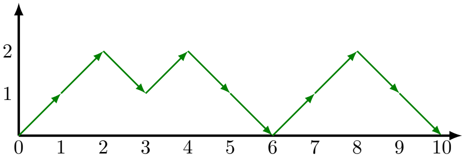
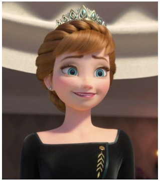
Sorting probability for Young diagrams

Swee Hong Chan (UCLA)

joint with Igor Pak and Greta Panova

1	2	4	7	8
3	5	6	9	10





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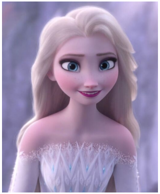

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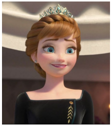
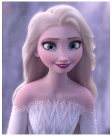
 <  <  <  <  < 

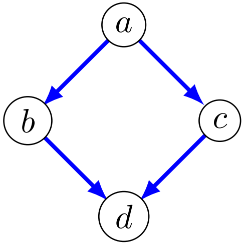
Partially ordered set

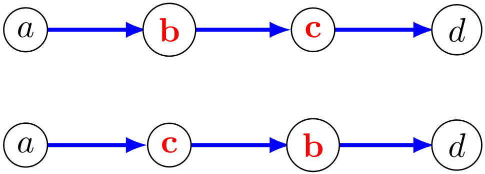
A poset P is a set X with a partial order $\preccurlyeq$ on X.

$$\begin{tikzcd} & a \arrow[dl] \arrow[dr] & \\ b \arrow[dr] & & c \arrow[dl] \\ & d & \end{tikzcd}$$

Linear extension

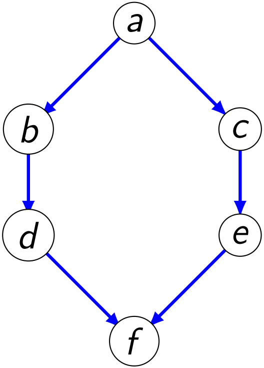
A linear extension L is a complete order of $\preccurlyeq$.





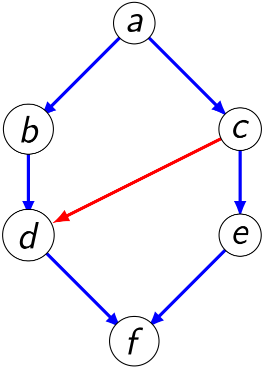
We write $e(P)$ for number of linear extensions of P.

How many steps needed to complete a partial order?



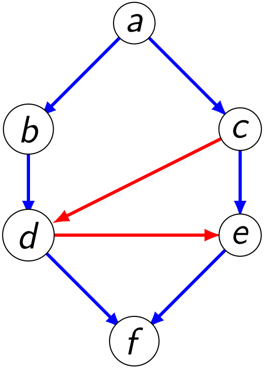
How many steps needed to complete a partial order?

We first compare c and d, and get $c \preccurlyeq d$.



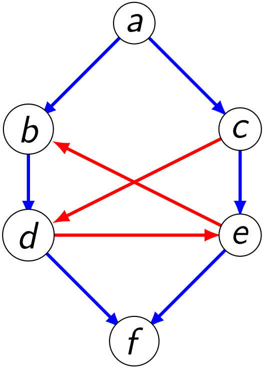
How many steps needed to complete a partial order?

We then compare d and e, and get $d \preccurlyeq e$.



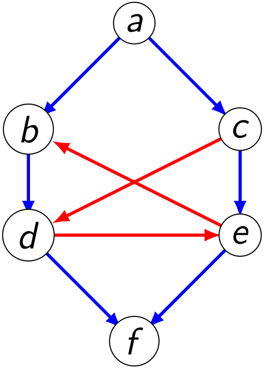
How many steps needed to complete a partial order?

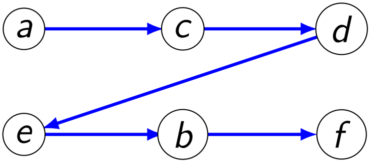
We continue with b and e, and get $e \preccurlyeq b$.



How many steps needed to complete a partial order?

Completing the partial order took 3 steps.





Strategy to complete the partial order

At each step, compare x and y that satisfies

$$\frac{1}{2} - c \quad \leq \quad \mathrm{P}\big[x \preccurlyeq y\big] \quad \leq \quad \frac{1}{2} + c\,,$$

where P is uniform on linear extensions of P.

Runtime is $\Theta(\log e(P))$ steps.

$\frac{1}{3} - \frac{2}{3}$ Conjecture

Conjecture (Kislitsyn '68, Fredman '75, Linial '84)

For every finite poset that is not completely ordered, there exists x, y:

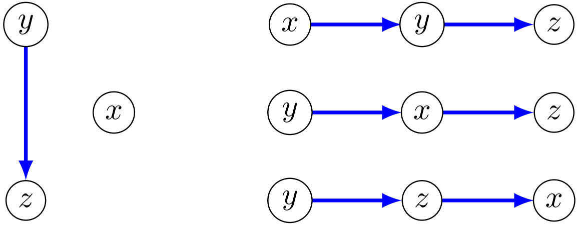
$$\frac{1}{3} \quad \leq \quad \mathrm{P}\big[x \preccurlyeq y\big] \quad \leq \quad \frac{2}{3}.$$

(Brightwell-Felsner-Trotter '95)

"This problem remains one of the most intriguing problems in the combinatorial theory of posets."

