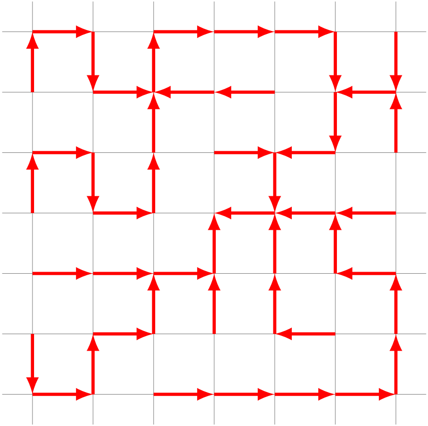
In between random walk and rotor walk

Swee Hong Chan
University of California, Los Angeles
Joint work with Lila Greco, Lionel Levine, Boyao Li



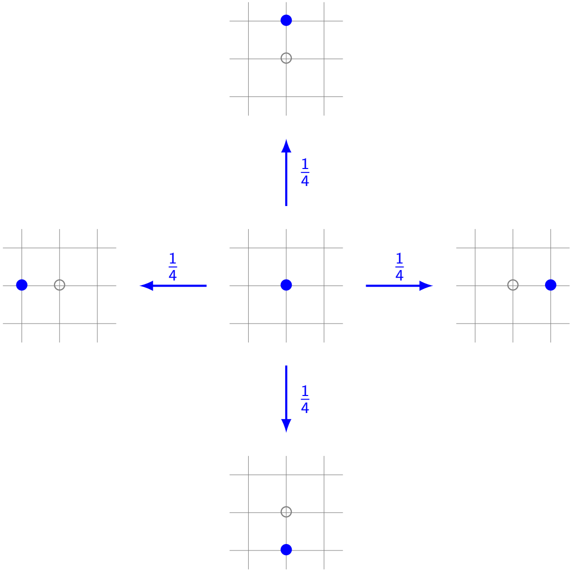

A BIG, FAT HIT!
SCHOOL OF ROCK
A SHEER TRIUMPH!
BIG BUS NEW YORK
THE LEGEND
THE
TDK
LG

Random walk

Rotor walk

Simple random walk on $\mathbb{Z}^2$

Simple random walk on $\mathbb{Z}^2$



Simple random walk on $\mathbb{Z}^2$

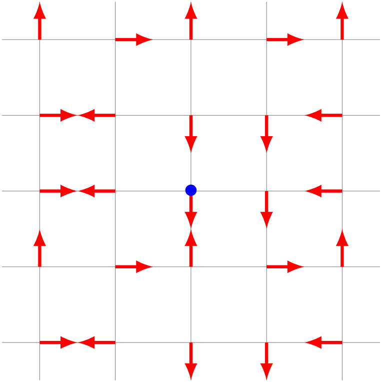
- Visits every site infinitely often? Yes!
- Scaling limit? The standard 2-D Brownian motion:

$$\Big(\frac{1}{\sqrt{n}} \underbrace{X_{[nt]}}_{\substack{\text{location of the}\\ \text{walker at time } [nt]}}\Big)_{t\geq 0} \overset{n\to\infty}{\Longrightarrow} \frac{1}{\sqrt{2}} \underbrace{(B_1(t), B_2(t))}_{\substack{\text{independent standard}\\ \text{Brownian motions}}}{}_{t\geq 0}.$$

Rotor walk on $\mathbb{Z}^2$

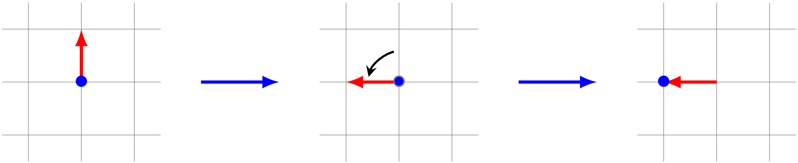
Rotor walk on $\mathbb{Z}^2$

Put a signpost at each site.



Rotor walk on $\mathbb{Z}^2$

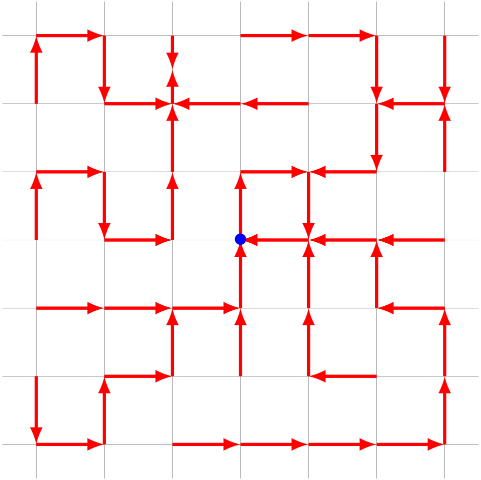
Turn the signpost 90° counterclockwise, then follow the signpost.



The signpost says:
"This is the way you went the last time you were here",
(assuming you ever were!)

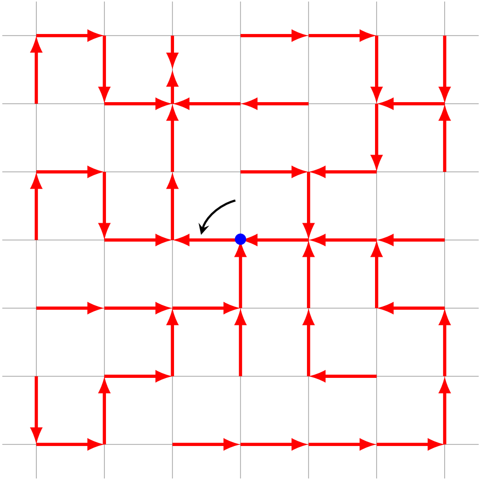
Rotor walk on $\mathbb{Z}^2$

Turn the signpost 90° counterclockwise, then follow the signpost.



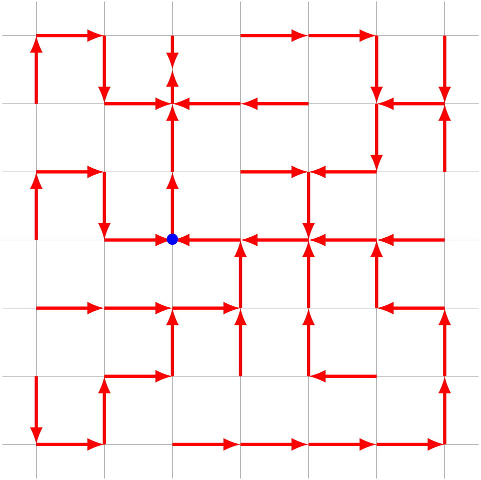
Rotor walk on $\mathbb{Z}^2$

Turn the signpost 90° counterclockwise, then follow the signpost.



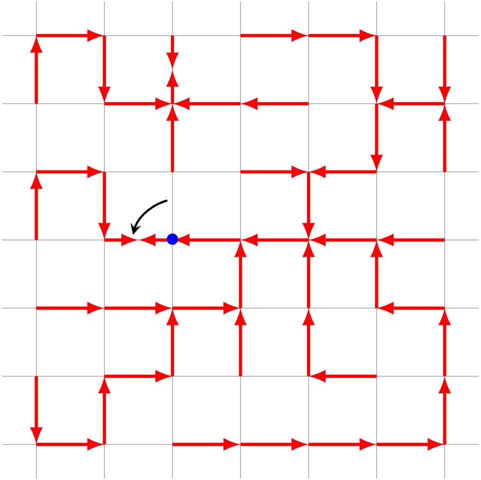
Rotor walk on $\mathbb{Z}^2$

Turn the signpost 90° counterclockwise, then follow the signpost.



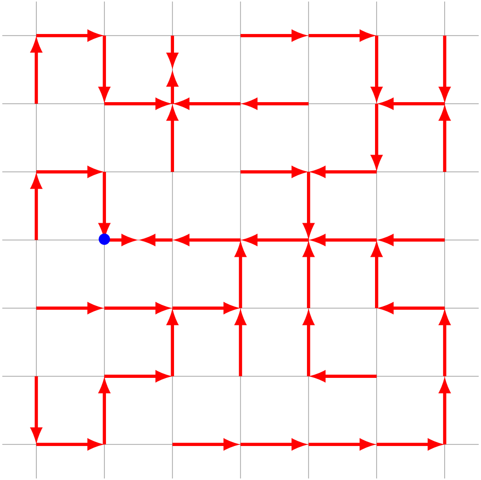
Rotor walk on $\mathbb{Z}^2$

Turn the signpost 90° counterclockwise, then follow the signpost.



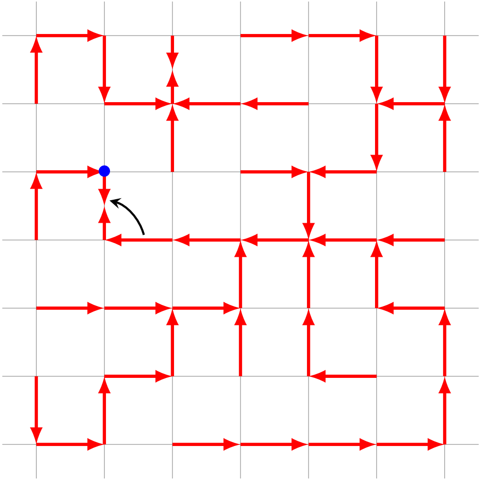
Rotor walk on $\mathbb{Z}^2$

Turn the signpost 90° counterclockwise, then follow the signpost.



