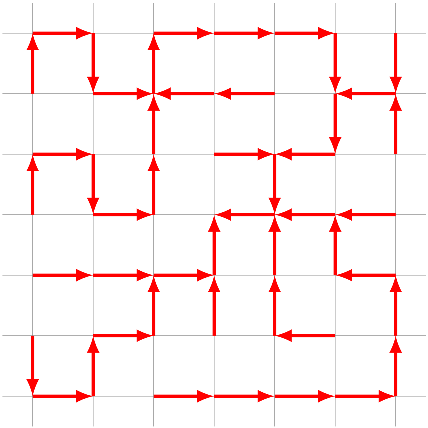
In between random walk and rotor walk

Swee Hong Chan
Cornell University
Joint work with Lila Greco, Lionel Levine, Boyao Li



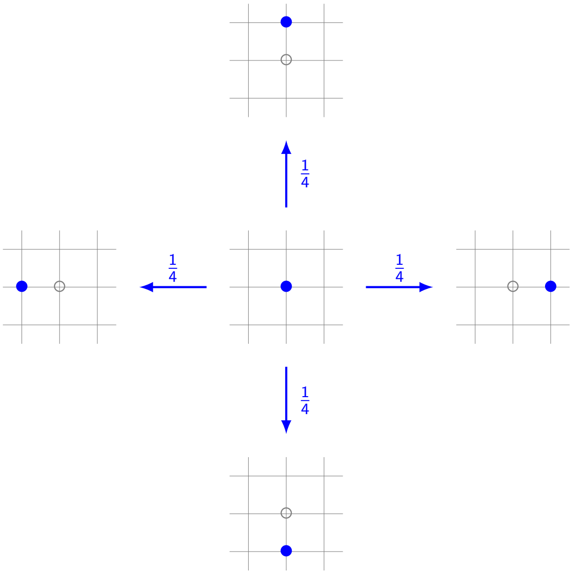

SPRING awakening
A BIG, FAT HIT!
SOMETHING ROTTEN!
SCHOOL of ROCK
BIG BUS NEW YORK
THE LEGEND
OF LIFE
THE
TDK
TOSHIBA

Random
walk

Rotor
walk

Simple random walk on $\mathbb{Z}^2$

Simple random walk on $\mathbb{Z}^2$



Simple random walk on $\mathbb{Z}^2$

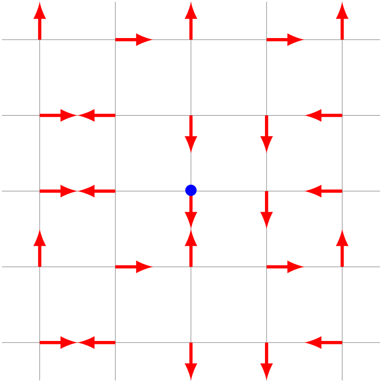
- Visits every site infinitely often? Yes!
- Scaling limit? The standard 2-D Brownian motion:

$$\Big(\frac{1}{\sqrt{n}}\underbrace{X_{[nt]}}_{\substack{\text{location of the}\\ \text{walker at time } [nt]}}\Big)_{t\geq 0} \overset{n\to\infty}{\Longrightarrow} \frac{1}{\sqrt{2}}\underbrace{(B_1(t), B_2(t))}_{\substack{\text{independent standard}\\ \text{Brownian motions}}}{}_{t\geq 0}.$$

Rotor walk on $\mathbb{Z}^2$

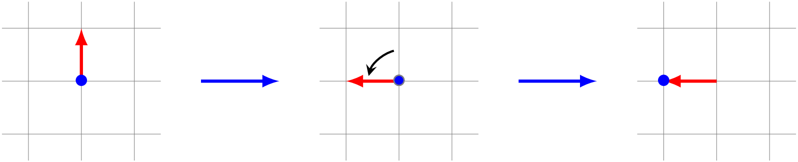
Rotor walk on $\mathbb{Z}^2$

Put a signpost at each site.



Rotor walk on $\mathbb{Z}^2$

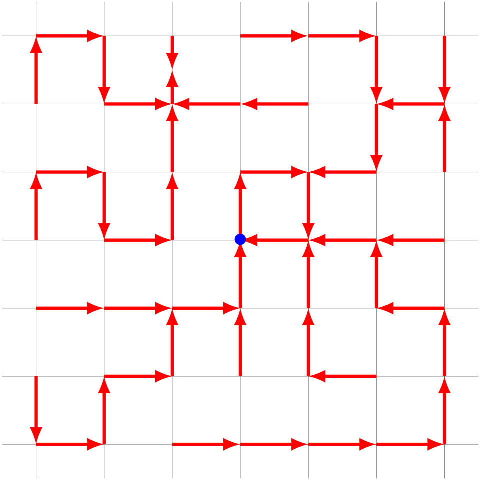
Turn the signpost 90° counterclockwise, then follow the signpost.



The signpost says:
"This is the way you went the last time you were here",
(assuming you ever were!)

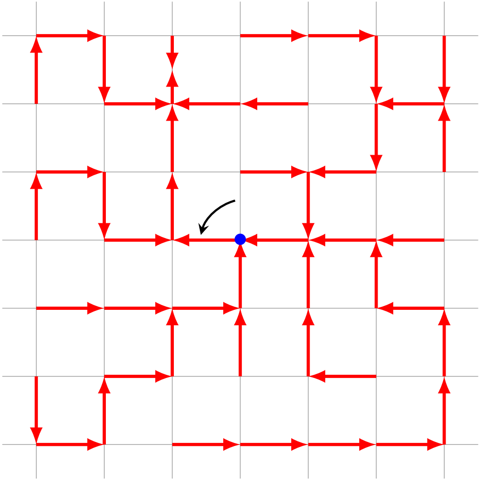
Rotor walk on $\mathbb{Z}^2$

Turn the signpost 90° counterclockwise, then follow the signpost.



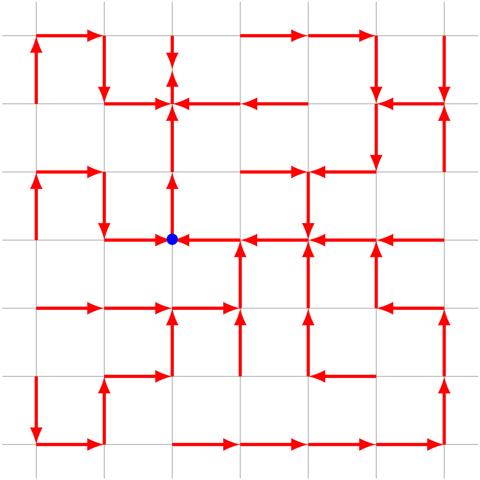
Rotor walk on $\mathbb{Z}^2$

Turn the signpost 90° counterclockwise, then follow the signpost.



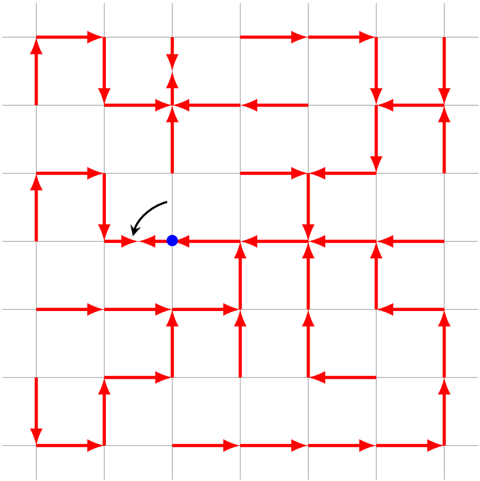
Rotor walk on $\mathbb{Z}^2$

Turn the signpost 90° counterclockwise, then follow the signpost.



