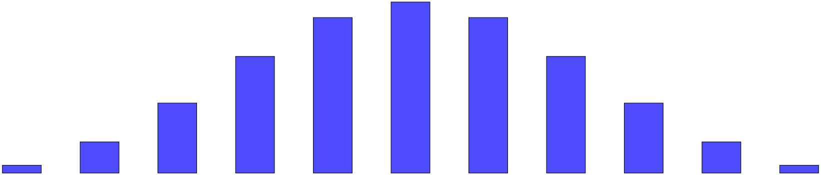
Equality cases of Alexandrov–Fenchel inequality are not in PH

Swee Hong Chan

joint with Igor Pak



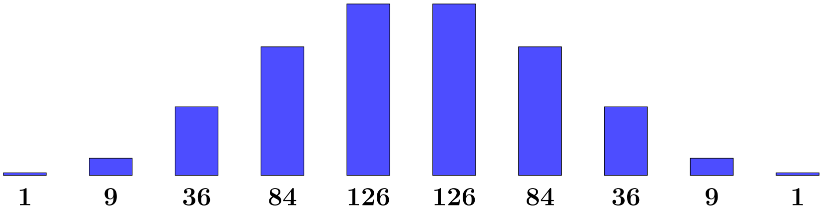
Log-concavity

A sequence $a_1, \ldots, a_n \in \mathbb{R}_{\geq 0}$ is log-concave if

$$a_k^2 \ \geq \ a_{k+1}\, a_{k-1} \qquad (1 < k < n).$$

Log-concavity (and positivity) implies unimodality:

$$a_1 \ \leq \ \cdots \ \leq \ a_m \ \geq \ \cdots \ \geq \ a_n \quad \text{for some } 1 \leq m \leq n.$$



Example: binomial coefficients

$$a_k \;=\; \binom{n}{k} \;=\; \frac{n!}{k!\,(n-k)!} \qquad k = 0, 1, \ldots, n.$$

This sequence is log-concave because

$$\frac{a_k^2}{a_{k+1}\,a_{k-1}} \;=\; \frac{\binom{n}{k}^2}{\binom{n}{k+1}\binom{n}{k-1}} \;=\; \left(1+\frac{1}{k}\right)\left(1+\frac{1}{n-k}\right),$$

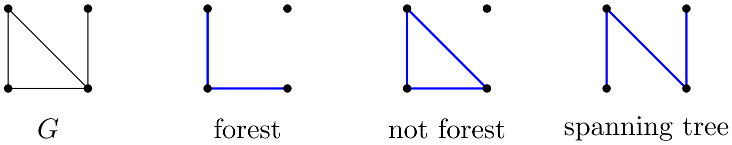
which is greater than 1.

Example: forests of a graph

a_k = number of forests with k edges of graph G.

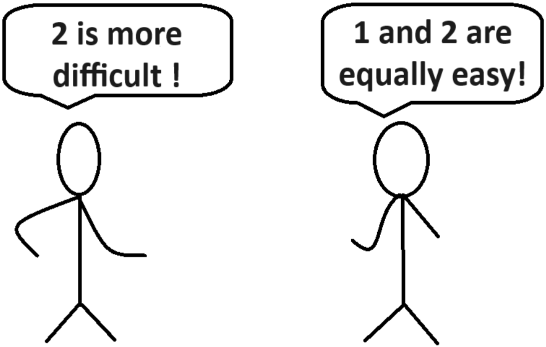
Forest is a subset of edges of G that has no cycles.

Log-concavity was conjectured for all matroids (Mason '72), and was proved through combinatorial Hodge theory (Huh '15).



Motivation

Which log-concave inequality is more "difficult"?



Our goal

We aim to differentiate simple log-concave inequalities from complex log-concave inequalities using **Complexity Theory**.

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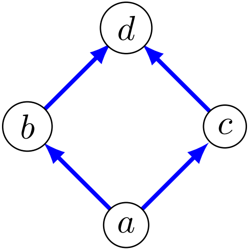
Today we focus on log-concave **poset inequalities**.