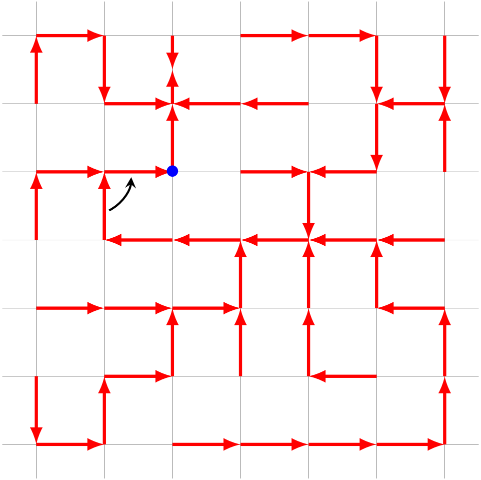
Rotor walk on $\mathbb{Z}^2$

Turn the signpost 90° counterclockwise, then follow the signpost.



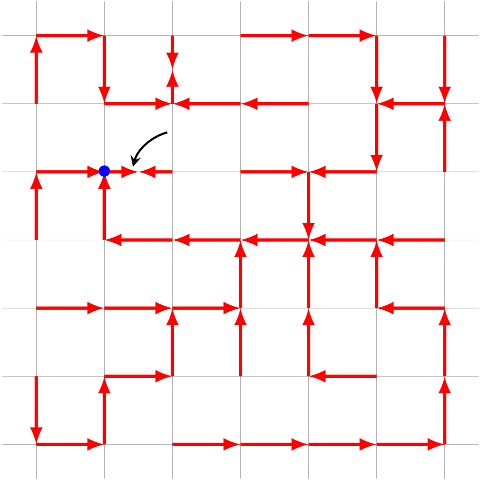
Rotor walk on $\mathbb{Z}^2$

Turn the signpost 90° counterclockwise, then follow the signpost.



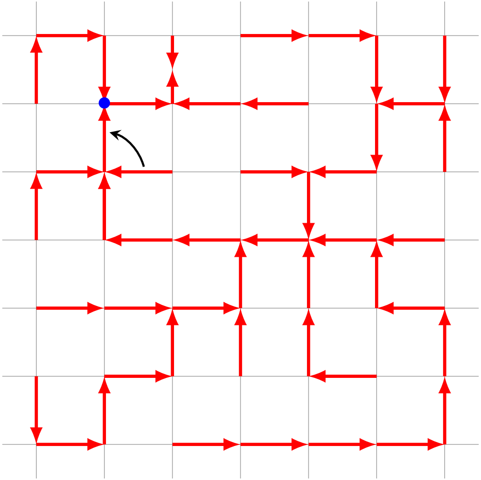
Rotor walk on $\mathbb{Z}^2$

Turn the signpost 90° counterclockwise, then follow the signpost.



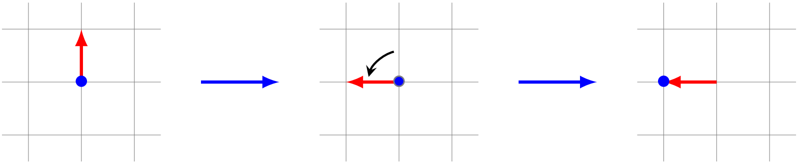
Rotor walk on $\mathbb{Z}^2$

Turn the signpost 90° counterclockwise, then follow the signpost.



Rotor walk on $\mathbb{Z}^2$

Turn the signpost 90° counterclockwise, then follow the signpost.



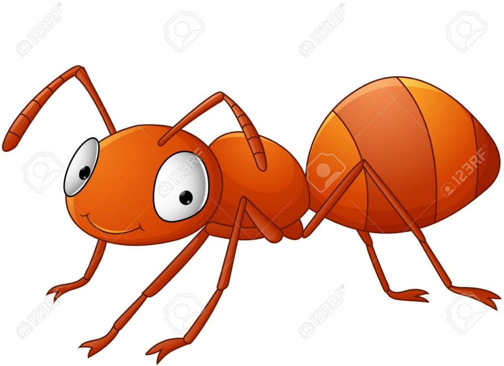
The signpost says:
"This is the way you went the last time you were here",
(assuming you ever were!)

Why rotor walk?

Randomness can be (was) expensive to simulate!

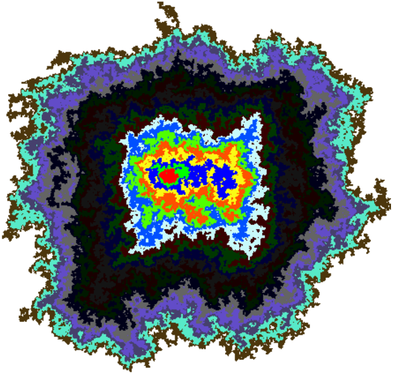
Why rotor walk?

As a model for ants' foraging strategy



Why rotor walk?

As a model of self organized criticality for statistical mechanics.



Visited sites after 80 returns to the origin (by Laura Florescu).

Conjectures for rotor walk on $\mathbb{Z}^2$

For initial signposts i.i.d. uniform among the four directions,

- (PDDK '96) Visits every site infinitely often?
- (PDDK '96) $\#\{X_1, \ldots, X_n\}$ is $\asymp n^{2/3}$?
  (compare with $n/\log n$ for the simple random walk.)
- (Kapri-Dhar '09) The asymptotic shape of $\{X_1, \ldots, X_n\}$ is a disc?

More randomness please!

Well studied

Many open problems

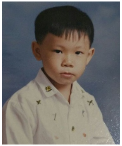
Random

Deterministic

More randomness please!

Well studied

Let's study this!!!



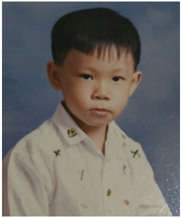
Many open problems

Random

Something in between

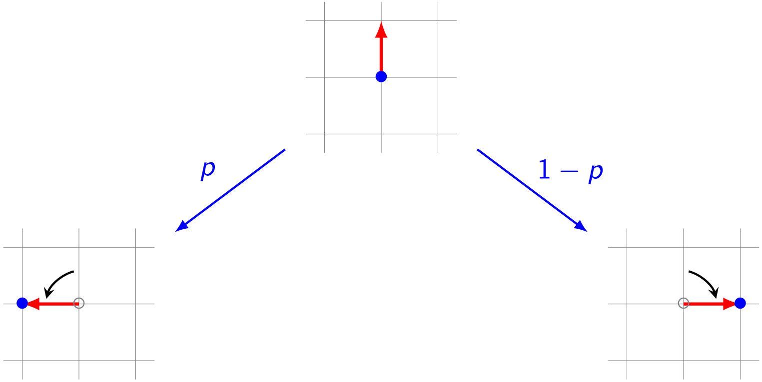
Deterministic

p-rotor walk on $\mathbb{Z}^2$



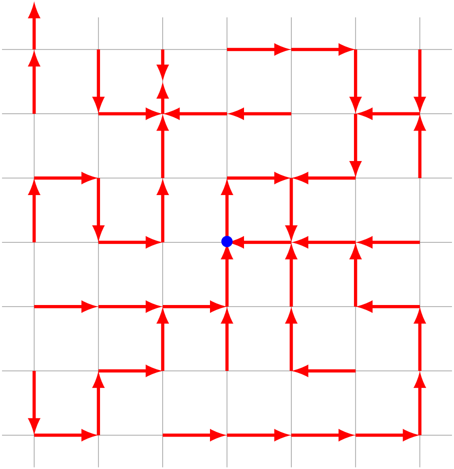
p-rotor walk on $\mathbb{Z}^2$

With probability p, turn the signpost 90° counter-clockwise.
With probability $1-p$, turn the signpost 90° clockwise.



