In between random walk and rotor walk

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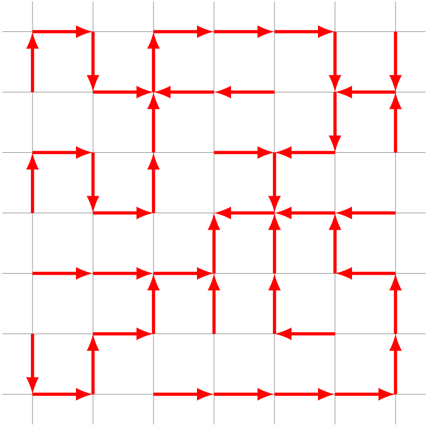
Rutgers University

Joint with Lila Greco, Lionel Levine, Peter Li



Motivation: Exploring Times Square

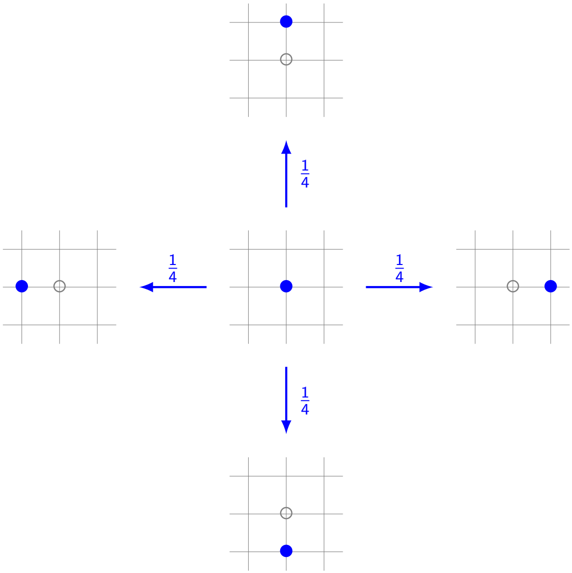
Random model vs deterministic model

Random walk

Rotor walk

Simple random walk on $\mathbb{Z}^2$

Simple random walk on $\mathbb{Z}^2$



Simple random walk on $\mathbb{Z}^2$

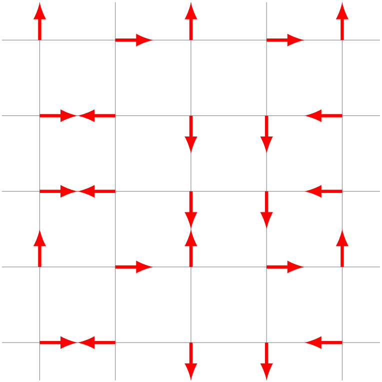
- Visits every site infinitely often? Yes!
- Number of distinct points visited in n steps is $\asymp n^{2/3}$.
- Scaling limit? The standard 2-D Brownian motion:

$$\Big(\frac{1}{\sqrt{n}} \underbrace{X_{[nt]}}_{\text{location of the walker at time } [nt]}\Big)_{t\geq 0} \overset{n\to\infty}{\Longrightarrow} \frac{1}{\sqrt{2}} \underbrace{(B_1(t), B_2(t))}_{\text{independent standard Brownian motions}}{}_{t\geq 0}.$$

Rotor walk on $\mathbb{Z}^2$

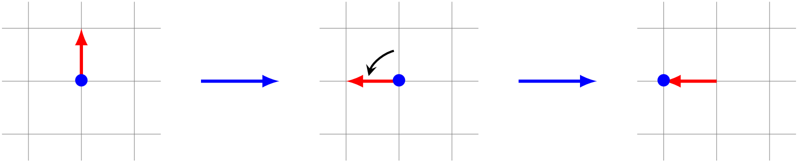
Rotor walk on $\mathbb{Z}^2$

Put a signpost at every vertex.



Rotor walk on $\mathbb{Z}^2$

Turn the signpost at your location 90° counterclockwise, then follow its new direction.



The signpost says:
"This is the way you went the last time you were here",
(assuming you ever were!)

Rotor walk on $\mathbb{Z}^2$

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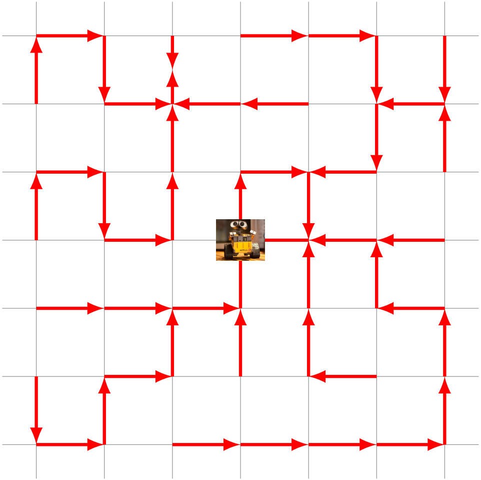

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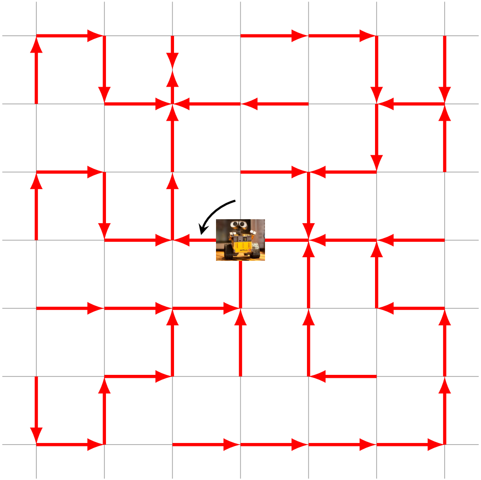

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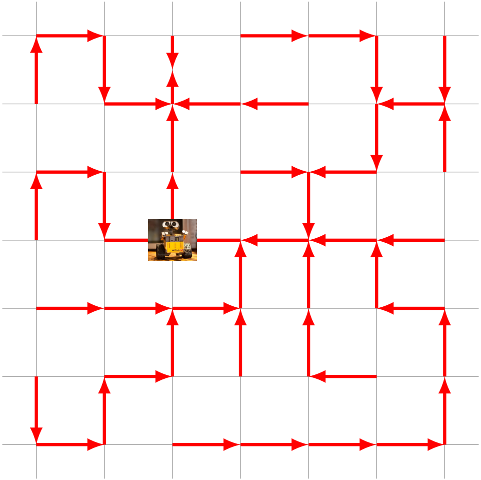

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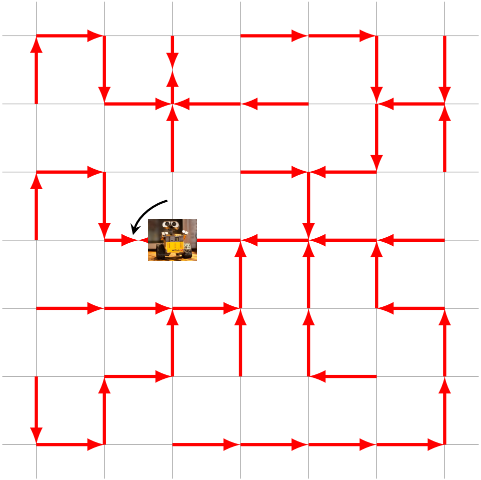

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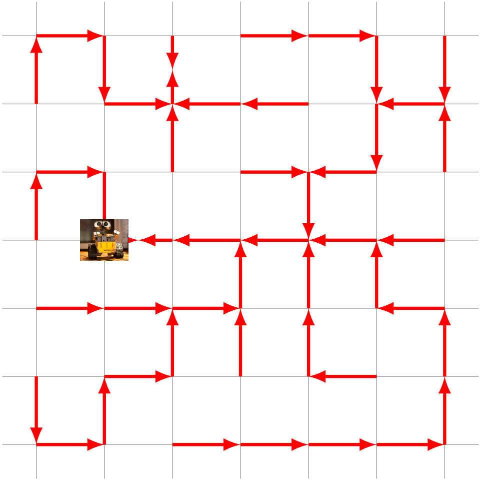

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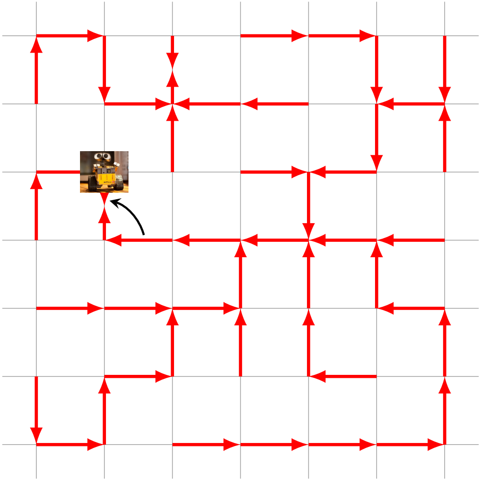

