Combinatorial Atlas for Log-concave Inequalities

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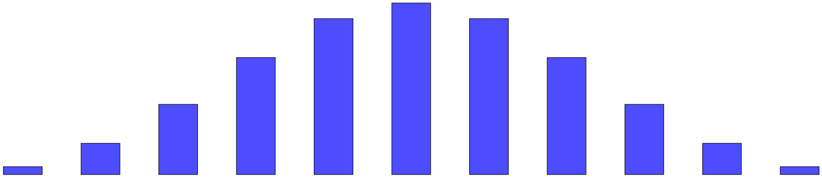
joint with Igor Pak



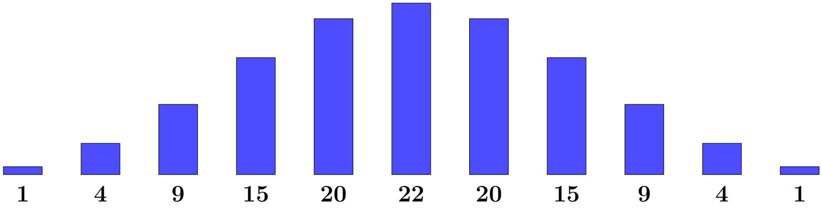
What is log-concavity?

A sequence $a_1, \ldots, a_n \in \mathbb{R}_{\geq 0}$ is log-concave if

$$a_k^2 \;\geq\; a_{k+1}\, a_{k-1} \qquad (1 \leq k < n).$$

Equivalently,

$$\log a_k \;\geq\; \frac{\log a_{k+1} + \log a_{k-1}}{2} \qquad (1 \leq k < n).$$



Example: binomial coefficients

$$a_k \;=\; \binom{n}{k} \qquad k = 0, 1, \ldots, n.$$

This sequence is log-concave because

$$\frac{a_k^2}{a_{k+1}\, a_{k-1}} \;=\; \frac{\binom{n}{k}^2}{\binom{n}{k+1}\binom{n}{k-1}} \;=\; \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right),$$

which is greater than 1.

Example: permutations with k inversions

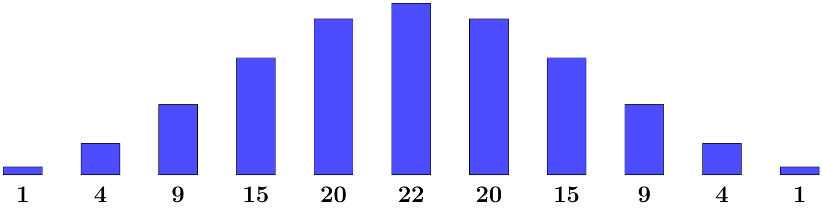
$$a_k \;=\; \text{number of } \pi \in S_n \text{ with } k \text{ inversions},$$

where inversion of π is pair $i < j$ s.t. $\pi_i > \pi_j$.

This sequence is log-concave because

$$\sum_{0 \le k \le \binom{n}{2}} a_k\, q^k \;=\; [n]_q! \;=\; (1+q)\,\ldots\,(1+q\ldots+q^{n-1})$$

is a product of log-concave polynomials.



Log-concavity appears in different objects
for different reasons.

Today we focus on reason for **matroids**.

Warmup: graphs and forests

Let $G = (V, E)$ be a graph.

A (spanning) forest $F = (V, E')$ with $E' \subseteq E$

is a subset of edges without cycles.

G forest not forest spanning tree

Log-concavity for forests

Theorem (Huh '15)

For every graph and $k \geq 1$,

$$I_k^{\,2} \;\geq\; I_{k+1}\, I_{k-1},$$

where I_k is the number of forests with k edges.

Proof used Hodge theory from algebraic geometry.

In fact, stronger inequalities for more general objects are true.

Object: Matroids

Matroid $\mathcal{M} = (X, \mathcal{I})$ is ground set X with collection of independent sets $\mathcal{I} \subseteq 2^X$.

Graphical matroids

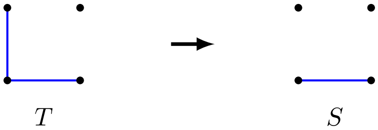
- X = edges of a graph G,
- $\mathcal{I}$ = forests in G.

Realizable matroids

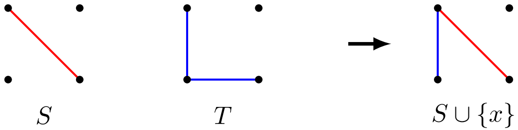
- X = finite set of vectors over field $\mathbb{F}$,
- $\mathcal{I}$ = sets of linearly independent vectors.

Matroids: Conditions

- $S \subseteq T$ and $T \in \mathcal{I}$ implies $S \in \mathcal{I}$.



- If $S, T \in \mathcal{I}$ and $|S| < |T|$, then there is $x \in T \setminus S$ such that $S \cup \{x\} \in \mathcal{I}$.