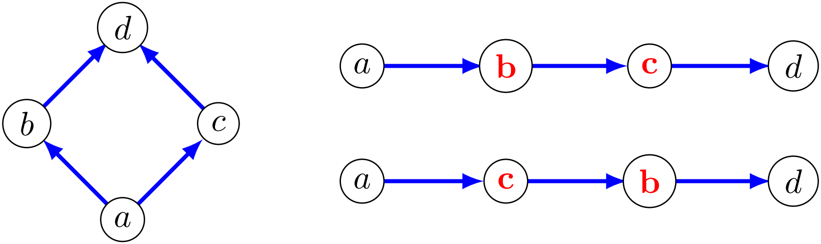
Partially ordered sets

A poset $\mathcal{P}$ is a set X with a partial order $\prec$ on X.



Linear extension

A linear extension L is a complete order of $\prec$.



We write $L(x) = k$ if x is k-th smallest in L.

Stanley (poset) inequality: simple form

Fix $x \in \mathcal{P}$.

$N(k) :=$ number of linear extensions with $L(x) = k$.

Theorem (Stanley '81)

$$N(k)^2 \geq N(k+1)\, N(k-1) \qquad (k \in \mathbb{N}).$$

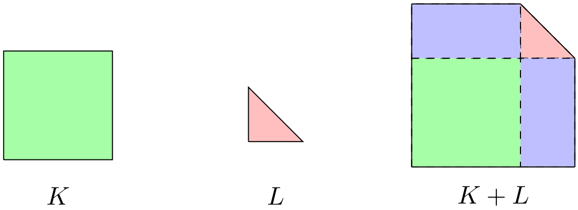
The inequality was initially conjectured by Chung-Fishburn-Graham, and was proved using Aleksandrov-Fenchel inequality for mixed volumes.

Mixed volumes: dimension 2

For convex bodies $K, L \subseteq \mathbb{R}^2$,

$$\mathrm{Vol}(aK+bL) = V(K,K)\,a^2 + V(L,L)\,b^2 + 2V(K,L)\,ab$$

is a quadratic polynomial in $a, b \geq 0$.



Coefficients $V(K,K)$, $V(L,L)$, $V(K,L)$

are mixed volumes.

Mixed volumes: dimension n

Theorem (Minkowski '03)

For convex bodies $K_1, \ldots, K_n \subseteq \mathbb{R}^n$, the function

$$(\lambda_1, \ldots \lambda_n) \mapsto \mathrm{Vol}(\lambda_1 K_1 + \ldots + \lambda_n K_n)$$

is a homogeneous polynomial in $\lambda_1, \ldots, \lambda_n \geq 0$.

Mixed volume $V(K_1, \ldots, K_n)$ is $\frac{1}{n!}$ of the coefficient of $\lambda_1 \cdots \lambda_n$ in the polynomial expansion of $\mathrm{Vol}(\lambda_1 K_1 + \ldots + \lambda_n K_n)$.

Alexandrov-Fenchel (AF) inequality

Theorem (Alexandrov '37, Fenchel '36)

For convex bodies $A, B, K_1, \ldots, K_{n-2} \subseteq \mathbb{R}^n$,

$$V^*(A,B)^2 \ \geq \ V^*(A,A)\, V^*(B,B),$$

where $V^*(A,B) \ := \ V(A,B,K_1,\ldots,K_{n-2})\,.$

Stanley inequality $N(k)^2 \geq N(k+1)\,N(k-1)$ follows by substituting $A, B, K_1, \ldots, K_{n-2}$ with slices of order polytopes of the poset.

Stanley (poset) inequality: true form

Fix $d \geq 0$, $x, y_1, \ldots, y_d \in \mathcal{P}$ and $\ell_1, \ldots, \ell_d \in \mathbb{N}$.

$$N_d(k) := \begin{array}{l}\text{number of linear extensions with} \\ L(x) = k, \quad L(y_i) = \ell_i \quad \text{for } i \in [d].\end{array}$$

Theorem (Stanley '81)

$$N_d(k)^2 \geq N_d(k+1)\, N_d(k-1) \qquad (k \in \mathbb{N}).$$

This form corresponds to imposing boundary conditions in PDE/statistical physics.

When is equality achieved?

Question (Stanley '81)

Find equality condition for [Stanley inequality].

