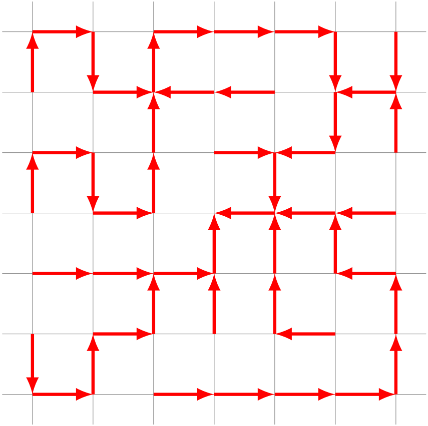
Rotor walk and escaping from prison

Swee Hong Chan
University of California, Los Angeles



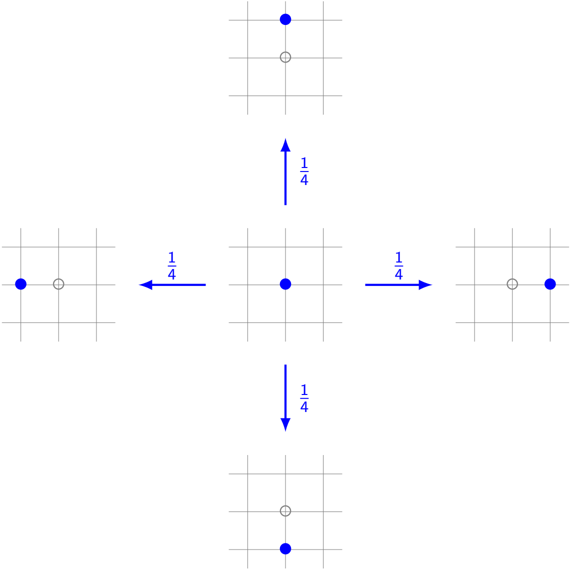

Spring awakening
A BIG, FAT HIT!
ROTTEN!
SCHOOL OF ROCK
BIG BUS NEW YORK
TOSHIBA
TDK
THE LEGEND
CIRCLE OF LIFE
THE

Random walk

Rotor walk

Simple random walk on $\mathbb{Z}^d$

Simple random walk on $\mathbb{Z}^d$



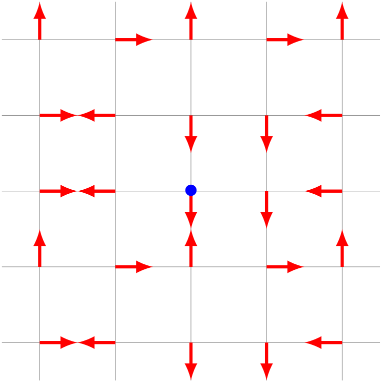
Simple random walk on $\mathbb{Z}^d$

- Visits every site infinitely often? Yes for $d = 2$, No for $d \geq 3$.
- Scaling limit? The standard Brownian motion on $\mathbb{Z}^d$.
- Escape probability? The value of discrete Green function at 0.

Rotor walk on $\mathbb{Z}^2$

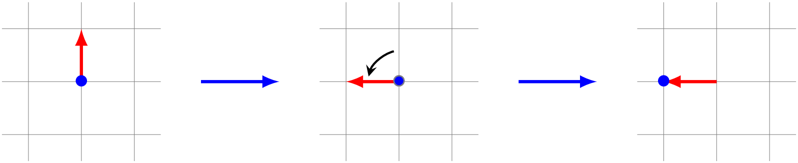
Rotor walk on $\mathbb{Z}^2$

Put a signpost at each site.



Rotor walk on $\mathbb{Z}^2$

Turn the signpost 90° counterclockwise, then follow the signpost.



The signpost says:
“This is the way you went the last time you were here“,
(assuming you ever were!)

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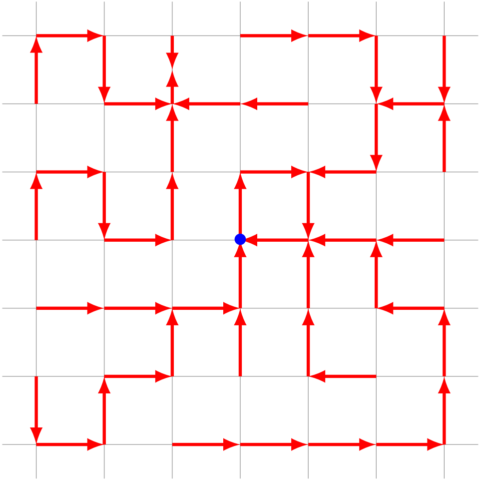

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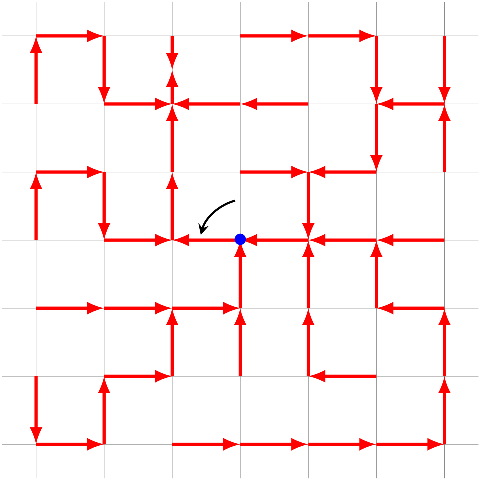

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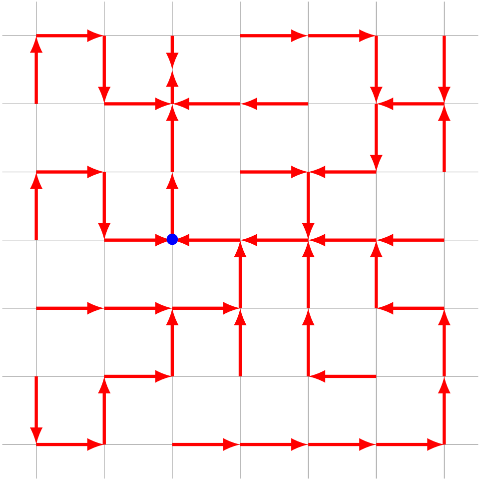

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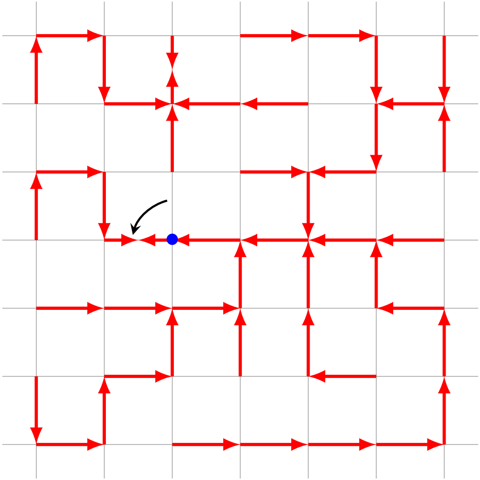

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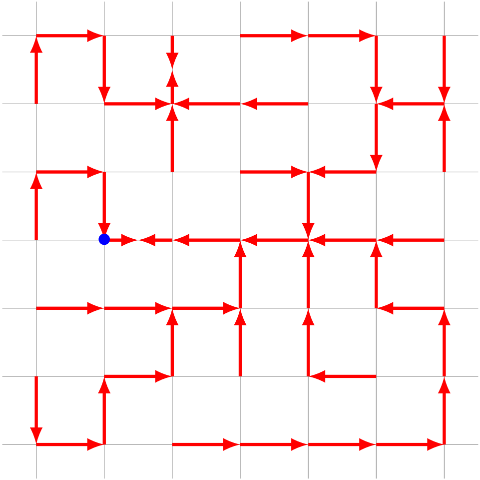