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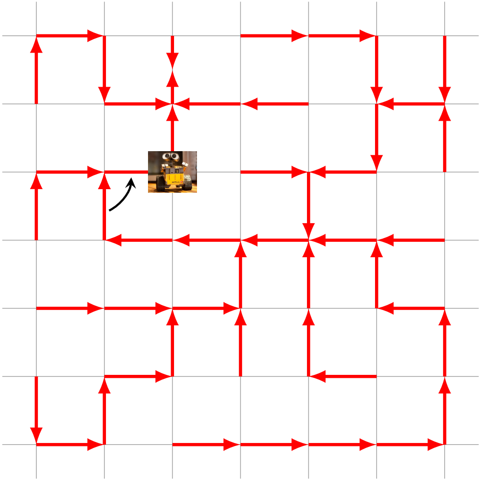

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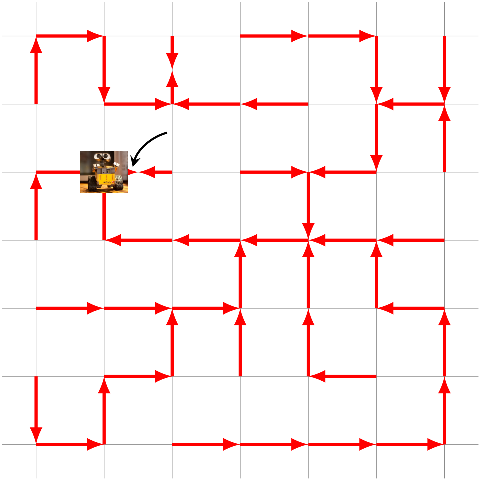

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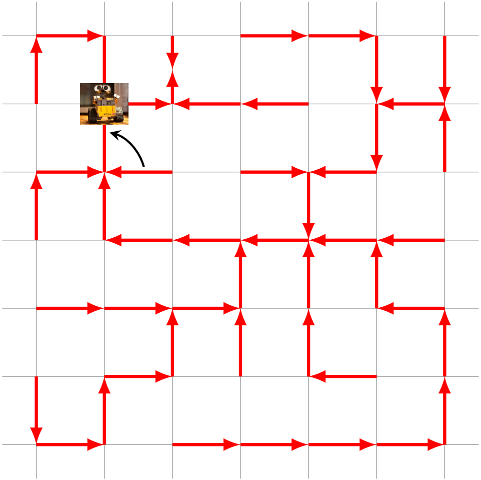

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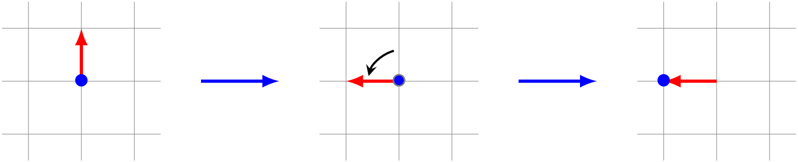


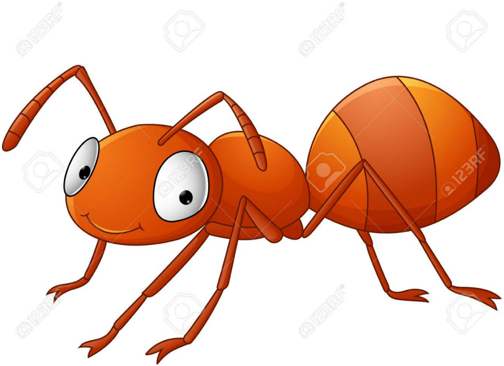
The signpost says:
"This is the way you went the last time you were here",
(assuming you ever were!)

Why rotor walk?

Randomness can be (was) expensive to simulate!

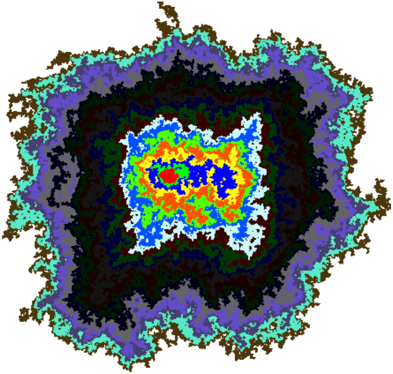
Why rotor walk?

As a model for ants' foraging strategy.



Why rotor walk?

As a model of self-organized criticality for statistical mechanics.



Visited sites after 80 returns to the origin (by Laura Florescu).

Conjectures for rotor walk on $\mathbb{Z}^2$

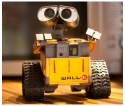
For initial signposts i.i.d. uniform among the four directions,

- (PDDK '96) Visits every site infinitely often?
- (PDDK '96) No. of points visited in n steps is $\asymp n^{2/3}$? (compare with $n/\log n$ for the simple random walk.)
- (Kapri-Dhar '09) The asymptotic shape of $\{X_1, \ldots, X_n\}$ is a disc?

More randomness please!

Well studied

Many open problems



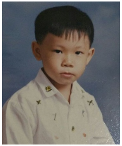
Random

Deterministic

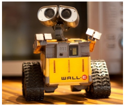
More randomness please!

Well studied

Let's study this!!!



Many open problems

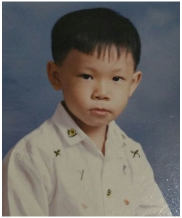


Random

Something in between

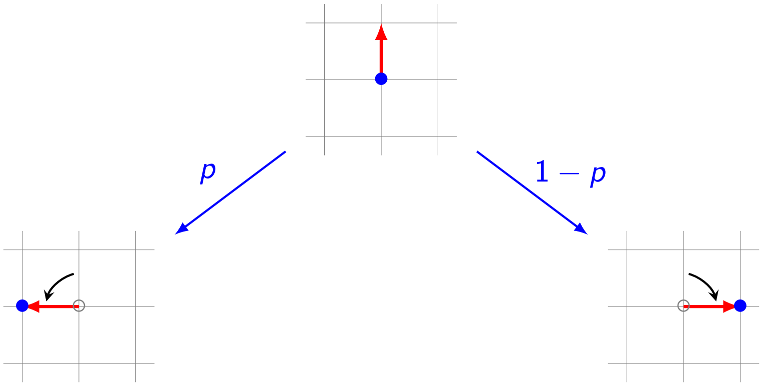
Deterministic

p-rotor walk on $\mathbb{Z}^2$



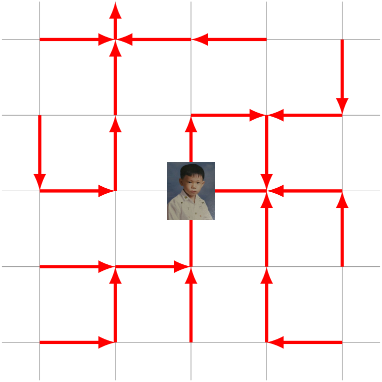
p-rotor walk on $\mathbb{Z}^2$

With probability p, turn the signpost 90° counter-clockwise.
With probability $1-p$, turn the signpost 90° clockwise.



p-rotor walk on $\mathbb{Z}^2$

Follow rotor walk rule with probability p,
do the opposite with probability $1 - p$.



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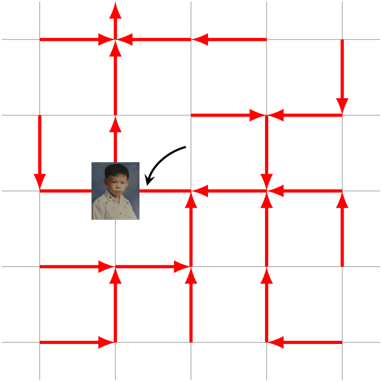

Follow the rule.

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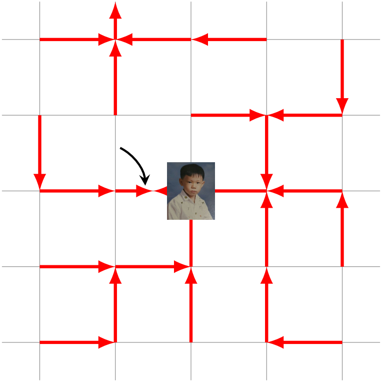

Do the opposite.

p-rotor walk on $\mathbb{Z}^2$

Follow rotor walk rule with probability p,
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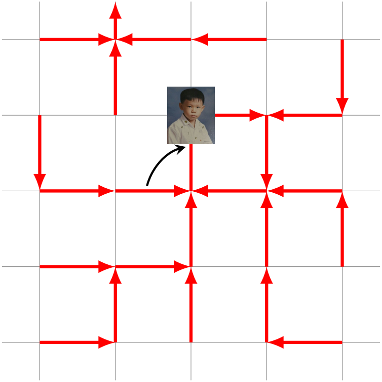
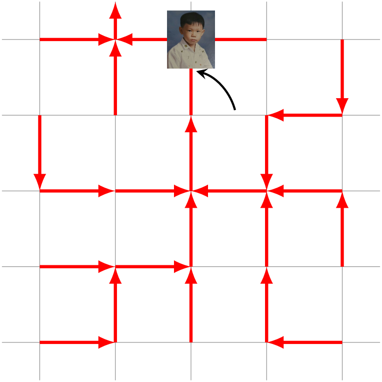


Do the opposite again.

p-rotor walk on $\mathbb{Z}^2$

Follow rotor walk rule with probability p,
do the opposite with probability $1 - p$.



