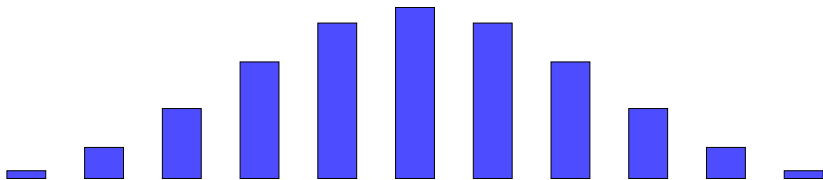


Complexity of Combinatorial Log-concave Inequalities

Swee Hong Chan

joint with Igor Pak



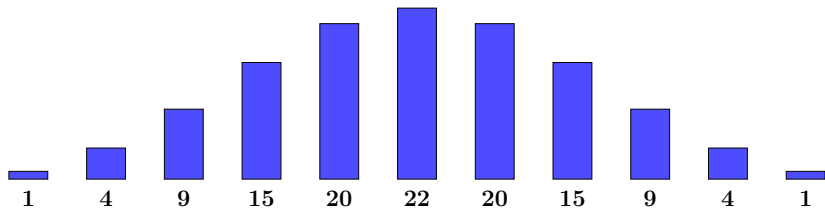
What is log-concavity?

A sequence $a_1, \dots, a_n \in \mathbb{R}_{\geq 0}$ is **log-concave** if

$$a_k^2 \geq a_{k+1} a_{k-1} \quad (1 < k < n).$$

Log-concavity (and positivity) implies **unimodality**:

$a_1 \leq \dots \leq a_m \geq \dots \geq a_n$ for some $1 \leq m \leq n$.



Example: binomial coefficients

$$a_k = \binom{n}{k} \quad k = 0, 1, \dots, n.$$

This sequence is **log-concave** because

$$\frac{a_k^2}{a_{k+1} a_{k-1}} = \frac{\binom{n}{k}^2}{\binom{n}{k+1} \binom{n}{k-1}} = \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right),$$

which is greater than 1.

Example: permutations with k inversions

a_k = number of $\pi \in S_n$ with k inversions,

where **inversion** of π is pair $i < j$ s.t. $\pi_i > \pi_j$.

This sequence is **log-concave** because

$$\sum_{0 \leq k \leq \binom{n}{2}} a_k q^k = [n]_q! = \prod_{i=1}^{n-1} (1 + q + q^2 + \dots + q^i)$$

is a product of log-concave polynomials.

Examples: forests of a graph

a_k = number of forests with k edges of graph G .

Forest is a subset of edges of G that has no cycles.

Log-concavity was conjectured for all **matroids** (Mason '72), and was proved through **combinatorial Hodge theory** (Huh '15).



G



forest



not forest



spanning tree

Log-concavity has been proved in various aspects of mathematics in varying degrees of difficulties.

We would like to rigorously formalize this **difference in difficulties**.

Log-concavity has been proved in various aspects of mathematics in varying degrees of difficulties.

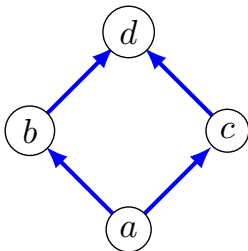
We would like to rigorously formalize this **difference in difficulties**.

We will start with log-concave **poset inequalities**.

Poset inequalities

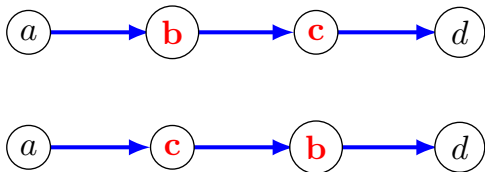
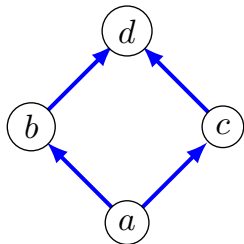
Partially ordered sets

A poset \mathcal{P} is a set X with a partial order \prec on X .



Linear extension

A linear extension L is a complete order of \prec .



We write $L(x) = k$ if x is k -th smallest in L .

Stanley (poset) inequality: simple form

Fix $x \in \mathcal{P}$.

$N(k) :=$ number of linear extensions with $L(x) = k$.

Theorem (Stanley '81)

$$N(k)^2 \geq N(k+1)N(k-1) \quad (k \in \mathbb{N}).$$

The inequality was initially conjectured by Chung-Fishburn-Graham, and was proved using Aleksandrov-Fenchel inequality for mixed volumes.