Quote (Gardner '02)

If inequalities are silver currency in mathematics, those that come along with precise equality conditions are gold.

Equality condition: $d = 0$

Theorem (Shenfeld-van Handel '23)

Suppose $d = 0$ *and* $N_d(k) > 0$. *Then*

$$N_d(k)^2 = N_d(k+1)\, N_d(k-1)$$

if and only if

$$\begin{aligned} |\mathcal{P}_{<z}| &> k && \text{for all } z \in \mathcal{P}_{>x}, \\ |\mathcal{P}_{>z}| &> |P| - k + 1 && \text{for all } z \in \mathcal{P}_{<x}, \end{aligned}$$

where $\mathcal{P}_{<z} :=$ *set of* $y \in \mathcal{P}$ *with* $y < z$.

This is a combinatorial condition, and can be checked in $O(|\mathcal{P}|^2)$ steps.

Main result

Consider the decision problem for
checking equality in Stanley inequality:

$$N_d(k)^2 \overset{?}{=} N_d(k+1)\, N_d(k-1).$$

Theorem 1 (C.–Pak '23+)

- $d \leq 1$: *combinatorial equality condition that is checkable in* $\text{poly}(|\mathcal{P}|)$ *steps.*
- $d \geq 2$: *not part of* **polynomial hierarchy**, *unless polynomial hierarchy collapses.*

Alexandrov–Fenchel equality condition

Consider the decision problem for checking equality in AF inequality for unimodular polytopes:

$$V^*(A,B)^2 \stackrel{?}{=} V^*(A,A)\,V^*(B,B).$$

Theorem 2 (C.–Pak '23+)

Problem is not in **PH**, *unless* PH *collapses.*

Shenfeld–van Handel ('23) obtained complete geometric description of AF equality, but those conditions are computationally intractable.

Recall our goal ...

We aim to differentiate **simple** log-concave inequalities from **complex** log-concave inequalities using Complexity Theory.

Consequence of main result

Theorem 3 (C.–Pak '23+)

For $d \geq 2$, the counting problem of determining

$$N_d(k)^2 \; - \; N_d(k+1)\, N_d(k-1)$$

is not in $\#P$, unless PH *collapses.*

Note: $N_d(k)^2$ and $N_d(k+1)\, N_d(k-1)$ are in $\#P$.

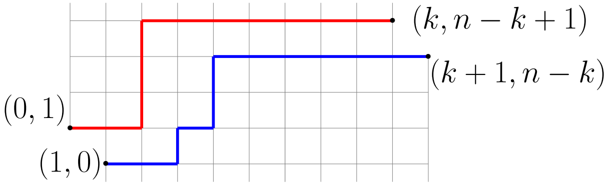
Comparison: binomial inequalities

For $1 < k < n$, the counting problem

$$\binom{n}{k}^2 - \binom{n}{k+1}\binom{n}{k-1}$$

can be determined in poly(n), and thus in $\#P$.

Combinatorial interpretation: number of pairs of non-intersecting north-east lattice paths.



Back to our goal

We compare two log-concave inequalities:

Binomial inequality: in $\#P$.

Stanley inequality: not in $\#P$, unless PH collapses.

This differentiates Stanley inequality from binomial inequality and many other combinatorial inequalities.

THANK YOU!

Preprint: www.arxiv.org/abs/2309.05764

Webpage: www.math.rutgers.edu/~sc2518/

Email: sweehong.chan@rutgers.edu

Example: defect of Sidorenko inequality

For a permutation $\sigma \in S_n$, let

$$\mathrm{LI}(\sigma) \; := \; \text{number of } \pi \in S_n \text{ such that } \pi \; \trianglelefteq \; \sigma,$$

where $\trianglelefteq$ is the weak Bruhat order on S_n.

Theorem (Sidorenko '91)

$$\mathrm{LI}(\sigma)\,\mathrm{LI}(\overline{\sigma}) \; \geq \; n!, \qquad (\sigma \in S_n),$$

where $\overline{\sigma}$ is the reverse of σ.

Proved by max-flow min-cut argument. Defect was shown to be in $\#P$ by C.–Pak–Panova '23.