Why $\frac{1}{3}$ and $\frac{2}{3}$?

The upper,lower bound are achieved by this poset:



$$\mathrm{P}\left[x \preccurlyeq y\right] = \frac{1}{3}; \qquad \mathrm{P}\left[y \preccurlyeq x\right] = \frac{2}{3}.$$

What is known so far

Theorem (Kahn-Saks '84)

For every finite poset, there always exists x, y*:*

$$\frac{3}{11} \quad \leq \quad \mathrm{P}\big[\, x \preccurlyeq y \,\big] \quad \leq \quad \frac{8}{11},$$

roughly between 0.273 *and* 0.727*.*

Proof is by applying mixed-volume inequalities to order polytopes.

What is known so far

Theorem (Brightwell-Felsner-Trotter '95)

For every finite poset, there always exists x, y:

$$\frac{5-\sqrt{5}}{10} \quad \leq \quad \mathrm{P}\big[\, x \preccurlyeq y \,\big] \quad \leq \quad \frac{5+\sqrt{5}}{10},$$

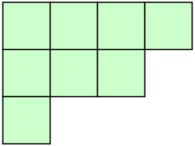
roughly between 0.276 *and* 0.724.

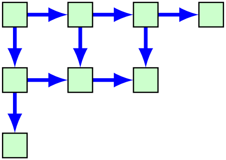
This bound cannot be improved for infinite posets.

Young diagrams

Elements of P_λ are cells of Young diagram of shape λ.

$x \preccurlyeq y \quad$ if y lies to the Southeast of x.





Young diagram of shape $\lambda = (4, 3, 1)$

We write n for number of cells of Young diagram.

Young diagrams

Linear extensions of P_λ correspond to standard Young tableau of the Young diagram.

1	2	5	6
3	4	7	
8			

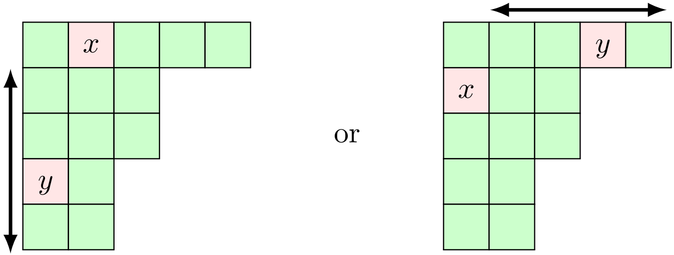
Linear extensions are counted by hook-length formulas.

What is known for Young diagrams

Theorem 1 (Olson–Sagan '18)

For Young diagrams, there always exists x, y:

$$\frac{1}{3} \quad \leq \quad \mathrm{P}\big[x \preccurlyeq y\big] \quad \leq \quad \frac{2}{3}\,.$$



What is known for Young diagrams

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We sketch an alternative proof for Young diagrams using Naruse hook-length formulas.

Hook-length formulas

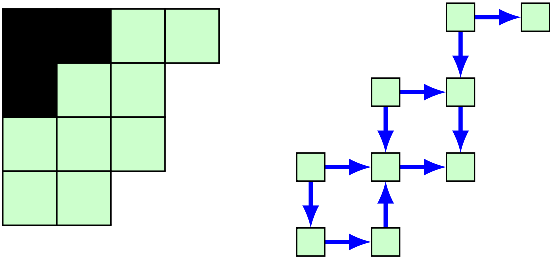
Number of standard Young tableau of shape λ is

$$f^{\lambda} := \frac{n!}{\prod_{x \in \lambda} h_{\lambda}(x)}.$$

7	6	4	1
5	4	2	
4	3	1	
2	1		

$$f^{\lambda} = \frac{12!}{7\,6\,4\,1\,5\,4\,2\,4\,3\,1\,2\,1} = 2970$$

Skew Young diagrams

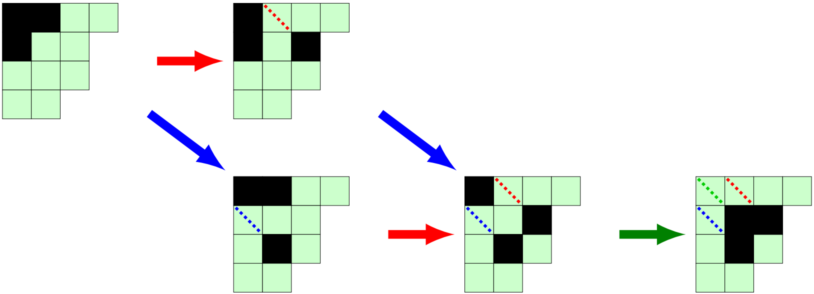


Skew Young diagram of shape λ/μ,
$\lambda = (5, 3, 3, 1)$ and $\mu = (2, 1)$.