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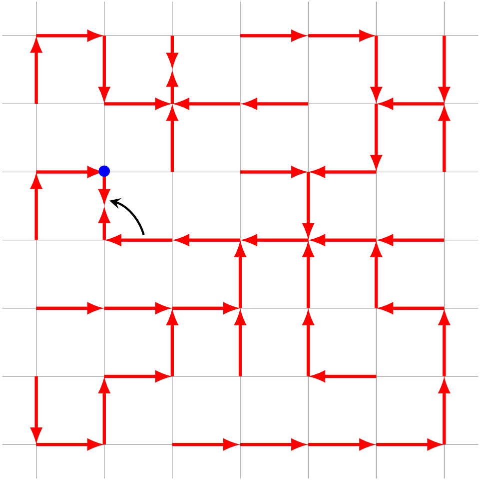

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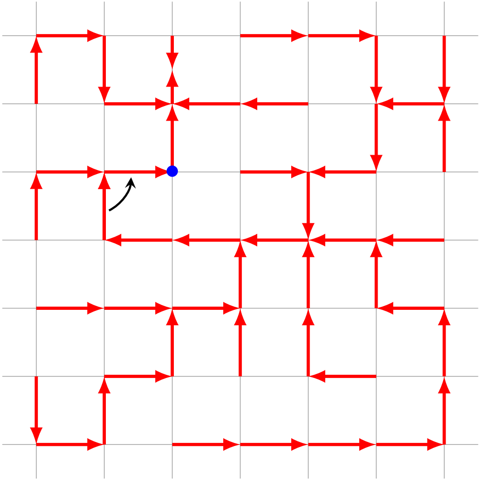

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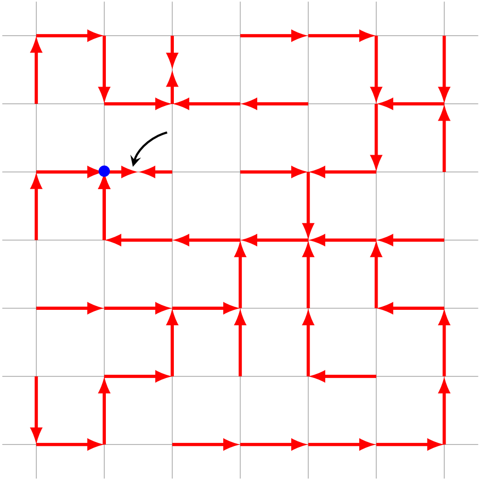

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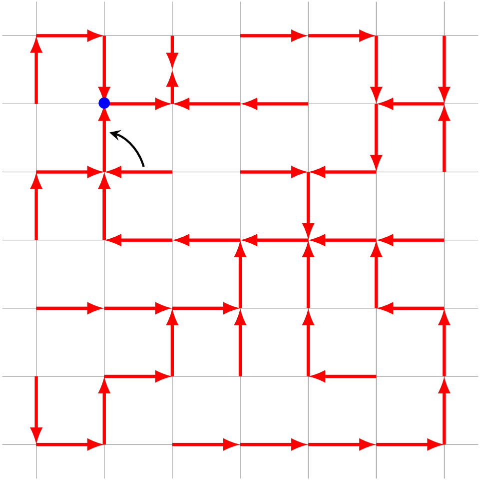

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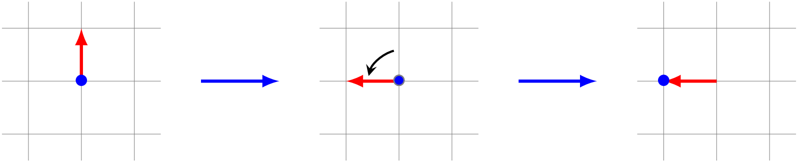


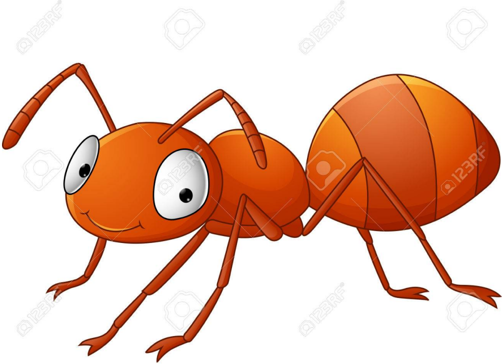
The signpost says:
"This is the way you went the last time you were here",
(assuming you ever were!)

Why rotor walk?

Randomness can be (was) expensive to simulate!

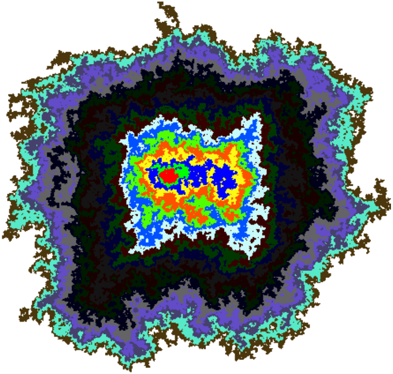
Why rotor walk?

As a model for ants' foraging strategy.



Why rotor walk?

As a model of self-organized criticality for statistical mechanics.



Visited sites after 80 returns to the origin (by Laura Florescu).

Conjectures for rotor walk on $\mathbb{Z}^d$

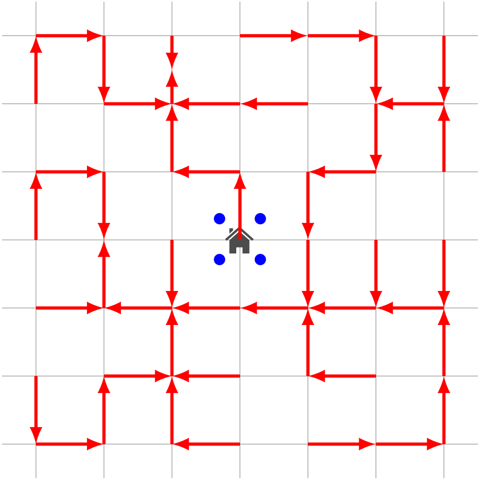
For initial signposts i.i.d. uniform among the $2d$ directions,

- (PDDK '96) Visits every site infinitely often?
- (Kapri-Dhar '09) The asymptotic shape of $\{X_1, \ldots, X_n\}$ is a disc?
- (FGLP '13) Escape rate (escape probability) of rotor walk?

Escape rate of rotor walk

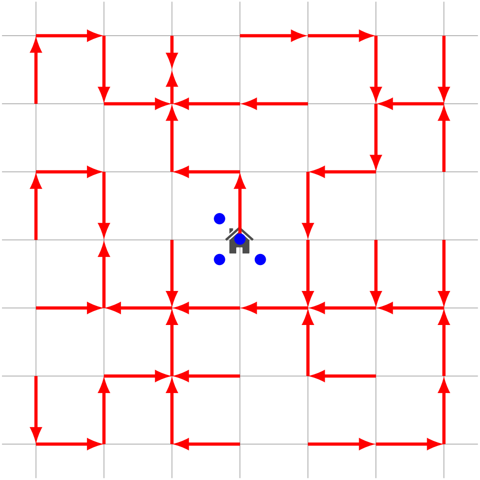
Prison break using rotor walk

Put n walkers at the origin (the prison).



Prison break using rotor walk

First walker performs rotor walk, remove if returns to prison.