Note: These are natural properties of sets of linearly independent vectors.

Mason's Conjecture (1972)

For every matroid and $k \geq 1$,

(1) ${I_k}^2 \; \geq \; I_{k+1}\, I_{k-1}$;

(2) ${I_k}^2 \; \geq \; \left(1 + \frac{1}{k}\right) I_{k+1}\, I_{k-1}$;

(3) ${I_k}^2 \; \geq \; \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right) I_{k+1}\, I_{k-1}$.

I_k is number of ind. sets of size k, and $n = |X|$.

Note: (3) $\Rightarrow$ (2) $\Rightarrow$ (1).

Why $\left(1+\frac{1}{k}\right)\left(1+\frac{1}{n-k}\right)$?

Mason (3) is equivalent to ultra/binomial log-concavity,

$$\frac{{I_k}^2}{\binom{n}{k}^2} \;\geq\; \frac{I_{k+1}}{\binom{n}{k+1}}\,\frac{I_{k-1}}{\binom{n}{k-1}}.$$

Equality occurs **if** every subset with $k+1$ elements is independent.

Solution to Mason (1)

Theorem (Adiprasito-Huh-Katz '18)

For every matroid and $k \geq 1$,

$$I_k^{\,2} \;\geq\; I_{k+1}\, I_{k-1}.$$

Proof used combinatorial Hodge theory for matroids.

Solution to Mason (2)

Theorem (Huh-Schröter-Wang '18)

For every matroid and $k \geq 1$,

$$I_k^{\,2} \;\geq\; \left(1 + \frac{1}{k}\right) I_{k+1}\, I_{k-1}.$$

Proof used combinatorial Hodge theory for correlation inequality on matroids.

Solution to Mason (3)

Theorem

(Anari-Liu-Oveis Gharan-Vinzant, Brändén-Huh '20)

For every matroid and $k \geq 1$,

$$I_k{}^2 \;\geq\; \left(1+\frac{1}{k}\right)\left(1+\frac{1}{n-k}\right) I_{k+1}\, I_{k-1}.$$

Proof used theory of strong log-concave polynomials / Lorentzian polynomials.

Solution to Mason (3)

Theorem

(Anari-Liu-Oveis Gharan-Vinzant, Brändén-Huh '20)

For every matroid and $k \geq 1$,

$$I_k{}^2 \;\geq\; \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right) I_{k+1}\, I_{k-1}.$$

Theorem (Murai-Nagaoka-Yazawa '21)

Equality occurs if and only if every subset with $k+1$ *elements is independent.*

Our contribution

Method: Combinatorial atlas

Results: Log-concave inequalities, and
if and only if conditions for equality

- Matroids (refined);
- Morphism of matroids (refined);
- Discrete polymatroids;
- Stanley's poset inequality (refined);
- Poset antimatroids;
- Branching greedoid (log-convex);
- Interval greedoids.

Method: Combinatorial atlas

Results: Log-concave inequalities, and

if and only if conditions for equality

- **Matroids (refined)**;
- Morphism of matroids (refined);
- Discrete polymatroids;
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- Branching greedoid (log-convex);
- Interval greedoids.

Combinatorial atlas application: Matroids

Warmup: graphical matroids refinement

Corollary (C.-Pak)

For graphical matroid of simple connected graph $G = (V, E)$, *and* $k = |V| - 2$,

$$(I_k)^2 \;\geq\; \frac{3}{2}\left(1 + \frac{1}{k}\right) I_{k+1}\, I_{k-1},$$

with equality if and only if G *is cycle graph.*

Numerically better than Mason (3), because

$$\frac{3}{2} \;\geq\; 1 + \frac{1}{n-k} \;=\; 1 + \frac{1}{|E| - |V| + 2}$$

for G that is not tree.

Comparison with Mason (3)

Our bound gives

$$\frac{(I_k)^2}{I_{k+1}\, I_{k-1}} \quad \geq \quad \frac{3}{2} \qquad \text{when} \;\; |E| - |V| \to \infty,$$

Meanwhile, Mason (3) bound only gives

$$\frac{(I_k)^2}{I_{k+1}\, I_{k-1}} \quad \geq \quad 1 \qquad \text{when} \;\; |E| - |V| \to \infty.$$

Our bound is better numerically and asymptotically.

