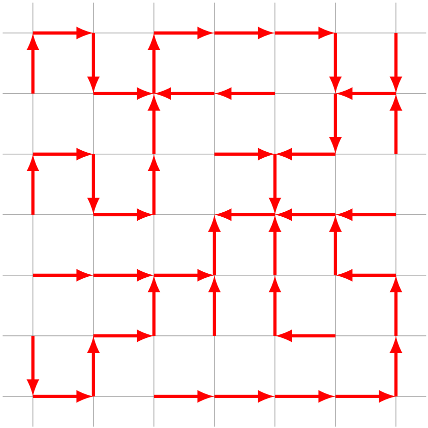
In between random walk and rotor walk

Swee Hong Chan
Cornell University
Joint work with Lila Greco, Lionel Levine, Boyao Li

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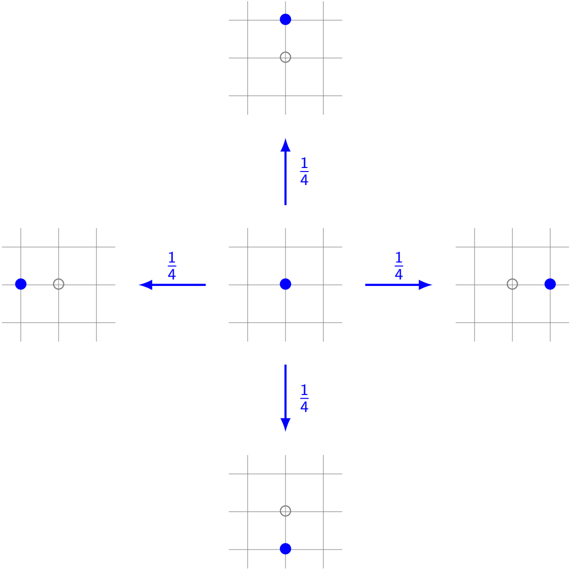
Random
walk

Rotor
walk

Simple random walk on $\mathbb{Z}^2$

Simple random walk on $\mathbb{Z}^2$



Simple random walk on $\mathbb{Z}^2$

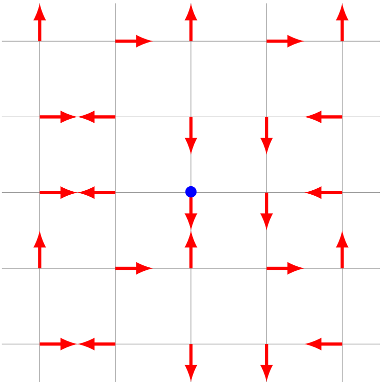
- Visits every site infinitely often? Yes!
- Scaling limit? The standard 2-D Brownian motion:

$$\Big(\frac{1}{\sqrt{n}}\underbrace{X_{[nt]}}_{\substack{\text{location of the}\\ \text{walker at time } [nt]}}\Big)_{t\geq 0} \stackrel{n\to\infty}{\Longrightarrow} \frac{1}{\sqrt{2}}\big(\underbrace{B_1(t), B_2(t)}_{\substack{\text{independent standard}\\ \text{Brownian motions}}}\big)_{t\geq 0}.$$

Rotor walk on $\mathbb{Z}^2$

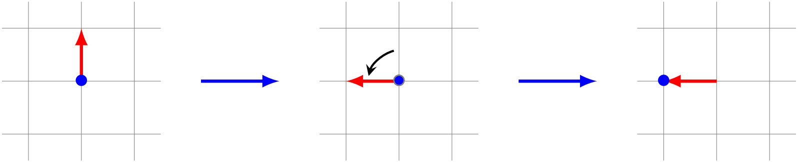
Rotor walk on $\mathbb{Z}^2$

Put a signpost at each site.



Rotor walk on $\mathbb{Z}^2$

Turn the signpost 90° counterclockwise, then follow the signpost.



The signpost says:
"This is the way you went the last time you were here",
(assuming you ever were!)