Stanley (poset) inequality: consequence

Weak Bruhat order on permutation group S_n is

$\pi \leq \sigma$ if some reduced word of π is a left subword of some reduced word of σ .

For $\sigma \in S_n$, let

$N^\sigma(k) :=$ number of $\pi \in S_n$ such that
 $\pi \leq \sigma$ and $\pi(1) = k$.

Corollary

Sequence $N^\sigma(1), \dots, N^\sigma(n)$ is *log-concave*.

Stanley (poset) inequality: true form

Fix $d \geq 0$, $x, y_1, \dots, y_d \in \mathcal{P}$ and $\ell_1, \dots, \ell_d \in \mathbb{N}$.

$N_d(k) :=$ number of linear extensions with
 $L(x) = k, \quad L(y_i) = \ell_i \quad \text{for } i \in [d].$

Theorem (Stanley '81)

$$N_d(k)^2 \geq N_d(k+1) N_d(k-1) \quad (k \in \mathbb{N}).$$

This form corresponds to **imposing boundary conditions** in PDE/statistical physics.

When is equality achieved?

Question (Stanley '81)

Find *equality condition* for [Stanley inequality].

Quote (Gardner '02)

If inequalities are *silver* currency in mathematics, those that come along with precise equality conditions are *gold*.

Equality condition: $d = 0$ (numerical)

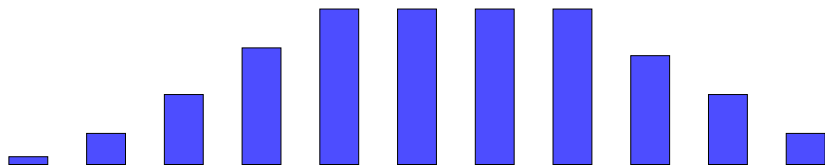
Theorem (Shenfeld-van Handel '23)

Suppose $d = 0$ and $N_d(k) > 0$. Then

$$N_d(k)^2 = N_d(k+1) N_d(k-1)$$

if and only if

$$N_d(k) = N_d(k+1) = N_d(k-1).$$



Equality condition: $d = 0$ (combinatorial)

Theorem (Shenfeld-van Handel '23)

Suppose $d = 0$ and $N_d(k) > 0$. Then

$$N_d(k)^2 = N_d(k+1) N_d(k+1)$$

if and only if

$$|\mathcal{P}_{<z}| > k \quad \text{for all } z \in \mathcal{P}_{>x},$$

$$|\mathcal{P}_{>z}| > |\mathcal{P}| - k + 1 \quad \text{for all } z \in \mathcal{P}_{<x},$$

where $\mathcal{P}_{<z} :=$ set of $y \in \mathcal{P}$ with $y < z$.

This a **combinatorial condition**, and
can be checked in $O(|\mathcal{P}|^2)$ steps.

Equality condition: $d \geq 1$ (numerical)

Theorem (Ma–Shenfeld '22+)

Suppose $d \geq 1$ and $N_d(k) > 0$. Then

$$N_d(k)^2 = N_d(k+1) N_d(k-1)$$

if and only if

$$N_d(k) = N_d(k+1) = N_d(k-1).$$

Question (Ma–Shenfeld '22+)

Find a *combinatorial condition* for $d \geq 1$.

Main result

Consider the decision problem for checking equality in Stanley inequality:

$$N_d(k)^2 \stackrel{?}{=} N_d(k+1) N_d(k+1).$$

Theorem 1 (C.–Pak '23+)

- $d \leq 1$: *combinatorial equality condition that is checkable in $\text{poly}(|\mathcal{P}|)$ steps.*
- $d \geq 2$: *not part of **polynomial hierarchy**, unless polynomial hierarchy collapses.*

Polynomial hierarchy

Decision vs counting

Decision problem: answer is either 'Yes' or 'No'.

Counting problem: answer is a nonnegative integer.

Example (3-colorings of graph G)

- *Decision problem: Check if there exists a proper 3-coloring of G .*
- *Counting problem: Find the number of proper 3-colorings of G .*

Polynomial hierarchy is a subclass of **decision** problems.

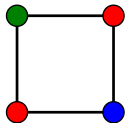
Complexity class P

$P := \left\{ \begin{array}{l} \text{Decision problems solvable by } \text{deterministic} \\ \text{Turing machine in } \text{polynomial} \text{ time} \end{array} \right\}$

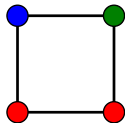
Example

Check if a given 3-coloring of a graph G is proper.