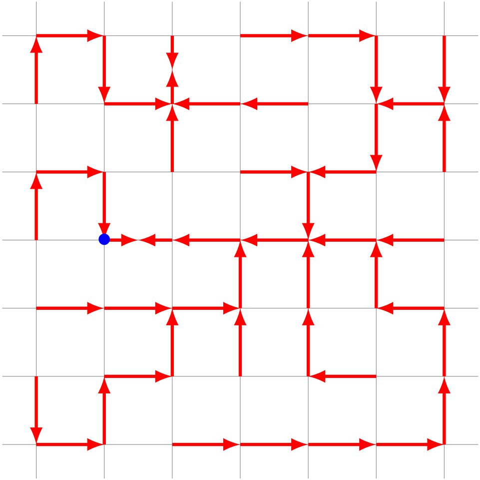
Rotor walk on $\mathbb{Z}^2$

Turn the signpost 90° counterclockwise, then follow the signpost.



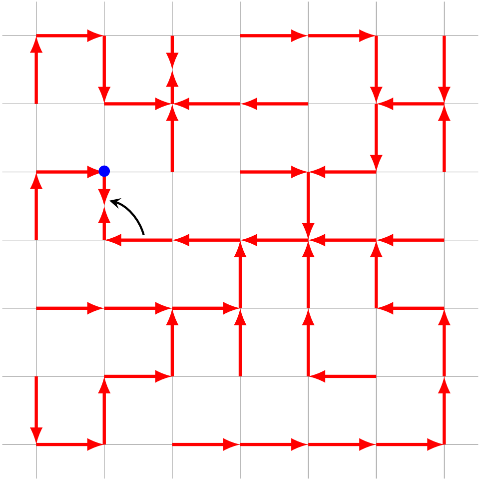
Rotor walk on $\mathbb{Z}^2$

Turn the signpost 90° counterclockwise, then follow the signpost.



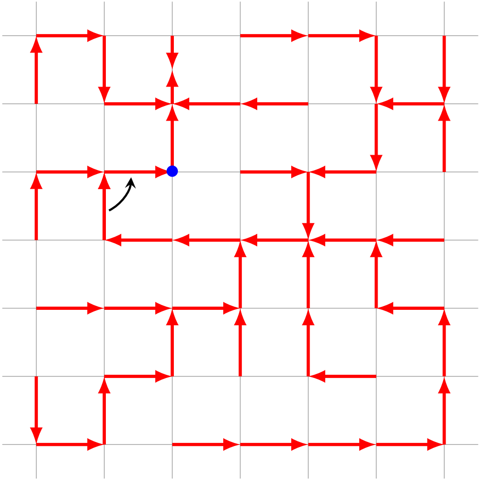
Rotor walk on $\mathbb{Z}^2$

Turn the signpost 90° counterclockwise, then follow the signpost.



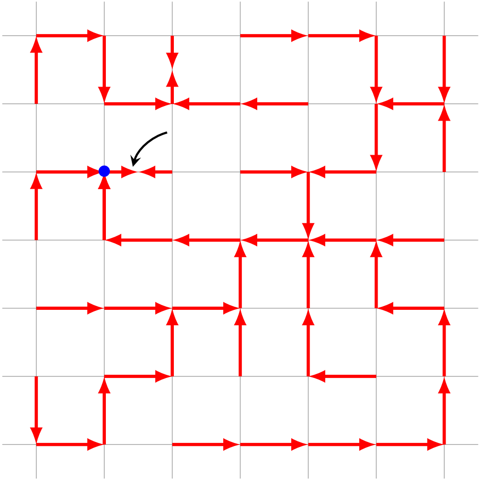
Rotor walk on $\mathbb{Z}^2$

Turn the signpost 90° counterclockwise, then follow the signpost.



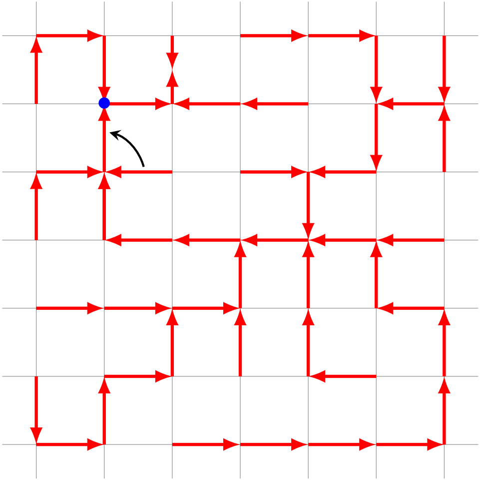
Rotor walk on $\mathbb{Z}^2$

Turn the signpost 90° counterclockwise, then follow the signpost.



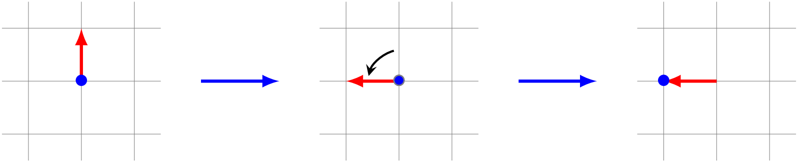
Rotor walk on $\mathbb{Z}^2$

Turn the signpost 90° counterclockwise, then follow the signpost.



Rotor walk on $\mathbb{Z}^2$

Turn the signpost 90° counterclockwise, then follow the signpost.



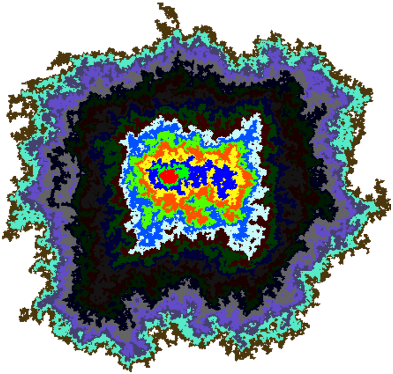
The signpost says:
"This is the way you went the last time you were here",
(assuming you ever were!)

Why rotor walk?

Randomness can be (was) expensive to simulate!

Why rotor walk?

As a model of self-organized criticality for statistical mechanics.



Visited sites after 80 returns to the origin (by Laura Florescu).

Conjectures for rotor walk on $\mathbb{Z}^2$

If the initial signposts are i.i.d. uniform among the four directions, then

- (PDDK '96) Visits every site infinitely often?
- (PDDK '96) $\#\{X_1, \ldots, X_n\}$ is $\asymp n^{2/3}$?
  (compare with $n/\log n$ for the simple random walk.)
- (Kapri-Dhar '09) The asymptotic shape of $\{X_1, \ldots, X_n\}$ is a disc?

More randomness please!

Well studied

Many open problems

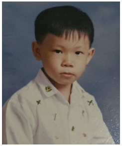
Random ⟷ Deterministic

More randomness please!

Well studied

Random

Let's study this!!!

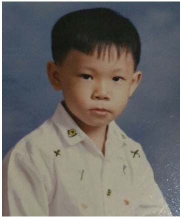


Something in between

Many open problems

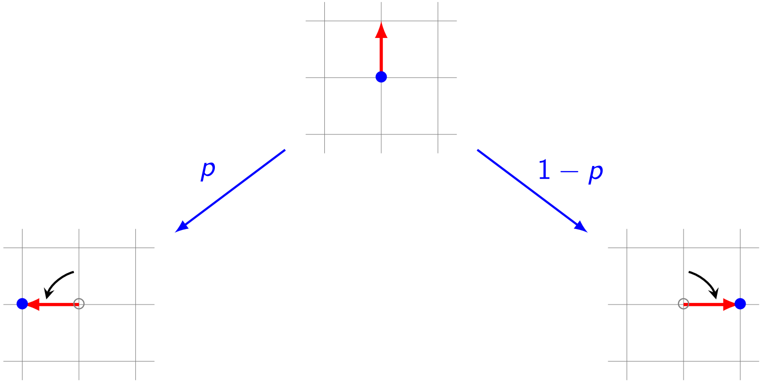
Deterministic

p-rotor walk on $\mathbb{Z}^2$



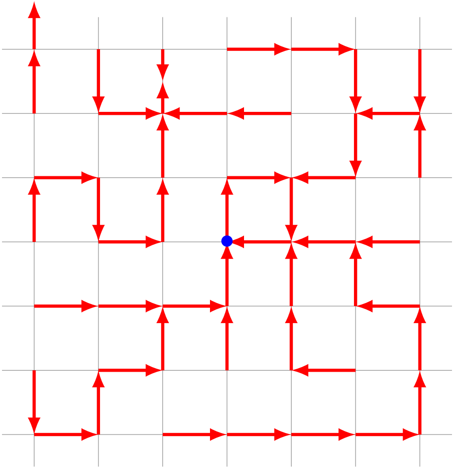
p-rotor walk on $\mathbb{Z}^2$

With probability p, turn the signpost 90° counter-clockwise.
With probability $1 - p$, turn the signpost 90° clockwise.



