Equality cases of Alexandrov–Fenchel inequality are not in PH

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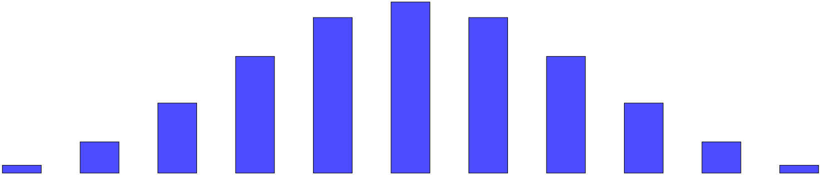
joint with Igor Pak



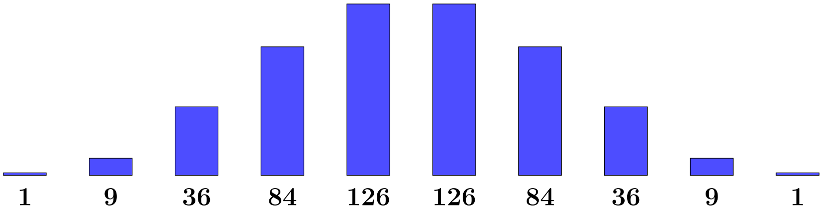
Log-concavity

A sequence $a_1, \ldots, a_n \in \mathbb{R}_{\geq 0}$ is log-concave if

$$a_k^2 \;\geq\; a_{k+1}\, a_{k-1} \qquad (1 < k < n).$$

Log-concavity (and positivity) implies unimodality:

$a_1 \;\leq\; \cdots \;\leq\; a_m \;\geq\; \cdots \;\geq\; a_n$ for some $1 \leq m \leq n$.



Example: binomial coefficients

$$a_k \;=\; \binom{n}{k} \;=\; \frac{n!}{k!\,(n-k)!} \qquad k = 0, 1, \ldots, n.$$

This sequence is log-concave because

$$\frac{a_k^2}{a_{k+1}\,a_{k-1}} \;=\; \frac{\binom{n}{k}^2}{\binom{n}{k+1}\binom{n}{k-1}} \;=\; \left(1+\frac{1}{k}\right)\left(1+\frac{1}{n-k}\right),$$

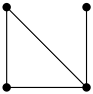
which is greater than 1.

Example: forests of a graph

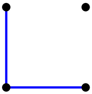
a_k = number of forests with k edges of graph G.

Forest is a subset of edges of G that has no cycles.

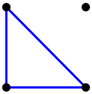
Log-concavity was conjectured for all matroids (Mason '72), and was proved through combinatorial Hodge theory (Huh '15).



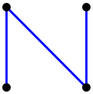
G



forest



not forest



spanning tree

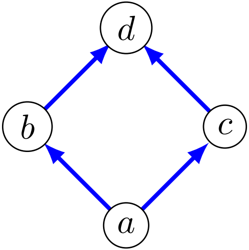
Log-concavity has been observed in different areas of mathematics and proved through different methods.

Today we focus on log-concavity that arises from **Convex Geometry**.

Stanley's poset inequality

Partially ordered sets

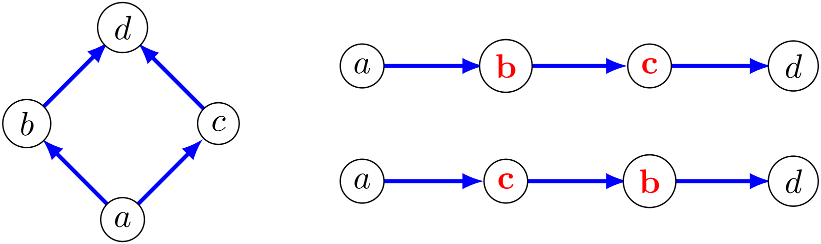
A poset $\mathcal{P}$ is a set X with a partial order $\prec$ on X.



We write $n := |X|$.

Linear extension

A linear extension L is a complete order of $\prec$.



We write $L(x) = k$ if x is k-th smallest in L.

Stanley's (poset) inequality: simple form

Fix $x \in \mathcal{P}$.

$N(k) :=$ number of linear extensions with $L(x) = k$.