Refinement for Mason (3)

Theorem 1 (C.-Pak)

For every matroid and $k \geq 1$,

$$I_k{}^2 \;\geq\; \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{\mathrm{prl}_{\mathcal{M}}(k-1) - 1}\right) I_{k+1}\, I_{k-1}.$$

This refines Mason (3),

$$I_k{}^2 \;\geq\; \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right) I_{k+1}\, I_{k-1},$$

since

$$\mathrm{prl}_{\mathcal{M}}(k-1) \;\leq\; n - k + 1.$$

Refinement for different matroids

- For all matroids,

  $$I_k{}^2 \;\geq\; \left(1+\tfrac{1}{k}\right)\left(1+\tfrac{1}{n-k}\right) I_{k+1}\, I_{k-1}.$$

- Graphical matroids and $k = |V| - 2$,

  $$I_k{}^2 \;\geq\; \left(1+\tfrac{1}{k}\right)\tfrac{3}{2}\, I_{k+1}\, I_{k-1}.$$

- Realizable matroids over $\mathbb{F}_q$,

  $$I_k{}^2 \;\geq\; \left(1+\tfrac{1}{k}\right)\left(1+\tfrac{1}{q^{m-k+1}-2}\right) I_{k+1}\, I_{k-1}.$$

- (k, m, n)-Steiner system matroid,

  $$I_k{}^2 \;\geq\; \left(1+\tfrac{1}{k}\right)\tfrac{n-k+1}{n-m}\, I_{k+1}\, I_{k-1}.$$

Refinement for Mason (3)

Theorem 2 (C.-Pak)

For every matroid and $k \geq 1$,

$$I_k{}^2 \;\geq\; \left(1+\frac{1}{k}\right)\left(1+\frac{1}{\mathrm{prl}_{\mathcal{M}}(k-1)-1}\right) I_{k+1}\, I_{k-1}.$$

This refines Mason (3),

$$I_k{}^2 \;\geq\; \left(1+\frac{1}{k}\right)\left(1+\frac{1}{n-k}\right) I_{k+1}\, I_{k-1},$$

since

$$\mathrm{prl}_{\mathcal{M}}(k-1) \;\leq\; n-k+1.$$

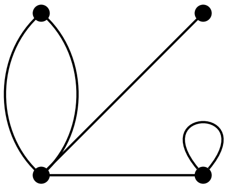
Parallel classes of matroid $\mathcal{M}$

Loop is $x \in X$ such that $\{x\} \notin \mathcal{I}$.

Non-loops x, y are parallel if $\{x, y\} \notin \mathcal{I}$.

Parallelship equiv. relation: $x \sim y$ if $\{x, y\} \notin \mathcal{I}$.

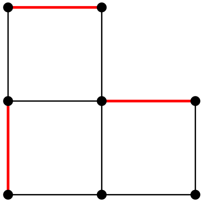
Parallel class $=$ equivalence class of $\sim$.

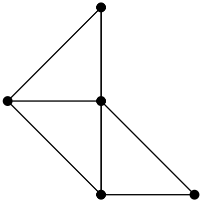


Matroid contraction

Contraction of $S \in \mathcal{I}$ is matroid $\mathcal{M}_S$ with

$$X_S \;=\; X \setminus S, \qquad \mathcal{I}_S \;=\; \{T \setminus S \,:\, S \subseteq T\}.$$



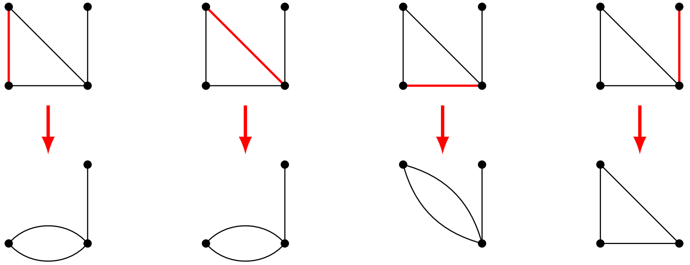


$$\mathrm{prl}(S) \;:=\; \text{number of parallel classes of } \mathcal{M}_S$$

Parallel number

The *k-parallel number* is

$$\mathrm{prl}_{\mathcal{M}}(k) \;:=\; \max\{\mathrm{prl}(S) \mid S \in \mathcal{I} \text{ with } |S| = k\}.$$



$\mathrm{prl}(S) = 2$ $\mathrm{prl}(S) = 2$ $\mathrm{prl}(S) = 2$ $\mathrm{prl}(S) = 3$

$\mathrm{prl}_{\mathcal{M}}(1) = 3$

Refinement for Mason (3)

Theorem 3 (C.-Pak)

For every matroid and $k \geq 1$,

$$I_k^{\,2} \;\geq\; \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{\mathrm{prl}_{\mathcal{M}}(k-1) - 1}\right) I_{k+1}\, I_{k-1}.$$

This refines Mason (3),

$$I_k^{\,2} \;\geq\; \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right) I_{k+1}\, I_{k-1},$$

since

$$\mathrm{prl}_{\mathcal{M}}(k-1) \;\leq\; n - k + 1.$$

When is equality achieved?