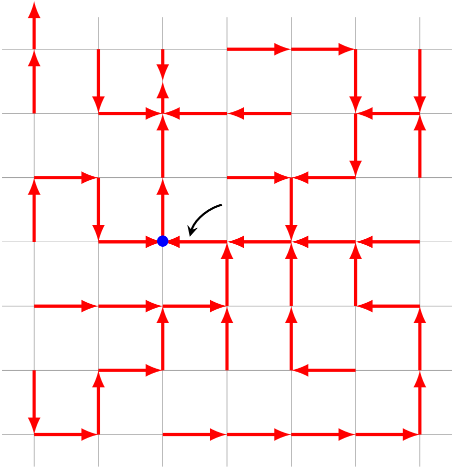
p-rotor walk on $\mathbb{Z}^2$

Follow rotor walk rule with prob. p, do the opposite with prob. $1 - p$



p-rotor walk on $\mathbb{Z}^2$

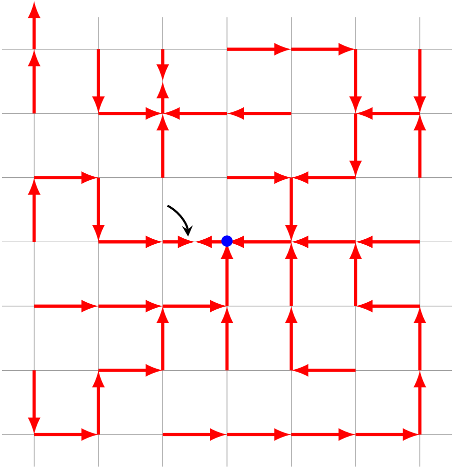
Follow rotor walk rule with prob. p, do the opposite with prob. $1 - p$



Follow the rule.

p-rotor walk on $\mathbb{Z}^2$

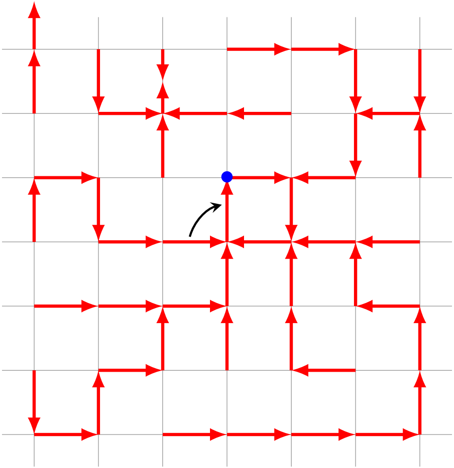
Follow rotor walk rule with prob. p, do the opposite with prob. $1 - p$



Do the opposite.

p-rotor walk on $\mathbb{Z}^2$

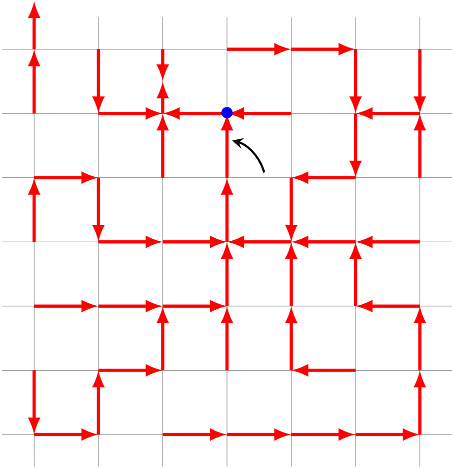
Follow rotor walk rule with prob. p, do the opposite with prob. $1 - p$



Do the opposite again.

p-rotor walk on $\mathbb{Z}^2$

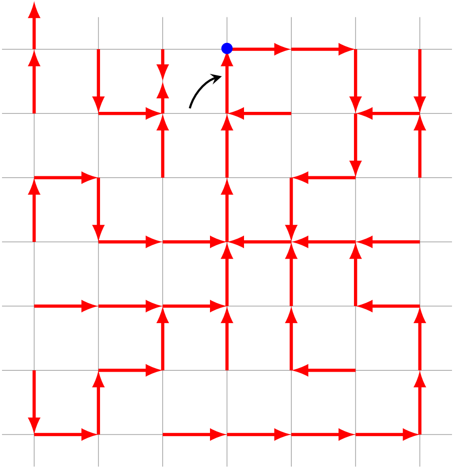
Follow rotor walk rule with prob. p, do the opposite with prob. $1 - p$



Follow the rule.

p-rotor walk on $\mathbb{Z}^2$

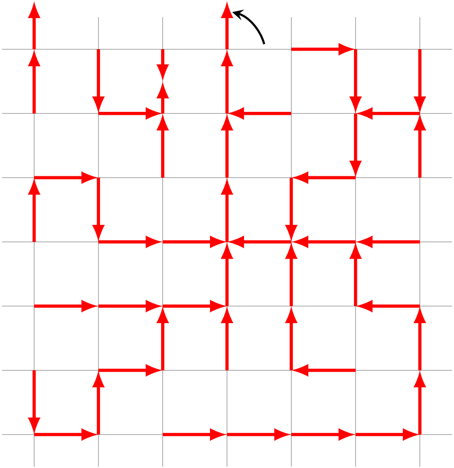
Follow rotor walk rule with prob. p, do the opposite with prob. $1-p$



Do the opposite.

p-rotor walk on $\mathbb{Z}^2$

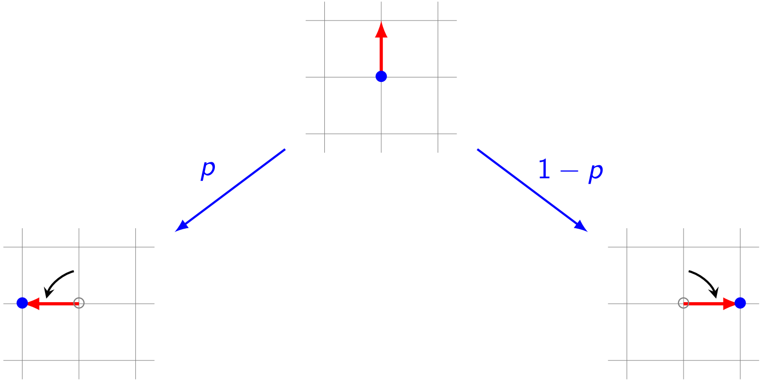
Follow rotor walk rule with prob. p, do the opposite with prob. $1 - p$



Ops...

p-rotor walk on $\mathbb{Z}^2$

With probability p, turn the signpost 90° counter-clockwise.
With probability $1 - p$, turn the signpost 90° clockwise.

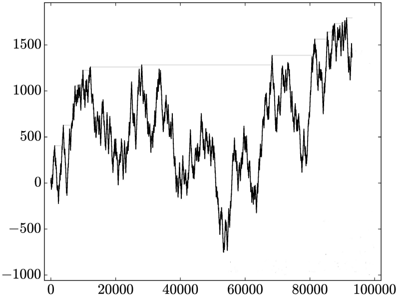


Recover the rotor walk if $p = 1$.

Scaling limit for p-rotor walk on $\mathbb{Z}$

(Huss, Levine, Sava-Huss 18) The scaling limit for p-rotor walk on $\mathbb{Z}$ is a perturbed Brownian motion $(Y(t))_{t\geq 0}$,

$$Y(t) = \underbrace{B(t)}_{\substack{\text{standard}\\\text{Brownian}\\\text{motion}}} + \underbrace{a \sup_{0\leq s\leq t} Y(s)}_{\substack{\text{perturbation at}\\\text{maximum}}} + \underbrace{b \inf_{0\leq s\leq t} Y(s)}_{\substack{\text{perturbation at}\\\text{minimum}}}, \qquad t \geq 0.$$



$Y(t)$ for $a = -0.998$, and $b = 0$ (by Wilfried Huss).