This can be solved in $O(n^2)$ time by checking the color of endpoints of every edge.



YES

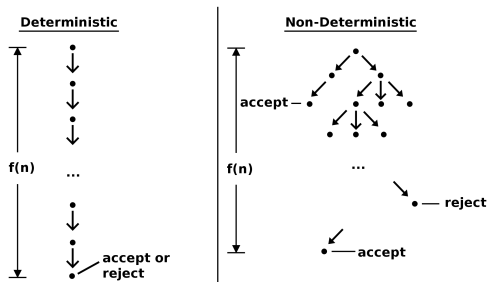


NO

Complexity class NP

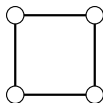
NP := { Decision problems solvable by **nondeterministic** Turing machine in **polynomial** time }

- Can split into many parallel **branches**;
- Output 'YES' if **one of the branches** said 'YES';
- Output 'NO' if **all branches** said 'NO'.

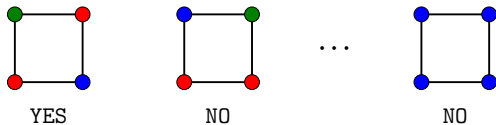


Complexity class NP: example

Problem: Check if graph G has a proper 3-coloring.



Each branch corresponds to checking if a particular 3-coloring of G is proper.

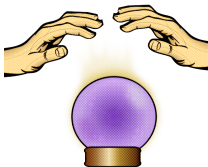


Output to this example is 'YES'.

Turing machine with an oracle

At each step, this machine can either:

- Perform usual **nondeterministic** Turing machine operation; or
- Ask an **oracle** that is able to answer any instance of a given computational problem.



Turing machine with an oracle: example

Problem: Check if there is an induced subgraph of G of size $\lceil n/2 \rceil$ that is not 3-colorable.

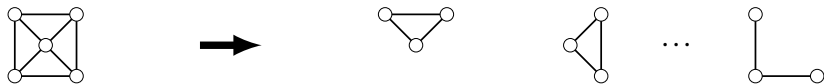
Oracle: Can check if a graph is 3-colorable.

Turing machine with an oracle: example

Problem: Check if there is an induced subgraph of G of size $\lceil n/2 \rceil$ that is not 3-colorable.

Oracle: Can check if a graph is 3-colorable.

Each branch of the machine corresponds to an induced subgraph of G of size $\lceil n/2 \rceil$.



For every branch, **oracle** checks if subgraph is 3-colorable.

Complexity class Σ_i^P

The first two classes are

$$\Sigma_0^P := P; \quad \Sigma_1^P := NP.$$

For $i \geq 1$, the class $\Sigma_i^P := NP^{\Sigma_{i-1}^P}$ is

{ Decision problems solvable by **nondeterministic** Turing machine in **polynomial** time
with an **oracle** for problem from Σ_{i-1}^P . }

Note that

$$\Sigma_0^P \subseteq \Sigma_1^P \subseteq \Sigma_2^P \subseteq \Sigma_3^P \subseteq \dots$$

Complexity class Σ_i^P : example

Problem A: Check if a 3-coloring of G is proper.

Problem A is in $\Sigma_0^P = P$.

Problem B: Check if G has a proper 3-coloring.

Problem B is in $\Sigma_1^P = NP$.

Problem C: Check if there is an induced subgraph of G of size $\lceil n/2 \rceil$ that is not 3-colorable.

Problem C is in $\Sigma_2^P = NP^{NP}$.

Polynomial hierarchy (PH)

Polynomial hierarchy is the union of all Σ_i^P 's,

$$\text{PH} := \bigcup_{i=0}^{\infty} \Sigma_i^P.$$

Conjecture

Polynomial hierarchy does not collapse,

$$\Sigma_0^P \subsetneq \Sigma_1^P \subsetneq \Sigma_2^P \subsetneq \Sigma_3^P \subsetneq \dots$$

- $\Sigma_0^P = \Sigma_1^P$ is equivalent to $P = NP$.
- $\Sigma_1^P = \Sigma_2^P$ is equivalent to $NP = \text{coNP}$.

Back to main result

Consider the decision problem for checking equality in Stanley inequality:

$$N_d(k)^2 \stackrel{?}{=} N_d(k+1) N_d(k+1).$$

Theorem (C.-Pak '23+)

- $d \leq 1$: Problem is in P .
- $d \geq 2$: Problem is *not* in PH , unless PH collapses.