Follow the rule.

p-rotor walk on $\mathbb{Z}^2$

Follow rotor walk rule with probability p,
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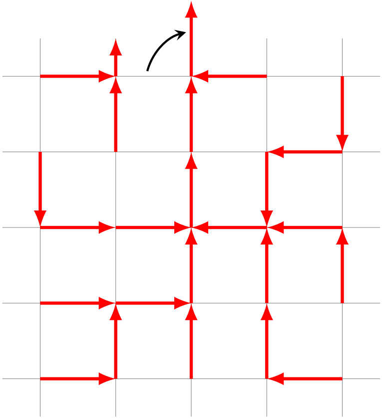
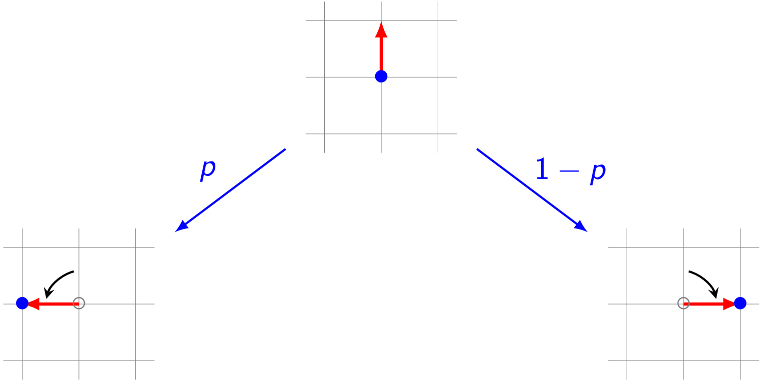


Ops...

p-rotor walk on $\mathbb{Z}^2$

With probability p, turn the signpost 90° counter-clockwise.
With probability $1 - p$, turn the signpost 90° clockwise.



Recover the rotor walk if $p = 1$.

Recurrence result for p-rotor walk

Recurrence for p-rotor walk on $\mathbb{Z}^2$

Theorem (C., '23)

Let $p = \frac{1}{2}$ and let the i.i.d uniform among four directions be the initial signpost configuration. Then the p-rotor walk visits every vertex infinitely often almost surely.

Proof of recurrence for the simple random walk

Consider the following martingale:

$$M(t) := \underbrace{a(X(t))}_{\substack{\text{potential}\\ \text{kernel}}} - \underbrace{N(t)}_{\substack{\#\text{ of times}\\ \text{leaving } o}}.$$

Use the optional stopping theorem:

$$0 = \mathbb{E}[M(\underbrace{\tau(r)}_{\substack{\text{hitting time}\\ \text{of } \partial B_r \cup \{o\}}})] \approx \frac{2}{\pi} \ln r \, (1 - \underbrace{p_{\mathrm{ret}}(r)}_{\substack{\text{prob. of return}\\ \text{before hitting } \partial B_r}}) - 1.$$

Proof of recurrence for the simple random walk (ctd.)

We rewrite the equation to

$$\underbrace{p_{\text{ret}}(r)}_{\substack{\text{prob. of return} \\ \text{before hitting } \partial B_r}} \approx 1 - \frac{\pi}{2\ln r},$$

and we then conclude that

$$\underbrace{p_{\text{rec}}}_{\substack{\text{recurrence} \\ \text{probability}}} = 1 - \lim_{r\to\infty} \frac{\pi}{2\ln r} = 1.$$

Proof of recurrence for p-rotor walk

Consider the following martingale:

$$M(t) := a(X(t)) - N(t) + \underbrace{\sum_{x \in \{X_0, \ldots, X_t\}} \mathtt{w}(x; \rho_t)}_{\text{compensator}}.$$

By the same argument as before,

$$\underbrace{p_{\text{rec}}}_{\substack{\text{recurrence}\\ \text{probability}}} = 1 - \lim_{r \to \infty} \frac{\pi}{2 \ln r} \left(\sum_{|x| \leq r} \mathbb{E}[\mathtt{w}(x; \rho_{\tau(r)})] \right).$$

Proof of recurrence for p-rotor walk (ctd.)

We can estimate the terms in the compensator locally by

$$\left|\mathbb{E}[\mathtt{w}(x;\rho_{\tau(r)})]\right| \;\le\; \left(1-\frac{1}{2^{70}}\right)\frac{2}{\pi|x|^2}.$$

Plugging this estimate into previous equation,

$$p_{\mathrm{rec}} \ge 1 - \lim_{r\to\infty}\frac{\pi}{2\ln r}\left(\sum_{|x|\le r}\left(1-\frac{1}{2^{70}}\right)\frac{2}{\pi|x|^2}\right) \;=\; \frac{1}{2^{70}} > 0.$$

By Kolmogorov zero-one law, the recurrence probability is 1.

So we have proved ...

Theorem (C., '23)

Let $p = \frac{1}{2}$ and let the i.i.d uniform among four directions be the initial signpost configuration. Then the p-rotor walk visits every vertex infinitely often almost surely.

Open problem

Conjecture

Let $p \neq \frac{1}{2}$. Prove that p-rotor walk with i.i.d. uniform signpost configuration is recurrent.

Obstacle: Need a good estimate for the compensator.

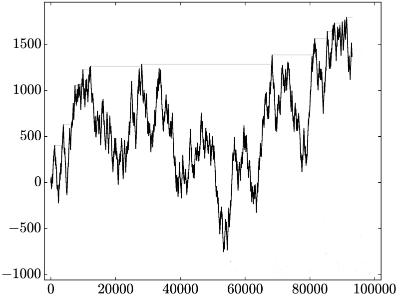
$$\underbrace{M(t)}_{\text{martingale}} := a(X(t)) \; - \; N(t) \; + \underbrace{\sum_{x \in \{X_0, \ldots, X_t\}} \mathtt{w}(x; \rho_t)}_{\text{compensator}} .$$

Scaling limit result for p-rotor walk

Scaling limit for p-rotor walk on $\mathbb{Z}$

(Huss, Levine, Sava-Huss 18) The scaling limit for p-rotor walk on $\mathbb{Z}$ is a perturbed Brownian motion $(Y(t))_{t\geq 0}$,

$$Y(t) = \underbrace{B(t)}_{\substack{\text{standard}\\ \text{Brownian}\\ \text{motion}}} + \underbrace{a \sup_{0\leq s\leq t} Y(s)}_{\substack{\text{perturbation at}\\ \text{maximum}}} + \underbrace{b \inf_{0\leq s\leq t} Y(s)}_{\substack{\text{perturbation at}\\ \text{minimum}}}, \qquad t \geq 0.$$



$Y(t)$ for $a = -0.998$, and $b = 0$ (by Wilfried Huss).

Scaling limit for p-rotor walk on $\mathbb{Z}^2$

Question: Is the scaling limit for p-rotor walk on $\mathbb{Z}^2$ a "2-D perturbed Brownian motion"?

Problem: How to define "2-D perturbed Brownian motion"?.

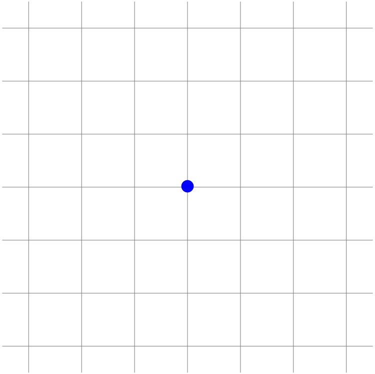
Scaling limit for p-rotor walk on $\mathbb{Z}^2$

Question: Is the scaling limit for p-rotor walk on $\mathbb{Z}^2$ a "2-D perturbed Brownian motion"?

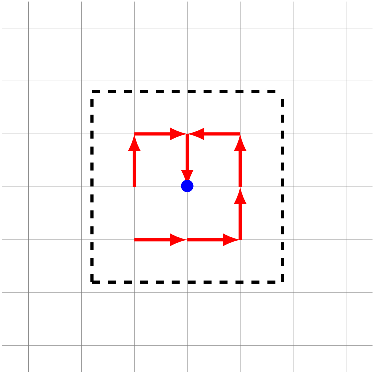
Problem: How to define "2-D perturbed Brownian motion"?.

