Log-concavity, Cross Product Conjectures, and FKG Inequalities in Order Theory

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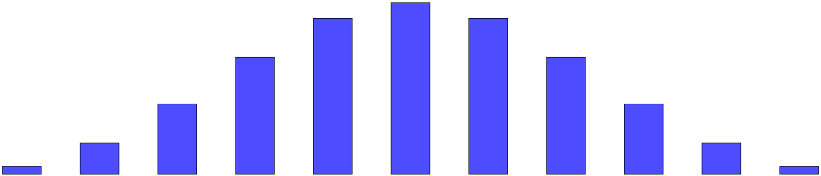
joint with Igor Pak and Greta Panova



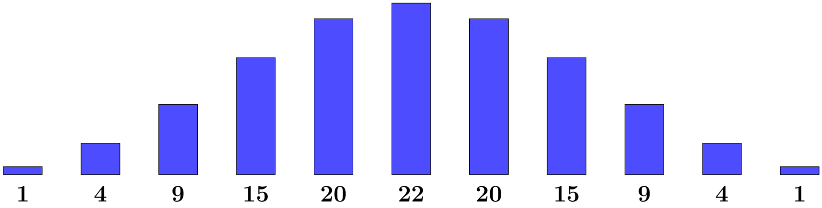
What is log-concavity?

A sequence $a_1, \ldots, a_n \in \mathbb{R}_{\geq 0}$ is log-concave if

$$a_k^2 \;\geq\; a_{k+1}\, a_{k-1} \qquad \text{for all} \;\; 1 < k < n.$$

Log-concavity (and positivity) implies unimodality:

$$a_1 \;\leq\; \cdots \;\leq\; a_m \;\geq\; \cdots \;\geq\; a_n \quad \text{for some} \;\; 1 \leq m \leq n.$$



Example: binomial coefficients

$$a_k = \binom{n}{k} \qquad k = 0, 1, \ldots, n.$$

This sequence is log-concave because

$$\frac{a_k^2}{a_{k+1}\, a_{k-1}} = \frac{\binom{n}{k}^2}{\binom{n}{k+1}\binom{n}{k-1}} = \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right),$$

which is greater than 1.

Example: permutations with k inversions

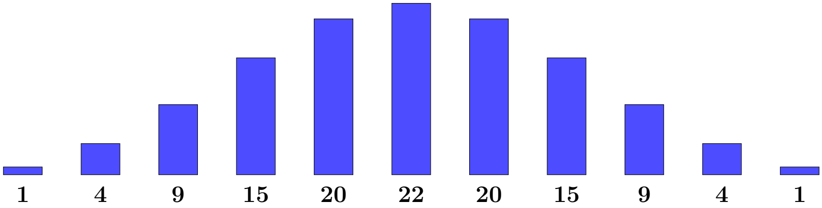
$$a_k \;=\; \text{number of } \pi \in S_n \text{ with } k \text{ inversions},$$

where inversion of π is pair $i < j$ s.t. $\pi_i > \pi_j$.

This sequence is log-concave because

$$\sum_{0 \le k \le \binom{n}{2}} a_k\, q^k \;=\; [n]_q! \;=\; (1+q)\,\ldots\,(1+q\ldots+q^{n-1})$$

is a product of log-concave polynomials.

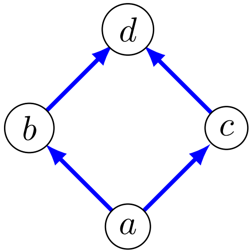


Log-concavity is a widespread phenomenon observed in numerous subjects in mathematics.

Today we focus on log-concavity for **probabilities in posets**.

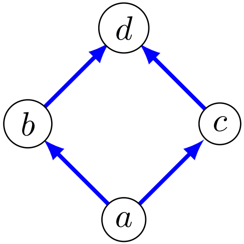
Partially ordered sets

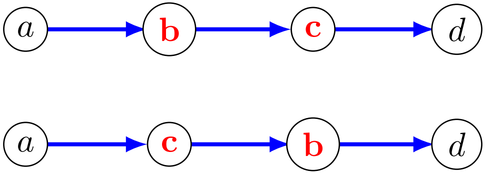
A poset P is a set X with a partial order $\prec$ on X.