Theorem (Stanley '81)

For $k \geq 1$,

$$N(k)^2 \geq N(k+1)\,N(k-1).$$

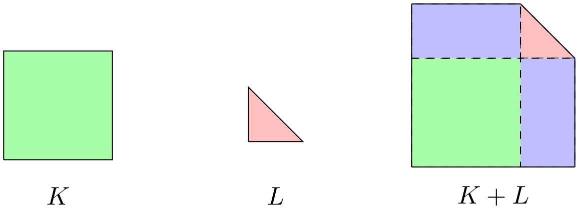
The inequality was initially conjectured by Chung-Fishburn-Graham, and was proved using Aleksandrov-Fenchel inequality for mixed volumes.

Mixed volumes: dimension 2

For convex bodies $K, L \subseteq \mathbb{R}^2$,

$$\mathrm{Vol}(aK+bL) = V(K,K)\,a^2 + V(L,L)\,b^2 + 2V(K,L)\,ab$$

is a quadratic polynomial in $a, b \geq 0$.



Coefficients $V(K,K)$, $V(L,L)$, $V(K,L)$

are mixed volumes.

Mixed volumes: dimension m

Theorem (Minkowski '03)

For convex bodies $C_1, \ldots, C_m \subseteq \mathbb{R}^m$, *the function*

$$(\lambda_1, \ldots \lambda_m) \mapsto \mathrm{Vol}(\lambda_1 C_1 + \ldots + \lambda_m C_m)$$

is a homogeneous polynomial in $\lambda_1, \ldots, \lambda_m \geq 0$.

Mixed volume $V(C_1, \ldots, C_m)$ is $\frac{1}{m!}$ of the coefficient of $\lambda_1 \cdots \lambda_m$ in the polynomial expansion of $\mathrm{Vol}(\lambda_1 C_1 + \ldots + \lambda_m C_m)$.

Alexandrov-Fenchel (AF) inequality

Theorem (Alexandrov '37, Fenchel '36)

For convex bodies $A, B, C_1, \ldots, C_{m-2} \subseteq \mathbb{R}^m$,

$$V^*(A,B)^2 \;\geq\; V^*(A,A)\, V^*(B,B),$$

where $V^*(A,B) \;:=\; V(A,B,C_1,\ldots,C_{m-2})\,.$

Stanley's inequality $N(k)^2 \geq N(k+1)\,N(k-1)$ follows by substituting $A, B, C_1, \ldots, C_{m-2}$ with slices of order polytopes of the poset.

Proof of Stanley's inequality

For poset $\mathcal{P}$, the order polytope is

$$\mathcal{O}(\mathcal{P}) \ := \ \big\{\mathbf{t} \in [0,1]^{\mathcal{P}} \ \mid \ t_y \le t_z \ \text{ if } \ y \prec z \text{ in } \mathcal{P}\big\}.$$

For $x \in \mathcal{P}$, the slices are:

$$K \ := \ \big\{\mathbf{t} \in \mathcal{O}(\mathcal{P}) \ \mid \ t_x = 0\big\},$$
$$L \ := \ \big\{\mathbf{t} \in \mathcal{O}(\mathcal{P}) \ \mid \ t_x = 1\big\}.$$

Let $n := |\mathcal{P}|$, and set

$$A \ := \ K, \quad B \ := \ L,$$
$$C_1, \ldots, C_{n-3} \ := \ \underbrace{K, \ldots, K}_{n-k-1}, \underbrace{L, \ldots, L}_{k-2}.$$

Proof of Stanley's inequality

Then

$$V^*\big(A,B\big) = \tfrac{1}{(n-1)!}\,N(k), \quad V^*\big(A,A\big) = \tfrac{1}{(n-1)!}\,N(k-1),$$

$$V^*\big(B,B\big) = \tfrac{1}{(n-1)!}\,N(k+1).$$

Thus

$$V^*(A,B)^2 \ \geq\ V^*(A,A)\,V^*(B,B) \qquad \text{(AF)}$$

implies

$$N(k)^2 \ \geq\ N(k+1)\,N(k-1) \qquad \text{(Stanley)}.$$

Stanley's (poset) inequality: true form

Fix $d \geq 0$, $x, y_1, \ldots, y_d \in \mathcal{P}$ and $\ell_1, \ldots, \ell_d \in \mathbb{N}$.

$N_d(k) :=$ number of linear extensions with $L(x) = k, \quad L(y_i) = \ell_i$ for $i \in [d]$.

