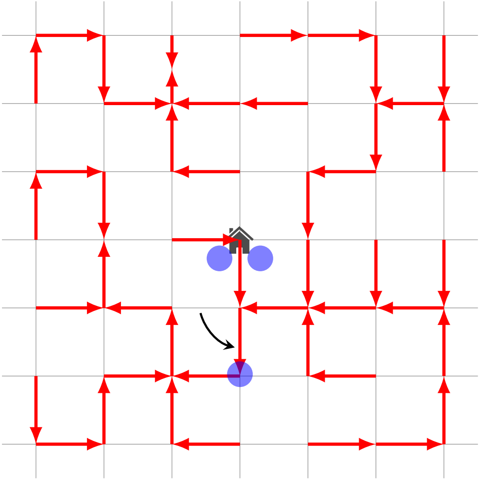

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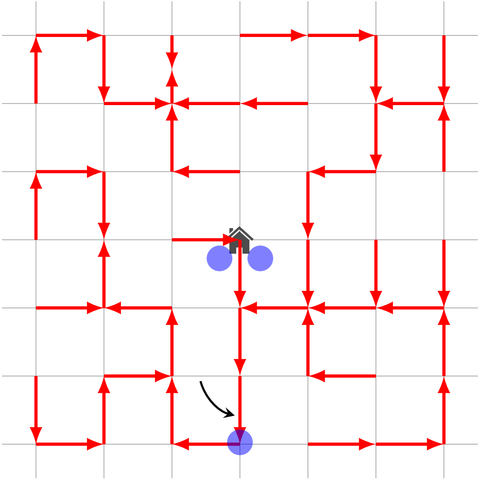

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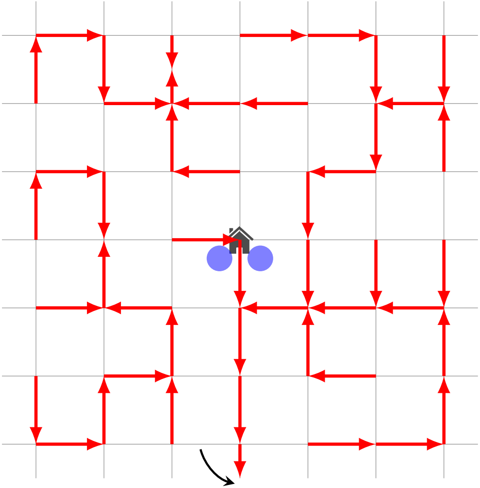
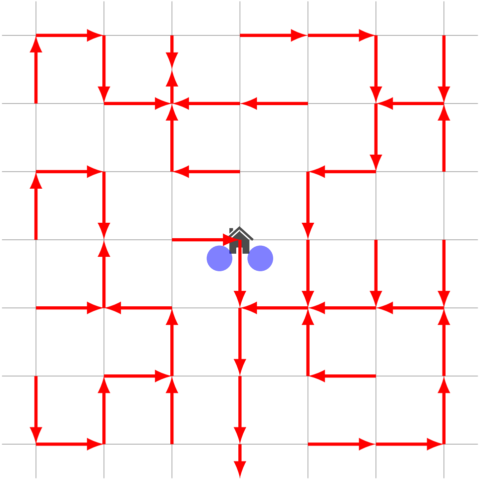


Note: Rotors are never reset between escape attempts.

Prison break using rotor walk

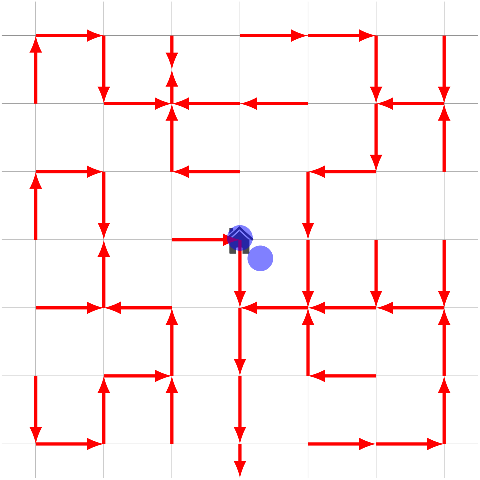
Second walker never returns to origin.



Note: Rotors are never reset between escape attempts.

Prison break using rotor walk

Third walker performs rotor walk to escape.