- When every $(k+1)$-subset is independent,
  $\mathrm{prl}_{\mathcal{M}}(k-1) \; = \; n-k+1.$
- Graphical matroid when G is a cycle,
  $\mathrm{prl}_{\mathcal{M}}(k-1) \; = \; 3.$
- Realizable matroids of every m-vectors over $\mathbb{F}_q$,
  $\mathrm{prl}_{\mathcal{M}}(k-1) \; = \; q^{m-k+1}-1.$
- (k,m,n)-Steiner system matroid,
  $$\mathrm{prl}_{\mathcal{M}}(k-1) \; = \; \frac{n-k+1}{m-k+1}.$$

Equality conditions

Theorem 4 (C.-Pak)

For every matroid and $k \geq 1$,

$$I_k{}^2 = \left(1+\frac{1}{k}\right)\left(1+\frac{1}{\mathrm{prl}_{\mathcal{M}}(k-1)-1}\right) I_{k+1}\, I_{k-1}$$

if and only if

for every $S \in \mathcal{I}$ *with* $|S| = k-1$,

- $\mathcal{M}_S$ *has* $\mathrm{prl}_{\mathcal{M}}(k-1)$ *parallel classes; and*
- *Every parallel class of* $\mathcal{M}_S$ *has same size.*

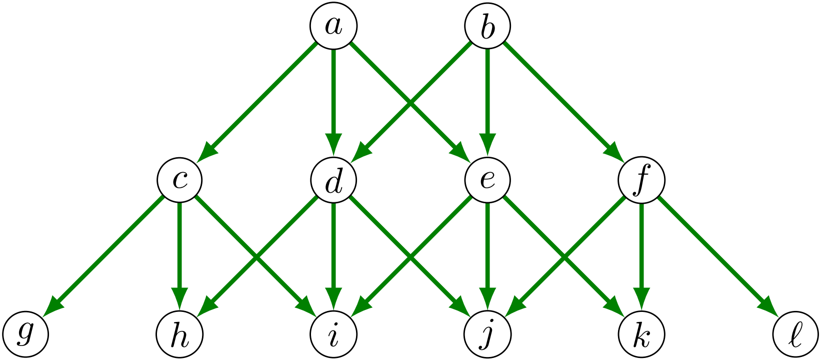
Combinatorial atlas: the method

Combinatorial atlas

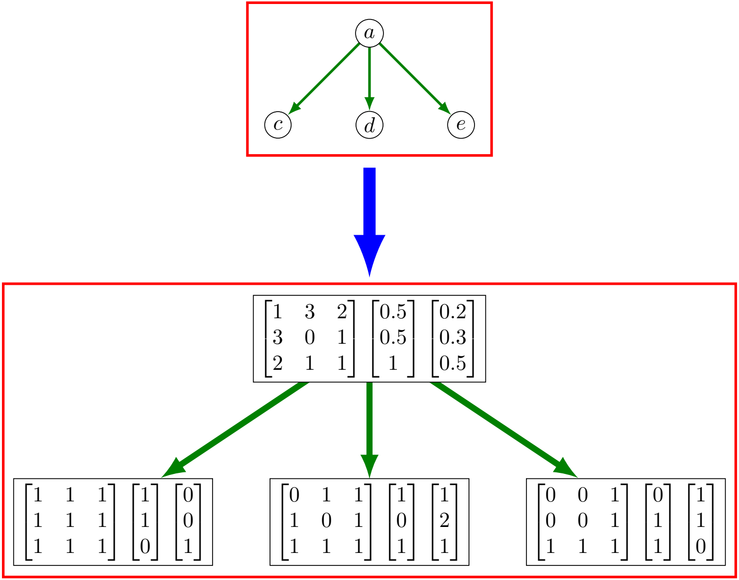
Input: Acyclic digraph $\mathcal{A}$, where each vertex v is associated with

- Symmetric matrix $\boldsymbol{M}$ with nonnegative entries;
- Vector $\boldsymbol{g}, \boldsymbol{h}$ with nonnegative entries.

Atlas: example

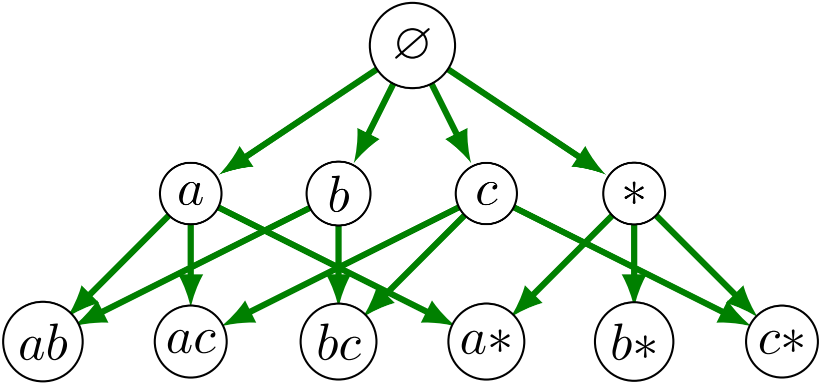


Atlas: example (zoomed in)



Atlas example: matroid (simplified)

For matroid with $X = \{a, b, c\}$, the atlas for $k = 2$ is



Atlas example: matroid (simplified)

The matrix for the top vertex is

$$
\begin{aligned}
\boldsymbol{M}_{a,b} &= (k+1)! \times \text{number of independent sets} \\
&\quad \text{of size } k+1 \text{ containing } a, b \\
\boldsymbol{M}_{a,*} &= k! \times \text{number of independent sets} \\
&\quad \text{of size } k \text{ containing } a \\
\boldsymbol{M}_{*,*} &= (k-1)! \times \text{number of independent sets} \\
&\quad \text{of size } k-1
\end{aligned}
$$

Combinatorial atlas

Input: Acyclic digraph $\mathcal{A}$, where each vertex v is associated with

- Symmetric matrix $\boldsymbol{M}$ with nonnegative entries;
- Vector $\boldsymbol{g}, \boldsymbol{h}$ with nonnegative entries.