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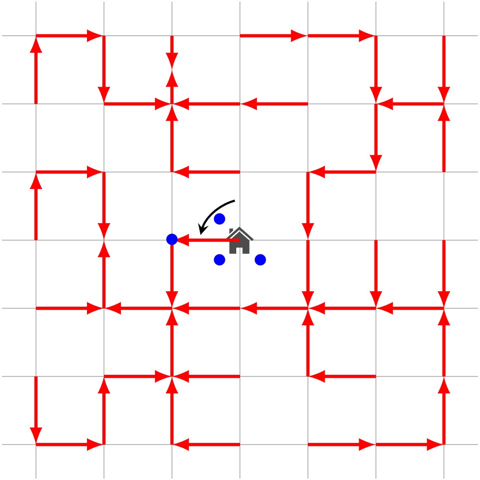

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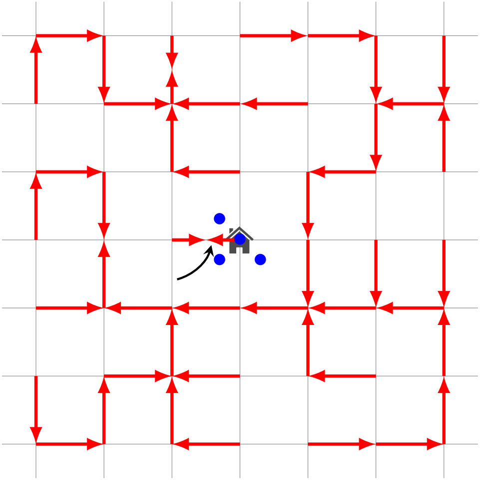
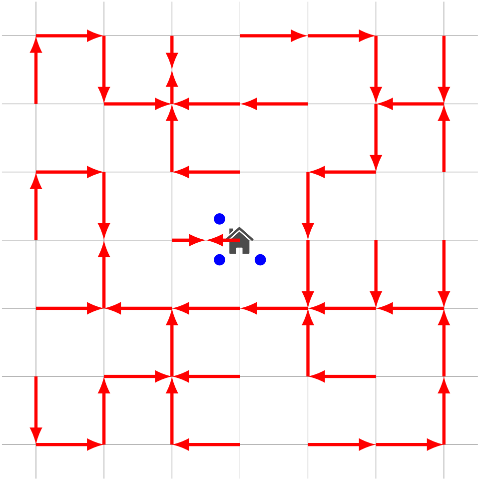


Prison break using rotor walk

First walker returns to prison, and is removed.



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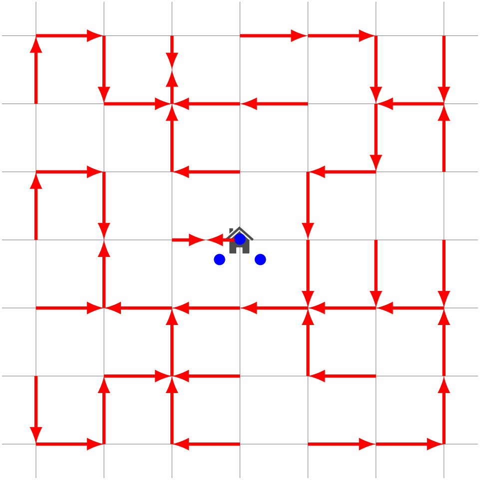

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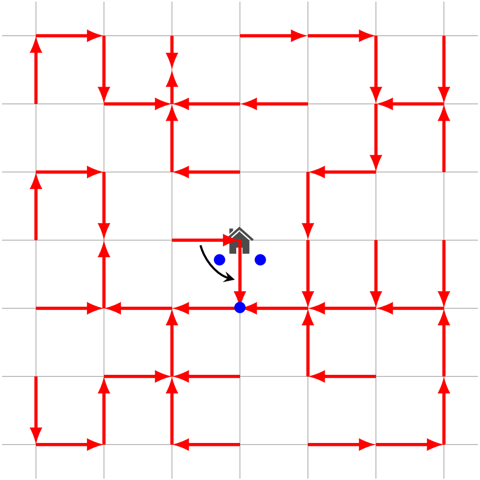

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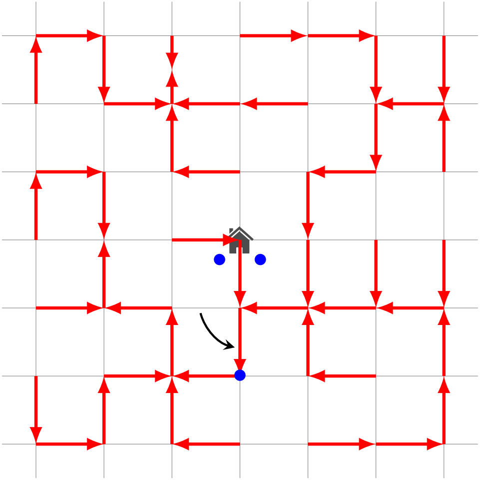

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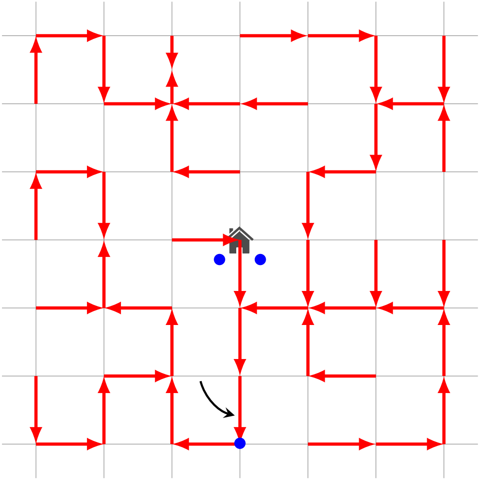

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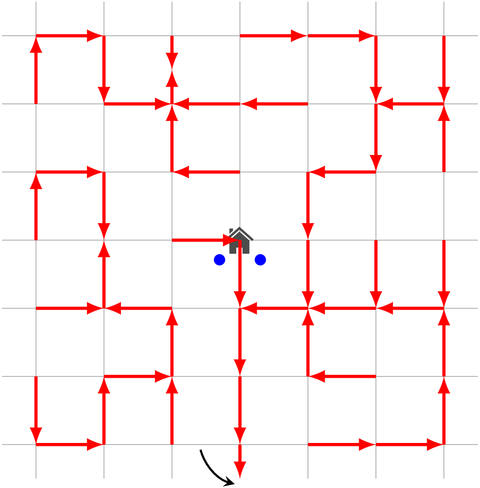
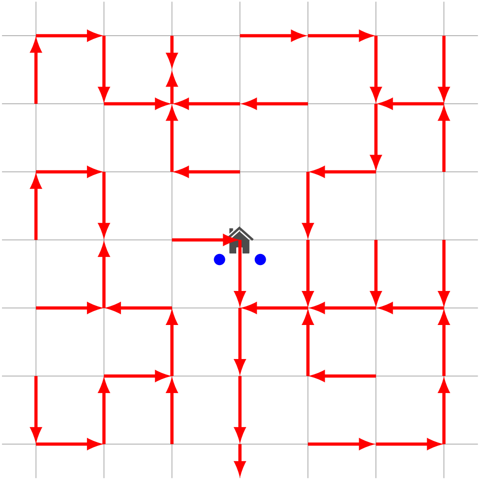


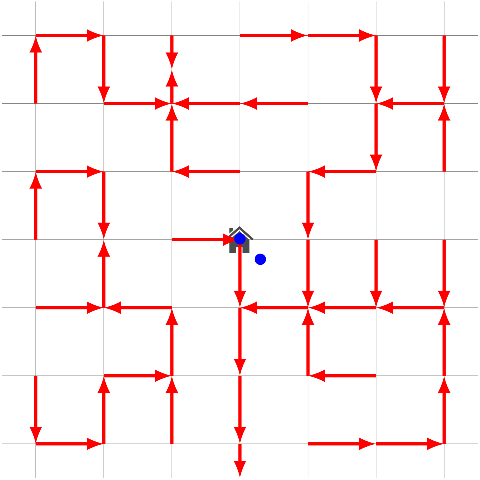
Prison break using rotor walk

Second walker never returns to origin.