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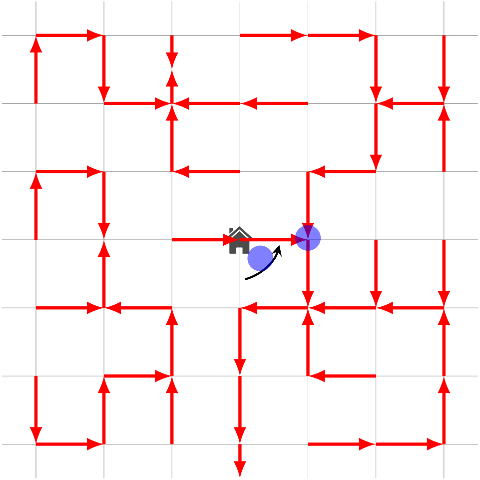

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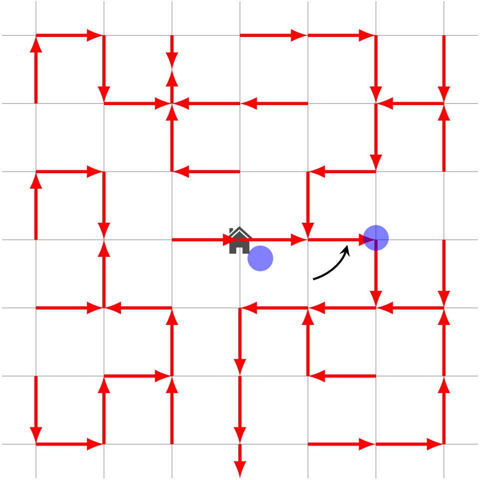

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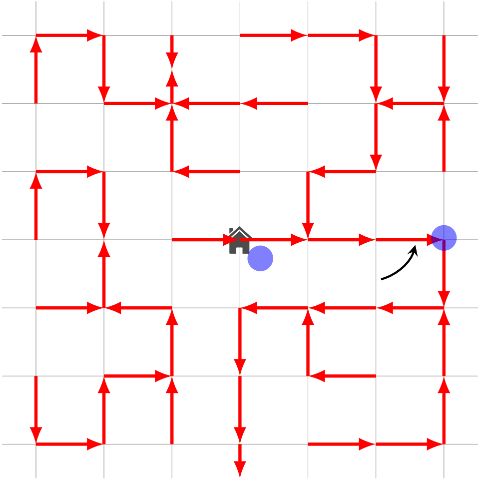

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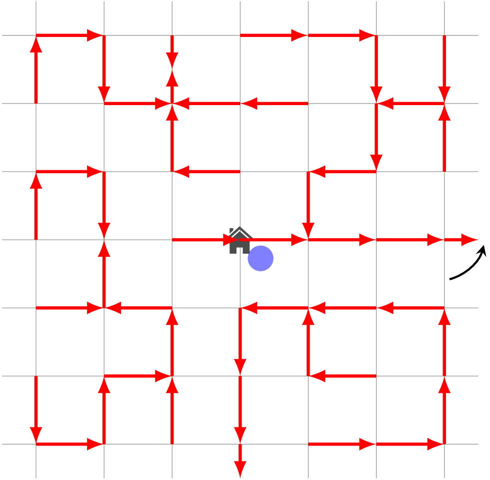
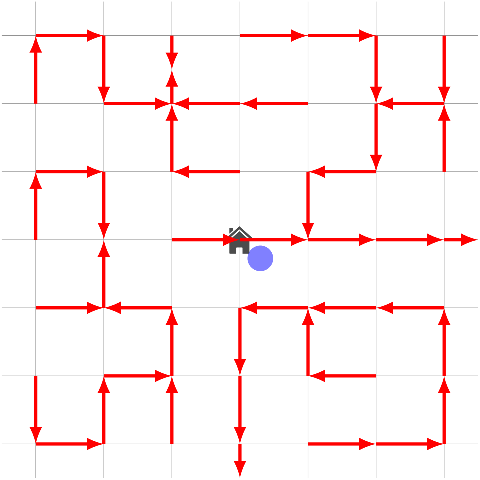


Note: Rotors are never reset between escape attempts.