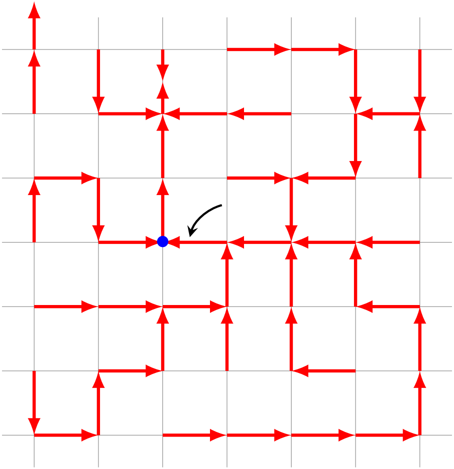
p-rotor walk on $\mathbb{Z}^2$

Follow rotor walk rule with prob. p, do the opposite with prob. $1-p$



p-rotor walk on $\mathbb{Z}^2$

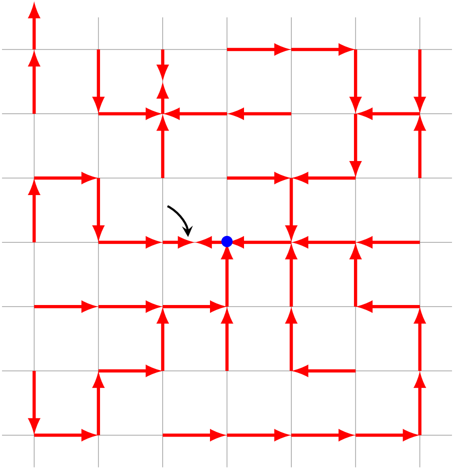
Follow rotor walk rule with prob. p, do the opposite with prob. $1-p$



Follow the rule.

p-rotor walk on $\mathbb{Z}^2$

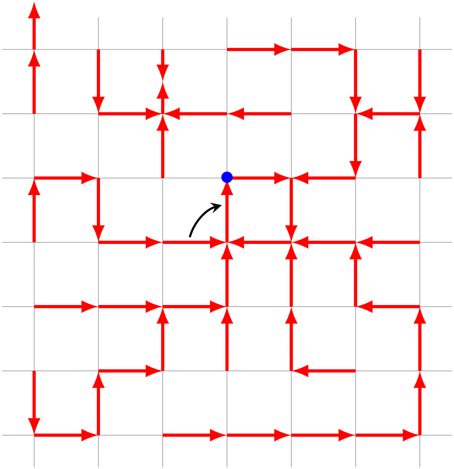
Follow rotor walk rule with prob. p, do the opposite with prob. $1-p$



Do the opposite.

p-rotor walk on $\mathbb{Z}^2$

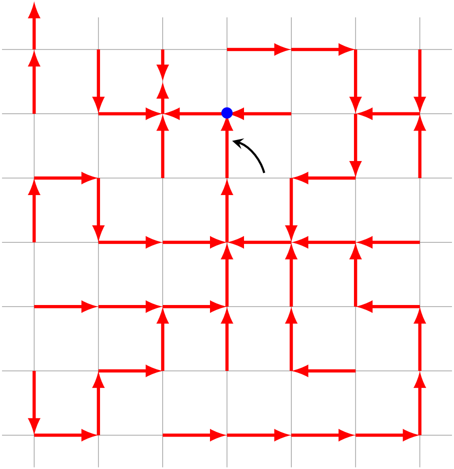
Follow rotor walk rule with prob. p, do the opposite with prob. $1-p$



Do the opposite again.

p-rotor walk on $\mathbb{Z}^2$

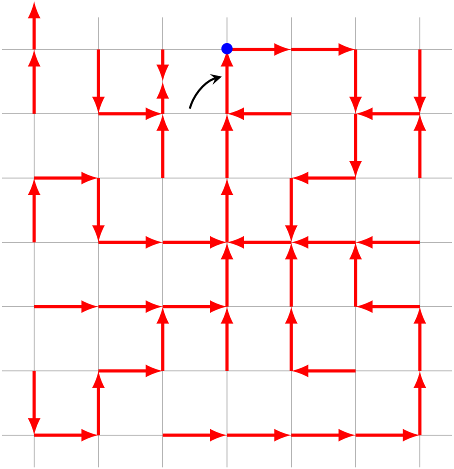
Follow rotor walk rule with prob. p, do the opposite with prob. $1 - p$



Follow the rule.

p-rotor walk on $\mathbb{Z}^2$

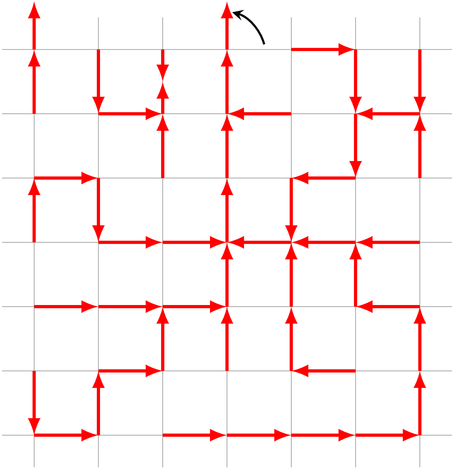
Follow rotor walk rule with prob. p, do the opposite with prob. $1-p$



Do the opposite.

p-rotor walk on $\mathbb{Z}^2$

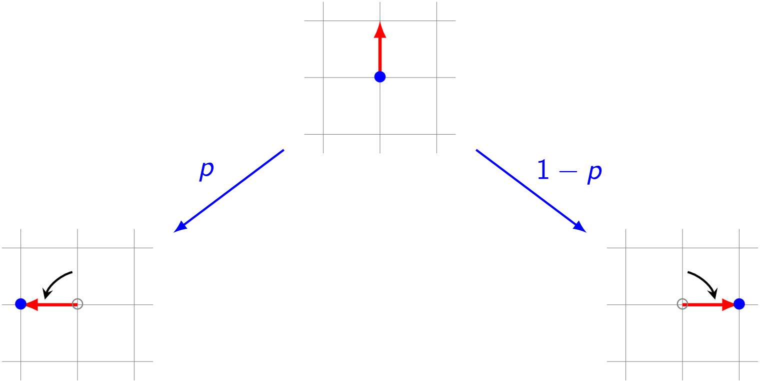
Follow rotor walk rule with prob. p, do the opposite with prob. $1-p$



Ops...

p-rotor walk on $\mathbb{Z}^2$

With probability p, turn the signpost 90° counter-clockwise.
With probability $1-p$, turn the signpost 90° clockwise.



Recover the rotor walk if $p = 1$.

Recurrence result for p-rotor walk

Recurrence for p-rotor walk on $\mathbb{Z}^2$

Theorem (C., Greco, Levine, Li '19+)

Let $p = \frac{1}{2}$ *and let the i.i.d uniform among four directions be the initial signpost configuration. Then the p-rotor walk visits every vertex infinitely often almost surely.*

Proof of recurrence for the simple random walk

Consider the following martingale:

$$M(t) := \underbrace{a(X(t))}_{\substack{\text{potential}\\ \text{kernel}}} - \underbrace{N(t)}_{\substack{\#\text{ of times}\\ \text{leaving } o}}.$$

Use the optional stopping theorem:

$$0 = \mathbb{E}[M(\underbrace{\tau(r)}_{\substack{\text{hitting time}\\ \text{of } \partial B_r \cup \{o\}}})] \approx \frac{2}{\pi} \ln r \, (1 - \underbrace{p_{\mathrm{ret}}(r)}_{\substack{\text{prob. of return}\\ \text{before hitting } \partial B_r}}) - 1.$$

Proof of recurrence for the simple random walk (ctd.)

We rewrite the equation to

$$\underbrace{p_{\text{ret}}(r)}_{\substack{\text{prob. of return}\\ \text{before hitting } \partial B_r}} \approx 1 - \frac{\pi}{2\ln r},$$

and we then conclude that

$$\underbrace{p_{\text{rec}}}_{\substack{\text{recurrence}\\ \text{probability}}} = 1 - \lim_{r\to\infty} \frac{\pi}{2\ln r} = 1.$$

Proof of recurrence for p-rotor walk

Consider the following martingale:

$$M(t) := a(X(t)) \; - \; N(t) \; + \; \underbrace{\sum_{x \in \{X_0, \ldots, X_t\}} \mathtt{w}(x; \rho_t)}_{\text{compensator}} .$$

By the same argument as before,

$$\underbrace{p_{\text{rec}}}_{\substack{\text{recurrence}\\ \text{probability}}} \; = \; 1 - \lim_{r \to \infty} \frac{\pi}{2 \ln r} \left(\sum_{|x| \le r} \mathbb{E}[\mathtt{w}(x; \rho_{\tau(r)})] \right) .$$

Proof of recurrence for p-rotor walk (ctd.)

We can estimate the terms in the compensator locally by

$$\left|\mathbb{E}[\mathtt{w}(x;\rho_{\tau(r)})]\right| \lesssim \left(1-\frac{1}{2^{100}}\right)\frac{2}{\pi|x|^2}.$$

Plugging this estimate into previous equation,

$$p_{\text{rec}} \geq 1 - \lim_{r\to\infty}\frac{\pi}{2\ln r}\left(\sum_{|x|\leq r}\left(1-\frac{1}{2^{100}}\right)\frac{2}{\pi|x|^2}\right) = \frac{1}{2^{100}} > 0.$$

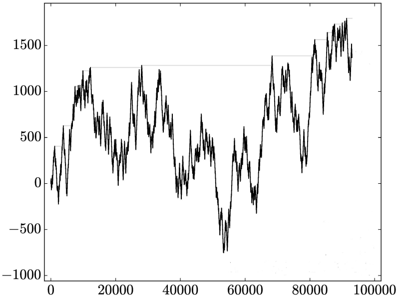
By Kolmogorov zero-one law, the recurrence probability is 1.