Linear extension

A linear extension L is a complete order of $\prec$.





We write $L(x) = k$ if x is k-th smallest in L.

Stanley's inequality

Fix $z \in P$.

$$N_k \text{ is probability that } \mathcal{L}(z) = k,$$

where $\mathcal{L}$ is uniform random linear extension of P.

Theorem (Stanley '81)

For every poset and $k \geq 1$,

$$N_k{}^2 \;\geq\; N_{k+1}\, N_{k-1}.$$

The inequality was initially conjectured by Chung-Fishburn-Graham, and was proved using Aleksandrov-Fenchel inequality for mixed volumes.

Our contribution

Problem

(Folklore, Graham '83, Biró-Trotter '11, Stanley '14)

Give a combinatorial proof of Stanley's inequality.

Answer (C.–Pak '21+)

More combinatorial proof for Stanley's inequality, with generalizations to weighted version.

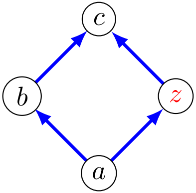
Order-reversing weight

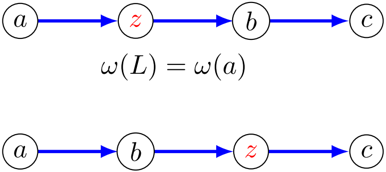
A weight $\omega : X \to \mathbb{R}_{>0}$ is order-reversing if

$$\omega(x) \;\geq\; \omega(y) \qquad \text{whenever} \qquad x \prec y.$$

Weight of linear extension L is

$$\omega(L) \;:=\; \prod_{L(x)<L(z)} \omega(x).$$





$\omega(L) = \omega(a)$

$\omega(L) = \omega(a)\omega(b)$

Weighted Stanley's inequality

Let $N_{\omega,k}$ be probability that $\mathcal{L}(z) = k$,

where $\mathcal{L}_\omega$ is ω-weighted random linear extension.

Theorem 1 (C.–Pak '21+)

For every poset and $k \geq 1$,

$$N_{\omega,k}{}^2 \; \geq \; N_{\omega,k+1}\, N_{\omega,k-1}.$$

Proof used combinatorial atlas method,

a new tool to establish log-concave inequalities.

Applications of log-concavity

$\frac{1}{3} - \frac{2}{3}$ Conjecture

Conjecture (Kislitsyn '68, Fredman '75, Linial '84)

For finite poset that is not completely ordered, there exist elements x, y:

$$\frac{1}{3} \quad \leq \quad \mathbb{P}\big[\mathcal{L}(x) < \mathcal{L}(y)\big] \quad \leq \quad \frac{2}{3},$$

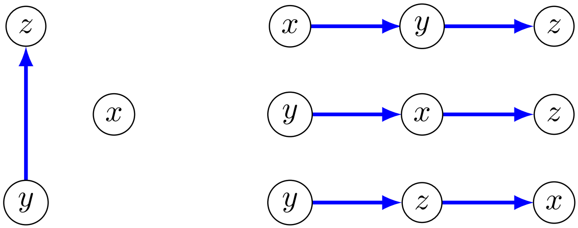
where $\mathcal{L}$ is uniform random linear extension of P.

Quote (Brightwell-Felsner-Trotter '95)

"This problem remains one of the most intriguing problems in the combinatorial theory of posets."

Why $\frac{1}{3}$ and $\frac{2}{3}$?

The upper,lower bound are achieved by this poset:



$$\mathbb{P}\big[\mathcal{L}(x) < \mathcal{L}(y)\big] \;=\; \frac{1}{3}\,; \qquad \mathbb{P}\big[\mathcal{L}(y) < \mathcal{L}(x)\big] \;=\; \frac{2}{3}\,.$$

The big breakthrough

Theorem (Kahn-Saks '84)

For poset that is not completely ordered, there exist elements x, y:

$$\frac{3}{11} \quad \leq \quad \mathbb{P}\big[\mathcal{L}(x) < \mathcal{L}(y)\big] \quad \leq \quad \frac{8}{11},$$

roughly between 0.273 *and* 0.727*.*

Proof used log-concavity as a crucial component.