Prison break using rotor walk

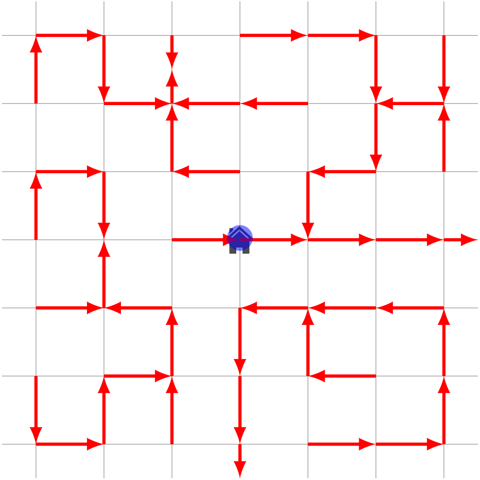
Third walker never returns to prison.



Note: Rotors are never reset between escape attempts.

Prison break using rotor walk

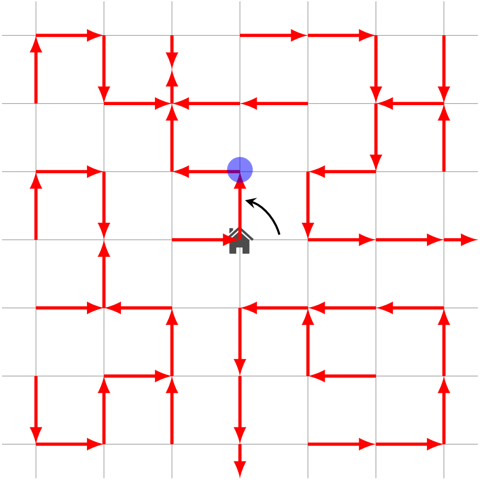
Fourth walker performs rotor walk.



Note: Rotors are never reset between escape attempts.

Prison break using rotor walk

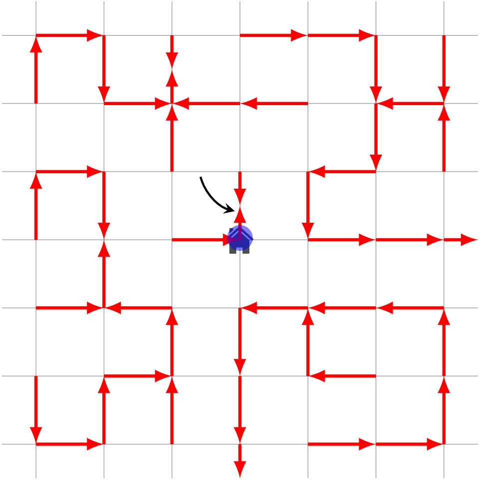
Fourth walker performs rotor walk.



Note: Rotors are never reset between escape attempts.

Prison break using rotor walk

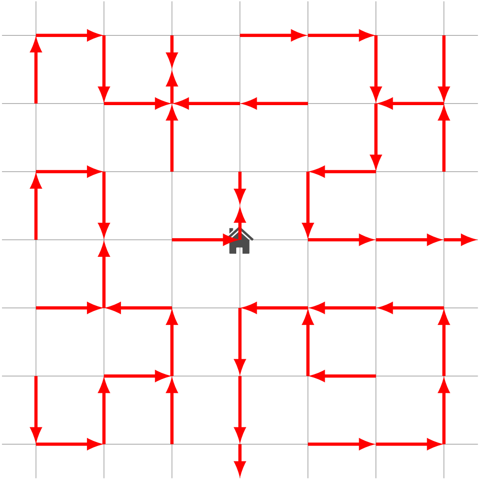
Fourth walker performs rotor walk.



Note: Rotors are never reset between escape attempts.

Prison break using rotor walk

Fourth walker returns to prison, and is removed.



Note: Rotors are never reset between escape attempts.

Escape rate of rotor walk

The escape rate of n rotor walkers with initial rotor ρ is

$$r_{\text{esc}}(\rho, n) := \frac{\text{number of escaped walkers}}{n}.$$

The escape rate of rotor walk is a deterministic counterpart of the escape probability of simple random walk.

What was known about escape rate

Theorem (Schramm '10 (posthumous))

For any initial rotor configuration ρ,

$$\limsup_{n\to\infty} \underbrace{r_{esc}(\rho, n)}_{\substack{\text{escape rate} \\ \text{of rotor walk}}} \quad \leq \quad \underbrace{p_{esc}(SRW)}_{\substack{\text{escape prob.} \\ \text{of SRW}}}.$$

Takeaway: Rotor walk can never be better than simple random walk for prison escape.