Scaling limit for p-rotor walk on $\mathbb{Z}^2$

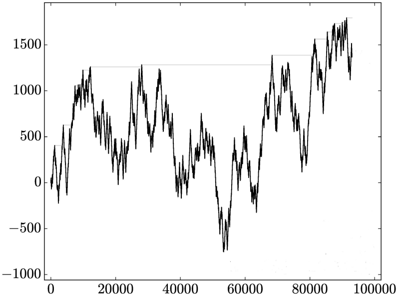
Question: Is the scaling limit for p-rotor walk on $\mathbb{Z}^2$ is a "2-D perturbed Brownian motion"?

Problem: How to define "2-D perturbed Brownian motion"?.

Scaling limit for p-rotor walk on $\mathbb{Z}$

(Huss, Levine, Sava-Huss 18) The scaling limit for p-rotor walk on $\mathbb{Z}$ is a perturbed Brownian motion $(Y(t))_{t\geq 0}$,

$$Y(t) = \underbrace{B(t)}_{\substack{\text{standard}\\ \text{Brownian}\\ \text{motion}}} + \underbrace{a \sup_{0\leq s\leq t} Y(s)}_{\substack{\text{perturbation at}\\ \text{maximum}}} + \underbrace{b \inf_{0\leq s\leq t} Y(s)}_{\substack{\text{perturbation at}\\ \text{minimum}}}. \qquad t \geq 0.$$



$Y(t)$ for $a = -0.998$, and $b = 0$ (by Wilfried Huss).

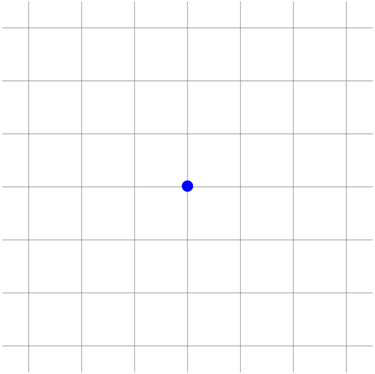
Scaling limit for p-rotor walk on $\mathbb{Z}^2$

Question: Is the scaling limit for p-rotor walk on $\mathbb{Z}^2$ is a "2-D perturbed Brownian motion"?

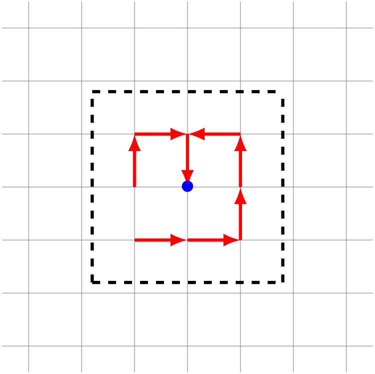
Problem: How to define "2-D perturbed Brownian motion"?.

Conjecture: The scaling limit for p-rotor walk on $\mathbb{Z}^2$ when $p = \frac{1}{2}$ is the standard 2-D Brownian motion.

Uniform spanning tree plus one edge (UST$^+$)

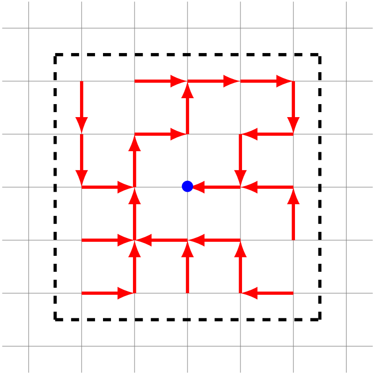


Uniform spanning tree plus one edge (UST$^+$)



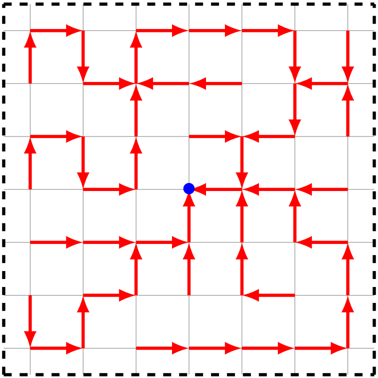
Pick a spanning tree of the black box directed to the origin (uniformly at random).

Uniform spanning tree plus one edge (UST$^+$)



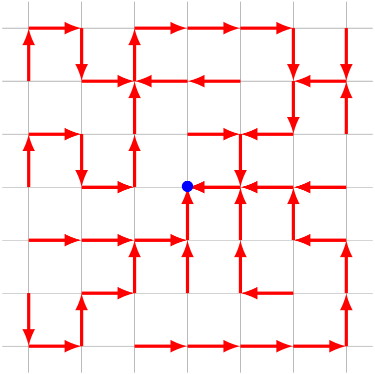
Take the limit as the black box grows until it covers $\mathbb{Z}^2$.

Uniform spanning tree plus one edge (UST$^+$)



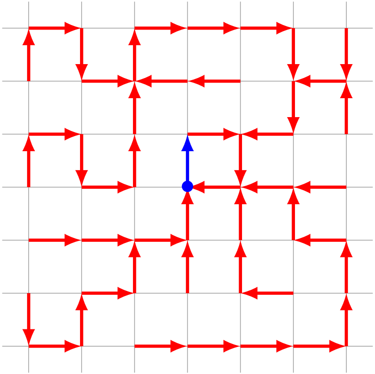
Take the limit as the black box grows until it covers $\mathbb{Z}^2$.

Uniform spanning tree plus one edge (UST$^+$)



Take the limit as the black box grows until it covers $\mathbb{Z}^2$.

Uniform spanning tree plus one edge (UST$^+$)



Add a signpost from the origin, uniform among the four directions.

Scaling limit for p-rotor walk on $\mathbb{Z}^2$

Theorem (C., Greco, Levine, Li '18+)

Let $p = \frac{1}{2}$ and let the uniform spanning tree plus one edge be the initial signpost configuration. Then, with probability 1, the p-rotor walk on $\mathbb{Z}^2$ scales to the standard 2-D Brownian motion:

$$\frac{1}{\sqrt{n}}(\underbrace{X_{[nt]}}_{\substack{\text{location of the} \\ \text{walker at time } [nt]}})_{t \geq 0} \overset{n \to \infty}{\Longrightarrow} \frac{1}{\sqrt{2}}(\underbrace{B_1(t), B_2(t)}_{\substack{\text{independent} \\ \text{Brownian motions}}})_{t \geq 0}.$$

Main ideas of the proof

- How does $p = \frac{1}{2}$ help?

- How does uniform spanning tree plus one edge help?

Main ideas of the proof

- How does $p = \frac{1}{2}$ help?

  Because then the p-rotor walk is a martingale:

$$\mathbb{E}[\underbrace{X_{t+1}}_{\text{location of the walker}} \mid \mathcal{F}_t] = X_t + \Big(p \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{90^\circ \text{ rotation matrix}} + (1-p) \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{-90^\circ \text{ rotation matrix}}\Big) \underbrace{\rho_t(X_t)}_{\text{signpost of } X_t \text{ at time } t}$$

$$= X_t.$$

- How does uniform spanning tree plus one edge help?