Conjecture: The scaling limit for p-rotor walk on $\mathbb{Z}^2$ when $p = \frac{1}{2}$ is the standard 2-D Brownian motion.

Uniform spanning forest plus one edge (USF^+)

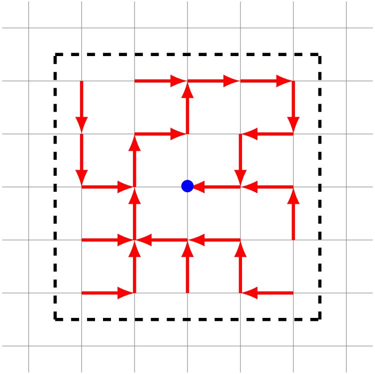


Uniform spanning forest plus one edge (USF$^+$)



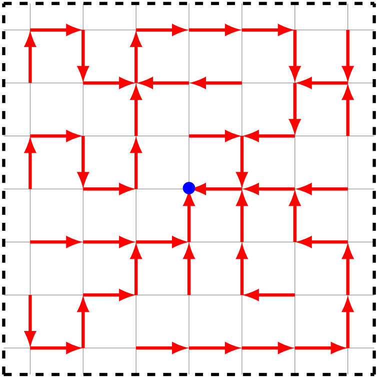
Pick a spanning tree of the black box directed to the origin (uniformly at random).

Uniform spanning forest plus one edge (USF^+)



Take the limit as the black box grows until it covers $\mathbb{Z}^2$.

Uniform spanning forest plus one edge (USF^+)



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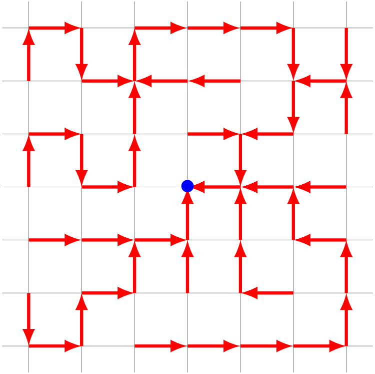

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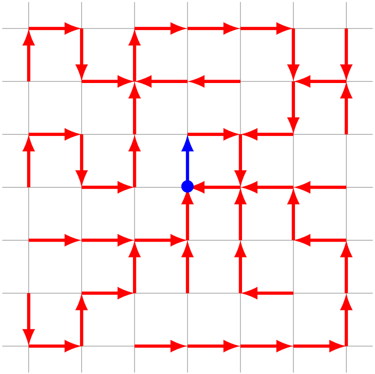


Add a signpost from the origin, uniform among the four directions.

Scaling limit for p-rotor walk on $\mathbb{Z}^2$

Theorem (C., Greco, Levine, Li '21)

Let $p = \frac{1}{2}$ and let the uniform spanning forest plus one edge be the initial signpost configuration. Then, with probability 1, the p-rotor walk on $\mathbb{Z}^2$ scales to the standard 2-D Brownian motion:

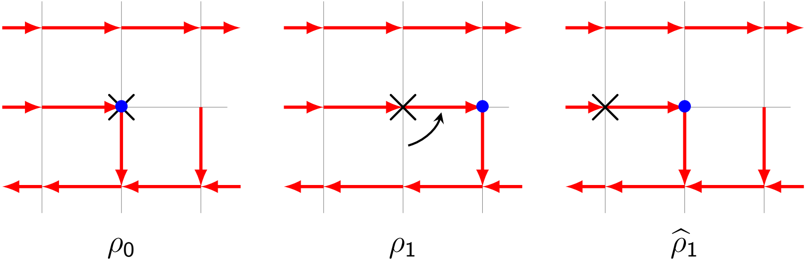
$$\frac{1}{\sqrt{n}}(\underbrace{X_{[nt]}}_{\substack{\text{location of the} \\ \text{walker at time } [nt]}})_{t\geq 0} \overset{n\to\infty}{\Longrightarrow} \frac{1}{\sqrt{2}}(\underbrace{B_1(t), B_2(t)}_{\substack{\text{independent} \\ \text{Brownian motions}}})_{t\geq 0}.$$

Disclaimer: Proof in the paper was for h-v walks, not p-rotor walks.

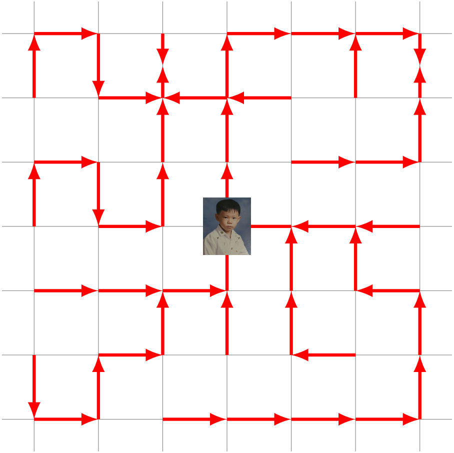
Stationarity from the walker's POV

A signpost configuration $(\rho_0(x))_{x\in\mathbb{Z}^2}$ is stationary in time from the walker's point of view if

$$\underbrace{(\widehat{\rho}_1(x))_{x\in\mathbb{Z}^2}}_{\substack{\text{signpost conf. at} \\ \text{time 1 from walker's POV}}} := (\rho_1(x - X_1))_{x\in\mathbb{Z}^2} \overset{d}{=} \underbrace{(\rho_0(x))_{x\in\mathbb{Z}^2}}_{\substack{\text{signpost conf.} \\ \text{at time 0}}}.$$



Why is USF$^+$ stationary from walker's POV?



The signposts at previously visited vertices
form a tree oriented toward the walker.

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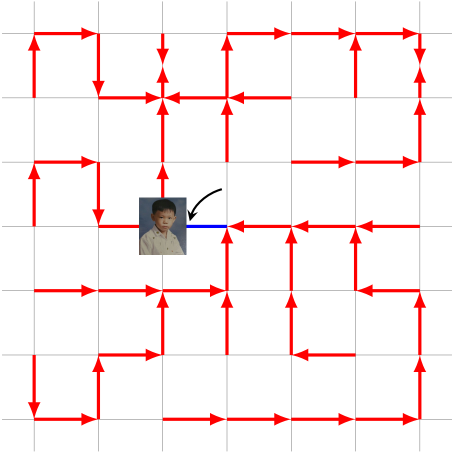

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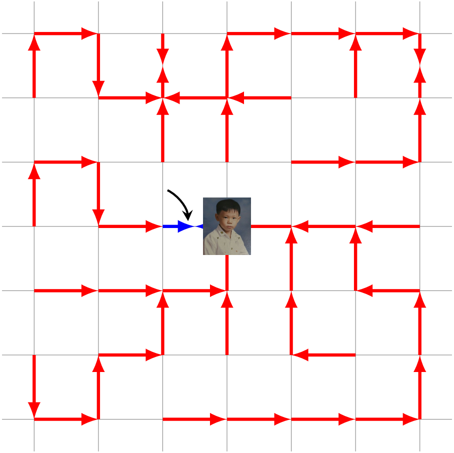

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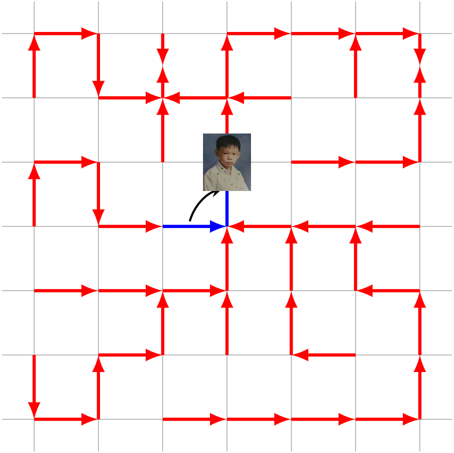

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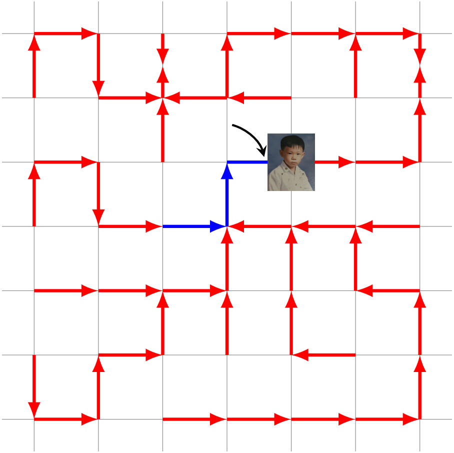