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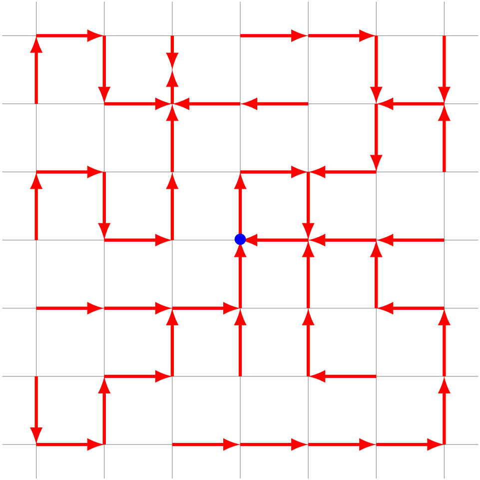

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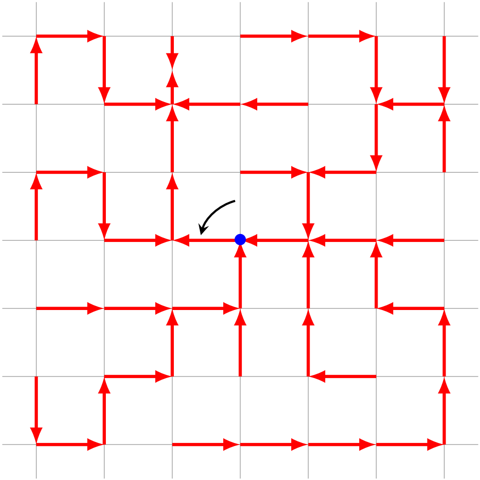

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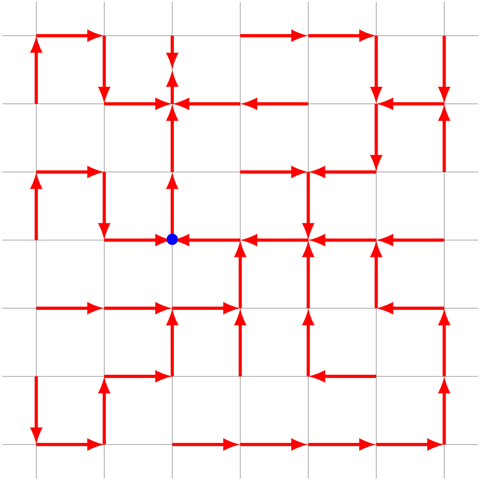

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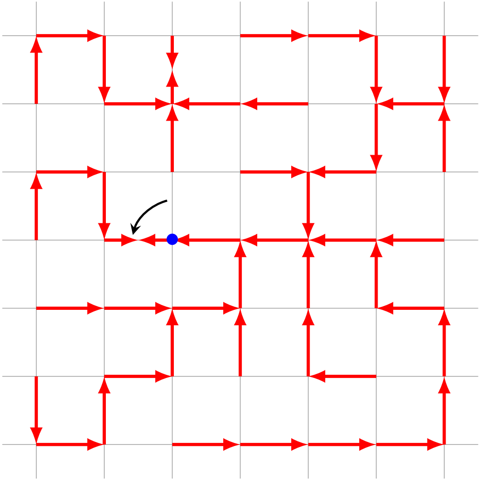

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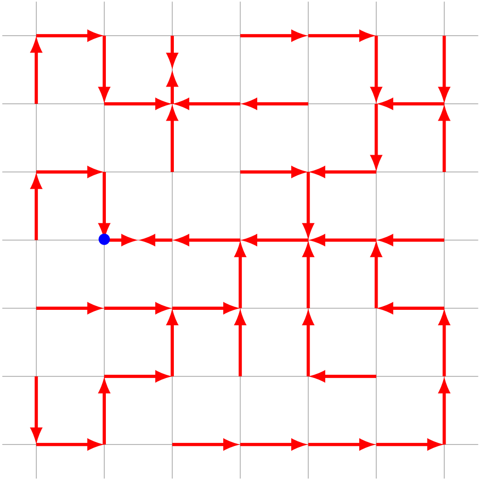

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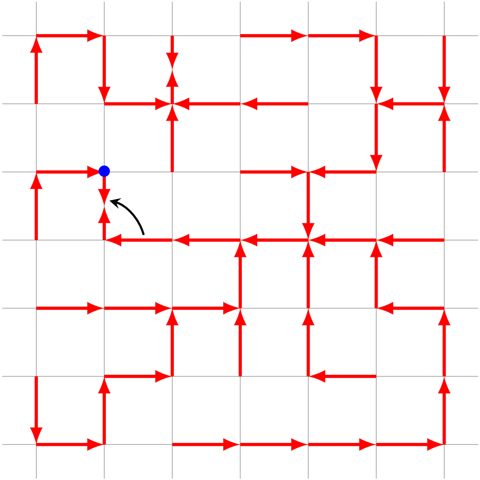