Prison break using rotor walk

Third walker performs rotor walk, remove if returns to prison.



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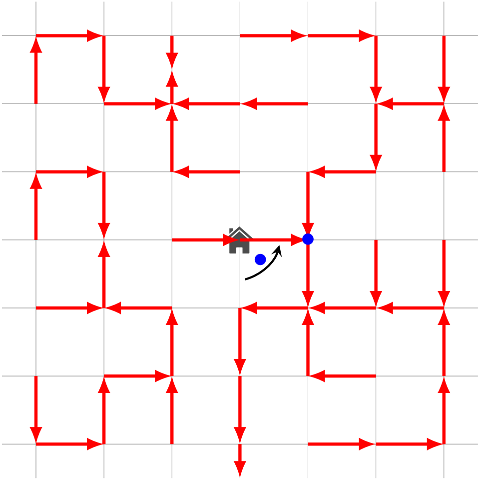

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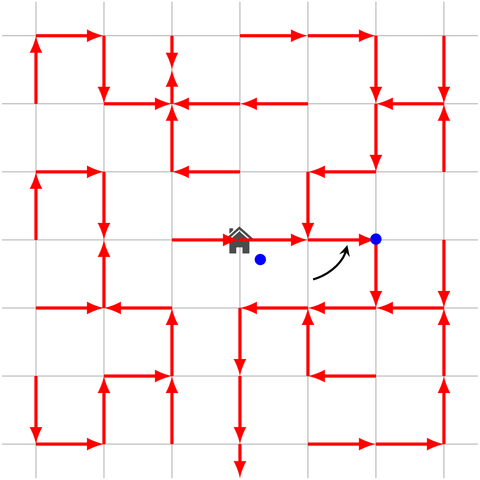

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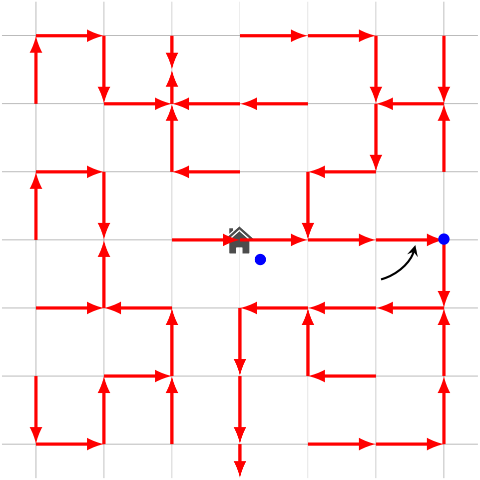

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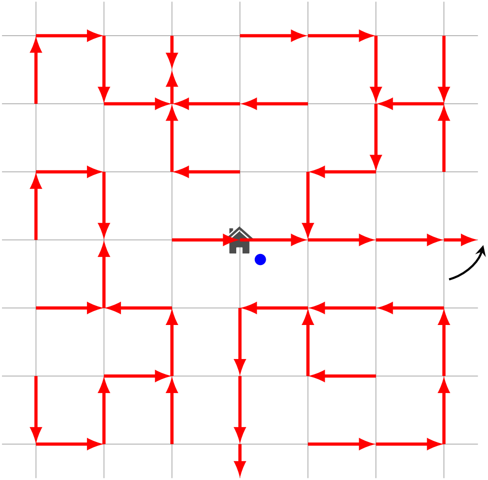

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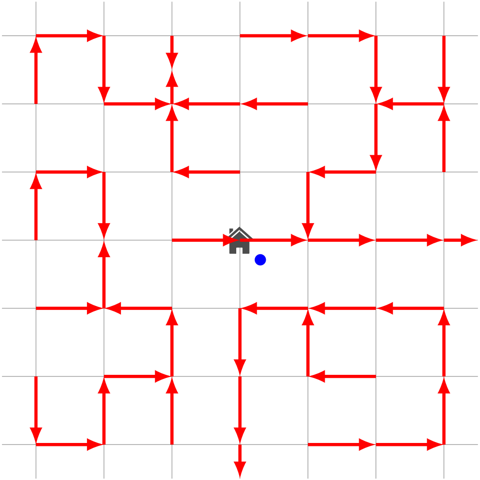
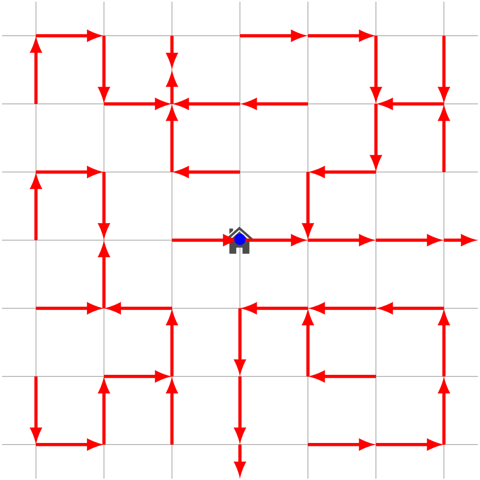


Prison break using rotor walk

Fourth walker performs rotor walk, remove if returns to prison.



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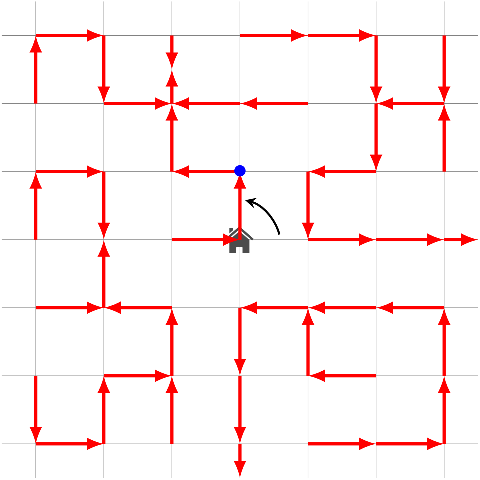

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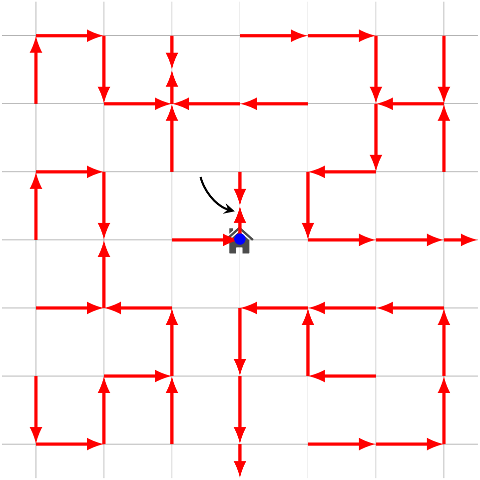
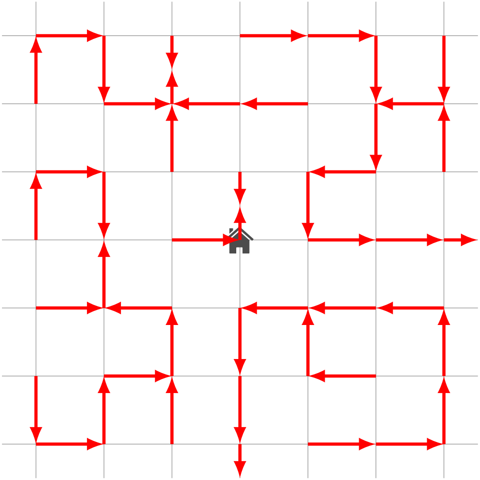


Prison break using rotor walk

Fourth walker returns to prison, and is removed.