Proof sketch of Kahn-Saks Theorem

Find $x, y \in P$ such that

$$|h(y) - h(x)| \ \leq \ 1,$$

where $h(x) := \mathbb{E}[\mathcal{L}(x)]$ and $h(y) := \mathbb{E}[\mathcal{L}(y)]$.

Let F_k be probability that $\mathcal{L}(y) - \mathcal{L}(x) = k$.

$$\mathbb{P}\big[\mathcal{L}(x) < \mathcal{L}(y)\big] \ = \ F_1 + F_2 + \cdots + F_n,$$

$$\mathbb{P}\big[\mathcal{L}(y) < \mathcal{L}(x)\big] \ = \ F_{-1} + F_{-2} + \cdots + F_{-n}.$$

Proof sketch of Kahn-Saks Theorem

Since $|h(y) - h(x)|$ is small,

$$F_1 + 2F_2 + \cdots + nF_n \approx F_{-1} + 2F_{-2} + \cdots + nF_{-n}.$$

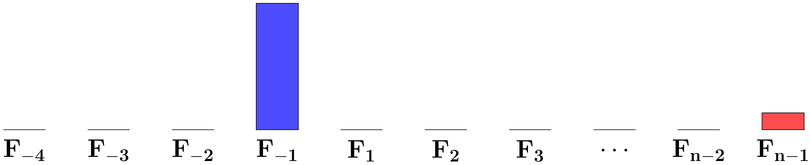
One can hope this implies

$$F_1 + F_2 + \cdots + F_n \approx F_{-1} + F_{-2} + \cdots + F_{-n},$$

which would then imply

$$\mathbb{P}[\mathcal{L}(x) < \mathcal{L}(y)] \approx \mathbb{P}[\mathcal{L}(y) < \mathcal{L}(x)] \approx 0.5.$$

But things can go really wrong:



Log-concavity comes to rescue

Theorem (Kahn–Saks '84)

For $k \neq 0$,

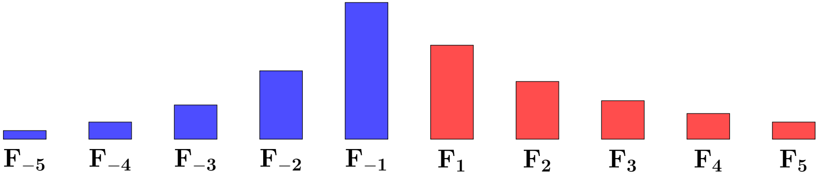
$$F_k{}^2 \;\geq\; F_{k+1}\, F_{k-1},$$
$$F_{-k}{}^2 \;\geq\; F_{-(k+1)}\, F_{-(k-1)}.$$

This generalizes Stanley's inequality, and was proved by Aleksandrov-Fenchel inequality.

Proof sketch of Kahn-Saks Theorem

Log-concavity (and other ineqs.) imply:

- $\mathbb{P}[\mathcal{L}(x) < \mathcal{L}(y)]$ is maximized (resp. minimized) when $F_1, F_2, \ldots, F_n$ is geometric sequence,
- $\mathbb{P}[\mathcal{L}(y) < \mathcal{L}(x)]$ is minimized (resp. maximized) when $F_{-1}, F_{-2}, \ldots, F_{-n}$ is geometric sequence.



Combined with $|h(y) - h(x)| \leq 1$, the result follows.

Best known bound for $\frac{1}{3} - \frac{2}{3}$ Conjecture

Theorem (Brightwell-Felsner-Trotter '95)

For poset that is not completely ordered, there exist elements x, y:

$$\frac{5-\sqrt{5}}{10} \quad \leq \quad \mathbb{P}\big[\mathcal{L}(x) < \mathcal{L}(y)\big] \quad \leq \quad \frac{5+\sqrt{5}}{10},$$

roughly between 0.276 *and* 0.724*.*

Note: Kahn–Saks bound was 0.273 and 0.727.