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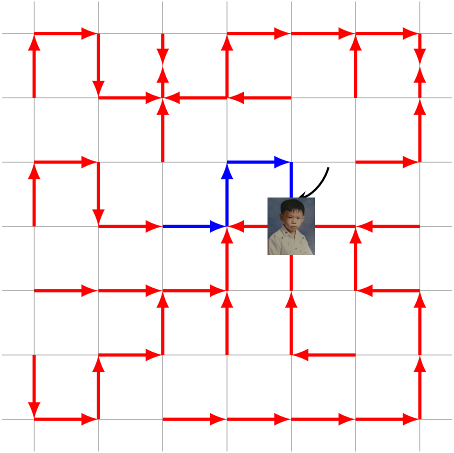

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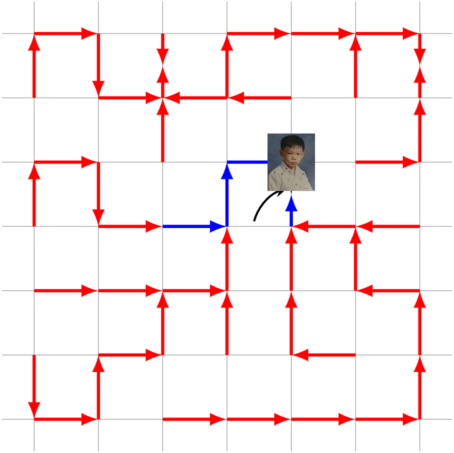

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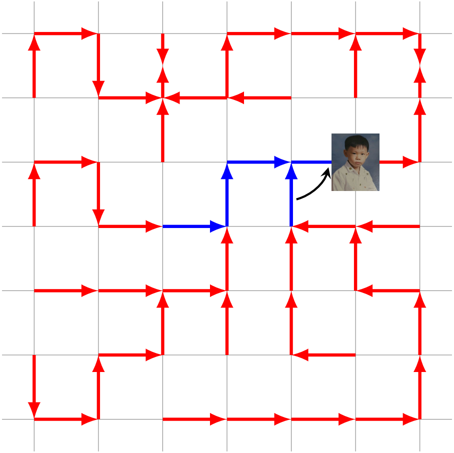

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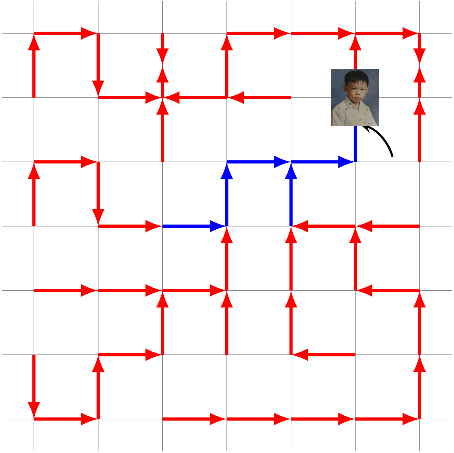
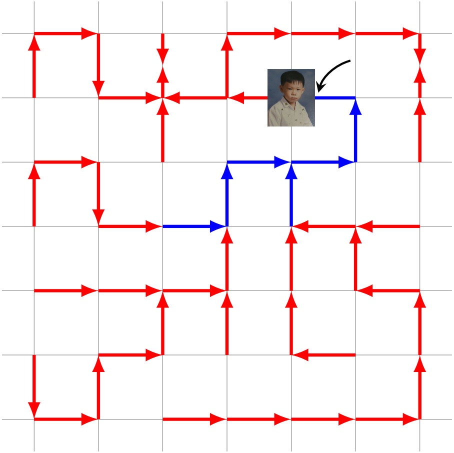


The signposts at previously visited vertices
form a tree oriented toward the walker.

Why is USF$^+$ stationary from walker's POV?



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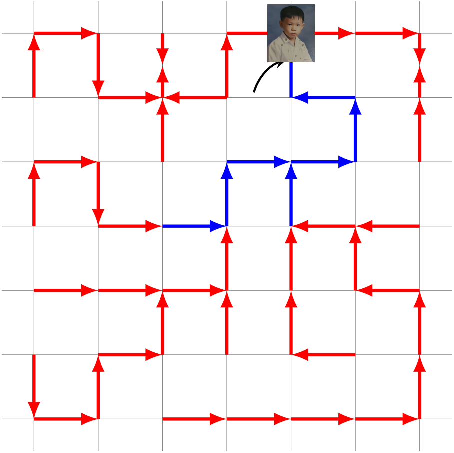

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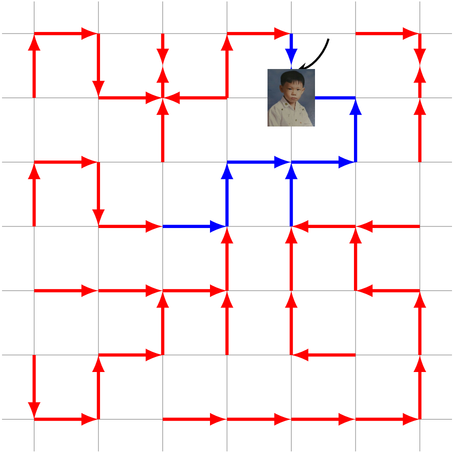

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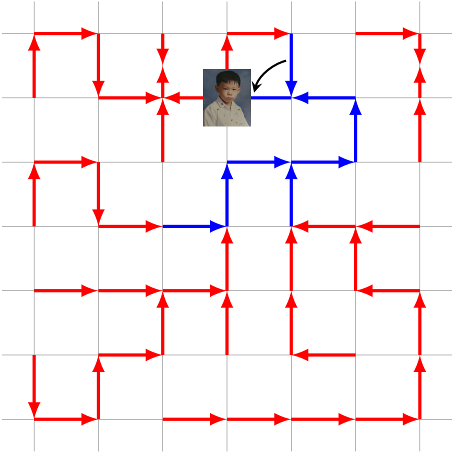
Why is USF$^+$ stationary from walker's POV?



The signposts at previously visited vertices
form a tree oriented toward the walker.

Sketch of the scaling limit proof

Scaling limit

Martingale CLT

Encounters vertical sign-posts half the time a.s.

Ergodic theorem

Stationarity and ergodicity of USF^+ from walker's POV

So we have proved...

Theorem (C., Greco, Levine, Li '21)

Let $p = \frac{1}{2}$ and let the uniform spanning forest plus one edge be the initial signpost configuration. Then, with probability 1, the p-rotor walk on $\mathbb{Z}^2$ scales to the standard 2-D Brownian motion:

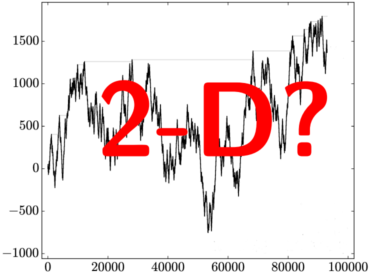
$$\frac{1}{\sqrt{n}}(\underbrace{X_{[nt]}}_{\substack{\text{location of the}\\ \text{walker at time } [nt]}})_{t\geq 0} \overset{n\to\infty}{\Longrightarrow} \frac{1}{\sqrt{2}}(\underbrace{B_1(t), B_2(t)}_{\substack{\text{independent}\\ \text{Brownian motions}}})_{t\geq 0}.$$

Open Problem

Problem

Find the scaling limit for the p-rotor walk with i.i.d. uniform signpost configuration.

Obstacle: Definition of “2-D perturbed Brownian motion (?)”.



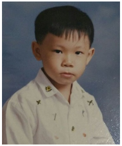
Back to our motivation

Simple random walk	p-rotor walk	Rotor walk
Well studied	Know a little bit now	Many open problems

Well studied

Simple random walk

Know a little bit now



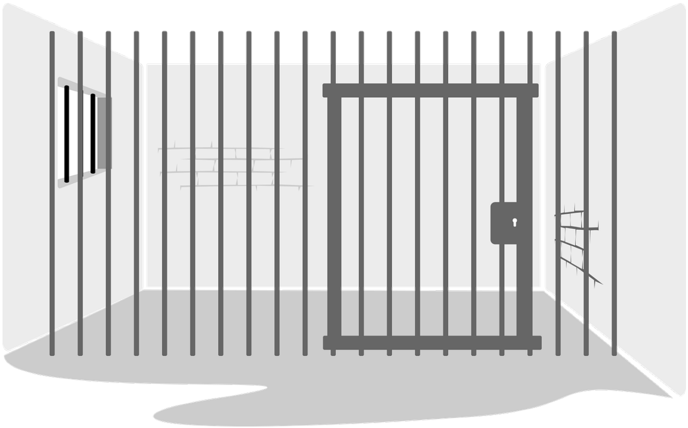
p-rotor walk

Many open problems

Rotor walk

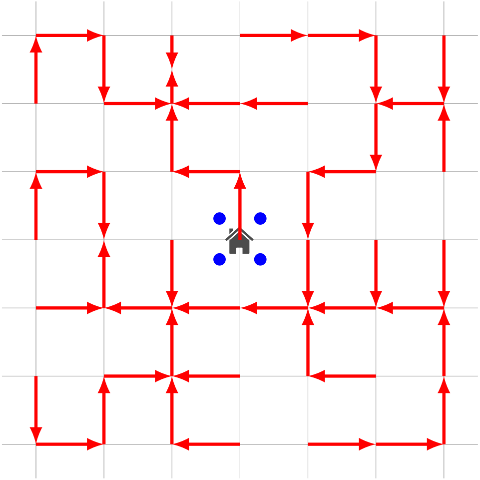
Let's apply what we have learnt to rotor walk.