Escape rate of rotor walk

The escape rate of n rotor walkers with initial signpost ρ is

$$r_{\text{esc}}(\rho, n) := \frac{\text{number of escaped walkers}}{n}.$$

The escape rate of rotor walk is a deterministic counterpart of the escape probability of simple random walk.

What was known about escape rate

Theorem (Schramm '10 (posthumous))

For any initial signpost ρ,

$$\limsup_{n\to\infty} \underbrace{r_{esc}(\rho, n)}_{\substack{\text{escape rate} \\ \text{of rotor walk}}} \quad \leq \quad \underbrace{p_{esc}(SRW)}_{\substack{\text{escape prob.} \\ \text{of SRW}}}.$$

Corollary

On $\mathbb{Z}^2$, for any initial signpost ρ,

$$\lim_{n\to\infty} r_{esc}(\rho, n) \;=\; p_{esc}(SRW) = 0.$$

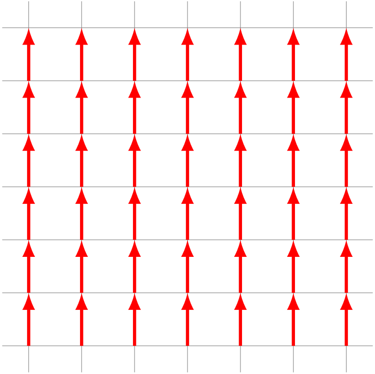
In fact, this is true for all recurrent graphs.

What was known about escape rate

Theorem (Florescu Ganguly Levine Peres '13)

On $\mathbb{Z}^d$ with $d \geq 3$, for the one-directional initial signpost ρ,

$$\liminf_{n\to\infty} r_{esc}(\rho, n) > 0.$$



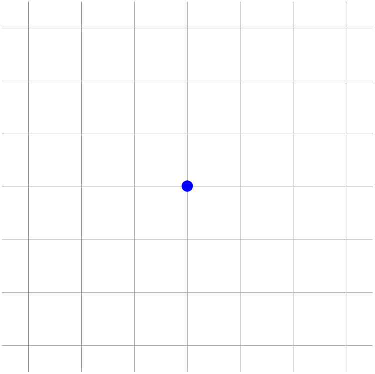
Escape rate conjecture

Conjecture (FGLP '13)

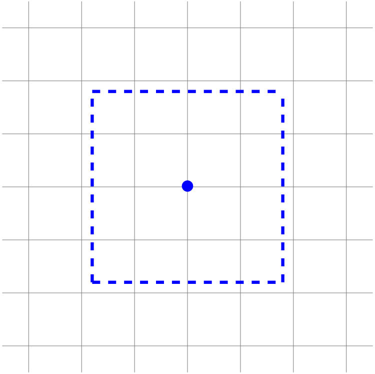
For any transient graph, there exists an initial signpost ρ for which

$$\lim_{n\to\infty} r_{esc}(\rho, n) \;=\; p_{esc}(SRW).$$

Wired spanning forest plus one edge (WSF^+)

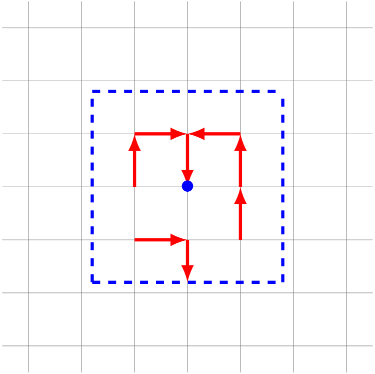


Wired spanning forest plus one edge (WSF$^+$)



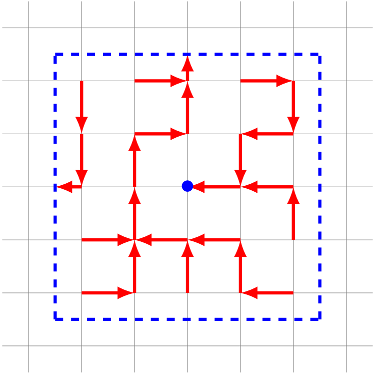
Pick a blue box around the origin, then identify the blue box with the origin.

Wired spanning forest plus one edge (WSF$^+$)



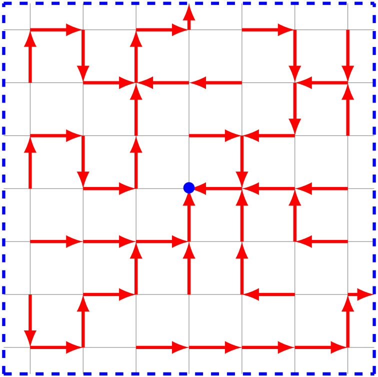
Pick a spanning tree of this graph directed to the origin (uniformly at random).

Wired spanning forest plus one edge (WSF$^+$)



Repeat the same steps with larger blue boxes.

Wired spanning forest plus one edge (WSF$^+$)



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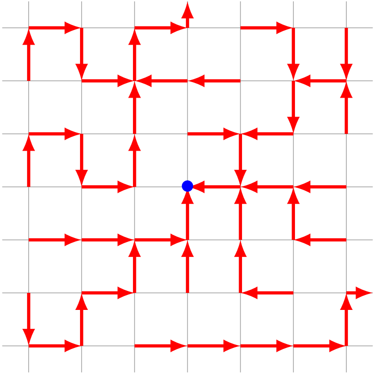
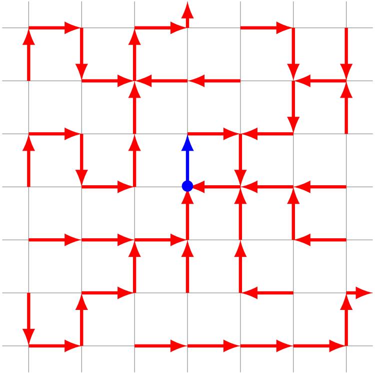


Take the limit as the blue box grows to cover $\mathbb{Z}^d$.

Wired spanning forest plus one edge (WSF$^+$)



Add a signpost to the origin, uniform among $2d$ directions.

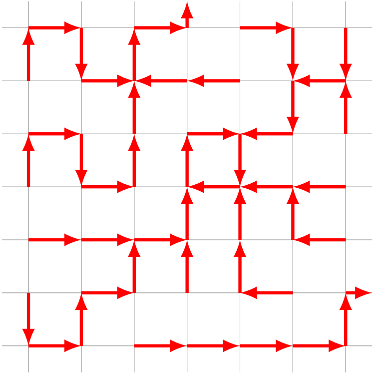
Answering the escape rate conjecture

Theorem (C. arXiv '18)

On $\mathbb{Z}^d$, the initial signpost ρ sampled from WSF$^+$ *satisfies*

$$\lim_{n \to \infty} r_{esc}(\rho, n) \;=\; p_{esc}(SRW).$$

In fact, this is true for all vertex-transitive graphs.