We write *n* for number of cells in λ,
and *m* for number of cells in μ.

Excited diagrams

Black boxes can move on SouthEast direction.

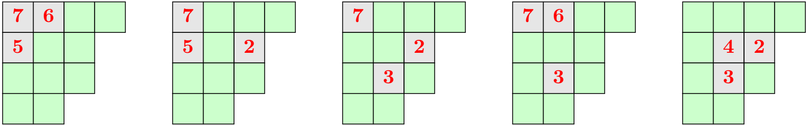


Naruse hook-length formulas

Theorem (Naruse '14, Morales-Pak-Panova '17)
Number of skew Young tableau of shape λ/μ *is*

$$f^{\lambda/\mu} \;:=\; f^{\lambda}\frac{(n-m)!}{n!} \sum_{\substack{\text{excited}\\ \text{diagrams } B}} \; \prod_{\substack{\text{black cells}\\ x\in B}} h_\lambda(x)\,.$$

Naruse hook-length formulas

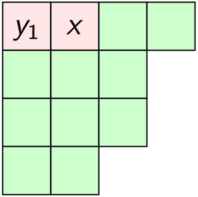


The number of SYT of shape λ/μ is equal to

$$2970\ \frac{9!}{12!}\left(7\cdot 6\cdot 5 + 7\cdot 5\cdot 2 + 7\cdot 2\cdot 3 + 7\cdot 6\cdot 3\ +\ 4\cdot 2\cdot 3\right)$$

$$=\ 1062\,.$$

Proof of Theorem Olson–Sagan

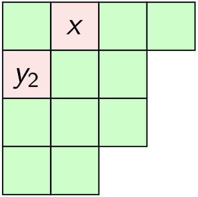


$P\left[\, x \preccurlyeq y_1 \,\right] \;=$

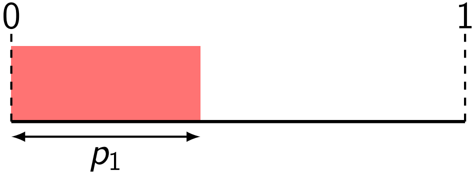
The jump probabilities are

$$p_i \;:=\; P\left[\, y_i \preccurlyeq x \preccurlyeq y_{i+1} \,\right]$$

Proof of Theorem Olson–Sagan



$$\mathrm{P}\big[\, x \preccurlyeq y_2 \,\big] \;=$$



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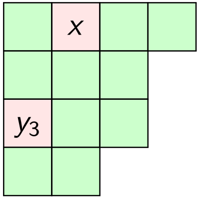
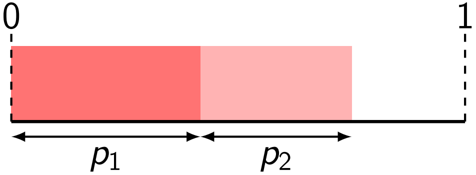


$$\mathrm{P}\left[\, x \preccurlyeq y_3 \,\right] \;=$$



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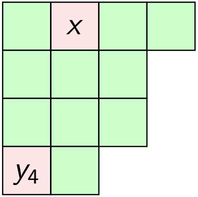
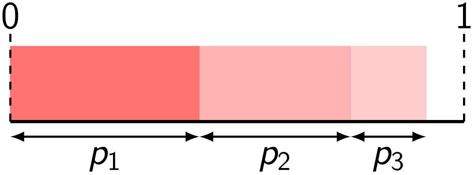


$$\mathrm{P}\big[\, x \preccurlyeq y_4 \,\big] \;=$$



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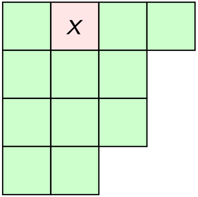
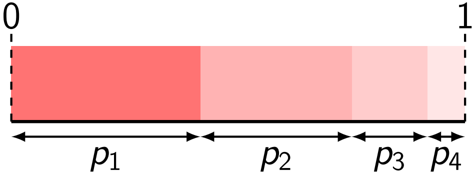
Proof of Theorem Olson–Sagan



$$\mathrm{P}\big[\, x \preccurlyeq y_5 \,\big] \;=$$



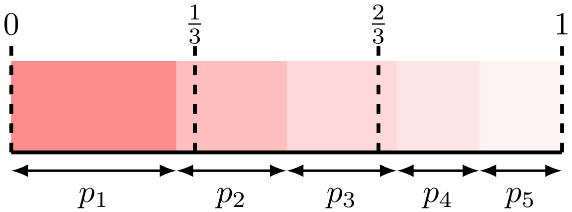
The jump probabilities are

$$p_i \;:=\; \mathrm{P}\big[\, y_i \preccurlyeq x \preccurlyeq y_{i+1} \,\big]$$

Linial-type argument

Supppose that $p_1, p_2, \ldots, p_\ell$ are all $< \frac{1}{3}$.