Martingale CLT

If $(X_t)_{t\geq 0}$ is a martingale with bounded differences in $\mathbb{R}^2$, then

$$\frac{1}{\sqrt{n}}(X_{[nt]})_{t\geq 0} \overset{n\to\infty}{\Longrightarrow} \frac{1}{\sqrt{2}}(\underbrace{B_1(t), B_2(t)}_{\substack{\text{independent} \\ \text{Brownian motions}}})_{t\geq 0},$$

provided that:

$$\frac{1}{n}\sum_{t=0}^{n-1}(\underbrace{X_{t+1}-X_t}_{\substack{\text{martingale} \\ \text{difference}}})(X_{t+1}-X_t)^\top \xrightarrow[P]{n\to\infty} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}. \qquad \text{(LLN)}$$

Martingale CLT

If $(X_t)_{t\geq 0}$ is a martingale with bounded differences in $\mathbb{R}^2$, then

$$\frac{1}{\sqrt{n}}(X_{[nt]})_{t\geq 0} \overset{n\to\infty}{\Longrightarrow} \frac{1}{\sqrt{2}}\underbrace{(B_1(t), B_2(t))}_{\substack{\text{independent} \\ \text{Brownian motions}}}{}_{t\geq 0},$$

provided that:

$$\frac{1}{n}\sum_{t=0}^{n-1} \mathbf{1}\{\underbrace{\widehat{\rho}_t(0)}_{\substack{\text{walker's signpost} \\ \text{at time } t}} = \text{vertical}\} \xrightarrow[P]{n\to\infty} \frac{1}{2}. \qquad \text{(LLN)}$$

In our case, (LLN) means the fraction of vertical signposts encountered by the walker converges (in probability) to one-half.

Martingale CLT

If $(X_t)_{t\geq 0}$ is a martingale with bounded differences in $\mathbb{R}^2$, then

$$\frac{1}{\sqrt{n}}(X_{[nt]})_{t\geq 0} \overset{n\to\infty}{\Longrightarrow} \frac{1}{\sqrt{2}}\underbrace{(B_1(t), B_2(t))}_{\substack{\text{independent} \\ \text{Brownian motions}}}{}_{t\geq 0},$$

provided that:

$$\frac{1}{n}\sum_{t=0}^{n-1} \mathbf{1}\{\underbrace{\widehat{\rho}_t(0)}_{\substack{\text{walker's signpost} \\ \text{at time } t}} = \text{vertical}\} \xrightarrow[P]{n\to\infty} \frac{1}{2}. \qquad \text{(LLN)}$$

In our case, (LLN) means the fraction of vertical signposts encountered by the walker converges (in probability) to one-half.

Main ideas of the proof

- How does $p = \frac{1}{2}$ help?

  Because then the p-rotor walk is a martingale.

- How does uniform spanning tree plus one edge help?

Main ideas of the proof

- How does $p = \frac{1}{2}$ help?

  Because then the p-rotor walk is a martingale.

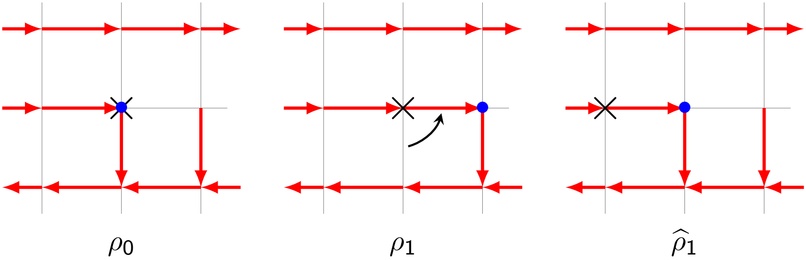
- How does uniform spanning tree plus one edge help?

  Because it is stationary and ergodic from the walker's POV.

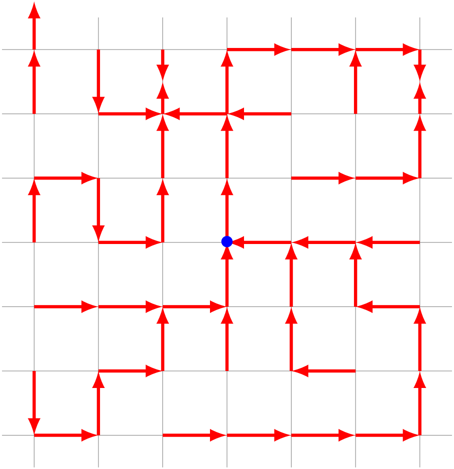
Stationarity from the walker's POV

A signpost configuration $(\rho_0(x))_{x\in\mathbb{Z}^2}$ is stationary in time from the walker's point of view if

$$\underbrace{(\widehat{\rho}_1(x))_{x\in\mathbb{Z}^2}}_{\substack{\text{signpost conf. at} \\ \text{time 1 from walker's POV}}} := (\rho_1(x - X_1))_{x\in\mathbb{Z}^2} \overset{d}{=} \underbrace{(\rho_0(x))_{x\in\mathbb{Z}^2}}_{\substack{\text{signpost conf.} \\ \text{at time 0}}}.$$

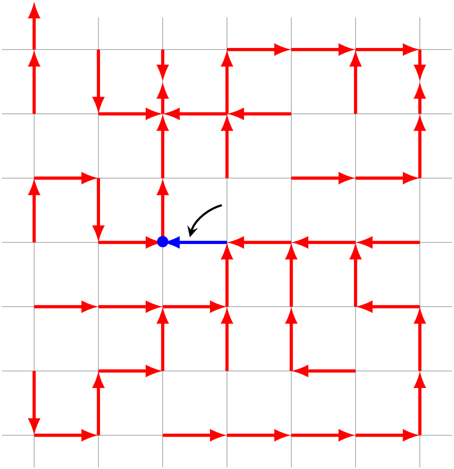


Why is UST$^+$ stationary?



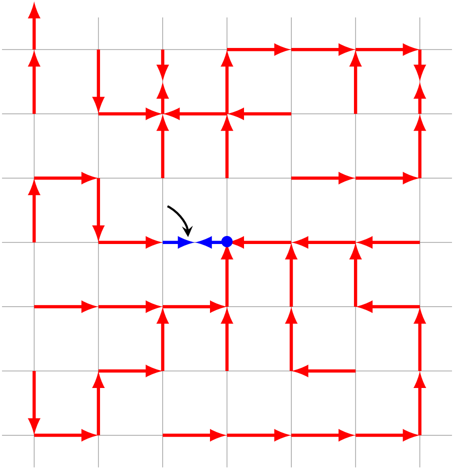
The signposts at previously visited sites form a tree oriented towards the walker.

Why is UST$^+$ stationary?



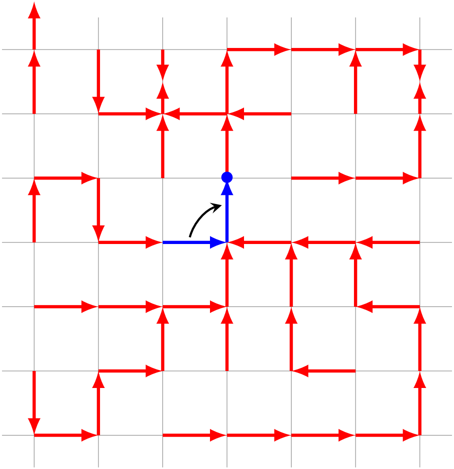
The signposts at previously visited sites form a tree oriented towards the walker.

Why is UST$^+$ stationary?



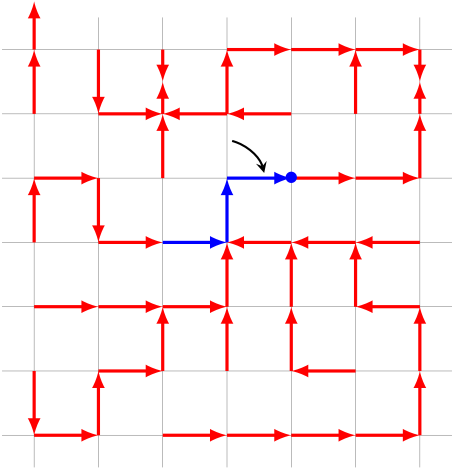
The signposts at previously visited sites form a tree oriented towards the walker.

Why is UST$^+$ stationary?