Scaling limit result for p-rotor walk

Scaling limit for p-rotor walk on $\mathbb{Z}$

(Huss, Levine, Sava-Huss 18) The scaling limit for p-rotor walk on $\mathbb{Z}$ is a perturbed Brownian motion $(Y(t))_{t\geq 0}$,

$$Y(t) = \underbrace{B(t)}_{\substack{\text{standard}\\\text{Brownian}\\\text{motion}}} + \underbrace{a \sup_{0\leq s\leq t} Y(s)}_{\substack{\text{perturbation at}\\\text{maximum}}} + \underbrace{b \inf_{0\leq s\leq t} Y(s)}_{\substack{\text{perturbation at}\\\text{minimum}}}, \qquad t \geq 0.$$



$Y(t)$ for $a = -0.998$, and $b = 0$ (by Wilfried Huss).

Scaling limit for p-rotor walk on $\mathbb{Z}^2$

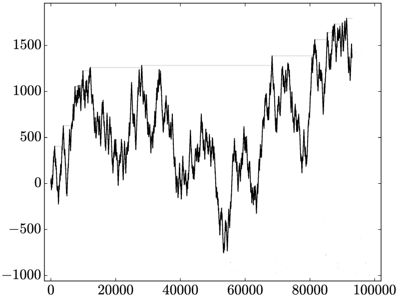
Question: Is the scaling limit for p-rotor walk on $\mathbb{Z}^2$ a "2-D perturbed Brownian motion"?

Problem: How to define "2-D perturbed Brownian motion"?.

Scaling limit for p-rotor walk on $\mathbb{Z}$

(Huss, Levine, Sava-Huss 18) The scaling limit for p-rotor walk on $\mathbb{Z}$ is a perturbed Brownian motion $(Y(t))_{t \geq 0}$,

$$Y(t) = \underbrace{B(t)}_{\text{standard Brownian motion}} + \underbrace{a \sup_{0 \leq s \leq t} Y(s)}_{\text{perturbation at maximum}} + \underbrace{b \inf_{0 \leq s \leq t} Y(s)}_{\text{perturbation at minimum}}. \qquad t \geq 0.$$



$Y(t)$ for $a = -0.998$, and $b = 0$ (by Wilfried Huss).

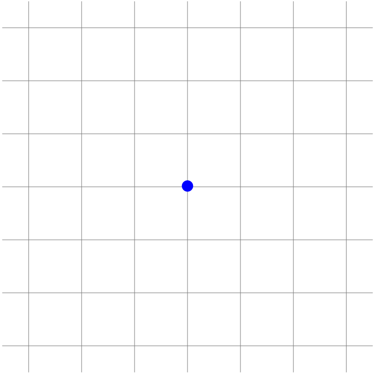
Scaling limit for p-rotor walk on $\mathbb{Z}^2$

Question: Is the scaling limit for p-rotor walk on $\mathbb{Z}^2$ a "2-D perturbed Brownian motion"?

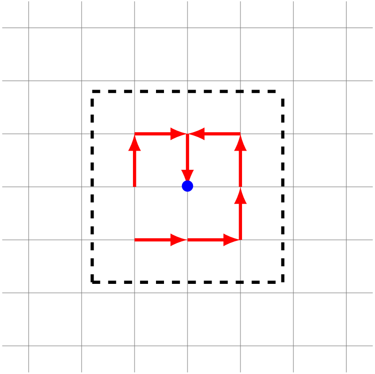
Problem: How to define "2-D perturbed Brownian motion"?.

Conjecture: The scaling limit for p-rotor walk on $\mathbb{Z}^2$ when $p = \frac{1}{2}$ is the standard 2-D Brownian motion.

Uniform spanning forest plus one edge (USF$^+$)

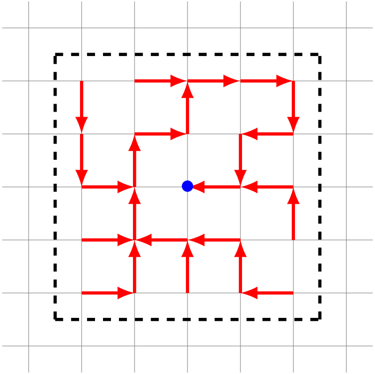


Uniform spanning forest plus one edge (USF$^+$)



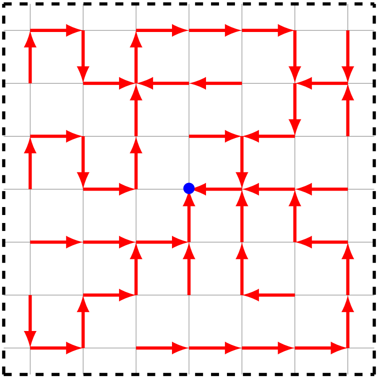
Pick a spanning tree of the black box directed to the origin (uniformly at random).

Uniform spanning forest plus one edge (USF$^+$)



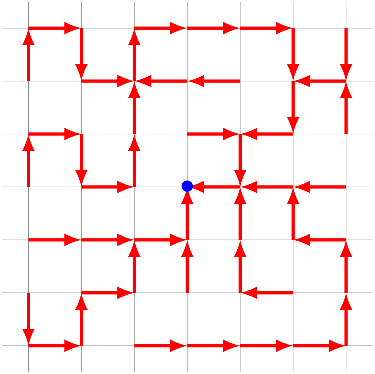
Take the limit as the black box grows until it covers $\mathbb{Z}^2$.

Uniform spanning forest plus one edge (USF$^+$)



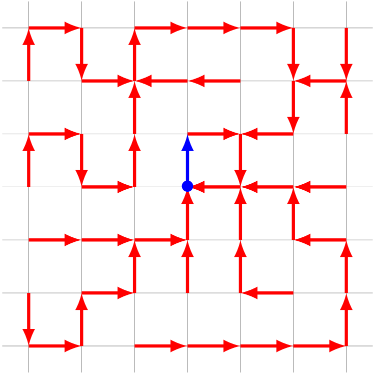
Take the limit as the black box grows until it covers $\mathbb{Z}^2$.

Uniform spanning forest plus one edge (USF$^+$)



Take the limit as the black box grows until it covers $\mathbb{Z}^2$.

Uniform spanning forest plus one edge (USF$^+$)



Add a signpost from the origin, uniform among the four directions.

Scaling limit for p-rotor walk on $\mathbb{Z}^2$

Theorem (C., Greco, Levine, Li '19+)

Let $p = \frac{1}{2}$ and let the uniform spanning forest plus one edge be the initial signpost configuration. Then, with probability 1, the p-rotor walk on $\mathbb{Z}^2$ scales to the standard 2-D Brownian motion:

$$\frac{1}{\sqrt{n}}(\underbrace{X_{[nt]}}_{\substack{\text{location of the} \\ \text{walker at time } [nt]}})_{t\geq 0} \overset{n\to\infty}{\Longrightarrow} \frac{1}{\sqrt{2}}(\underbrace{B_1(t), B_2(t)}_{\substack{\text{independent} \\ \text{Brownian motions}}})_{t\geq 0}.$$

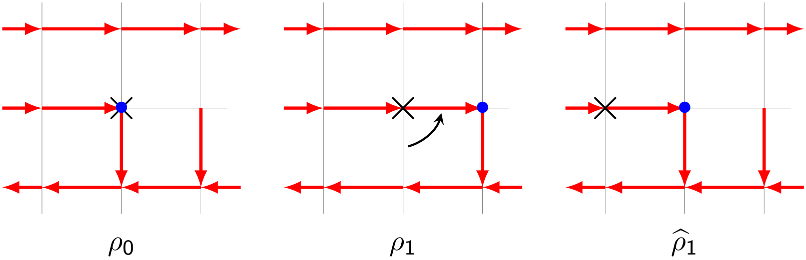
Sketch of the scaling limit proof

$$
\begin{array}{c}
\textbf{Scaling limit} \\
\Big\downarrow \;\textbf{Martingale CLT} \\
\textbf{Encounters vertical} \\
\textbf{signposts half the time} \\
\Big\downarrow \;\textbf{Ergodic theorem} \\
\textbf{Stationarity of USF}^{+} \\
\textbf{from walker's POV}
\end{array}
$$

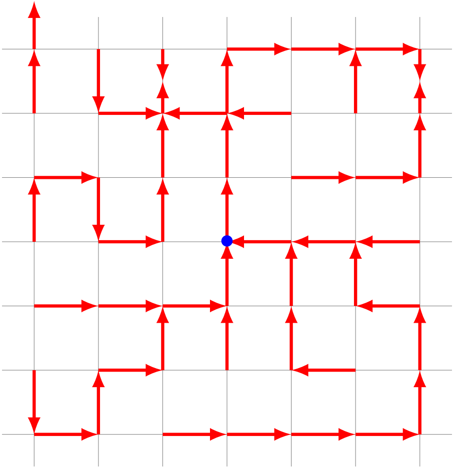
Stationarity from the walker's POV

A signpost configuration $(\rho_0(x))_{x\in\mathbb{Z}^2}$ is stationary in time from the walker's point of view if

$$\underbrace{(\widehat{\rho}_1(x))_{x\in\mathbb{Z}^2}}_{\substack{\text{signpost conf. at}\\ \text{time 1 from walker's POV}}} := (\rho_1(x - X_1))_{x\in\mathbb{Z}^2} \overset{d}{=} \underbrace{(\rho_0(x))_{x\in\mathbb{Z}^2}}_{\substack{\text{signpost conf.}\\ \text{at time 0}}}.$$

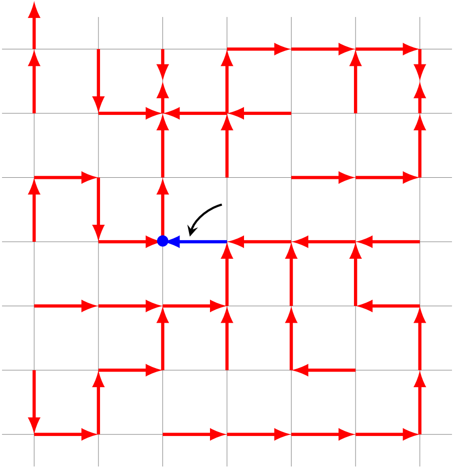


Why is USF$^+$ stationary?