Recall our goal ...

Log-concavity has been proved in various aspects of mathematics in varying degrees of difficulties.

We would like to rigorously formalize this **difference in difficulties**.

Complexity class $\#P$

Complexity class $\#P$

$\#P := \left\{ \begin{array}{l} \text{Counting problems realizable as number} \\ \text{of 'YES' branches in some nondetermi-} \\ \text{nistic Turing machine.} \end{array} \right\}.$

Example

Count number of proper 3-colorings of graph G .

Example

Count number of linear extensions $N_d(k)$ of poset \mathcal{P} .

Main result

Theorem 2 (C.–Pak '23+)

For $d \geq 2$, the *defect* of Stanley inequality

$$N_d(k)^2 - N_d(k+1)N_d(k+1)$$

is *not* in $\#P$, unless PH collapses.

Note: $N_d(k)^2$ and $N_d(k+1)N_d(k+1)$ are in $\#P$.

Theorem is a *consequence* of previous main result.

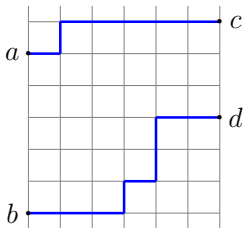
Example: defect of binomial inequalities

$$\binom{n}{k}^2 \geq \binom{n}{k+1} \binom{n}{k-1} \quad (1 < k < n).$$

This inequality has a **lattice path interpretation**:

$K(a \rightarrow c, b \rightarrow d) :=$ no. of pairs of north-east lattice paths from a to c and b to d ,

for $a, b, c, d \in \mathbb{Z}^2$.



Example: defect of binomial inequalities

Let

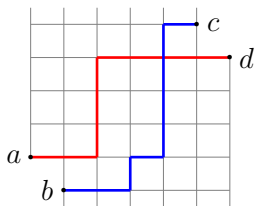
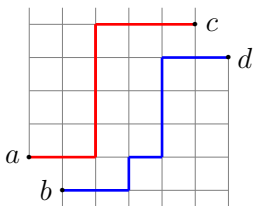
$$a = (0, 1), \quad c = (k, n - k + 1),$$

$$b = (1, 0), \quad d = (k + 1, n - k).$$

Then

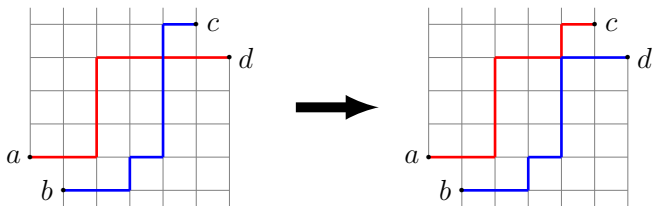
$$K(a \rightarrow c, b \rightarrow d) = \binom{n}{k},$$

$$K(a \rightarrow d, b \rightarrow c) = \binom{n}{k-1} \binom{n}{k+1}.$$



Example: defect of binomial inequalities

Note $K(a \rightarrow c, b \rightarrow d) \geq K(a \rightarrow d, b \rightarrow c)$ by
path-swapping injections.



$K(a \rightarrow c, b \rightarrow d) - K(a \rightarrow d, b \rightarrow c)$ is
number of pairs of north-east lattice paths
from a to c , b to d , that **do not intersect**.

This number is thus in $\#P$.

Example: Edge correlation for spanning trees

Let G be a graph, let e, f be distinct edges of G .

$\mathcal{T} :=$ no. of spanning trees of G ,

$\mathcal{T}_e :=$ no. of spanning trees of G containing e ,

$\mathcal{T}_{e,f} :=$ no. of spanning trees of G containing e and f .

Theorem

$$\mathcal{T}_e \mathcal{T}_f \geq \mathcal{T} \mathcal{T}_{e,f}.$$

Defect can be computed in **polynomial** time by **matrix tree theorem**. This number is thus in $\#P$.

Example: defect of Sidorenko inequality

For a permutation $\sigma \in S_n$, let

$LI(\sigma) :=$ number of $\pi \in S_n$ such that $\pi \leq \sigma$,

where \leq is the **weak Bruhat order** on S_n .

Theorem (Sidorenko '91)

$$LI(\sigma) LI(\bar{\sigma}) \geq n!, \quad (\sigma \in S_n),$$

where $\bar{\sigma}$ is the reverse of σ .