$\mathbf{P}\left[\, x \preccurlyeq y_i \,\right] \;=$



Look at when the probability exceeds $\frac{1}{3}$. Then

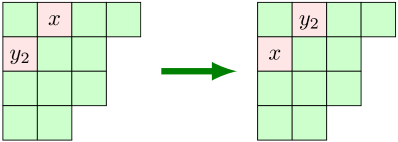
$$\frac{1}{3} \quad \leq \quad \mathsf{P}\left[x \preccurlyeq y_{i+1}\right] \quad \leq \quad \frac{2}{3}\,.$$

$p_1 < \frac{1}{3}$

- If $\frac{1}{3} \le p_1 \le \frac{2}{3}$, then

$$\frac{1}{3} \quad \le \quad p_1 \;=\; \mathrm{P}\big[x \preccurlyeq y_2\big] \quad \le \quad \frac{2}{3}.$$

- If $p_1 > \frac{2}{3}$, then conjugate to get $p_1 < \frac{1}{3}$.

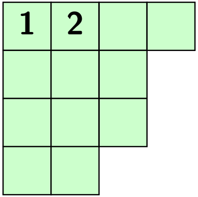


- So we assume $p_1 < \frac{1}{3}$.

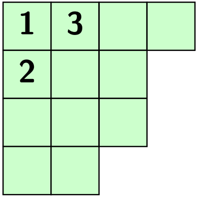
Skew diagrams enter the scene

It suffices to show $p_1 \geq p_2 \geq \ldots \geq p_\ell$.

$$p_1 \;=\; \mathrm{P}\big[\, y_1 \preccurlyeq x \preccurlyeq y_2 \,\big] \;=\; \frac{\#\text{ of SYTs of}}{f^\lambda}$$



$$p_2 \;=\; \mathrm{P}\big[\, y_2 \preccurlyeq x \preccurlyeq y_3 \,\big] \;=\; \frac{\#\text{ of SYTs of}}{f^\lambda}$$



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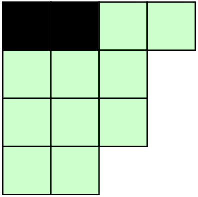

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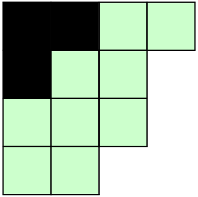
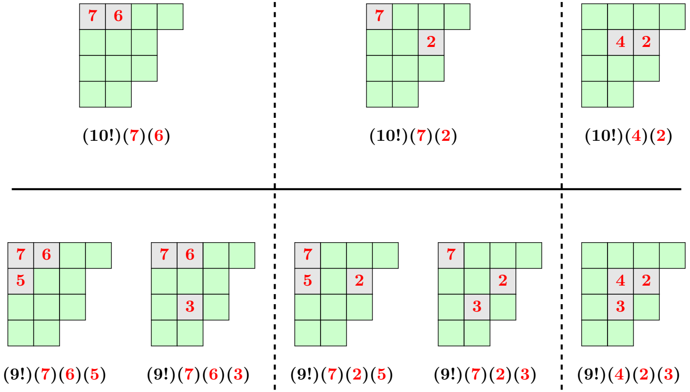


We can now use **NHLF**.

p_1 is greater than p_2



$$p_1 = \frac{\left(10! \cdot 7 \cdot 6 + 10! \cdot 7 \cdot 2 + 10! \cdot 4 \cdot 2\right)}{12!} = \frac{9!}{12!}\; 640\,.$$

$$p_2 = \frac{\left(9! \cdot 7 \cdot 6 \cdot 8 + 9! \cdot 7 \cdot 2 \cdot 8 + 9! \cdot 4 \cdot 2 \cdot 3\right)}{12!} = \frac{9!}{12!}\; 472\,.$$

Thus we complete the proof of this theorem.

Theorem (Olson–Sagan '18)

There always exists x, y:

$$\frac{1}{3} \quad \leq \quad \mathrm{P}\big[\, x \preccurlyeq y \,\big] \quad \leq \quad \frac{2}{3},$$

for poset P_λ of Young diagram of shape λ.

What we will do next

Previously, we want to find x, y:

$$\frac{1}{3} \quad \leq \quad \mathsf{P}\big[x \preccurlyeq y\big] \quad \leq \quad \frac{2}{3},$$

Now, we want to find x, y:

$$\frac{1}{2} - \delta \quad \leq \quad \mathsf{P}\big[x \preccurlyeq y\big] \quad \leq \quad \frac{1}{2} + \delta,$$

Sorting probability

Sorting probability of a poset P is

$$\delta(P) \;:=\; \min_{\text{distinct } x,y} \Big| \mathrm{P}\,[x \prec y] - \mathrm{P}\,[y \prec x] \Big|.$$

In particular, there exists x, y:

$$\frac{1}{2} - \frac{\delta(P)}{2} \quad\leq\quad \mathrm{P}\big[\, x \preccurlyeq y \,\big] \quad\leq\quad \frac{1}{2} + \frac{\delta(P)}{2}.$$

Kahn–Saks Conjecture

Conjecture (Kahn-Saks '84)

For every finite poset,

$$\delta(P) \to 0 \quad \textit{as} \quad \textit{width}(P) \to \infty.$$

Here width(P) is the largest size of anti-chains in P.