What was known about escape rate

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Takeaway: Rotor walk can never be better than simple random walk for prison escape.

Corollary

On $\mathbb{Z}^2$, for any initial rotor configuration ρ,

$$\lim_{n\to\infty} r_{esc}(\rho, n) \;=\; p_{esc}(SRW) = 0.$$

In fact, this is true for all recurrent graphs.

What was known about escape rate

Theorem (Angel Holroyd '09)

On $\mathbb{Z}^d$ with $d \geq 3$, there exists an initial rotor ρ so that

$$\lim_{n\to\infty} r_{esc}(\rho, n) \;=\; 0.$$

Theorem (Florescu Ganguly Levine Peres '13)

On $\mathbb{Z}^d$ with $d \geq 3$, for the one-directional initial rotor ρ,

$$\liminf_{n\to\infty} r_{esc}(\rho, n) \;>\; 0.$$

Takeaway: For $d \geq 3$, the escape rate heavily depends on the initial rotor configuration ρ.

Escape rate conjecture

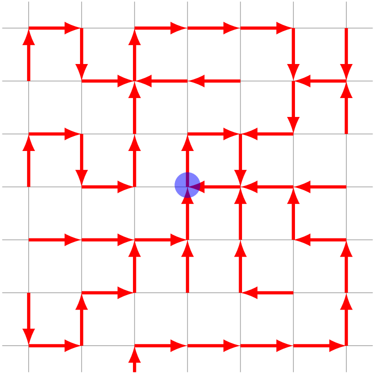
Conjecture (FGLP '13)

For any transient graph, there exists an initial rotor ρ for which

$$\lim_{n\to\infty} r_{esc}(\rho, n) \;=\; p_{esc}(SRW).$$

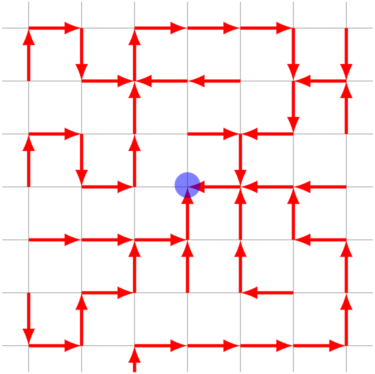
That is, there exists an initial rotor configuration so rotor walk performs just as well as random walk for prison escape.

Uniform spanning forest oriented to infinity (USF$^\infty$)



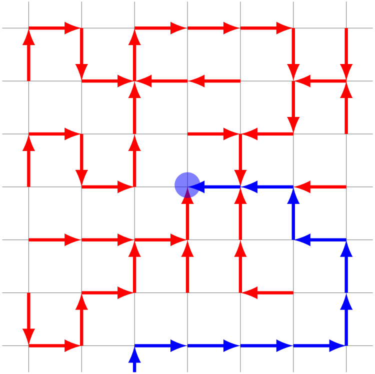
Start with uniform spanning forest plus one edge from before.

Uniform spanning forest oriented to infinity (USF$^\infty$)



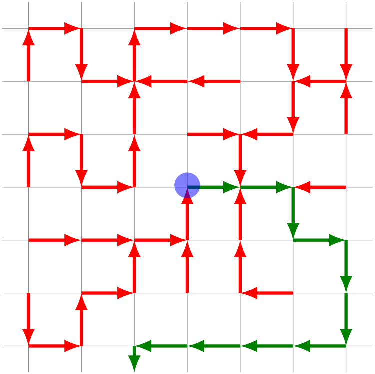
Remove the rotor at the origin.

Uniform spanning forest oriented to infinity (USF$^\infty$)



Find the unique infinite path oriented to origin.

Uniform spanning forest oriented to infinity (USF$^\infty$)



Reverse the orientation of this infinite path.

Answering the escape rate conjecture

Theorem (C. '19)

On $\mathbb{Z}^d$, almost every ρ sampled from USF^∞ satisfies

$$\lim_{n\to\infty} r_{esc}(\rho, n) \;=\; p_{esc}(SRW).$$

Proof sketch

On one hand, for all initial rotor ρ,

$$r_{\text{esc}}(\rho, n) \quad \leq \quad p_{\text{esc}}(\text{SRW}) + \frac{C}{n^2} \qquad \text{(A)}$$

for some $C > 0$, by Schramm's inequality.