Escape rate of rotor walk



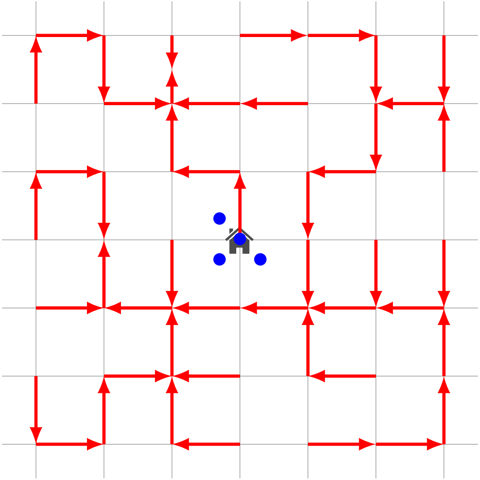
Prison break using rotor walk

Put n walkers at the origin (the prison).



Prison break using rotor walk

First walker performs rotor walk, remove if returns to prison.



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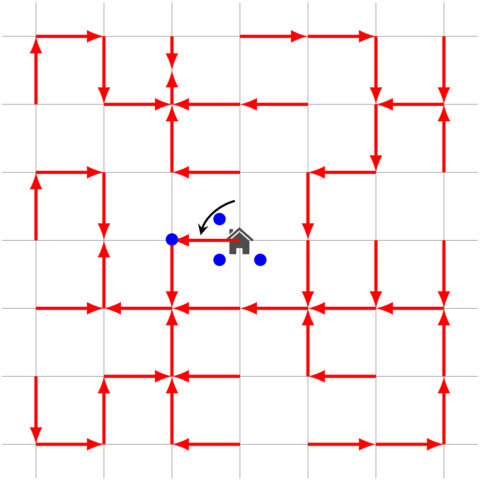

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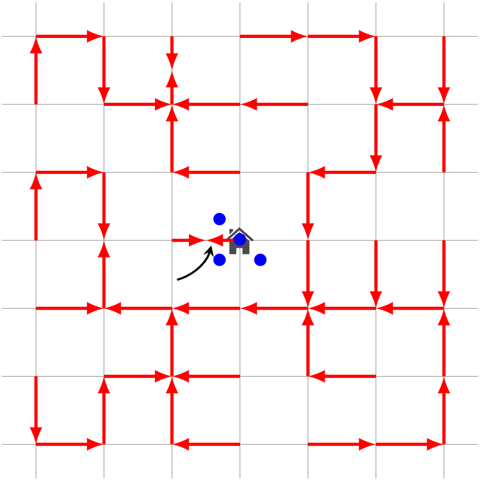
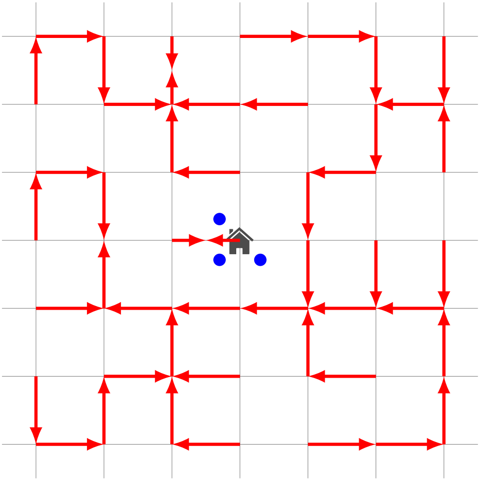


Prison break using rotor walk

First walker returns to prison, and is removed.



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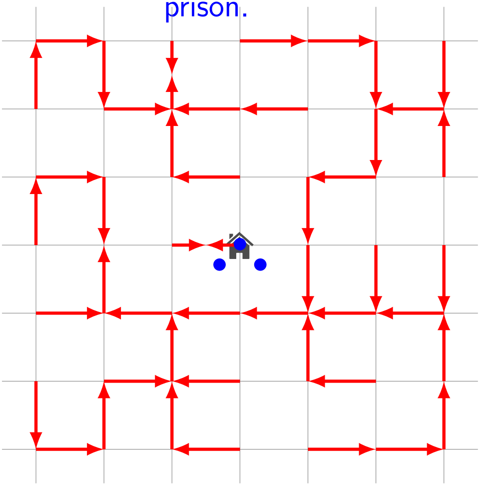

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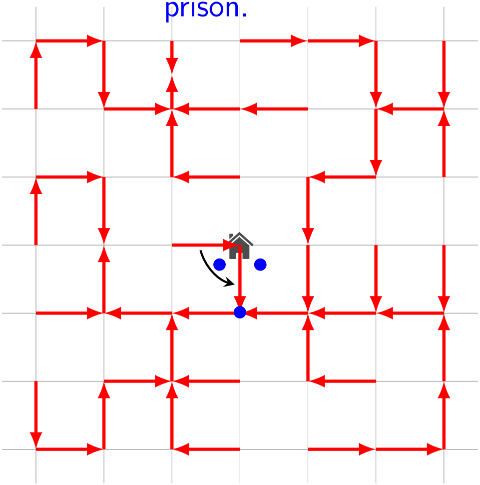

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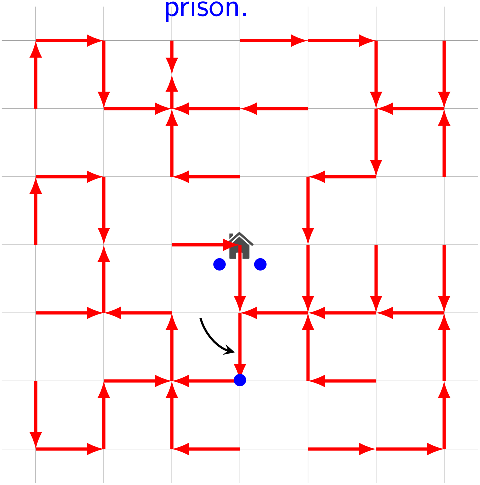

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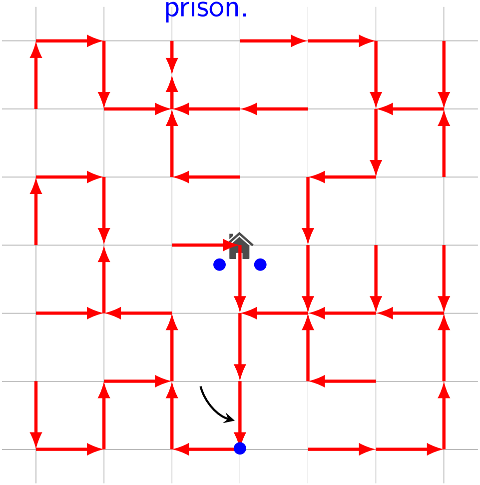

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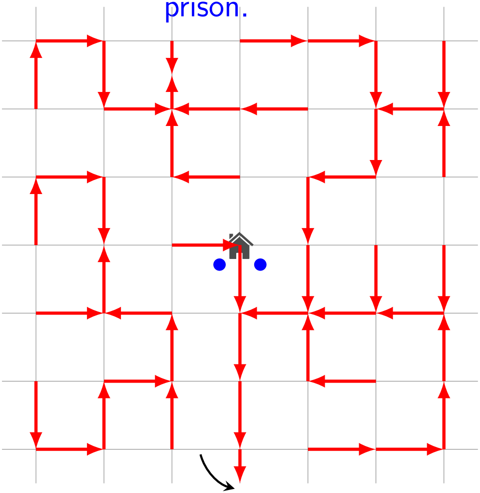
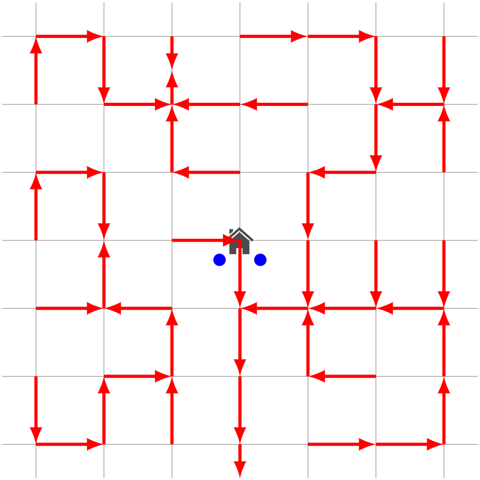


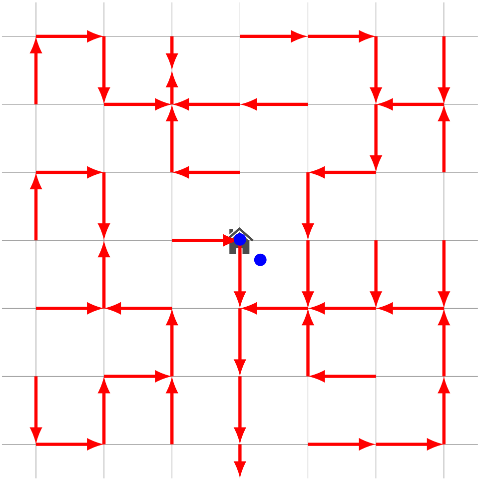
Prison break using rotor walk

Second walker never returns to origin.