Komlós '90 proved such a result for posets with $\Omega(\frac{n}{\log n})$ minimal elements.

Our results

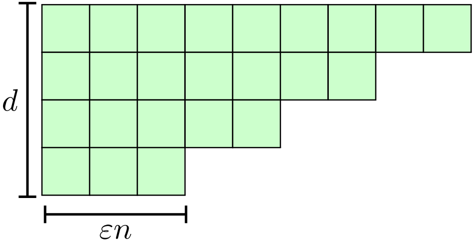
First result

Theorem (C.-Pak-Panova '20+)

Let $\lambda_1 \geq \ldots \geq \lambda_d \geq \varepsilon n$. For poset P_λ of Young diagram of λ,

$$\delta(P_\lambda) \ \leq \ \frac{C}{\sqrt{n}},$$

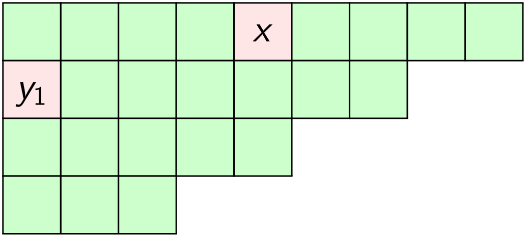
for some $C = C(d, \varepsilon) > 0$.



Where is the improvement?

x is fixed at the middle of the first row,

y varies across the second row.



$$P\left[x \preccurlyeq y_1\right] =$$

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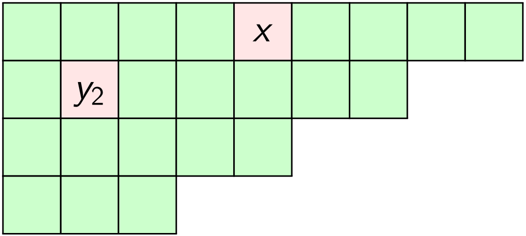
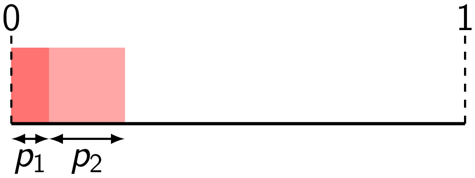


$P\left[x \preccurlyeq y_2\right] =$



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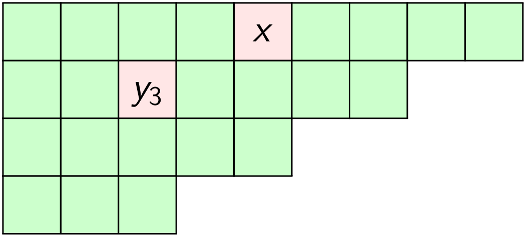
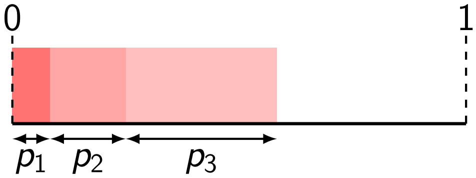


$\mathrm{P}\big[x \preccurlyeq y_3\big] \;=$



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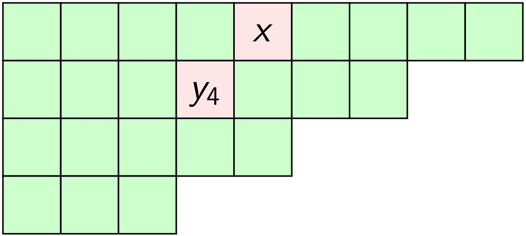
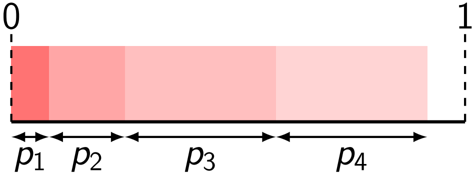


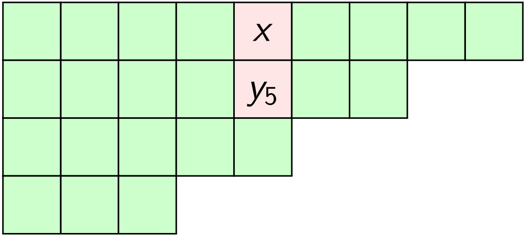
$$\mathrm{P}\left[x \preccurlyeq y_4\right] =$$



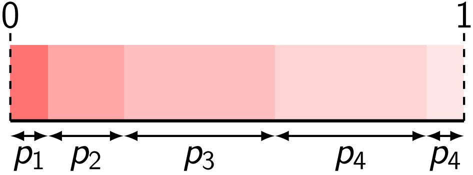
Where is the improvement?

x is fixed at the middle of the first row,

y varies across the second row.