This bound cannot be improved for infinite posets.

Cross Product Conjecture

New ingredient: Cross Product Conjecture

Fix $x, y, z \in P$. Let $F(k, \ell)$ be probability that

$$\mathcal{L}(y) - \mathcal{L}(x) = k \quad \text{and} \quad \mathcal{L}(z) - \mathcal{L}(y) = \ell.$$

Conjecture (Brightwell-Felsner-Trotter '95)

For $k, \ell \geq 1$,

$$F(k, \ell)\, F(k+1, \ell+1) \ \leq \ F(k+1, \ell)\, F(k, \ell+1).$$

Equivalently,

$$\det \begin{bmatrix} F(k, \ell) & F(k, \ell+1) \\ F(k+1, \ell) & F(k+1, \ell+1) \end{bmatrix} \ \leq \ 0.$$

What was known

Conjecture (Brightwell-Felsner-Trotter '95)

For $k, \ell \geq 1$,

$$F(k,\ell)\,F(k+1,\ell+1) \;\leq\; F(k+1,\ell)\,F(k,\ell+1).$$

Brightwell-Felsner-Trotter proved the case $k = \ell = 1$ by Ahlswede–Daykin inequality.

Combined with Kahn–Saks proof, this gives the $\frac{5\pm\sqrt{5}}{10}$ bound for $\frac{1}{3} - \frac{2}{3}$ Conjecture.

What was known

Conjecture (Brightwell-Felsner-Trotter '95)

For $k, \ell \geq 1$,

$$F(k,\ell)\,F(k+1,\ell+1) \;\leq\; F(k+1,\ell)\,F(k,\ell+1).$$

Quote (Brightwell-Felsner-Trotter '95)

"Something more powerful seems to be needed to prove general form of Cross Product Conjecture."

Our results

Theorem 2 (C.-Pak-Panova '22)
Cross Product Conjecture is true for posets of width two.

Proved algebraically using matrix algebra argument and combinatorially through Lindström–Gessel–Viennot type argument.

Our results

Theorem 3 (C.-Pak-Panova '23+)

For every poset and $k, \ell \geq 1$,

$$F(k,\ell)\, F(k+1,\ell+1) \;<\; 2F(k+1,\ell)\, F(k,\ell+1).$$

Proof is based on Favard's inequality for mixed volumes, for which factor of 2 is tight for general geometric objects.

On the other hand, for specific classes of posets this factor of 2 can be improved.

A new protagonist

We now shift the attention

from linear extensions to **order-preserving maps**.

Order-preserving maps

Fix poset $P = (X, \prec)$.

A map $M : X \to \{1, \ldots, t\}$ is order-preserving if

$$x \prec y \qquad \text{implies} \qquad M(x) \leq M(y).$$

Linear extensions are order-preserving maps that are also bijections to $\{1, \ldots, |X|\}$.

Previously on linear extensions ...

- Log-concavity?
  Solved: Stanley '81, Kahn–Saks '84, C.-Pak
- Cross–product conjecture?
  Open: Brightwell–Felsner–Trotter '95, C.-Pak-Panova '22
- $\frac{1}{3}$–$\frac{2}{3}$ Conjecture?
  Open: Kahn–Saks '84, Brightwell–Felsner–Trotter '95

Can we **improve** on these results
for **order-preserving maps**?

Log-concavity for order-preserving maps

Graham's conjecture

Fix $z \in P$ and positive integer t.

$$G_k \text{ is probability that } \mathcal{M}(z) = k,$$

where $\mathcal{M}$ is uniform random ord.-pres. map $X \to [t]$.

Conjecture (Graham '83)

For every poset and $k \geq 1$,

$$G_k^2 \;\geq\; G_{k+1}\, G_{k-1}.$$

Graham's conjecture

Quote (Graham '83)

"It would seem that [*the conjecture*] *should have a proof based on the FKG or AD inequalities. However, such a proof has up to now* successfully eluded all attempts to find it*".*

What is Harris/FKG/AD inequalities?