Goal: Show every $\boldsymbol{M}$ has hyperbolic inequality.

Hyperbolic inequality

$\boldsymbol{M}$ has hyperbolic inequality property if

$$\langle \boldsymbol{x}, \boldsymbol{M}\boldsymbol{y} \rangle^2 \geq \langle \boldsymbol{x}, \boldsymbol{M}\boldsymbol{x} \rangle \langle \boldsymbol{y}, \boldsymbol{M}\boldsymbol{y} \rangle,$$

for every $\boldsymbol{x} \in \mathbb{R}^r$, $\boldsymbol{y} \in \mathbb{R}^r_{\geq 0}$.

This condition is equivalent to

$\boldsymbol{M}$ has at most one positive eigenvalue.

Note: Already known to be important in Lorentzian polynomials and Bochner's method proof of Aleksandrov-Fenchel inequality.

How to get log-concave inequalities?

Assume a_{k-1}, a_k, a_{k+1} can be computed by

$a_k = \langle \boldsymbol{g}, \boldsymbol{M}\boldsymbol{h} \rangle$, $a_{k+1} = \langle \boldsymbol{g}, \boldsymbol{M}\boldsymbol{g} \rangle$, $a_{k-1} = \langle \boldsymbol{h}, \boldsymbol{M}\boldsymbol{h} \rangle$,

for $\boldsymbol{M}, \boldsymbol{g}, \boldsymbol{h}$ from a top vertex of the atlas.

$$\langle \boldsymbol{g}, \boldsymbol{M}\boldsymbol{h} \rangle^2 \geq \langle \boldsymbol{g}, \boldsymbol{M}\boldsymbol{g} \rangle \langle \boldsymbol{h}, \boldsymbol{M}\boldsymbol{h} \rangle \quad \text{(hyperbolic ineq.)}$$

then implies

$$a_k^2 \geq a_{k+1} a_{k-1} \quad \text{(log-concave ineq.)}$$

Combinatorial atlas

Input: Acyclic digraph $\mathcal{A}$, where each vertex v is associated with

- Symmetric matrix $\boldsymbol{M}$ with nonnegative entries;
- Vector $\boldsymbol{g}, \boldsymbol{h}$ with nonnegative entries.

Goal: Show every $\boldsymbol{M}$ has hyperbolic inequality.

Method: Verify three conditions:

- **Irreducibility condition**;
- **Inheritance condition**;
- **Subdivergence condition**.

Irreducibility condition

- Matrix $\boldsymbol{M}$ associated to v is irreducible when restricted to its support;
- Vector $\boldsymbol{h}$ is associated to v is a positive vector.

For matroids, this means that the base exchange graph is connected.

This is a consequence of the exchange property.

Inheritance condition

Edge $e = (v, v_i)$ of v is associated with linear map $T_i : \mathbb{R}^r \to \mathbb{R}^r$ such that, for every $\boldsymbol{x} \in \mathbb{R}^r$,

$$i\text{-th coordinate of } \boldsymbol{Mx} \;=\; \langle T_i\boldsymbol{x}, \boldsymbol{M}_i T_i \boldsymbol{h} \rangle,$$

where $\boldsymbol{M}$ and $\boldsymbol{h}$ are associated to v, and $\boldsymbol{M}_i$ is associated to v_i.

For matroids with $X = \{e_1, \ldots, e_n\}$, this means

$$k \times \text{number of independent } k\text{-sets}$$

$$= \sum_{i=1}^{n} \text{number of independent } k\text{-sets containing } e_i.$$

Subdivergence condition

For every $\boldsymbol{x} \in \mathbb{R}^r$,

$$\sum_{i=1}^{r} h_i \, \langle T_i \boldsymbol{x}, \boldsymbol{M}_i T_i \boldsymbol{x} \rangle \;\geq\; \langle \boldsymbol{x}, \boldsymbol{M} \boldsymbol{x} \rangle,$$

where $h_i = i$-th coordinate of $\boldsymbol{h}$.

Note: Equality occurs for Lorentzian polynomials and for matroids.

For matroids, this is consequence of hereditary property.