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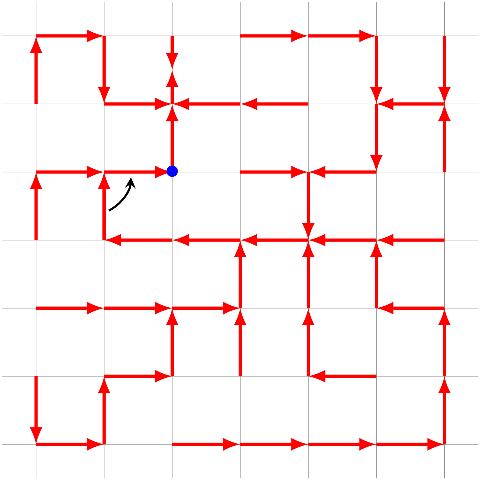

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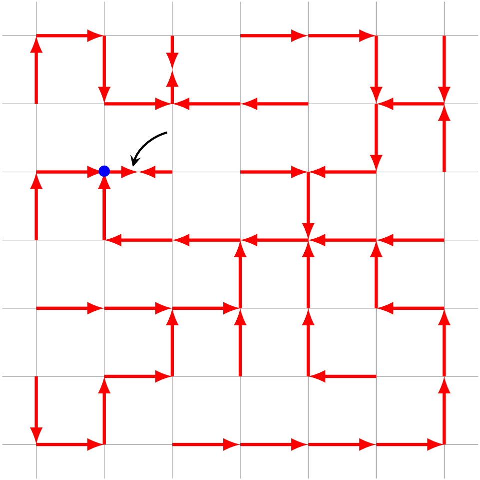

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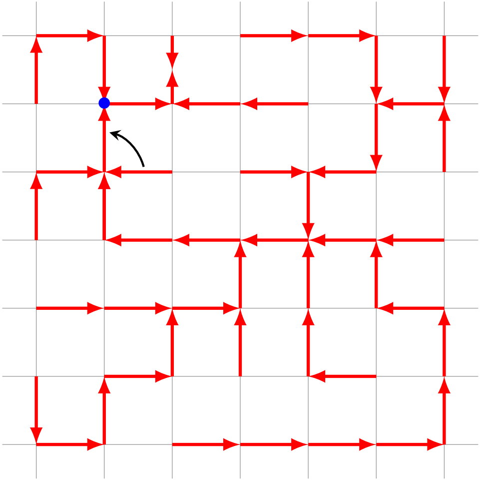

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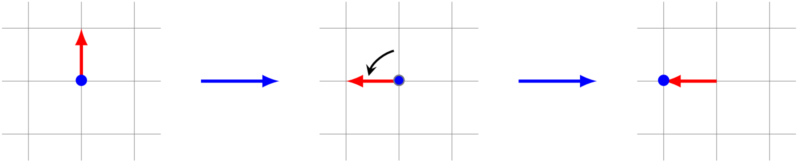

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Conjectures for rotor walk on $\mathbb{Z}^2$

If the initial signposts are i.i.d. uniform among the four directions, then

- (PDDK '96) Visits every site infinitely often?
- (PDDK '96) $\#\{X_1, \ldots, X_n\}$ is $\asymp n^{2/3}$?
  (compare with $n/\log n$ for the simple random walk.)
- (Kapri-Dhar '09) The asymptotic shape of $\{X_1, \ldots, X_n\}$ is a disc?

More randomness please!

Well studied

Random

Many open problems

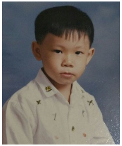
Deterministic

More randomness please!

Well studied

Random

Let's study this!!!