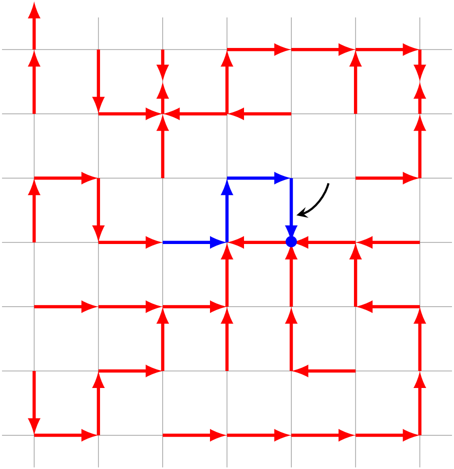
The signposts at previously visited sites form a tree oriented towards the walker.

Why is UST$^+$ stationary?



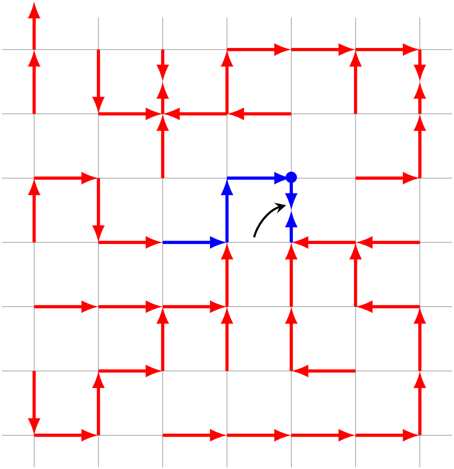
The signposts at previously visited sites form a tree oriented towards the walker.

Why is UST$^+$ stationary?



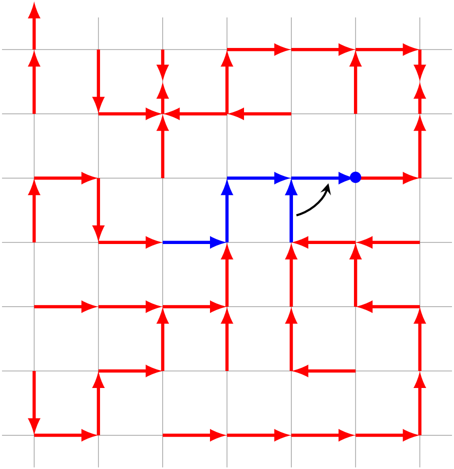
The signposts at previously visited sites form a tree oriented towards the walker.

Why is UST$^+$ stationary?



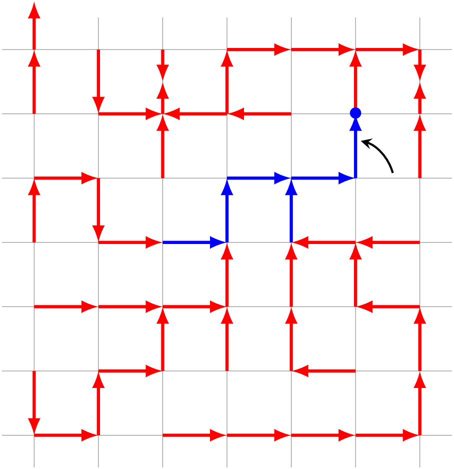
The signposts at previously visited sites form a tree oriented towards the walker.

Why is UST$^+$ stationary?



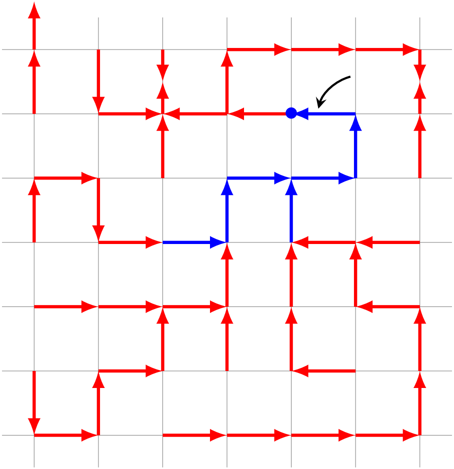
The signposts at previously visited sites form a tree oriented towards the walker.

Why is UST^+ stationary?



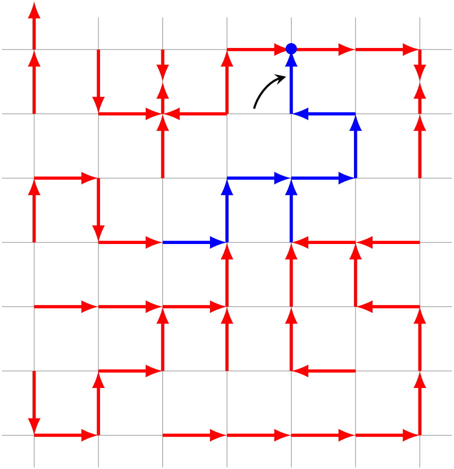
The signposts at previously visited sites form a tree oriented towards the walker.

Why is UST$^+$ stationary?



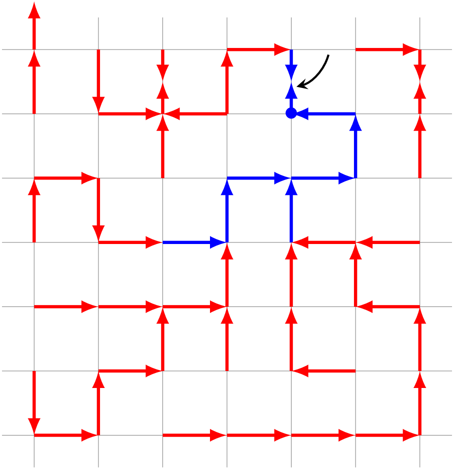
The signposts at previously visited sites form a tree oriented towards the walker.

Why is UST$^+$ stationary?



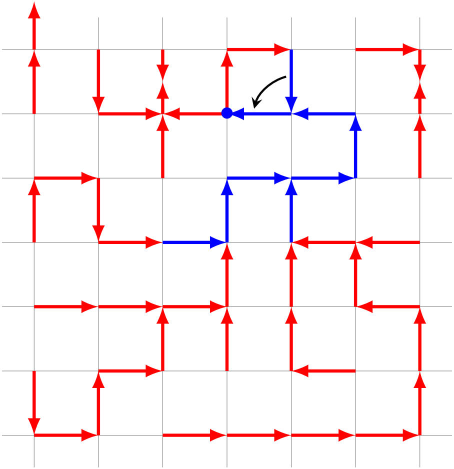
The signposts at previously visited sites form a tree oriented towards the walker.

Why is UST$^+$ stationary?



The signposts at previously visited sites form a tree oriented towards the walker.

Why is UST$^+$ stationary?



The signposts at previously visited sites form a tree oriented towards the walker.

Pointwise ergodic theorem

For a Markov chain (Ω, T, π) and any integrable function f,

$$\frac{1}{n} \sum_{t=0}^{n-1} f(T^t(x)) \xrightarrow[\pi-a.s.]{n\to\infty} \underbrace{\mathbb{E}[f \mid \mathcal{I}]}_{\substack{\text{conditioning on the} \\ \text{invariant } \sigma-\text{field}}}.$$

Pointwise ergodic theorem

For a Markov chain (Ω, T, π) and any integrable function f,

$$\frac{1}{n} \sum_{t=0}^{n-1} f(T^t(x)) \xrightarrow[\pi-a.s.]{n\to\infty} \underbrace{\mathbb{E}[f \mid \mathcal{I}]}_{\substack{\text{conditioning on the} \\ \text{invariant } \sigma-\text{field}}}.$$

This implies:

$$\frac{1}{n} \sum_{t=0}^{n-1} \mathbf{1}\{\underbrace{\widehat{\rho}_t(0)}_{\substack{\text{walker's signpost} \\ \text{at time } t}} = \text{vertical}\} \xrightarrow[P]{n\to\infty} \mathbb{E}[f \mid \mathcal{I}], \qquad \text{(LLN')}$$

Ω = set of signpost configurations;
T = One step of p-rotor walk + recentering;
$\pi = \text{UST}^+$;
$f = \mathbf{1}\{\rho(0) = \text{vertical}\}$.