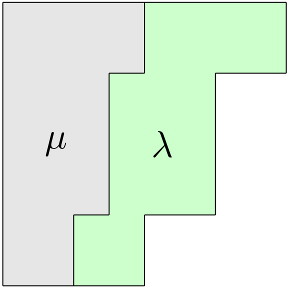
$$\mathrm{P}\left[x \preccurlyeq y_5\right] =$$



Sketch of proof

After reductions using Hoeffding's inequality,

$$\delta(P_\lambda) \;\le\; \sum_{\mu} \frac{\text{SYTs of}}{f^{\lambda}}$$



$$\text{with } \mu \;\approx\; \left(\frac{\lambda_1}{2} \pm \sqrt{n}, \ldots, \frac{\lambda_d}{2} \pm \sqrt{n}\right).$$

Right side is then upper-bounded via NHLF.

Back to first result

Theorem (C.-Pak-Panova '20+)
Let $\lambda_1 \geq \ldots \geq \lambda_d \geq \varepsilon n$. For poset P_λ of Young diagram of λ,

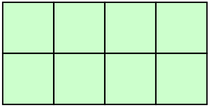
$$\delta(P_\lambda) \ \leq \ \frac{C}{\sqrt{n}},$$

for some $C = C(d, \varepsilon) > 0$.

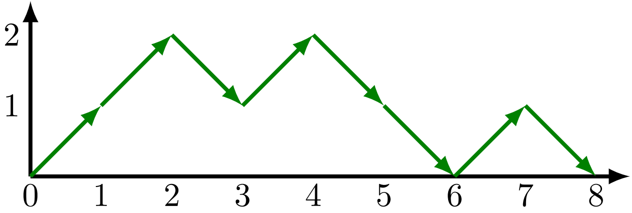
Next: better bound for Catalan posets.

Catalan posets, $\lambda = (\frac{n}{2}, \frac{n}{2})$

Young diagram is rectangle with 2 rows and n cells.



1	2	4	7
3	5	6	8



Second result

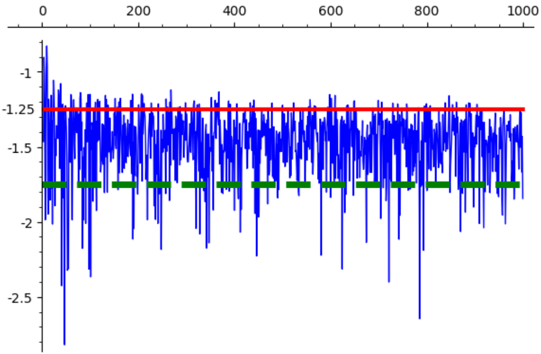
Theorem (C.-Pak-Panova '20+)

For Catalan posets with n cells,

$$\delta(P_\lambda) \;\le\; C n^{-\frac{5}{4}},$$

for some $C > 0$.

How good is this bound?



$$\frac{\log \delta(P_\lambda)}{n}$$

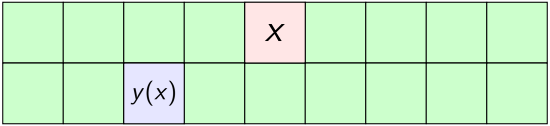
Open Problem

Show that

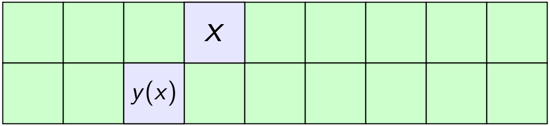
$$\limsup_{n\to\infty} \frac{\log \delta(P_\lambda)}{n} = -\frac{5}{4}; \quad \liminf_{n\to\infty} \frac{\log \delta(P_\lambda)}{n} < -\frac{5}{4}.$$

Where is the improvement?

Before: x is fixed at midpoint, only y is optimized.



Now: Optimize $y = y(x)$ for each x, then optimize x.



For each x, $y(x)$ is the element that minimizes

$$\delta(x, y(x)) \;:=\; \Big| \mathrm{P}\left[x \prec y(x)\right] - \mathrm{P}\left[y(x) \prec x\right] \Big| .$$

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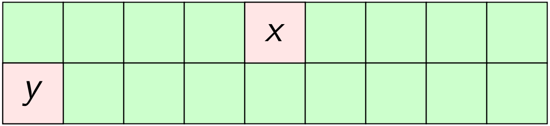

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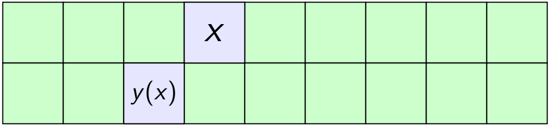

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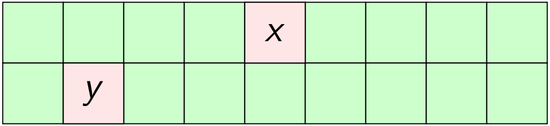

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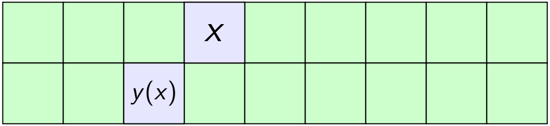