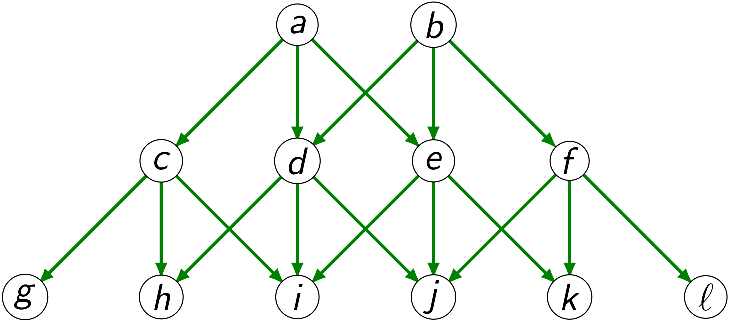
Bottom-to-top principle for hyperbolic inequalities

Proposition

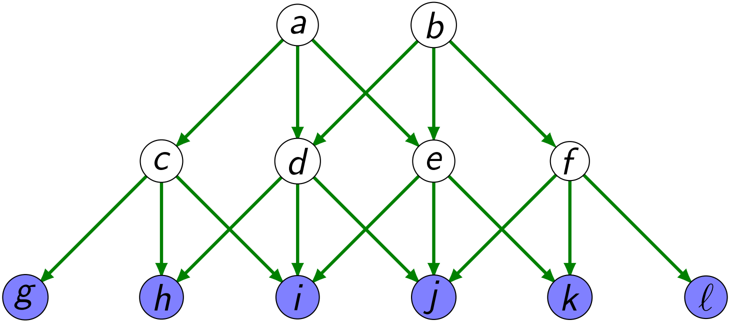
Assume irreducibility, inheritance, subdivergence. If every child vertex has hyperbolic inequality property, then so does the parent vertex.

Bottom-to-top principle reduces **Goal** to checking hyperbolic inequality only for sink vertices.

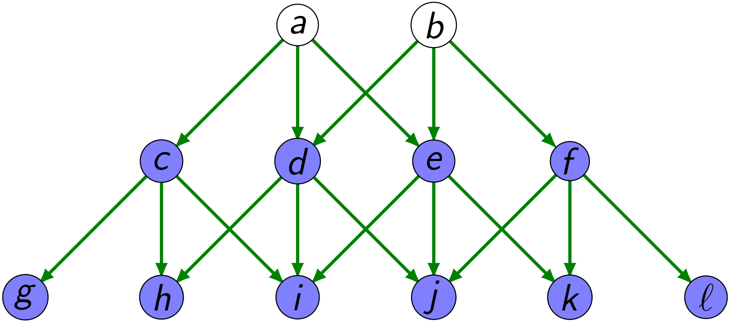
Bottom-to-top principle



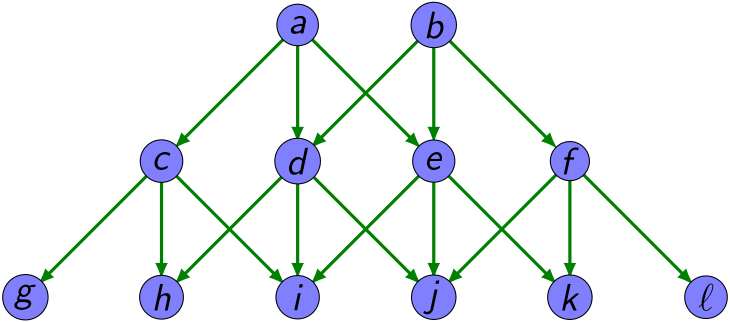
Bottom-to-top principle



Bottom-to-top principle



Bottom-to-top principle



How about equalities?

Combinatorial atlas equality

Input:

- An atlas $\mathcal{A}$ satisfying irreducibility, inheritance, subdivergence conditions.

Goal: Show "every" $\boldsymbol{M}$ has hyperbolic equality,

$$\langle \boldsymbol{g}, \boldsymbol{M}\boldsymbol{h} \rangle^2 = \langle \boldsymbol{g}, \boldsymbol{M}\boldsymbol{g} \rangle \langle \boldsymbol{h}, \boldsymbol{M}\boldsymbol{h} \rangle.$$