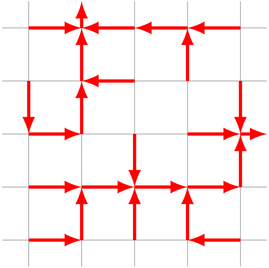


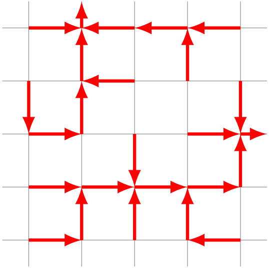
Why wired spanning forest plus one edge?

Why wired spanning forest plus one edge?

Because it is a stationary initial signpost distribution!

Stationary distribution for rotor walk





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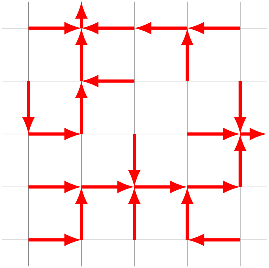
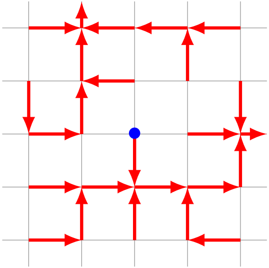
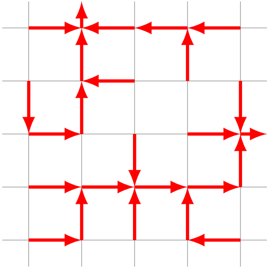
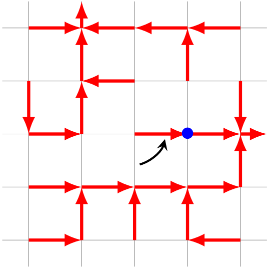




Drop a walker to the origin

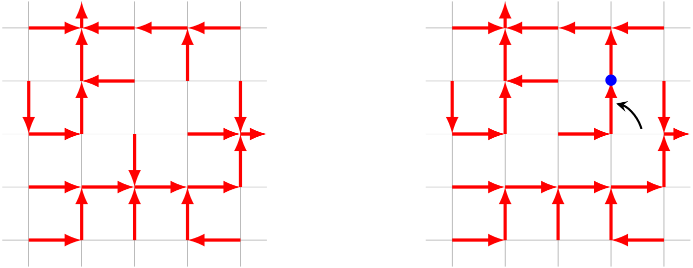
Stationary distribution for rotor walk





Perform rotor walk until walker reaches the origin or infinity.

Stationary distribution for rotor walk



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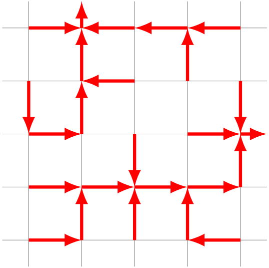

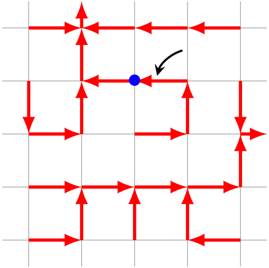

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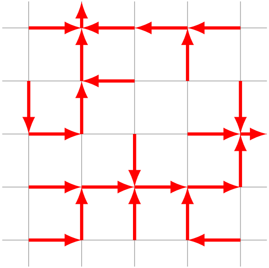

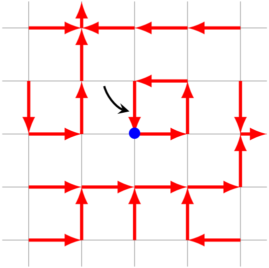

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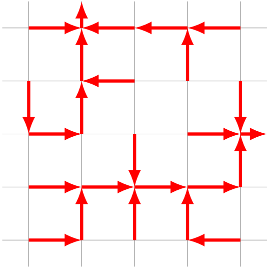

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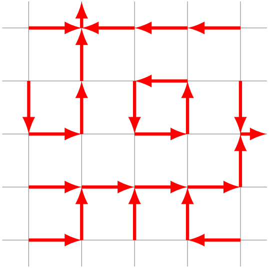
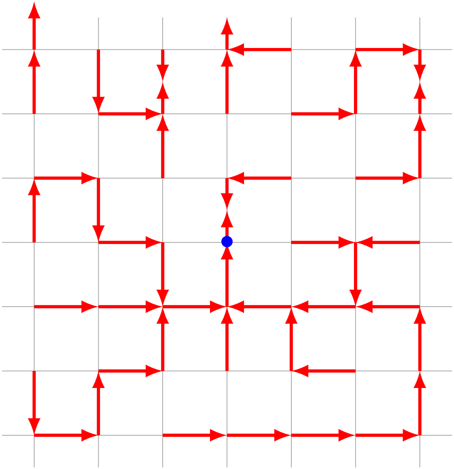


ρ'

The initial signpost distribution is stationary if

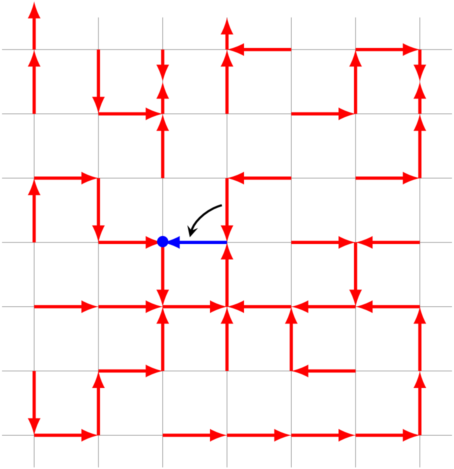
ρ' has the same distribution as ρ.

Why is WSF^{+} stationary?



The signposts at previously visited sites form a tree oriented toward the walker.

Why is WSF$^+$ stationary?



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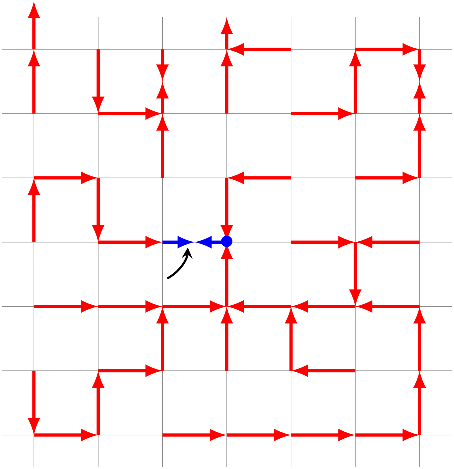

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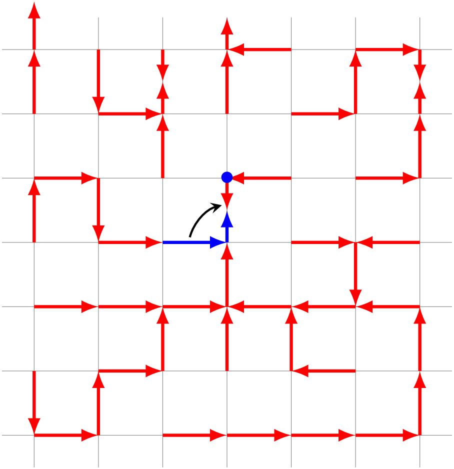

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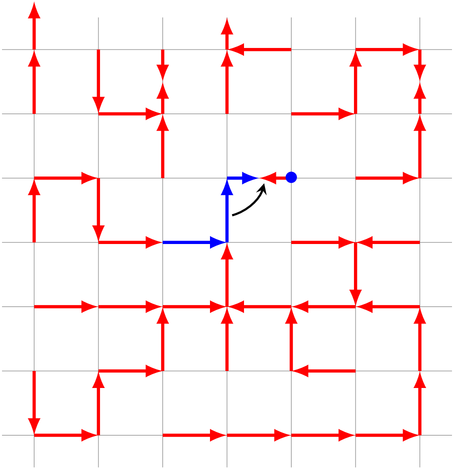