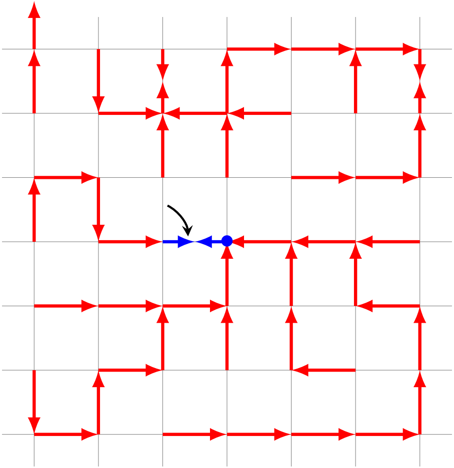
The signposts at previously visited sites form a tree oriented toward the walker.

Why is USF$^+$ stationary?



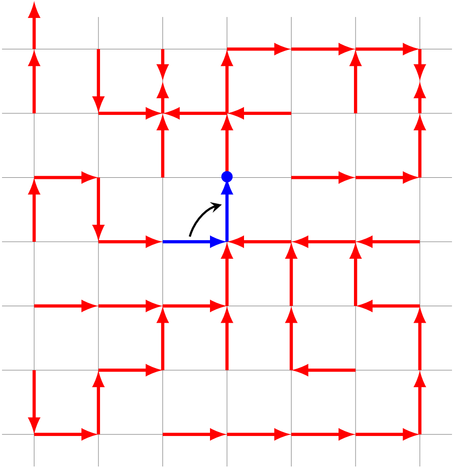
The signposts at previously visited sites form a tree oriented toward the walker.

Why is USF$^+$ stationary?



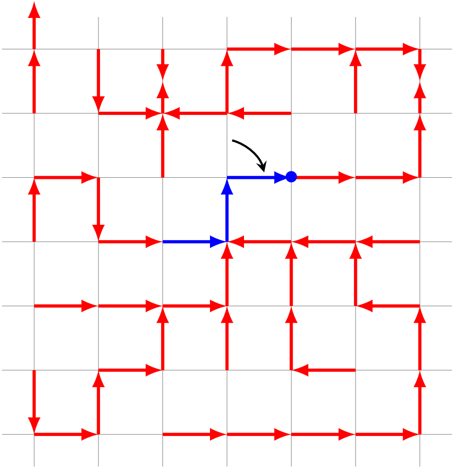
The signposts at previously visited sites form a tree oriented toward the walker.

Why is USF$^+$ stationary?



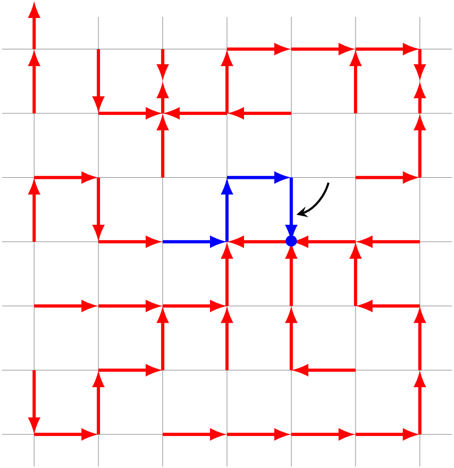
The signposts at previously visited sites form a tree oriented toward the walker.

Why is USF^+ stationary?



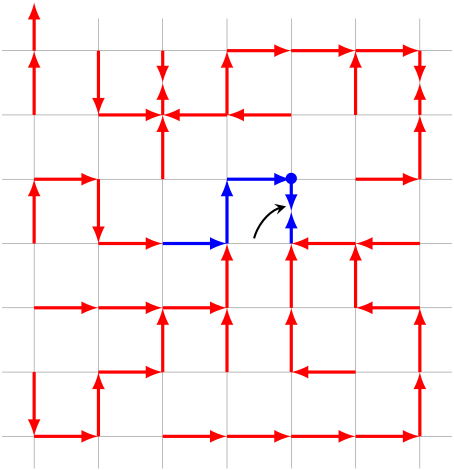
The signposts at previously visited sites form a tree oriented toward the walker.

Why is USF$^+$ stationary?



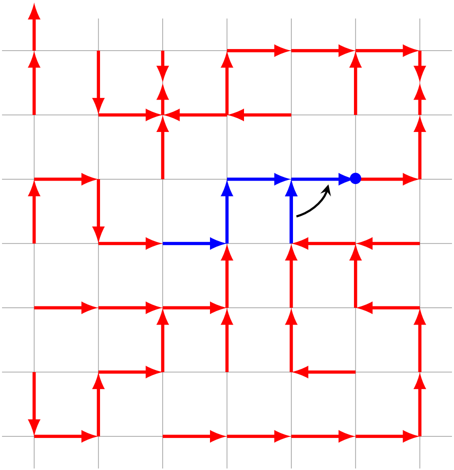
The signposts at previously visited sites form a tree oriented toward the walker.

Why is USF$^+$ stationary?



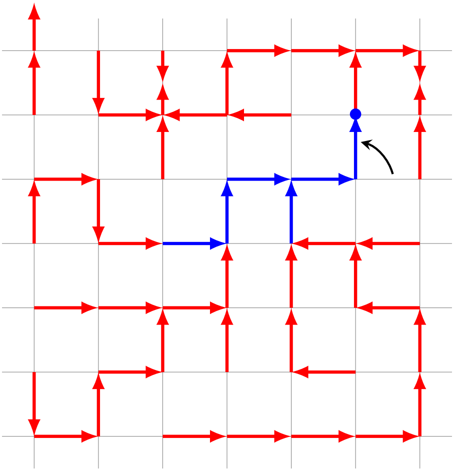
The signposts at previously visited sites form a tree oriented toward the walker.

Why is USF^{+} stationary?



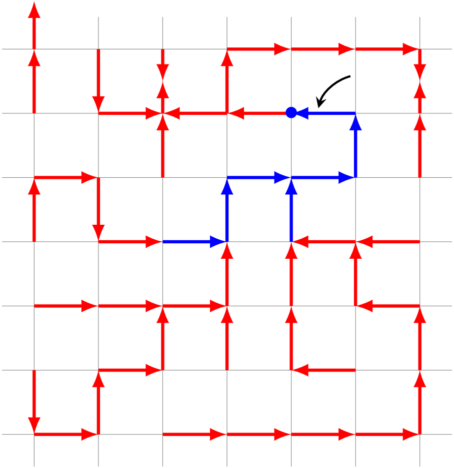
The signposts at previously visited sites form a tree oriented toward the walker.

Why is USF$^+$ stationary?



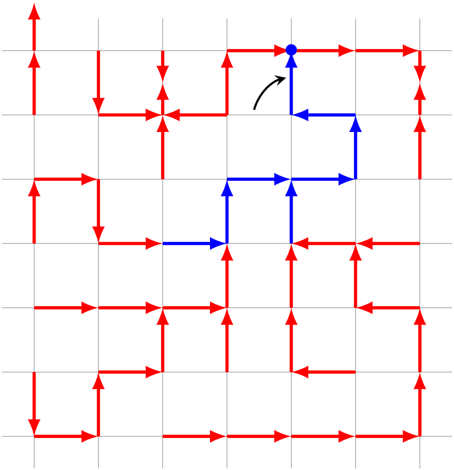
The signposts at previously visited sites form a tree oriented toward the walker.

Why is USF$^+$ stationary?



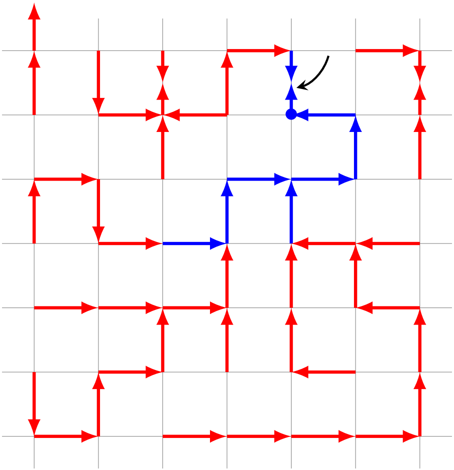
The signposts at previously visited sites form a tree oriented toward the walker.