Proved by **max-flow min-cut** argument. Defect was shown to be in $\#P$ by **C.–Pak–Panova '23**.

Back to main result

Theorem (C.–Pak '23+)

For $d \geq 2$, the *defect* of Stanley inequality

$$N_d(k)^2 - N_d(k+1)N_d(k+1)$$

is *not in #P*, unless PH collapses.

This *suggests* Stanley inequality can't be proved by

- 'Effective' *injection* arguments;
- *Determinantal process* arguments;
- 'Effective' *max-flow min-cut* arguments.

Matroids

Object: matroids

Matroid $\mathcal{M} = (X, \mathcal{I})$ is ground set X with collection of independent sets $\mathcal{I} \subseteq 2^X$.

Graphical matroids

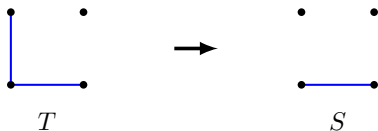
- X = edges of a graph G ,
- \mathcal{I} = forests in G .

Realizable matroids

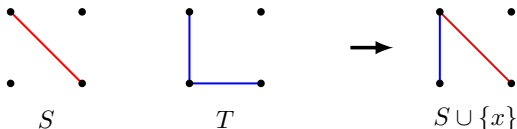
- X = finite set of vectors over field \mathbb{F} ,
- \mathcal{I} = sets of linearly independent vectors.

Matroids: conditions

- $S \subseteq T$ and $T \in \mathcal{I}$ implies $S \in \mathcal{I}$.



- If $S, T \in \mathcal{I}$ and $|S| < |T|$, then there is $x \in T \setminus S$ such that $S \cup \{x\} \in \mathcal{I}$.



A **basis** is a **maximal** independent set.

Rank r of matroid is the size of the bases.

Ultra log-concavity for bases

Fix $d \geq 0$, disjoint subsets S, S_1, \dots, S_d of X ,
and $\ell_1, \dots, \ell_d \in \mathbb{N}$.

$B_d(k) :=$ number of bases B of \mathcal{M} such that
 $|B \cap S| = k, |B \cap S_i| = \ell_i$ for $i \in [d]$,
divided by $\binom{r}{k, \ell_1, \dots, \ell_d}$.

Theorem (Basis ULC)

$$B_d(k)^2 \geq B_d(k+1)B_d(k-1) \quad (k \in \mathbb{N}).$$

Proved for **regular** matroids by (Stanley '81), and
for **all** matroids by (Brändén–Huh '20).

Basis ULC implies Mason inequality

$I(k) :=$ no. of independent sets with k elements.

$I(k)$ is no. of forest with k edges for graphic matroids.

Theorem (Mason inequality)

$$I(k)^2 \geq I(k+1)I(k-1).$$

Proof.

\mathcal{M} = given matroid with ground set X .

\mathcal{M}' = direct sum of \mathcal{M} with the free matroid.

Set $d = 0$ and $S = X$. Then

$$I(k) \text{ for } \mathcal{M} = B_d(k) \text{ for } \mathcal{M}'.$$

Main result

Consider the decision problem for checking equality in **Basis ULC** for **binary** matroids:

$$B_d(k)^2 \stackrel{?}{=} B_d(k+1)B_d(k-1).$$

Theorem 3 (C.-Pak)

- $d = 0$: Problem is in **coNP**.
- $d \geq 1$: Problem is *not in PH*, unless PH collapses.

Main result

Theorem 4 (C.–Pak)

For $d \geq 1$, *defect of Basis ULC for binary matroids*

$$B_d(k)^2 - B_d(k+1)B_d(k-1)$$

is not in $\#P$, unless PH collapses.

This *suggests* Basis ULC can't be proved by

- 'Effective' *injection* arguments;
- *Determinantal process* arguments;
- 'Effective' *max-flow min-cut* arguments.

What is next?

Conjecture

For $d = 0$, defect of *Stanley inequality*

$$N(k)^2 - N(k+1)N(k-1) \notin \#P.$$

For $d = 0$, defect of *Basis ULC*

$$B(k)^2 - B(k+1)B(k-1) \notin \#P.$$

For $d = 0$, defect of *Mason inequality*

$$I(k)^2 - I(k+1)I(k-1) \notin \#P.$$

THANK YOU!

Preprint: www.arxiv.org/abs/2309.05764

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