Back to (LLN)

Note that:

$$\frac{1}{n}\sum_{t=0}^{n-1} \mathbf{1}\{\underbrace{\widehat{\rho}_t(0)}_{\substack{\text{walker's signpost}\\ \text{at time } t}} = \text{vertical}\} \xrightarrow[P]{n\to\infty} \underbrace{\mathbb{E}[f \mid \mathcal{I}]}_{\substack{\text{conditioning on the}\\ \text{invariant } \sigma-\text{field}}}, \qquad \text{(LLN')}$$

implies

$$\frac{1}{n}\sum_{t=0}^{n-1} \mathbf{1}\{\widehat{\rho}_t(0) = \text{vertical}\} \xrightarrow[P]{n\to\infty} \frac{1}{2}, \qquad \text{(LLN)}$$

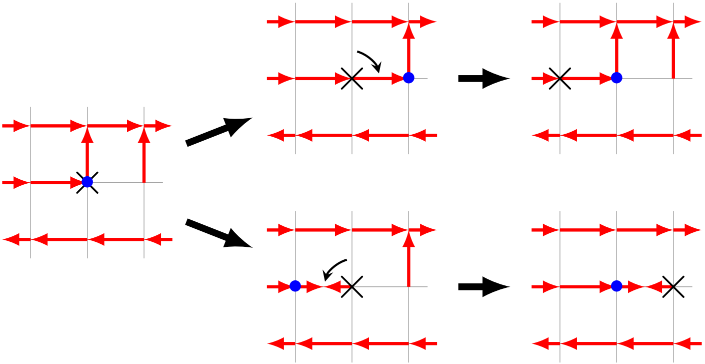
if the chain is ergodic,

$$A \text{ is invariant} \quad \Rightarrow \quad \mathbb{P}[A] \in \{0,1\}.$$

Invariant sets

A set of signpost configurations A is invariant if

$$\rho \in A \quad \Leftrightarrow \quad T(\rho) \in A \qquad \text{almost surely.}$$

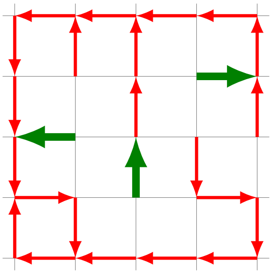


Tail triviality

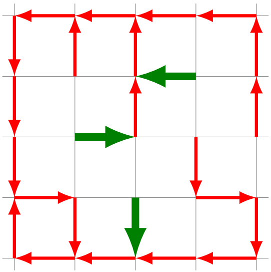
A set of signpost configurations A is a tail set if

$$\rho \in A \quad \Leftrightarrow \quad \rho' \in A,$$

for any ρ, ρ' that differ by finitely many edges.



ρ



ρ'

Tail triviality

A set of signpost configurations A is a tail set if

$$\rho \in A \quad \Leftrightarrow \quad \rho' \in A,$$

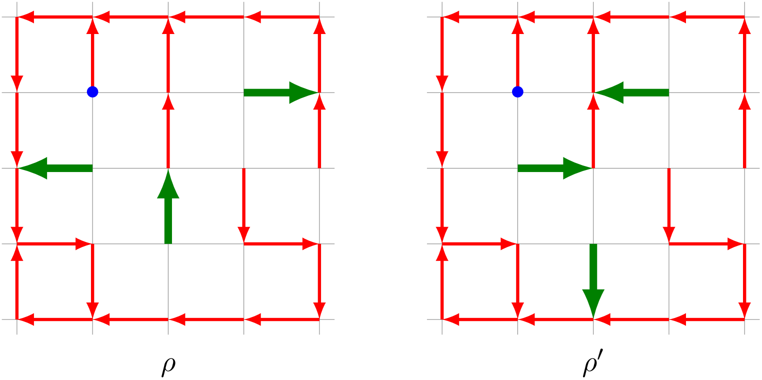
for any ρ, ρ' that differ by finitely many edges.

It is known that UST^+ is tail trivial,

$$A \text{ is a tail set} \quad \Rightarrow \quad \mathbb{P}[A] \in \{0, 1\}.$$

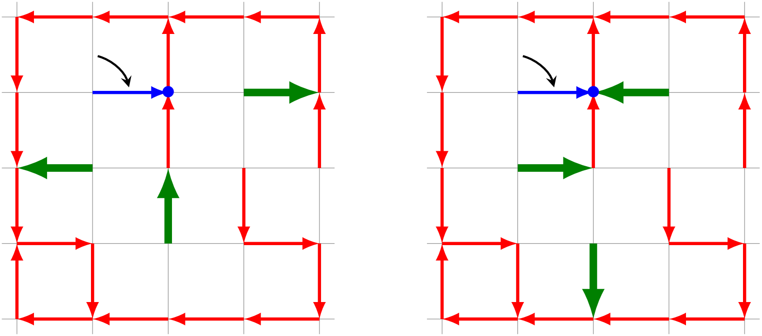
How tail triviality implies ergodicity (sketch)

Because “any” invariant set A is a tail set.



How tail triviality implies ergodicity (sketch)

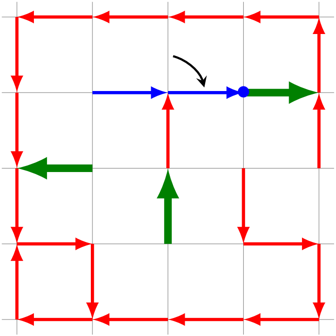
Because "any" invariant set A is a tail set.



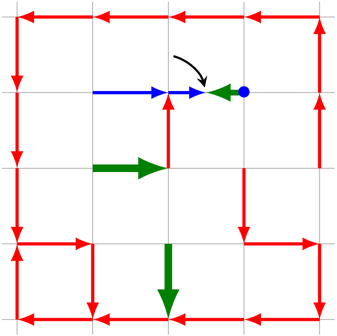
ρ_1 ρ_1'

How tail triviality implies ergodicity (sketch)

Because “any” invariant set A is a tail set.



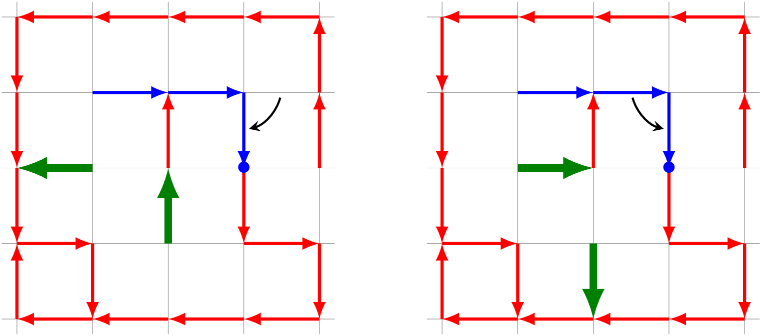
ρ_2



ρ_2'

How tail triviality implies ergodicity (sketch)

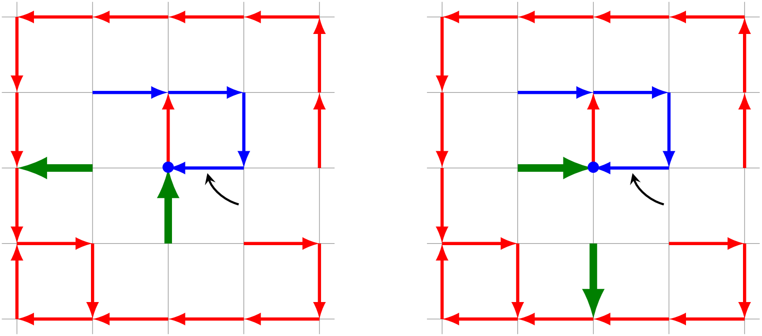
Because “any” invariant set A is a tail set.



ρ_3 $\quad\quad$ ρ_3'

How tail triviality implies ergodicity (sketch)

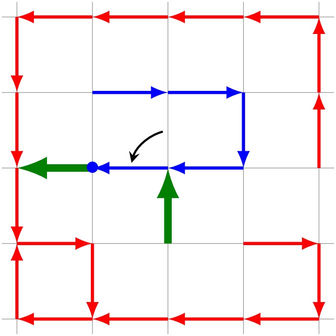
Because "any" invariant set A is a tail set.