Why is USF$^+$ stationary?



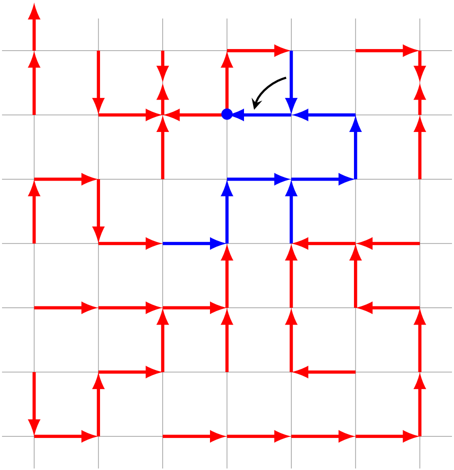
The signposts at previously visited sites form a tree oriented toward the walker.

Why is USF$^+$ stationary?



The signposts at previously visited sites form a tree oriented toward the walker.

Why is USF$^+$ stationary?

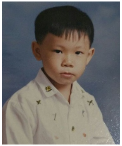


The signposts at previously visited sites form a tree oriented toward the walker.

Well studied

Simple random walk

Know a little bit now



p-rotor walk

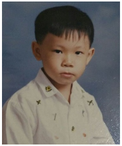
Many open problems

Rotor walk

Well studied

Simple random walk

Know a little bit now



p-rotor walk

Many open problems

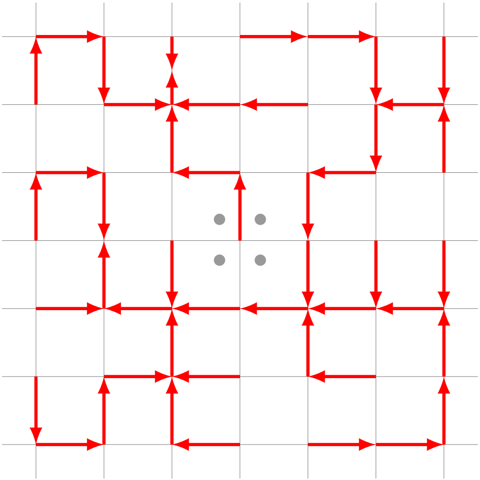
Rotor walk

Let's apply what we have learnt to rotor walk.

Prison escape

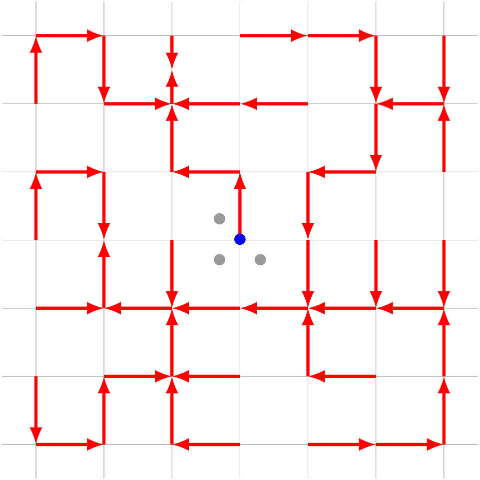
Prison escape

Put n walkers at the origin.



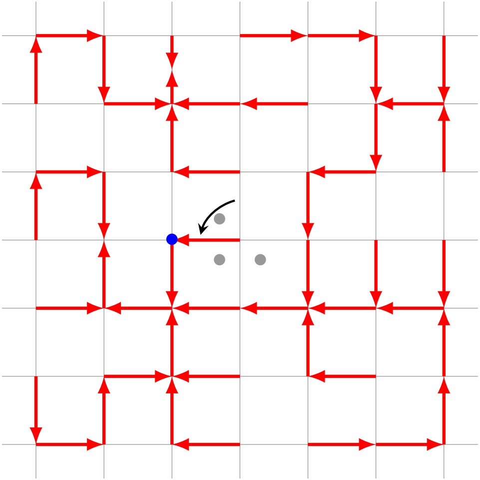
Prison escape

First walker performs rotor walk, remove if returns to origin.



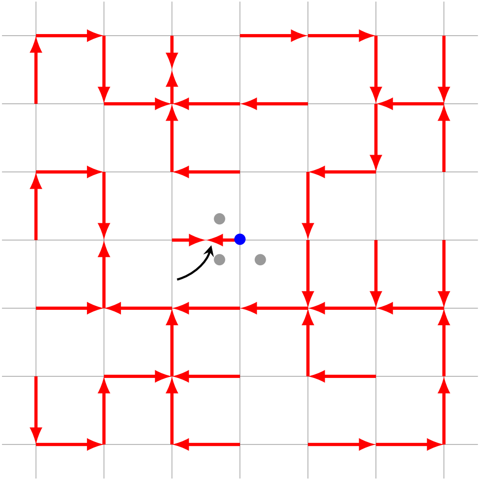
Prison escape

First walker performs rotor walk, remove if returns to origin.



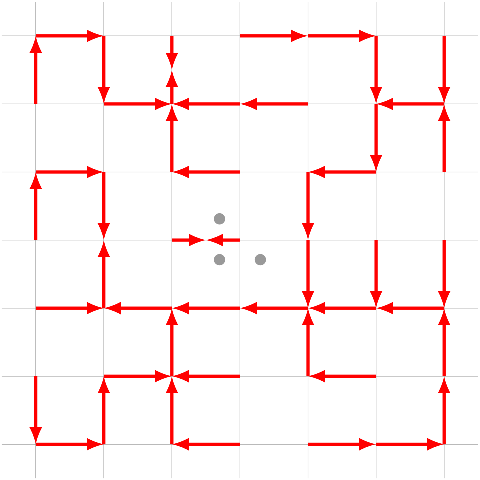
Prison escape

First walker performs rotor walk, remove if returns to origin.



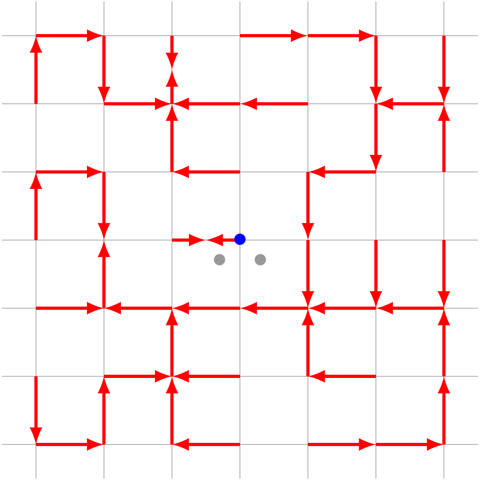
Prison escape

First walker returns to origin, and is removed.



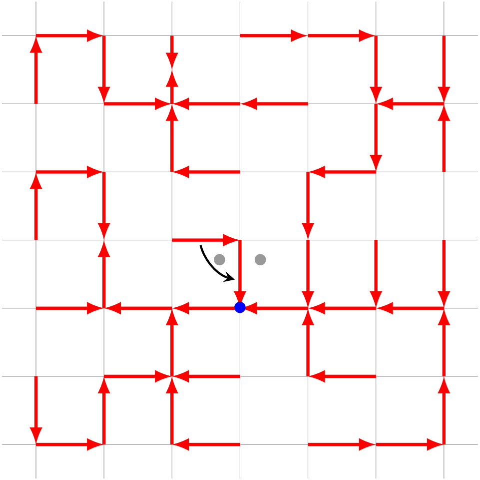
Prison escape

Second walker performs rotor walk, remove if returns to origin.



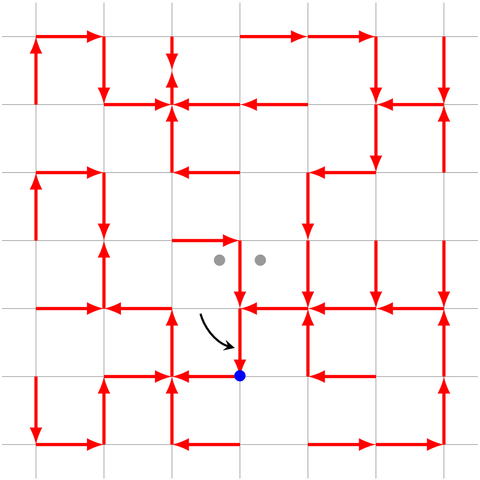
Prison escape

Second walker performs rotor walk, remove if returns to origin.