Theorem (Stanley '81)

For $k \geq 1$,

$$N_d(k)^2 \geq N_d(k+1)\, N_d(k-1).$$

But an elementary/combinatorial proof of Stanley's inequality remains elusive.

Equality conditions

When is equality achieved?

Question (Alexandrov '37)

When does AF inequality achieve equality?

Quote (Alexandrov '37)

"Serious difficulties occur in determining the conditions for equality to hold in AF inequality."

When is equality achieved?

Quote (Gardner '02)

If inequalities are silver currency in mathematics, those that come along with precise equality conditions are gold.

Example (Isoperimetric problem)

Among all closed curves in the plane of fixed perimeter, the curve that maximizes the area of its enclosed region is the circle.

(Informal) answer for convex polytopes

Theorem (Shenfeld-van Handel '23)

Let $A, B, C_1, \ldots, C_{m-2}$ *be convex polytopes. Then*

$$V^*(A,B)^2 = V^*(A,A)\,V^*(B,B)$$

arises from a combination of three mechanisms:

- *Translation and scaling;*
- *Relative positions of normal cones of boundaries of* $C_1, \ldots, C_{m-2}$*;*
- *Relative positions of affine hulls of* $C_1, \ldots, C_{m-2}$*.*

The key takeaway is the equality condition of AF inequality is extremely complicated.

The key takeaway is the equality condition of AF inequality is extremely complicated.

We can in fact prove this statement rigorously by using **Complexity Theory**.

Complexity theory perspective

Consider the decision problem:

Input: unimodular polytopes $A, B, C_1, \ldots, C_{m-2}$;

Output: - YES if $V^*(A, B)^2 = V^*(A, A)\, V^*(B, B)$;

- NO if $V^*(A, B)^2 > V^*(A, A)\, V^*(B, B)$.

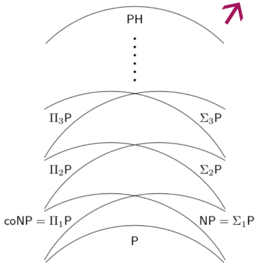
Theorem (C.–Pak '24)

This decision problem cannot be solved in polynomial time, unless NP = coNP.

A stronger result

Theorem (C.–Pak '24)

This decision problem does not belong to the polynomial hierarchy (PH), *unless* PH *collapses*.



Thus the geometric description of AF equality must be computationally intractable.

Back to posets

Consider the decision problem:

Input: poset $\mathcal{P}$, $x, y_1, \ldots, y_d \in \mathcal{P}$, $\ell_1, \ldots, \ell_d \in \mathbb{N}$.

Output: - YES if $N_d(k)^2 = N_d(k+1)\, N_d(k-1)$;

- NO if $N_d(k)^2 > N_d(k+1)\, N_d(k-1)$.

Theorem (C.–Pak '24)

This decision problem does not belong to the polynomial hierarchy (PH), unless PH collapses.

Application of main result

An injection $f : A \to B$ is combinatorial if both f and f^{-1} are poly-time computable.

Corollary (C.–Pak '24)

There is no combinatorial injective proof for Stanley's inequality

$$N_d(k)^2 \;\geq\; N_d(k+1)\,N_d(k-1)$$

unless PH *collapses*.

THANK YOU!

Preprint: www.arxiv.org/abs/2309.05764

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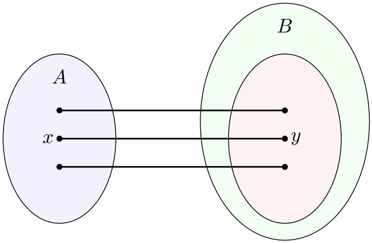
Combinatorial injection

An injection $f : A \to B$ is combinatorial if

- Given $x \in A$, the image $f(x)$ is computable in poly($|x|$) steps;
- Given $y \in B$, it takes poly($|y|$) steps to decide if y is in image of f; and if so, the pre-image $f^{-1}(y)$ is computable in poly($|y|$) steps.



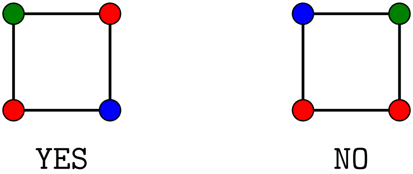
Complexity class P

$$\text{P} := \left\{ \begin{array}{l} \text{Decision problems solvable by deterministic} \\ \text{Turing machine in polynomial time} \end{array} \right\}$$

Example

Check if a given 3-coloring of a graph G is proper.