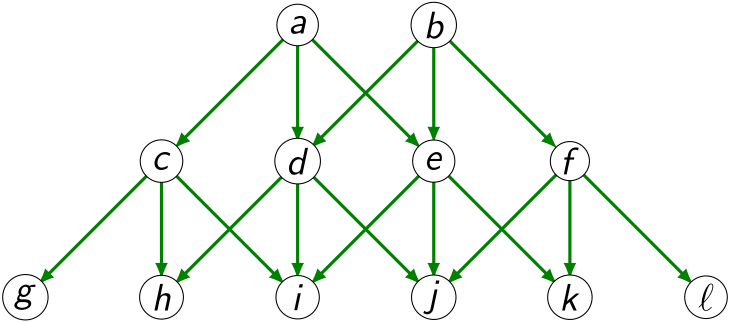
Top-to-bottom principle for equalities

Proposition

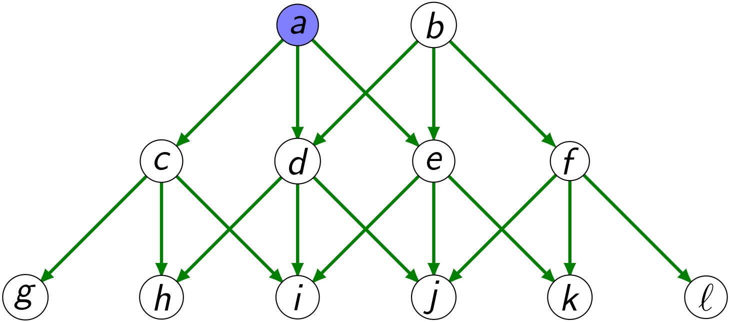
If parent vertex has hyperbolic equality property, then so does children vertices.

Top-to-bottom principle expands hyperbolic equality to sink vertices, and gives combinatorial characterizations.

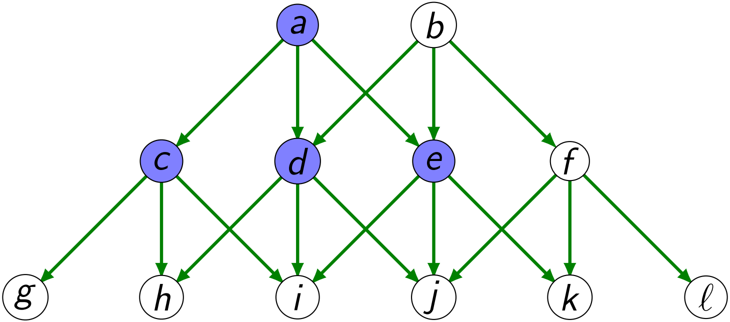
Top-to-bottom principle



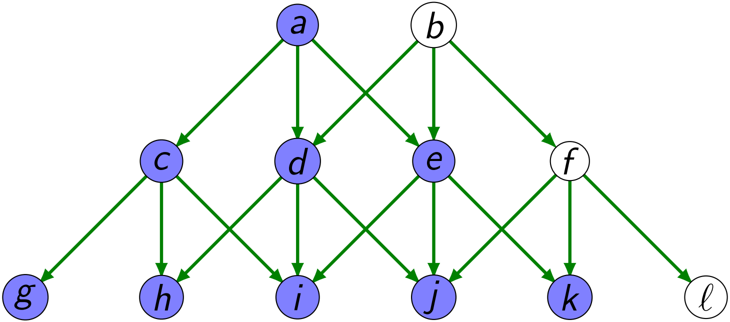
Top-to-bottom principle



Top-to-bottom principle



Top-to-bottom principle



Moral of the story

Problem: Log-concave inequalities and equalities.

Strategy:

- Build a combinatorial atlas;
- Verify the required conditions;
- Use hyperbolic inequality property to derive log-concave inequalities;
- Use hyperbolic equality property to derive log-concave equalities.

Other applications

Full version: `2110.10740` (71 pages)

Expository version: `2203.01533` (28 pages)

Results: Log-concave inequalities and equalities for

- Matroids (refined);
- Discrete polymatroids;
- Morphism of matroids (refined) (conjecture on equality conditions is resolved);
- Stanley's poset inequality (refined);
- Poset antimatroids;
- Branching greedoid (log-convex);
- Interval greedoids.

THANK YOU!

Preprint: www.arxiv.org/abs/2110.10740

www.arxiv.org/abs/2203.01533

Webpage: www.math.ucla.edu/~sweehong/

Email: sweehong@math.ucla.edu

Negative dependence for forests

Conjecture (Kahn '00, Grimmett-Winkler '04)
Let G be a graph, let e, f be distinct edges of G. Then

$$\mathrm{P}[e, f \in \mathcal{F}] \;\leq\; \mathrm{P}[e \in \mathcal{F}]\mathrm{P}[f \in \mathcal{F}],$$

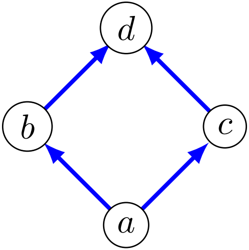
where $\mathcal{F}$ is uniform random forest of G.

- Known with extra factor of 2 in RHS by Lorentzian polynomials
- For matroids, the conjectured factor is $\frac{8}{7}$.

Combinatorial atlas application:
Stanley's poset inequality

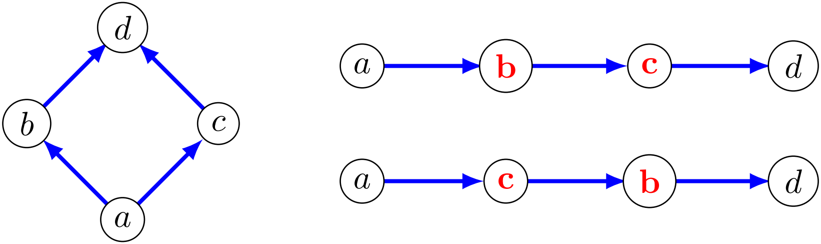
Partially ordered sets

A poset P is a set X with a partial order $\prec$ on X.