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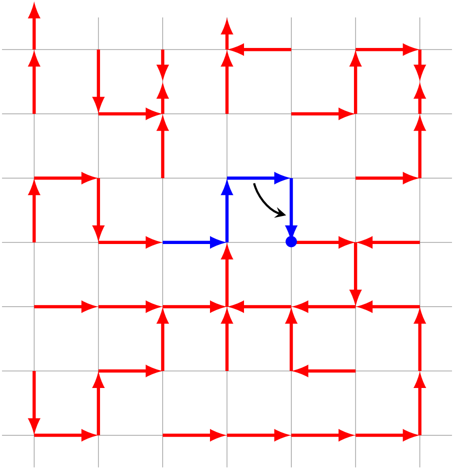

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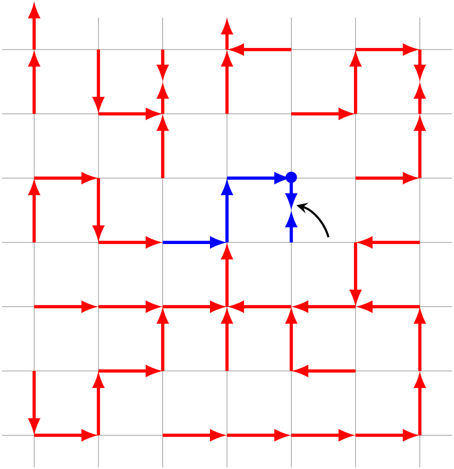

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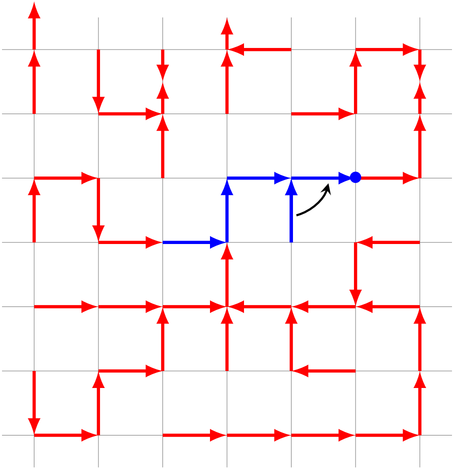

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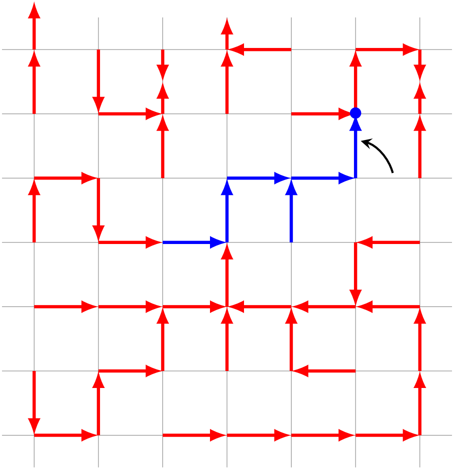

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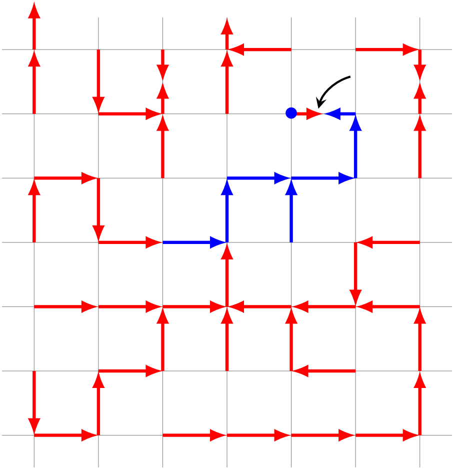

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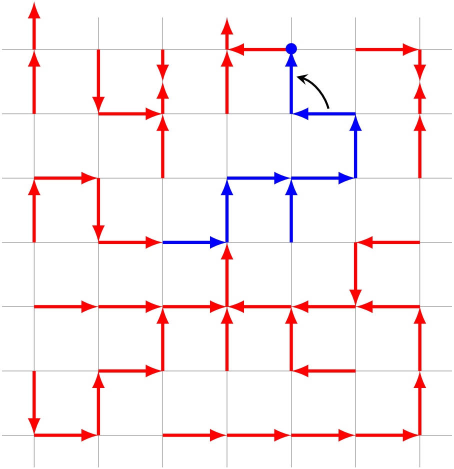

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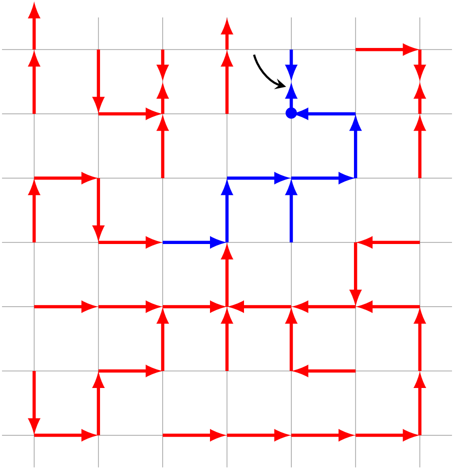

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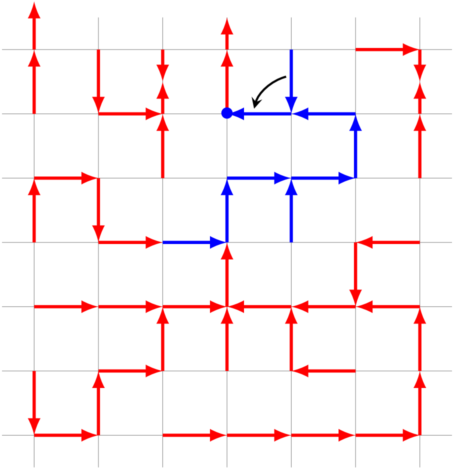


The signposts at previously visited sites form a tree oriented toward the walker.

Answering the escape rate conjecture

Theorem (C. arXiv '18)

On a vertex-transitive graph, the initial signpost ρ sampled from WSF$^+$ satisfies

$$\lim_{n\to\infty} r_{esc}(\rho, n) \; = \; p_{esc}(SRW).$$

Ingredients used in the proof:

- Stationarity of WSF$^+$;
- Ergodic theorem for Markov chains.

But ...

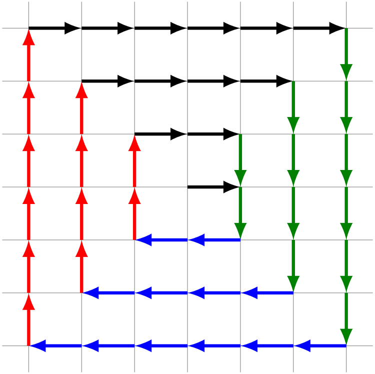
- The conjecture of FGLP '13 is for all transient graphs;
- There are already other constructions for the special case of $\mathbb{Z}^d$ (He '14) and trees (Angel Holroyd '11);
- Our construction of the initial signpost ρ is not deterministic.