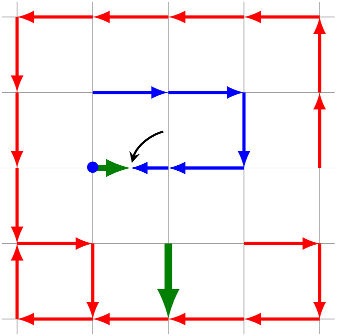
ρ_4 ρ_4'

How tail triviality implies ergodicity (sketch)

Because "any" invariant set A is a tail set.



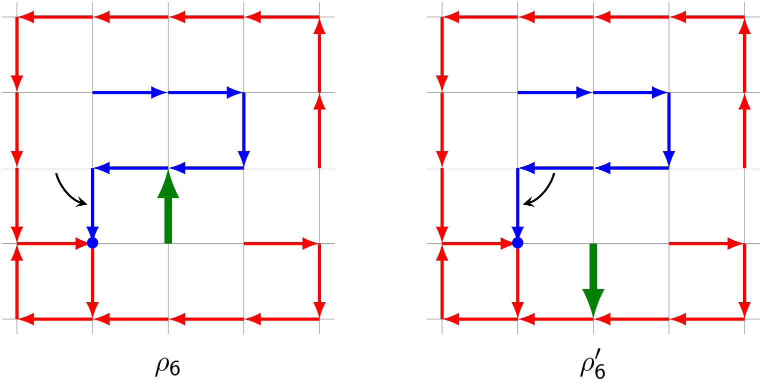
ρ_5



ρ'_5

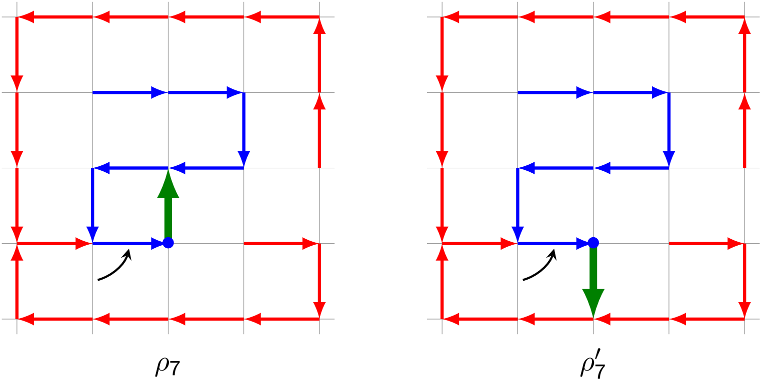
How tail triviality implies ergodicity (sketch)

Because “any” invariant set A is a tail set.



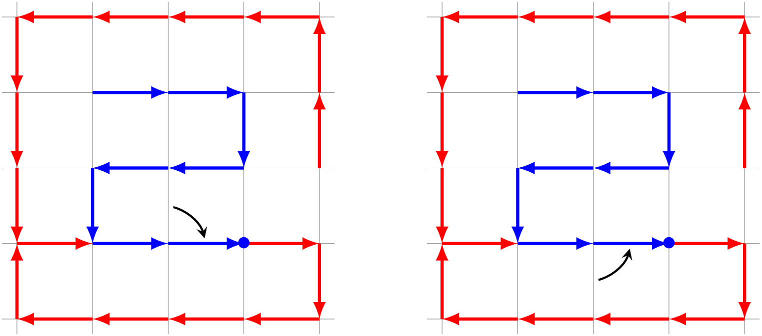
How tail triviality implies ergodicity (sketch)

Because “any” invariant set A is a tail set.



How tail triviality implies ergodicity (sketch)

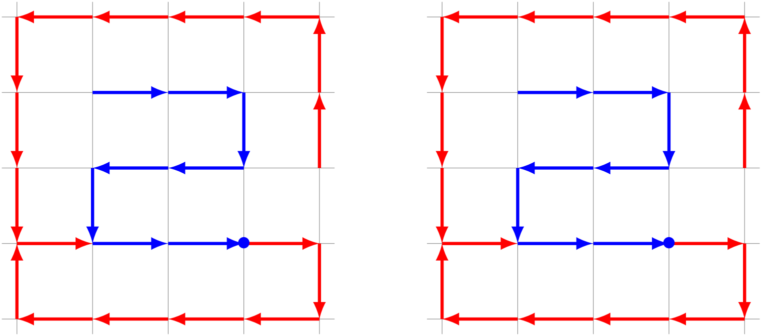
Because "any" invariant set A is a tail set.



ρ_8 $\qquad$ ρ'_8

How tail triviality implies ergodicity (sketch)

Because “any” invariant set A is a tail set.



ρ_8 $\qquad\qquad$ ρ'_8

$$\rho \in A \quad \Leftrightarrow \quad \rho_8 = \rho'_8 \in A \quad \Leftrightarrow \quad \rho' \in A.$$

Back to the scaling limit

Scaling limit

↓ **Martingale CLT**

(LLN)

↓ **Ergodic theorem**

Ergodicity

↓

Tail triviality ✓

Back to the scaling limit

Theorem (C., Greco, Levine, Li '18+)

Let $p = \frac{1}{2}$ and let the uniform spanning tree plus one edge be the initial signpost configuration. Then, with probability 1, the p-rotor walk on $\mathbb{Z}^2$ scales to the standard 2-D Brownian motion:

$$\frac{1}{\sqrt{n}}(\underbrace{X_{[nt]}}_{\substack{\text{location of the}\\ \text{walker at time } [nt]}})_{t\geq 0} \overset{n\to\infty}{\Longrightarrow} \frac{1}{\sqrt{2}}(\underbrace{B_1(t), B_2(t)}_{\substack{\text{independent}\\ \text{Brownian motions}}})_{t\geq 0}.$$

What is next?

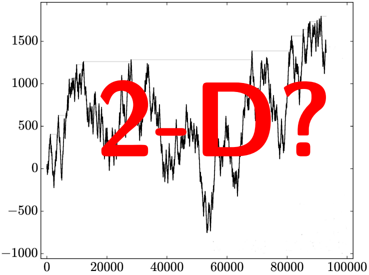
For p-rotor walk with UST^+ as the initial signpost configuration:

What is next?

For p-rotor walk with UST^+ as the initial signpost configuration:

Question: Prove scaling limit for when $p \neq \frac{1}{2}$?

Problem: Need to define the "2-D perturbed Brownian motion (?)".



What is next?

For p-rotor walk with UST^+ as the initial signpost configuration:

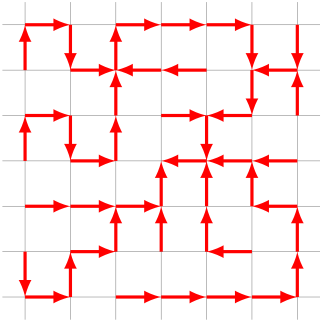
Question: Does the walk visit every site in $\mathbb{Z}^d$ infinitely often?

Answer for $p \in \{0, 1\}$: NO (Florescu, Levine, Peres 16):

Answer for $\mathbb{Z}$ with $p \in (0, 1)$: YES (Huss, Levine, Sava-Huss 18).

Answer for $\mathbb{Z}^d$ with $p = \frac{1}{2}$ and $d \geq 3$: NO.

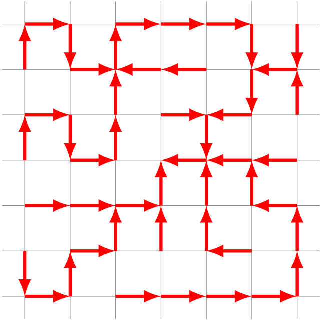
Open for $\mathbb{Z}^d$ with $p \in (0, 1)$ and $d = 2$.

Preprint: https://math.cornell.edu/∼sc2637/RWLM.pdf

Can be contacted at: sweehong@math.cornell.edu

THANK YOU!



Preprint: https://math.cornell.edu/∼sc2637/RWLM.pdf

Can be contacted at: sweehong@math.cornell.edu