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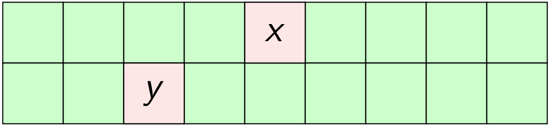

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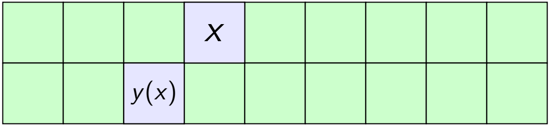

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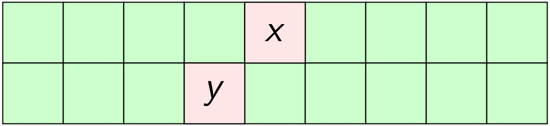

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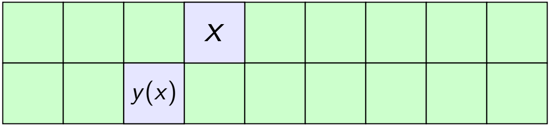

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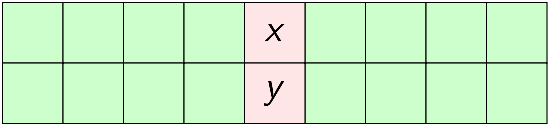

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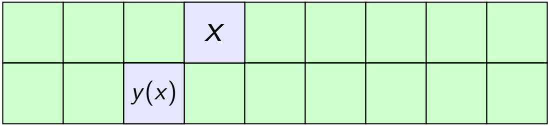

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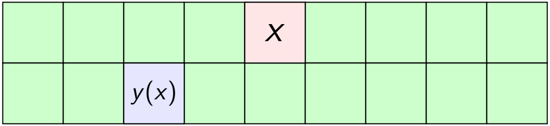

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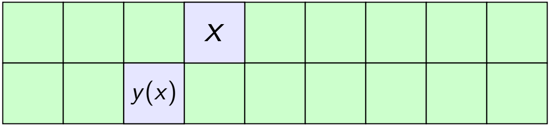

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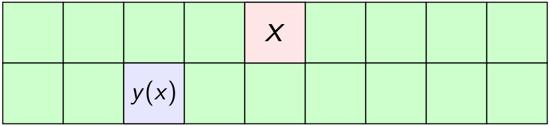

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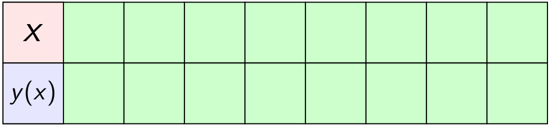

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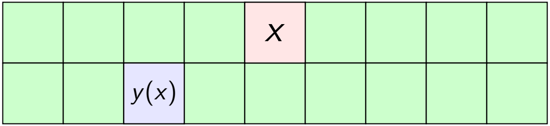

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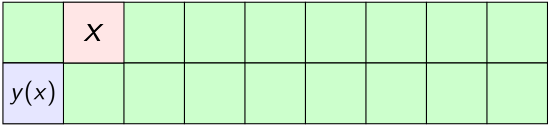

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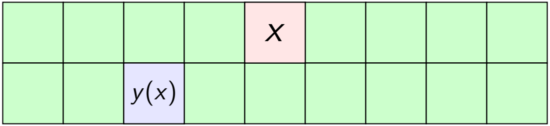

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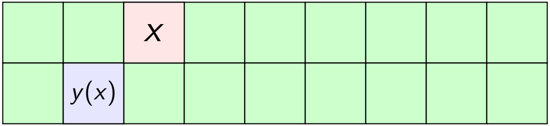

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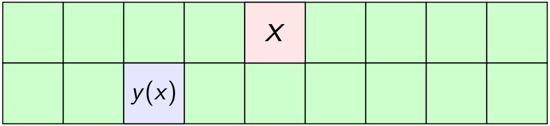

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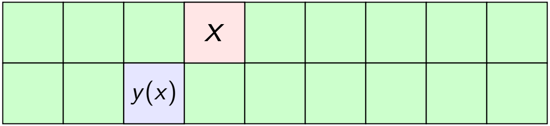

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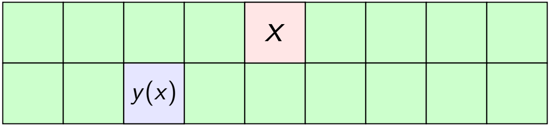

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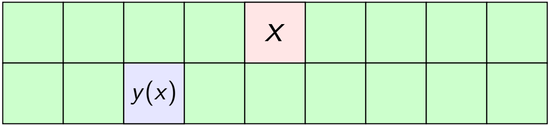

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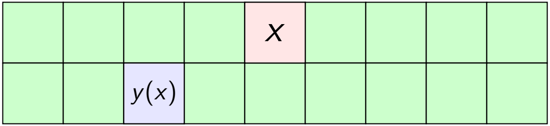