Complete answer to the escape rate conjecture

Theorem (C' arXiv '18)

For any transient graph, there exists an initial signpost ρ for which

$$\lim_{n\to\infty} r_{esc}(\rho, n) \;=\; p_{esc}(SRW).$$



Discrete Green function

The discrete Green function $\mathcal{G} : \mathbb{Z}^d \to \mathbb{R}$ is

$$\mathcal{G}(x) := \text{Expected number of visits to } x \text{ by SRW started at 0.}$$

Two important properties of $\mathcal{G}$:

- $\mathcal{G}$ is a solution to the discrete Poisson equation:

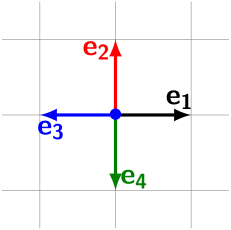
$$\mathcal{G}(x) - \frac{1}{2d} \sum_{\substack{x_1,\ldots,x_{2d} \\ \text{neighbors of } x}} \mathcal{G}(x_i) = \begin{cases} 1 & \text{if } x = 0; \\ 0 & \text{otherwise.} \end{cases}$$

- $$p_{\mathrm{esc}}(\mathrm{SRW}) = \frac{1}{\mathcal{G}(0)}.$$

Weight of a signpost

A signpost in $\mathbb{Z}^d$ points to one of the $2d$ directions:

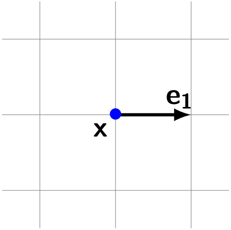
$$\begin{array}{cccc} e_1 = (1, 0, \ldots), & e_2 = (0, 1, \ldots, 0), & \cdots, & e_d = (0, \ldots, 1), \\ e_{d+1} = (-1, 0, \ldots), & e_{d+2} = (0, -1, \ldots, 0), & \cdots, & e_{2d} = (0, \ldots, -1). \end{array}$$



Weight of a signpost (continued)

The weight of a signpost at x is

$$W_x(e_i) := \sum_{k=1}^{2d} k\,\mathcal{G}(x + e_{i+k}).$$



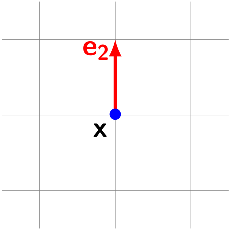
$$W_x(e_1) = \mathcal{G}(x + e_2) + 2\,\mathcal{G}(x + e_3) + 3\,\mathcal{G}(x + e_4) + 4\,\mathcal{G}(x + e_1).$$

IMPORTANT: For any x, there is a signpost direction that maximizes W_x.

Weight of a signpost (continued)

The weight of a signpost at x is

$$W_x(e_i) := \sum_{k=1}^{2d} k\, \mathcal{G}(x + e_{i+k}).$$



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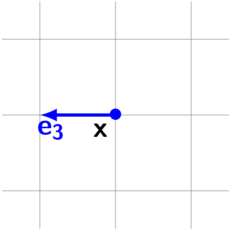

$$W_x(e_3) = \mathcal{G}(x + e_4) + 2\,\mathcal{G}(x + e_1) + 3\,\mathcal{G}(x + e_2) + 4\,\mathcal{G}(x + e_3).$$

IMPORTANT: For any x, there is a signpost direction that maximizes W_x.

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The weight of a signpost at x is

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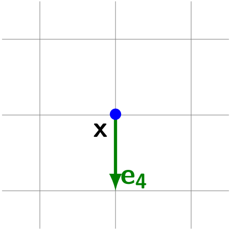


$$W_x(e_4) = \mathcal{G}(x + e_1) + 2\,\mathcal{G}(x + e_2) + 3\,\mathcal{G}(x + e_3) + 4\,\mathcal{G}(x + e_4).$$

IMPORTANT: For any x, there is a signpost direction that maximizes W_x.

Escape rate formula

Lemma

For any initial signpost ρ and number of walkers n,

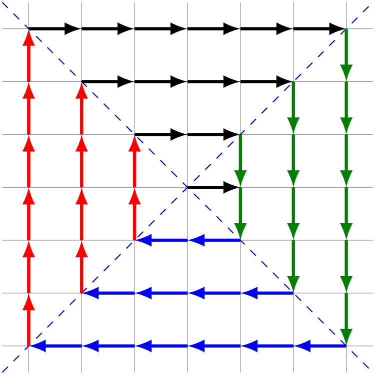
$$r_{esc}(\rho, n) = \frac{1}{\mathcal{G}(0)} - \frac{1}{2nd\,\mathcal{G}(0)} \left(\sum_{x \in \mathbb{Z}^d} W_x[\underbrace{\rho_n(x)}_{\substack{\text{signpost at } x \\ \text{after } n\text{-th walk}}}] - W_x[\underbrace{\rho(x)}_{\substack{\text{initial signpost} \\ \text{at } x}}] \right).$$

The proof is by recursion and discrete Poisson equation.

Our initial signpost configuration

The configuration ρ_{max} is constructed by choosing, for each x,

the direction $\rho_{\mathsf{max}}(x)$ that maximizes W_x.



Proof of the escape rate conjecture

- By the escape rate formula,

$$r_{\text{esc}}(\rho, n) = \frac{1}{\mathcal{G}(0)} - \frac{1}{2nd\mathcal{G}(0)} \left(\sum_{x \in \mathbb{Z}^d} W_x[\rho_n(x)] - W_x[\rho(x)] \right).$$

- By our choice of $\rho_{\max}$,

$$r_{\text{esc}}(\rho_{\max}, n) \;\geq\; \frac{1}{\mathcal{G}(0)} \;=\; p_{\text{esc}}(SRW).$$

- On the other hand, Schramm's inequality gives us

$$\limsup_{n \to \infty} r_{\text{esc}}(\rho_{\max}, n) \;\leq\; p_{\text{esc}}(SRW).$$

- Hence,

$$\lim_{n \to \infty} r_{\text{esc}}(\rho_{\max}, n) \;=\; p_{\text{esc}}(SRW).$$

Complete answer to the escape rate conjecture

Theorem (C' arXiv '18)

For any transient graph, the initial signpost $\rho_{\max}$ satisfies

$$\lim_{n\to\infty} r_{esc}(\rho_{\max}, n) \;=\; p_{esc}(SRW).$$

Ingredients used in the proof:

- Discrete Green function;
- Escape rate formula;
- Schramm's inequality.

Future direction: Transience for higher dimension

Conjecture (PDDK '96)

On $\mathbb{Z}^d$ with $d \geq 3$, the rotor walk with uniform i.i.d initial signpost visits each vertex only finitely many times.

Conjecture above can be verified by proving:

Conjecture

On $\mathbb{Z}^d$ with $d \geq 3$, the rotor walk with uniform i.i.d initial signpost satisfies

$$\lim_{n \to \infty} r_{esc}(\rho, n) \;=\; p_{esc}(SRW).$$

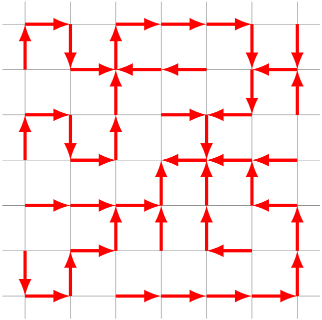
- Possible strategy: Use stationarity of WSF^+ to approximate escape rate of uniform i.i.d.?

Future direction: Recurrence for dimension 2

Conjecture (PDDK '96)

On $\mathbb{Z}^2$, the rotor walk with uniform i.i.d. initial signpost visits each vertex infinitely many times.

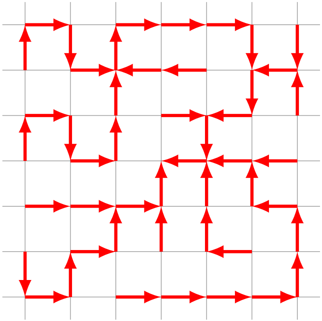
- For a randomized version of rotor walk, conjecture has been proved using escape rate formula (C. Greco Levine Li '20+).



Find preprints at: http://math.ucla.edu/~sweehong/

Email: sweehong@math.ucla.edu

THANK YOU!



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