Prison break using rotor walk

Third walker performs rotor walk, remove if returns to prison.



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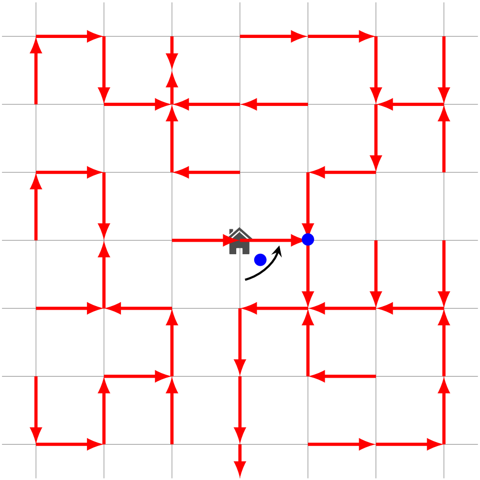

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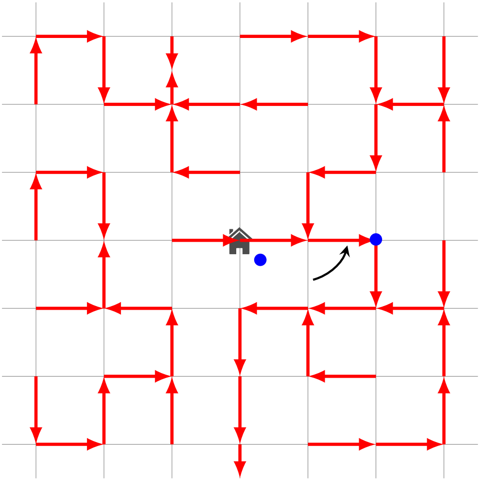

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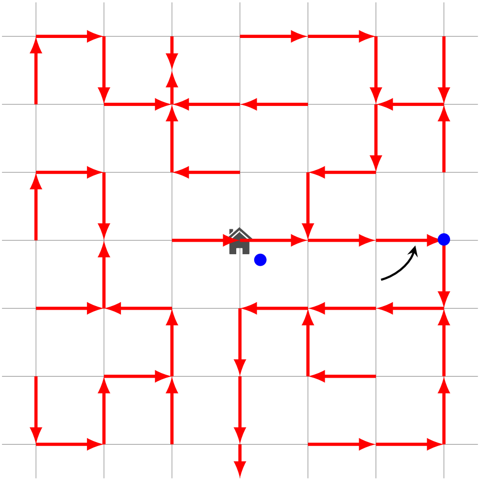

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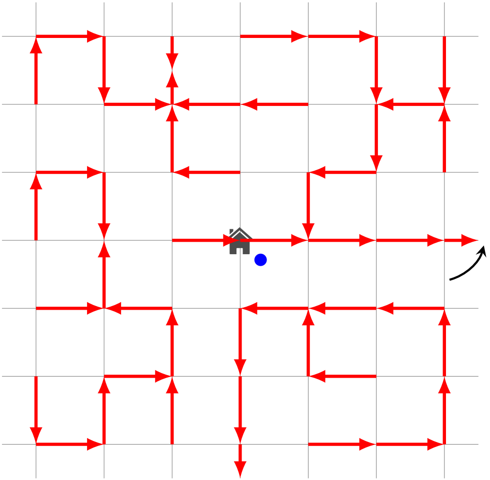
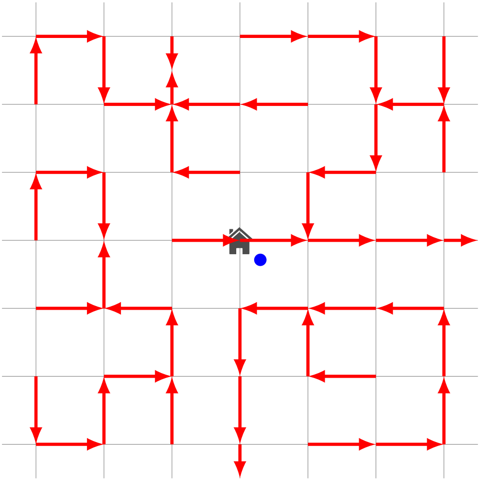


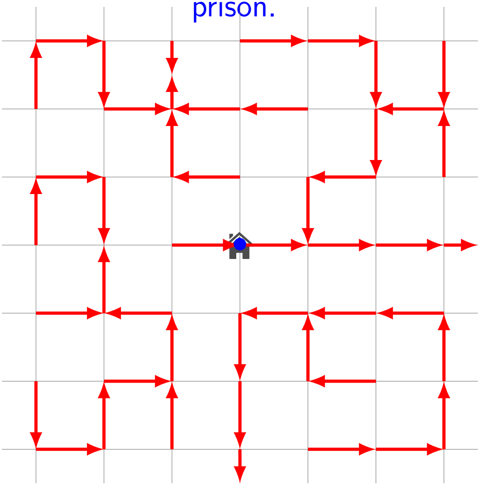
Prison break using rotor walk

Third walker never returns to prison.



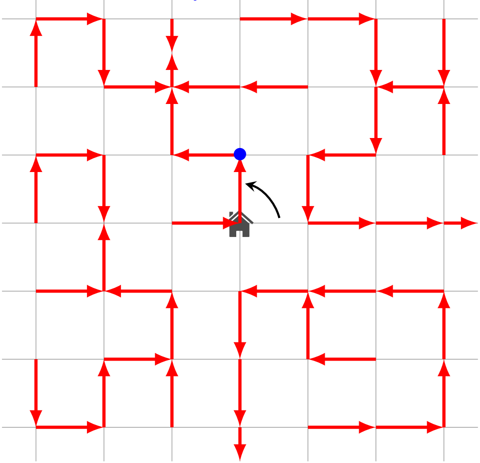
Prison break using rotor walk

Fourth walker performs rotor walk, remove if returns to prison.



Prison break using rotor walk

Fourth walker performs rotor walk, remove if returns to prison.



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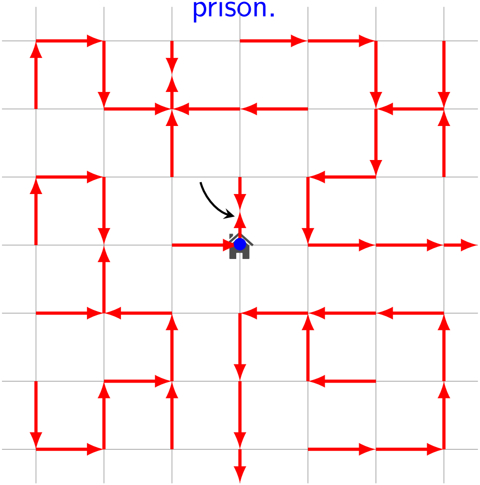
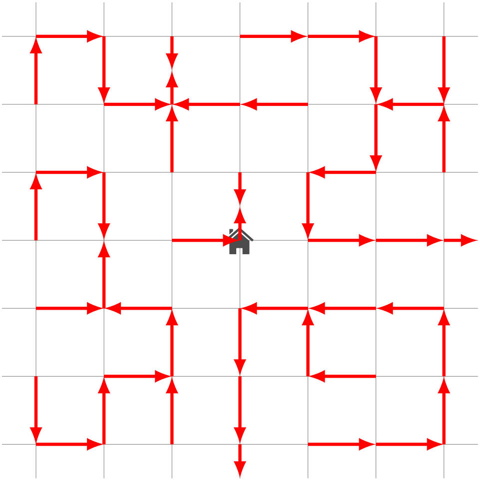


Prison break using rotor walk

Fourth walker returns to prison, and is removed.