On the other hand, for ρ sampled from USF^∞,

$$\mathbb{E}_{\rho \sim \text{USF}^\infty}\left[r_{\text{esc}}(\rho, n)\right] \quad \geq \quad p_{\text{esc}}(\text{SRW}) \qquad \text{(B)}$$

by infinite-step stationarity of USF^∞.

How to combine (A) and (B) to get

$$\lim_{n \to \infty} r_{\text{esc}}(\rho, n) \; = \; p_{\text{esc}}(\text{SRW})$$

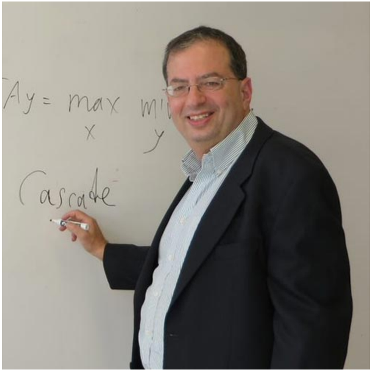
for almost every ρ sampled from USF^∞?

A Breakthrough Insight

After three months of being completely stuck, one conversation with Yuval Peres solved the whole problem.

一语惊醒梦中人

"One wise advice awakens the dreaming student"



Proof sketch (continued)

Fix $\varepsilon > 0$. Let A_n be set of initial rotors ρ such that

$$A_n \ := \ \left\{ \ \left| r_{\text{esc}}(\rho, n) \ - \ p_{\text{esc}}(\text{SRW}) \right| > \varepsilon \ \right\},$$

i.e. n-th rotor walk escape rate differs from escape probability of simple random walk by more than ε.

We need to show A_n occurs only finitely many times
for almost every ρ sampled from USF^∞.

By Borel–Cantelli lemma, it suffices to show

$$\sum_{n=1}^{\infty} \mathbb{P}[A_n] \quad < \quad \infty.$$

Proof sketch (continued)

By Markov's inequality,

$$\mathbb{P}[A_n] \quad \le \quad \frac{1}{\varepsilon}\mathbb{E}\left[\left| r_{\text{esc}}(\rho, n) \; - \; p_{\text{esc}}(\text{SRW})\right|\right].$$

By triangle inequality, RHS is less than

$$\frac{1}{\varepsilon}\mathbb{E}\left[\left| r_{\text{esc}}(\rho, n) \; - \; p_{\text{esc}}(\text{SRW}) - \frac{C}{n^2}\right|\right] \; + \; \frac{C}{\varepsilon n^2}.$$

Proof sketch (continued)

Recall that we already have

$$r_{\text{esc}}(\rho, n) \quad \leq \quad p_{\text{esc}}(\text{SRW}) + \frac{C}{n^2}. \qquad \text{(A)}$$

So the term inside $\mathbb{E}[\cdot]$ is negative,

$$\mathbb{P}[A_n] \quad \leq \quad \frac{1}{\varepsilon}\mathbb{E}\left[p_{\text{esc}}(\text{SRW}) + \frac{C}{n^2} - r_{\text{esc}}(\rho, n)\right] \; + \; \frac{C}{\varepsilon n^2}.$$

By linearity of expectation,

$$\mathbb{P}[A_n] \quad \leq \quad \frac{1}{\varepsilon}\left(p_{\text{esc}}(\text{SRW}) + \frac{C}{n^2} - \mathbb{E}\left[r_{\text{esc}}(\rho, n)\right]\right) \; + \; \frac{C}{\varepsilon n^2}.$$

Proof sketch (continued)

On the other hand, we already have

$$\mathbb{E}\left[r_{\mathrm{esc}}(\rho, n)\right] \quad \geq \quad p_{\mathrm{esc}}(\mathrm{SRW}). \tag{B}$$

So we can cancel these two terms,

$$\mathbb{P}[A_n] \quad \leq \quad \frac{1}{\varepsilon}\left(\cancel{p_{\mathrm{esc}}(\mathrm{SRW})} + \frac{C}{n^2} - \cancel{\mathbb{E}\left[r_{\mathrm{esc}}(\rho, n)\right]}\right) + \frac{C}{\varepsilon n^2}.$$

This gives us

$$\sum_{n=1}^{\infty} \mathbb{P}[A_n] \quad \leq \quad \sum_{n=1}^{\infty} \frac{2C}{\varepsilon n^2} \quad < \quad \infty.$$

So we have proved ...