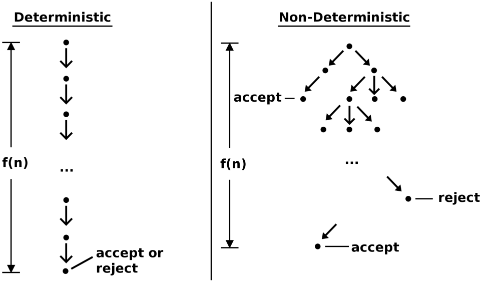
This can be solved in $O(n^2)$ time by checking the color of endpoints of every edge.



Complexity class NP

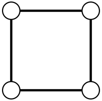
$$\text{NP} := \left\{ \begin{array}{l} \text{Decision problems solvable by nondetermi-} \\ \text{nistic Turing machine in polynomial time} \end{array} \right\}.$$

- Can split into many parallel branches;
- Output 'YES' if one of the branches said 'YES';
- Output 'NO' if all branches said 'NO'.

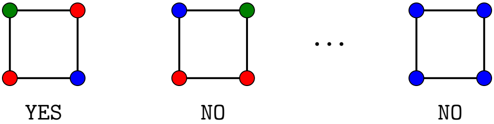


Complexity class NP: example

Problem: Check if graph G has a proper 3-coloring.



Each branch corresponds to checking if a particular 3-coloring of G is proper.

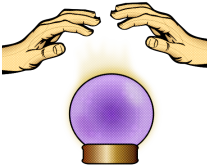


Output to this example is ‘YES’.

Turing machine with an oracle

At each step, this machine can either:

- Perform usual nondeterministic Turing machine operation; or
- Ask an oracle that is able to answer any instance of a given computational problem.

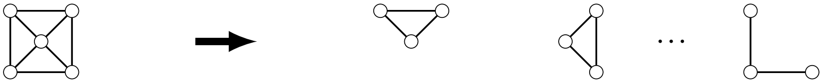


Turing machine with an oracle: example

Problem: Check if there is an induced subgraph of G of size $\lceil n/2 \rceil$ that is not 3-colorable.

Oracle: Can check if a graph is 3-colorable.

Each branch of the machine corresponds to an induced subgraph of G of size $\lceil n/2 \rceil$.



For every branch, oracle checks if subgraph is 3-colorable.

Complexity class Σ_i^{P}

The first two classes are

$$\Sigma_0^{\mathrm{P}} := \mathrm{P}; \qquad \Sigma_1^{\mathrm{P}} := \mathrm{NP}.$$

For $i \geq 1$, the class $\Sigma_i^{\mathrm{P}} := \mathrm{NP}^{\Sigma_{i-1}^{\mathrm{P}}}$ is

$$\left\{ \begin{array}{l} \text{Decision problems solvable by nondetermi-} \\ \text{nistic Turing machine in polynomial time} \\ \text{with an oracle for problem from } \Sigma_{i-1}^{\mathrm{P}}. \end{array} \right\}.$$

Note that

$$\Sigma_0^{\mathrm{P}} \subseteq \Sigma_1^{\mathrm{P}} \subseteq \Sigma_2^{\mathrm{P}} \subseteq \Sigma_3^{\mathrm{P}} \subseteq \cdots$$

Polynomial hierarchy (PH)

Polynomial hierarchy is the union of all Σ_i^{P}'s,

$$\mathsf{PH} := \bigcup_{i=0}^{\infty} \Sigma_i^{\mathsf{P}}.$$

Conjecture

Polynomial hierarchy does not collapse,

$$\Sigma_0^{\mathsf{P}} \subsetneq \Sigma_1^{\mathsf{P}} \subsetneq \Sigma_2^{\mathsf{P}} \subsetneq \Sigma_3^{\mathsf{P}} \subsetneq \cdots$$

- $\Sigma_0^{\mathsf{P}} = \Sigma_1^{\mathsf{P}}$ is equivalent to $\mathsf{P} = \mathsf{NP}$.
- $\Sigma_1^{\mathsf{P}} = \Sigma_2^{\mathsf{P}}$ is equivalent to $\mathsf{NP} = \mathsf{coNP}$.