They are fundamental inequalities in probability
that shows, in many random systems,
increasing events are positively correlated.

Example

For any $a, b, c, d \in \mathbb{Z}^d$ in bond percolation,

$$\mathbb{P}\big[a \leftrightarrow b,\, c \leftrightarrow d\big] \;\geq\; \mathbb{P}\big[a \leftrightarrow b\big]\, \mathbb{P}\big[c \leftrightarrow d\big],$$

where $a \leftrightarrow b$ is event that a and b are connected.

Presence of one path increases probability of other path.

Graham's conjecture is true

Theorem (Daykin–Daykin–Paterson '84)

For every poset and $k \geq 1$,

$$G_k^2 \ \geq \ G_{k+1}\, G_{k-1}.$$

Proof used an explicit injective argument,
not based on FKG/AD inequality.

Quote (Daykin–Daykin–Paterson '84)

"[*Proof using FKG or Ahlswede–Daykin inequality*]
have as yet eluded discovery".

Our results

Theorem 4 (C.–Pak ‘22+)
New proof of Daykin–Daykin–Paterson inequality based on Ahlswede–Daykin inequality, with generalization to multi-weighted version.

This proof validates Graham’s prediction.

Cross product conjecture for order-preserving maps

Our results

Fix $x, y, z \in P$ and $t \geq 1$. Let $G(k, \ell)$ be probability

$$\mathcal{M}(y) - \mathcal{M}(x) = k \quad \text{and} \quad \mathcal{M}(z) - \mathcal{M}(y) = \ell,$$

where $\mathcal{M}$ is uniform random ord.-pres. map $X \to [t]$.

Theorem 5 (C.–Pak '22+)

For all integers k, ℓ,

$$G(k, \ell)\, G(k+1, \ell+1) \;\leq\; G(k+1, \ell)\, G(k, \ell+1).$$

This proves cross product conjecture for order-preserving maps.

Our results

Theorem (C.–Pak '22+)

For all integers k, ℓ,

$$G(k,\ell)\,G(k+1,\ell+1) \ \leq \ G(k+1,\ell)\,G(k,\ell+1).$$

Proof is based on same approach discovered when proving Daykin–Daykin–Paterson inequality.

This approach does not work for linear extensions, where inequality is known with factor of 2 in RHS.

$\frac{1}{3}$–$\frac{2}{3}$ Conjecture for order-preserving maps

Conjecture

For finite poset that is not completely ordered, there exist elements x, y:

$$\frac{1}{3} \quad \leq \quad \lim_{t\to\infty} \mathbb{P}\big[\mathcal{M}_t(x) < \mathcal{M}_t(y)\big] \quad \leq \quad \frac{2}{3},$$

where $\mathcal{M}_t$ is uniform random o.p. map $X \to [t]$.

This is in fact equivalent to $\frac{1}{3}$–$\frac{2}{3}$ Conjecture for linear extensions.

All recent advances unfortunately do not improve known bounds for this conjecture.

Open problem

Kahn-Saks Conjecture

$\delta(P)$ is largest number such that there exist $x, y \in P$:

$$\delta(P) \;\leq\; \mathbb{P}\big[\mathcal{L}(x) < \mathcal{L}(y)\big] \;\leq\; 1 - \delta(P).$$

Note that $\frac{1}{3} - \frac{2}{3}$ Conjecture is equivalent to
$\delta(P) \geq \frac{1}{3}$ for P not completely ordered.

Conjecture (Kahn-Saks '84)

$$\delta(P) \to \frac{1}{2} \quad \textit{as} \quad \text{width}(P) \to \infty.$$

Kahn-Saks Conjecture

Conjecture (Kahn-Saks '84)

$$\delta(P) \to \frac{1}{2} \quad \textit{as} \quad \mathrm{width}(P) \to \infty.$$

Komlós '90 proved Conjecture for posets with $\Omega(\frac{n}{\log\log\log n})$ minimal elements.

C.-Pak-Panova '21 proved Conjecture for Young diagram posets with fixed width.

THANK YOU!

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