Escape rate of rotor walk

The escape rate of n rotor walkers with initial signpost ρ is

$$r_{\text{esc}}(\rho, n) := \frac{\text{number of escaped walkers}}{n}.$$

The escape rate of rotor walk is a deterministic counterpart of the escape probability of simple random walk.

What was known about escape rate

Theorem (Schramm '10 (posthumous))

For any initial signpost ρ,

$$\limsup_{n\to\infty} \underbrace{r_{esc}(\rho, n)}_{\substack{\text{escape rate}\\ \text{of rotor walk}}} \quad \leq \quad \underbrace{p_{esc}(SRW)}_{\substack{\text{escape prob.}\\ \text{of SRW}}}.$$

Corollary

On $\mathbb{Z}^2$, for any initial signpost ρ,

$$\lim_{n\to\infty} r_{esc}(\rho, n) \;=\; p_{esc}(SRW) = 0.$$

In fact, this is true for all recurrent graphs.

What was known about escape rate

Theorem (Angel Holroyd '09)

On $\mathbb{Z}^d$ with $d \geq 3$, there exists an initial signpost ρ so that

$$\lim_{n\to\infty} r_{esc}(\rho, n) \;=\; 0.$$

Theorem (Florescu Ganguly Levine Peres '13)

On $\mathbb{Z}^d$ with $d \geq 3$, for the one-directional initial signpost ρ,

$$\liminf_{n\to\infty} r_{esc}(\rho, n) \;>\; 0.$$

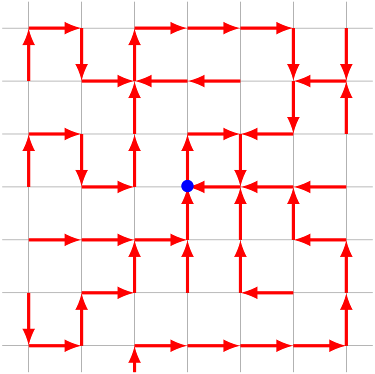
Escape rate conjecture

Conjecture (FGLP '13)

For any transient graph, there exists an initial signpost ρ *for which*

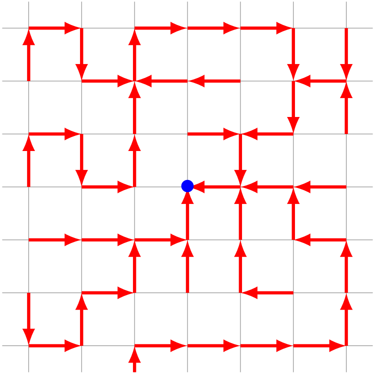
$$\lim_{n\to\infty} r_{esc}(\rho, n) \;=\; p_{esc}(SRW).$$

Uniform spanning forest oriented to infinity (USF$^\infty$)



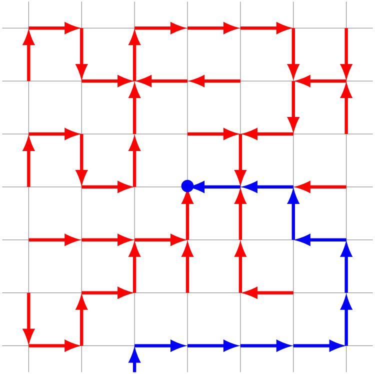
Start with uniform spanning forest plus one edge from before.

Uniform spanning forest oriented to infinity (USF^∞)



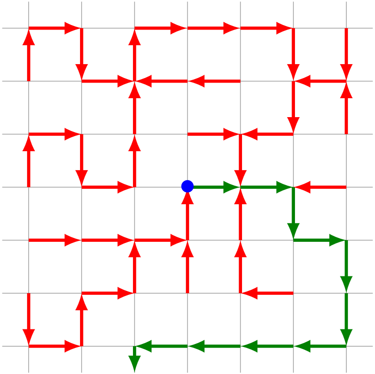
Remove the signpost at the origin.

Uniform spanning forest oriented to infinity (USF^∞)



Find the unique infinite path oriented to origin.

Uniform spanning forest oriented to infinity (USF^∞)



Reverse the orientation of this infinite path.

Answering the escape rate conjecture

Theorem (C. '19)

On $\mathbb{Z}^d$, almost every ρ sampled from USF^∞ satisfies

$$\lim_{n\to\infty} r_{esc}(\rho, n) \ = \ p_{esc}(SRW).$$

Remark: Similar result applies to all vertex-transitive graphs.

Except that ...

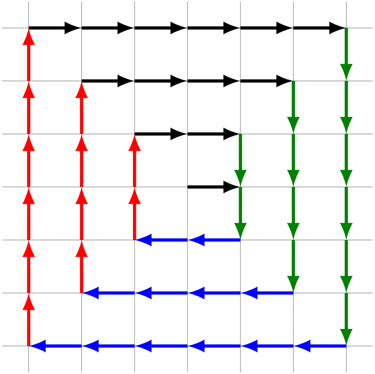
- The conjecture of FGLP '13 is for all transient graphs;
- There are already other constructions for the special case of $\mathbb{Z}^d$ (He '14) and trees (Angel Holroyd '11);
- Our construction of the initial signpost ρ is not deterministic.

Complete answer to the escape rate conjecture

Theorem (C., '20)

For any transient graph, the initial signpost $\rho_{\max}$ *satisfies*

$$\lim_{n\to\infty} r_{esc}(\rho_{\max}, n) \;=\; p_{esc}(SRW).$$



Escape rate formula

Lemma

For any initial signpost ρ and number of walkers n,

$$r_{esc}(\rho, n) = p_{esc}(SRW) - \sum_{x \in \mathbb{Z}^d} \Big(\mathtt{w}_x[\underbrace{\rho_n(x)}_{\substack{\text{signpost at } x \\ \text{after } n\text{-th walk}}}] - \mathtt{w}_x[\underbrace{\rho(x)}_{\substack{\text{initial signpost} \\ \text{at } x}}] \Big),$$

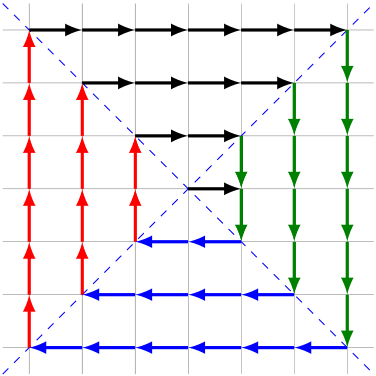
where $\mathtt{w}_x$ is a local compensator term.

The formula is inspired by the martingale used in proving recurrence for p-rotor walk.

Our initial signpost configuration

The configuration ρ_{max} is constructed by choosing, for each x,

the direction $\rho_{\mathsf{max}}(x)$ that maximizes compensator $\mathtt{w}_x$.



Proof of the escape rate conjecture

- By the escape rate formula,

$$r_{\text{esc}}(\rho, n) \;=\; p_{\text{esc}}(SRW) - \sum_{x \in \mathbb{Z}^d} \Big(\mathtt{w}_x[\rho_n(x)] - \mathtt{w}_x[\rho(x)] \Big),$$

- By our choice of ρ_{max},

$$r_{\text{esc}}(\rho_{\text{max}}, n) \;\geq\; p_{\text{esc}}(SRW).$$

- On the other hand, Schramm's inequality gives us

$$\limsup_{n\to\infty} r_{\text{esc}}(\rho_{\text{max}}, n) \;\leq\; p_{\text{esc}}(SRW).$$

- Hence,

$$\lim_{n\to\infty} r_{\text{esc}}(\rho_{\text{max}}, n) \;=\; p_{\text{esc}}(SRW).$$

So we have proved...

Theorem (C., '20)

For any transient graph, the initial signpost $\rho_{\max}$ satisfies

$$\lim_{n\to\infty} r_{esc}(\rho_{\max}, n) \;=\; p_{esc}(SRW).$$

Open problem

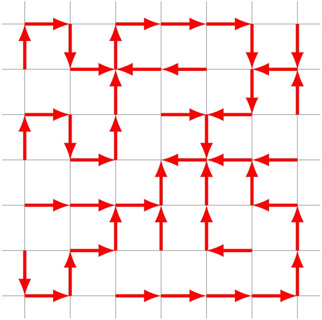
Conjecture

For any graph, the i.i.d. uniform signpost configuration has rotor walk escape rate equal to the escape probability of the SRW, i.e.,

$$\lim_{n\to\infty} r_{esc}(\rho, n) = p_{esc}(SRW).$$

Conjecture is known only for regular trees (Angel Holroyd '11).

THANK YOU!



Corresponding papers can be found in the webpage:

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