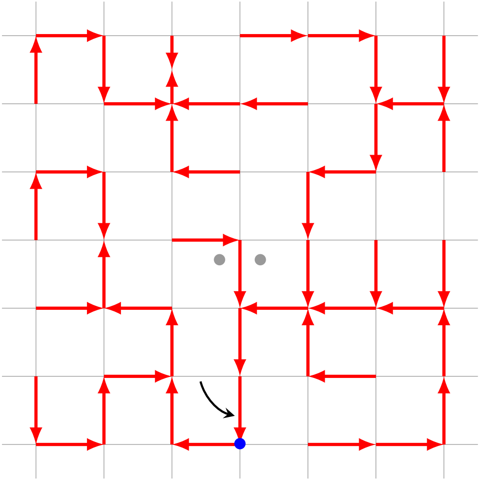
Prison escape

Second walker performs rotor walk, remove if returns to origin.



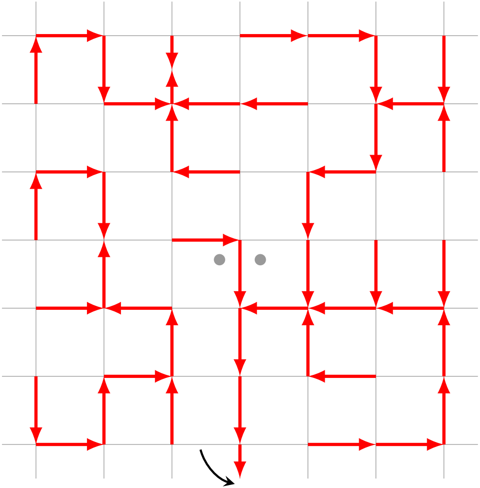
Prison escape

Second walker performs rotor walk, remove if returns to origin.



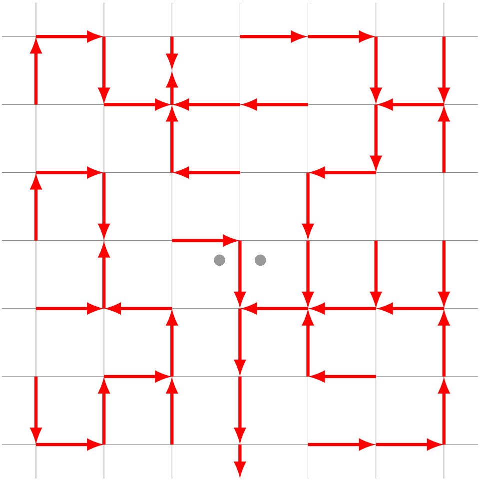
Prison escape

Second walker performs rotor walk, remove if returns to origin.



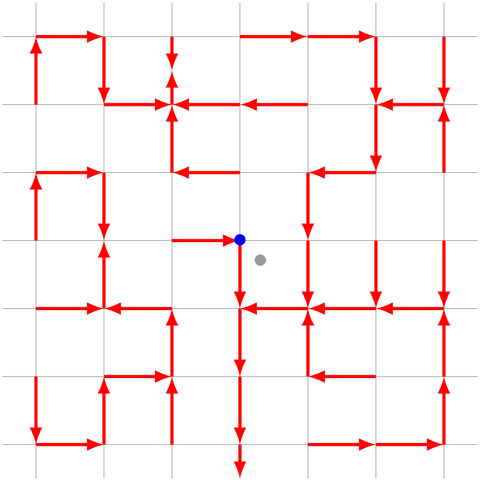
Prison escape

Second walker never returns to origin.



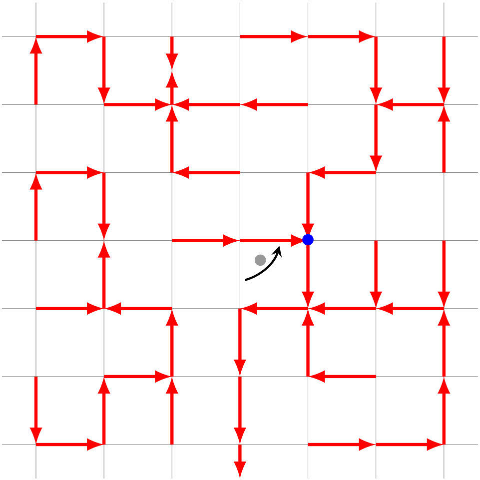
Prison escape

Third walker performs rotor walk, remove if returns to origin.



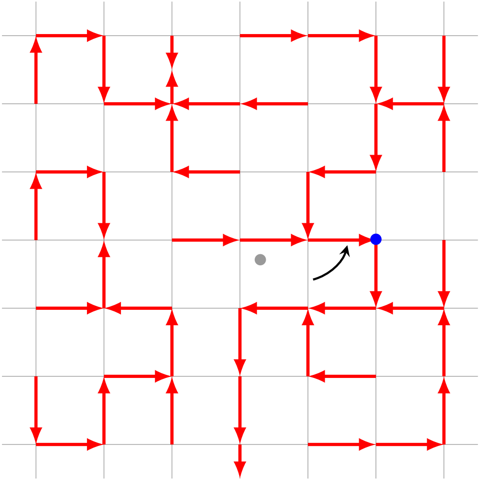
Prison escape

Third walker performs rotor walk, remove if returns to origin.



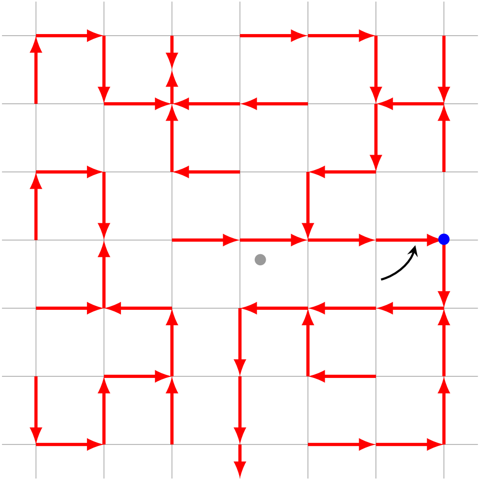
Prison escape

Third walker performs rotor walk, remove if returns to origin.



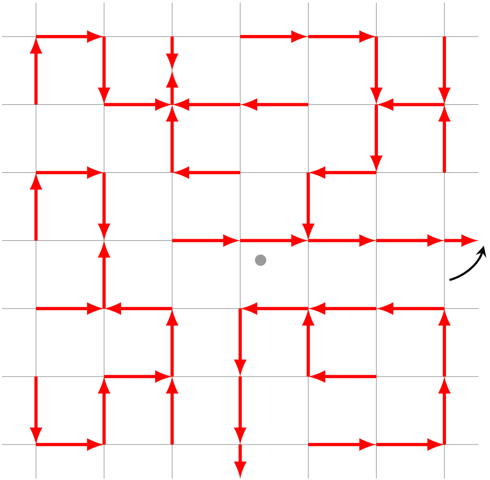
Prison escape

Third walker performs rotor walk, remove if returns to origin.



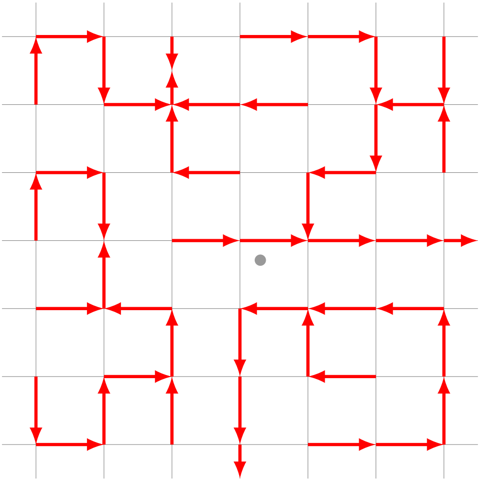
Prison escape

Third walker performs rotor walk, remove if returns to origin.



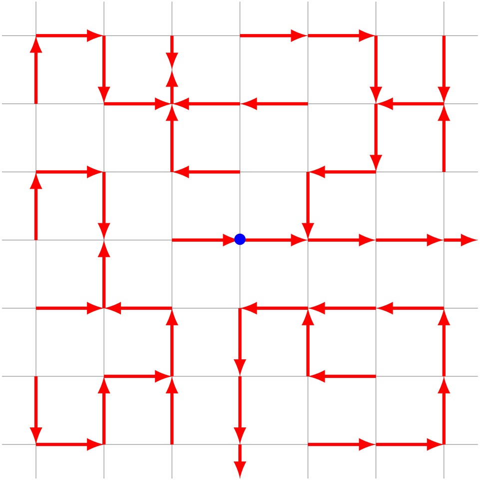
Prison escape

Third walker never returns to origin.



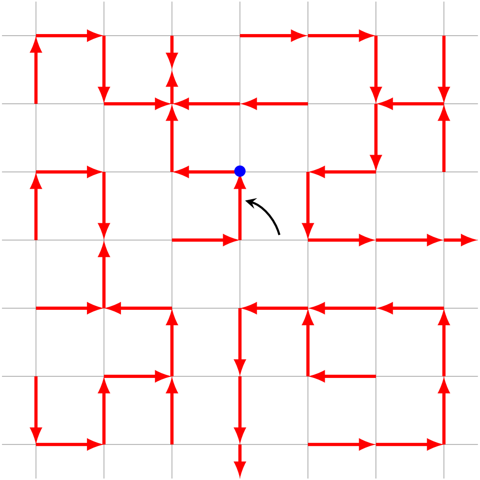
Prison escape

Fourth walker performs rotor walk, remove if returns to origin.