Theorem (C. '19)

On $\mathbb{Z}^d$, almost every ρ sampled from USF^∞ satisfies

$$\lim_{n\to\infty} r_{esc}(\rho, n) \;=\; p_{esc}(SRW).$$

Remark: Similar result applies to all vertex-transitive graphs.

Except that ...

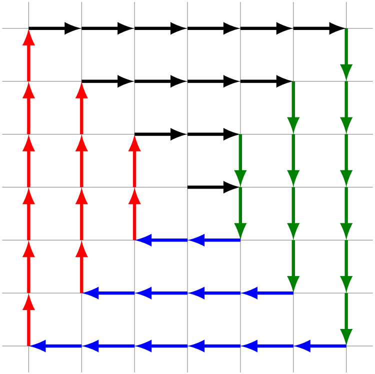
- The conjecture of FGLP '13 is for all transient graphs;
- There are already other constructions for the special case of $\mathbb{Z}^d$ (He '14) and trees (Angel Holroyd '11);
- Our construction of the initial rotor ρ is not deterministic.

Complete answer to the escape rate conjecture

Theorem (C., '20)

For any transient graph, the initial rotor $\rho_{\max}$ satisfies

$$\lim_{n\to\infty} r_{esc}(\rho_{\max}, n) \;=\; p_{esc}(SRW).$$



Escape rate formula

Lemma

For any initial rotor ρ and number of walkers n,

$$r_{esc}(\rho, n) = p_{esc}(SRW) - \sum_{x \in \mathbb{Z}^d} \left(\mathtt{w}_x[\underbrace{\rho_n(x)}_{\substack{\text{rotor at } x \\ \text{after } n\text{-th walk}}}] - \mathtt{w}_x[\underbrace{\rho(x)}_{\substack{\text{initial rotor} \\ \text{at } x}}] \right),$$

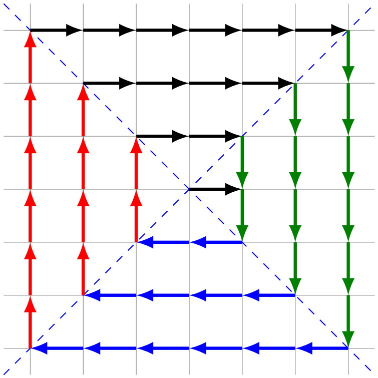
where $\mathtt{w}_x$ is a local compensator term.

The formula is inspired by the martingale used in proving recurrence for p-rotor walk.

Our initial rotor configuration

The configuration ρ_{max} is constructed by choosing, for each x,

the direction $\rho_{\mathsf{max}}(x)$ that maximizes compensator $\mathtt{w}_x$.



Proof of the escape rate conjecture

- By the escape rate formula,

$$r_{\text{esc}}(\rho, n) \;=\; p_{\text{esc}}(SRW) - \sum_{x \in \mathbb{Z}^d} \Big(\mathtt{w}_x[\rho_n(x)] - \mathtt{w}_x[\rho(x)] \Big),$$

- By our choice of $\rho_{\max}$,

$$r_{\text{esc}}(\rho_{\max}, n) \;\geq\; p_{\text{esc}}(SRW).$$

- On the other hand, Schramm's inequality gives us

$$\limsup_{n \to \infty} r_{\text{esc}}(\rho_{\max}, n) \;\leq\; p_{\text{esc}}(SRW).$$

- Hence,

$$\lim_{n \to \infty} r_{\text{esc}}(\rho_{\max}, n) \;=\; p_{\text{esc}}(SRW).$$

So we have proved...

Theorem (C., '20)

For any transient graph, the initial rotor $\rho_{\max}$ satisfies

$$\lim_{n\to\infty} r_{esc}(\rho_{\max}, n) \; = \; p_{esc}(SRW).$$

Open problem

Conjecture

For any graph, the i.i.d. uniform rotor configuration has rotor walk escape rate equal to the escape probability of the SRW, i.e.,

$$\lim_{n\to\infty} r_{esc}(\rho, n) = p_{esc}(SRW).$$

So far has only been proved for regular trees (Angel Holroyd '11).

感谢聆听！

THANK YOU!