Linear extension

A linear extension L is a complete order of $\prec$.



We write $L(x) = k$ if x is k-th smallest in L.

Stanley's inequality

Fix $z \in P$.

N_k *is* number of linear extensions with $L(z) = k$.

Theorem (Stanley '81)

For every poset and $k \geq 1$,

$${N_k}^2 \; \geq \; N_{k+1} \, N_{k-1}.$$

Proof used Aleksandrov-Fenchel inequality for mixed volumes.

When is equality achieved?

Theorem (Shenfeld-van Handel)

Suppose $N_k > 0$. Then

$$N_k{}^2 = N_{k+1}\,N_{k-1}$$

if and only if

$$N_k \;=\; N_{k+1} \;=\; N_{k-1}.$$

Proof used classifications of extremals of Aleksandrov-Fenchel inequality for convex polytopes.

Our contribution

Open Problem (Folklore)

Give a combinatorial proof to Stanley's inequality.

Answer (C.–Pak)

We give new combinatorial proof for Stanley's ineq. and extend to weighted version.

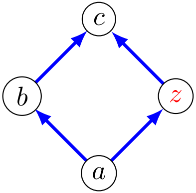
Order-reversing weight

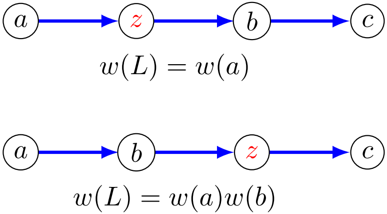
A weight $w : X \to \mathbb{R}_{>0}$ is order-reversing if

$$w(x) \;\geq\; w(y) \qquad \text{whenever} \qquad x \prec y.$$

Weight of linear extension L is

$$w(L) \;:=\; \prod_{L(x)<L(z)} w(x).$$





Weighted Stanley's inequality

Fix $z \in P$.

$N_{w,k}$ *is* w-weight of linear extensions with $L(z) = k$.

Theorem 5 (C. Pak)

For every poset and $k \geq 1$,

$$N_{w,k}{}^2 \;\geq\; N_{w,k+1}\, N_{w,k-1}.$$

When is equality achieved?

Theorem 6 (C.-Pak)

Suppose $N_{w,k} > 0$. *Then*

$$N_{w,k}{}^2 = N_{w,k+1}\, N_{w,k-1}$$

if and only if

for every linear extension L *with* $L(z) = k$,

$$w(L^{-1}(k+1)) \;=\; w(L^{-1}(k-1)) \;=:\; s,$$

and

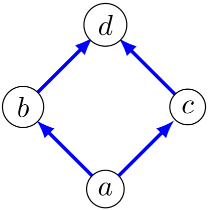
$$\frac{N_{w,k}}{s^k} \;=\; \frac{N_{w,k+1}}{s^{k+1}} \;=\; \frac{N_{w,k-1}}{s^{k-1}}.$$

Combinatorial atlas application: Poset antimatroids

Feasible words of a poset

A word $\alpha \in X^*$ is feasible if no repeating elements, and

y occurs in α and $x \prec y \quad \Rightarrow \quad x$ occurs in α before y.



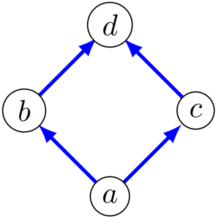
Feasible: $\varnothing$, a, ab, ac, abc, acb, $abcd$, $acbd$.

Not feasible: aa, bc, ba.

Chain weight

For $x \in P$, chain weight is

$\omega(x)$ = number of maximal chains that starts with x.

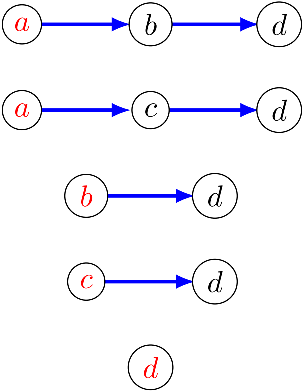


$$\omega(a) = 2$$

$$\omega(b) = 1$$

$$\omega(c) = 1$$

$$\omega(d) = 1$$



Weight of word α is $\omega(\alpha) := \omega(\alpha_1)\dots\omega(\alpha_\ell)$.

Log-concave inequality for poset antimatroids

$F_{\omega,k}$ is sum of ω-weight of feasible words of length k.

Theorem 7 (C.-Pak)

For every poset and $k \geq 1$,

$$F_{\omega,k}{}^2 \;\geq\; F_{\omega,k+1}\, F_{\omega,k-1}.$$

When is equality achieved?

Theorem 8 (C.-Pak)

Equality occurs for $k = 1, \ldots, \text{height}(P) - 1$

if and only if

Hasse diagram of P is a forest where every leaf is of the same level.