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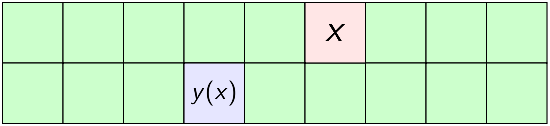

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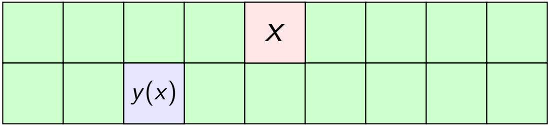

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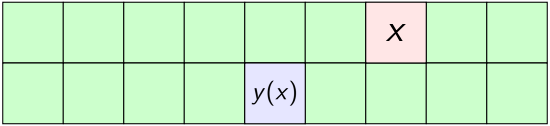

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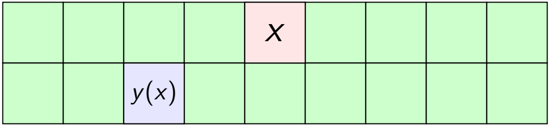

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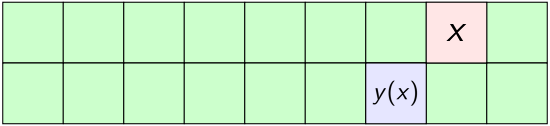

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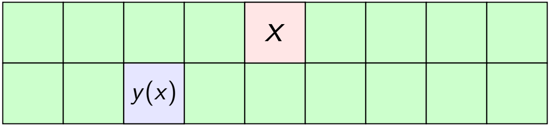

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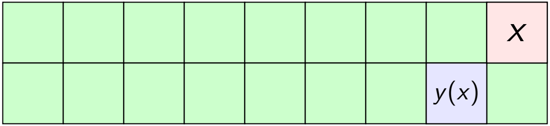

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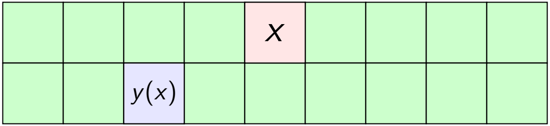

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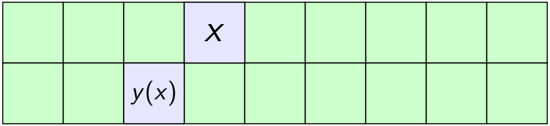

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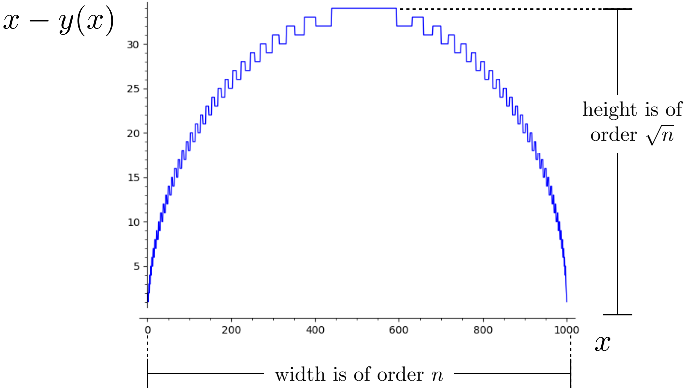
Location of $y(x)$ for $n = 2000$



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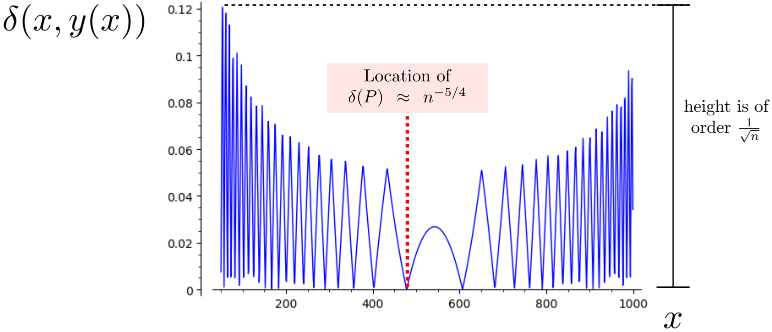
Sorting probability $\delta(P)$ for $n = 2000$



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Back to second result

Theorem (C.-Pak-Panova '20+)

For Catalan posets with n cells,

$$\delta(P_\lambda) \;\leq\; C n^{-\frac{5}{4}},$$

for some $C > 0$.

Important: Estimates are not done by NHLF,
but by direct computation.

Better upper bound for general Young diagrams
remain open.

What is next?

Theorem (C.-Pak-Panova '20+)

Let $\lambda_1 \geq \ldots \geq \lambda_d \geq \varepsilon n$. For poset P_λ of Young diagram of λ, there exists x, y:

$$\delta(P_\lambda) \to 0 \qquad \textit{as} \quad n \to \infty\,.$$

Open Problem

Prove same result for other families of posets, e.g., k-dimensional Young diagrams and periodic posets.

arXiv preprints: 2005.08390 and 2005.13686.

Webpage: http://math.ucla.edu/∼sweehong/

THANK YOU!

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