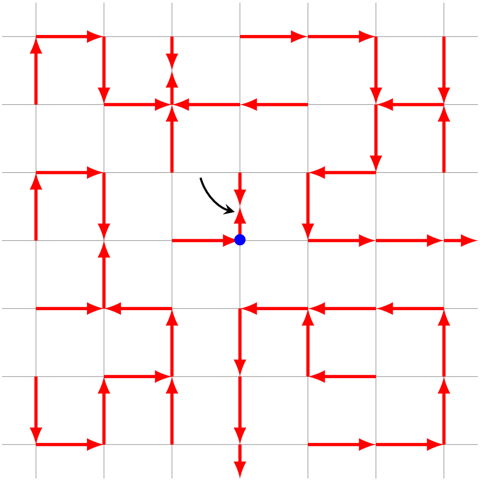
Prison escape

Fourth walker performs rotor walk, remove if returns to origin.



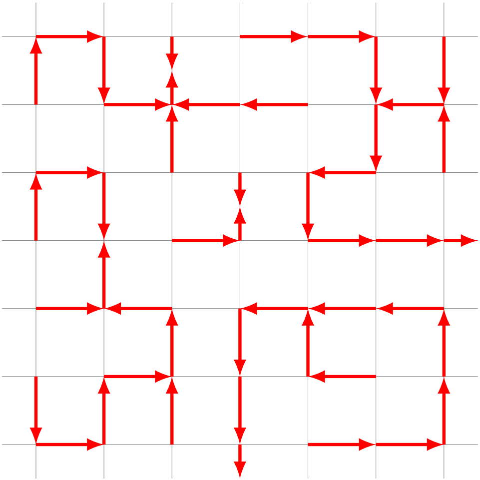
Prison escape

Fourth walker performs rotor walk, remove if returns to origin.



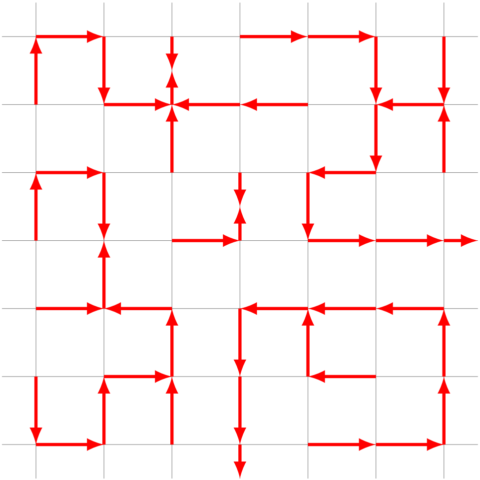
Prison escape

Fourth walker returns to origin, and is removed.



Prison escape

Objective is to maximize number of escapees.



Escape rate of rotor walk

Theorem (Schramm '10 (posthumous))

For any initial signpost ρ, the escape rate of rotor walk is at most equal to the escape probability of simple random walk, i.e.,

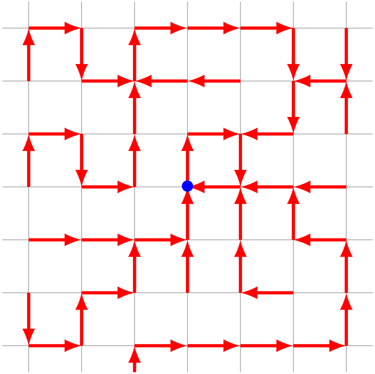
$$\limsup_{n\to\infty} \underbrace{\frac{I(\rho,n)}{n}}_{\substack{\text{proportion of}\\ \text{escaped walkers}}} \leq \underbrace{p_{esc}}_{\substack{\text{escape prob.}\\ \text{of SRW}}}.$$

Conjecture (Florescu Ganguly Levine Peres '13)

There exists an initial signpost ρ for which

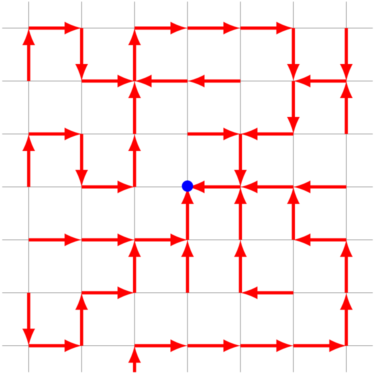
$$\lim_{n\to\infty} \frac{I(\rho,n)}{n} = p_{esc}.$$

Uniform spanning forest oriented to infinity (USF$^\infty$)



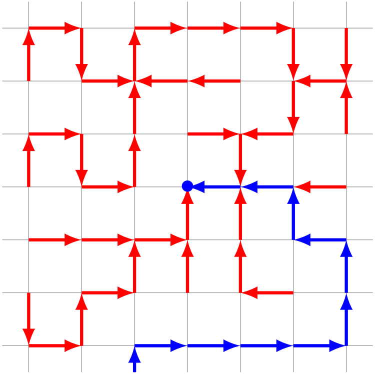
Start with uniform spanning forest plus one edge from before.

Uniform spanning forest oriented to infinity (USF$^\infty$)



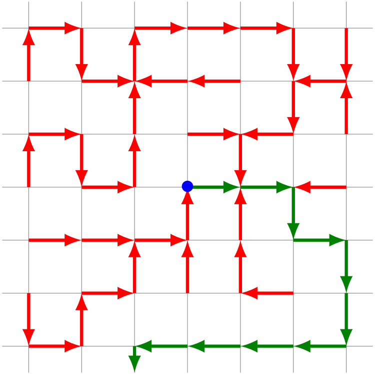
Remove the signpost at the origin.

Uniform spanning forest oriented to infinity (USF$^\infty$)



Find the unique infinite path oriented to origin.

Uniform spanning forest oriented to infinity (USF^∞)



Reverse the orientation of this infinite path.

Answers to the escape rate conjecture

Theorem (C. '19+)

For the graph $\mathbb{Z}^d$, the rotor walk escape rate of uniform spanning forest oriented to infinity is equal to p_{esc}.

This is proved using infinite-step stationarity for rotor walk.

Answers to the escape rate conjecture

Theorem (C. '19+)

For the graph $\mathbb{Z}^d$, the rotor walk escape rate of uniform spanning forest oriented to infinity is equal to p_{esc}.

This is proved using infinite-step stationarity for rotor walk.

Theorem (C. '19+)

For any graph, there exists an initial signpost configuration with rotor walk escape rate equal to p_{esc}.

This is proved using a variant of the martingale for recurrence.

What is next?

Conjecture

Let $p \neq \frac{1}{2}$. Prove that p-rotor walk with i.i.d. uniform signpost configuration is recurrent.

Obstacle: Need a good estimate for the compensator.

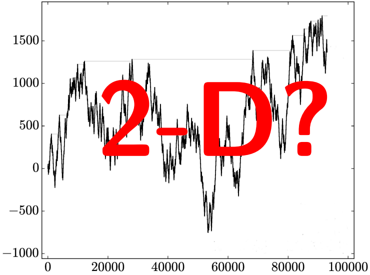
$$\underbrace{M(t)}_{\text{martingale}} := a(X(t)) \; - \; N(t) \; + \underbrace{\sum_{x \in \{X_0, \ldots, X_t\}} \mathtt{w}(x; \rho_t)}_{\text{compensator}}.$$

What is next?

Problem

Let $p \neq \frac{1}{2}$. Find the scaling limit for the p-rotor walk.

Obstacle: Need to define "2-D perturbed Brownian motion (?)".



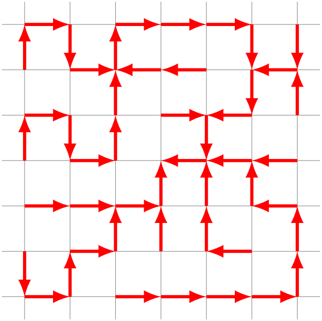
What is next?

Conjecture

For any graph, the i.i.d. uniform signpost configuration has rotor walk escape rate equal to the escape probability of the SRW, i.e.,

$$\lim_{n\to\infty} \frac{I(\rho, n)}{n} = p_{esc}.$$

So far has only been proved for regular trees (Angel Holroyd '11).

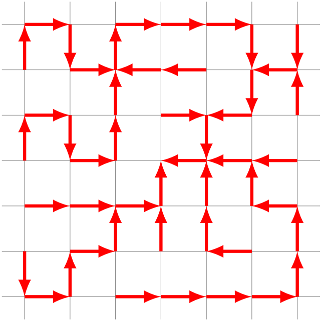
 

(Some of) the preprints can be found in my webpage:

http://math.ucla.edu/∼sweehong/

My email: sweehong@math.ucla.edu

THANK YOU!



(Some of) the preprints can be found in my webpage:

http://math.ucla.edu/∼sweehong/

My email: sweehong@math.ucla.edu