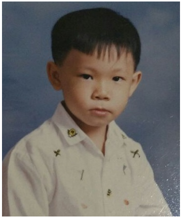


Something in between

Many open problems

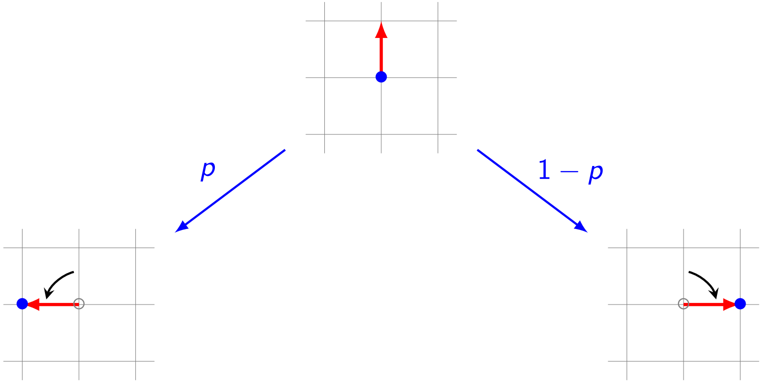
Deterministic

p-rotor walk on $\mathbb{Z}^2$



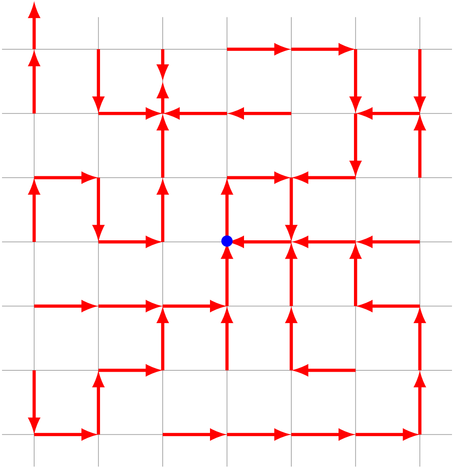
p-rotor walk on $\mathbb{Z}^2$

With probability p, turn the signpost 90° counter-clockwise.
With probability $1-p$, turn the signpost 90° clockwise.



p-rotor walk on $\mathbb{Z}^2$

Follow rotor walk rule with prob. p, do the opposite with prob. $1 - p$



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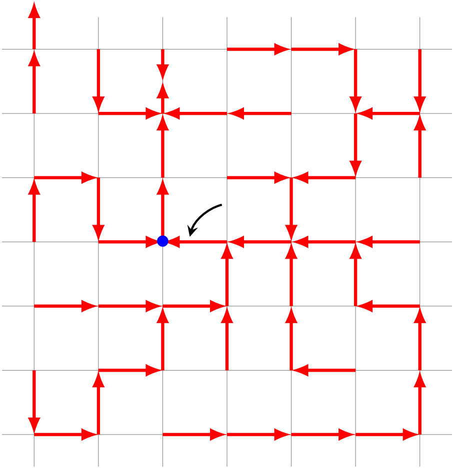

Follow the rule.

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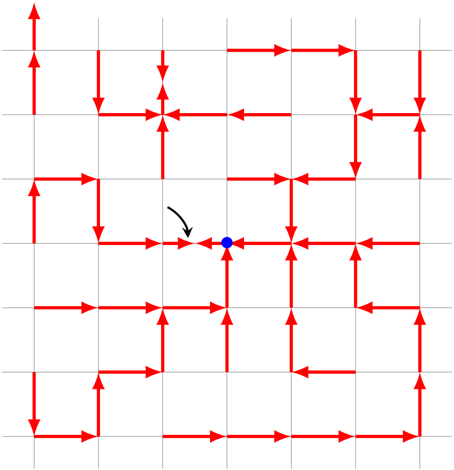

Do the opposite.

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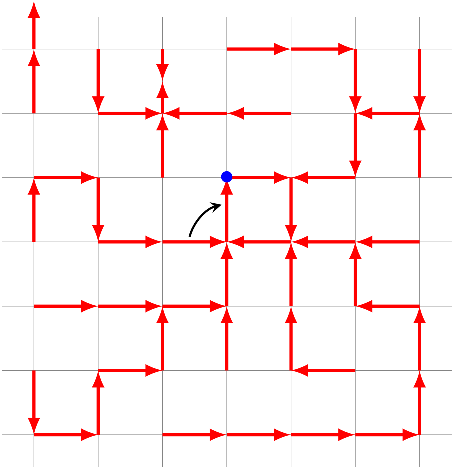
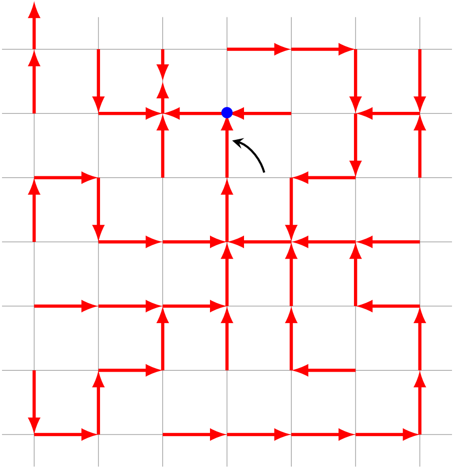


Do the opposite again.

p-rotor walk on $\mathbb{Z}^2$

Follow rotor walk rule with prob. p, do the opposite with prob. $1-p$



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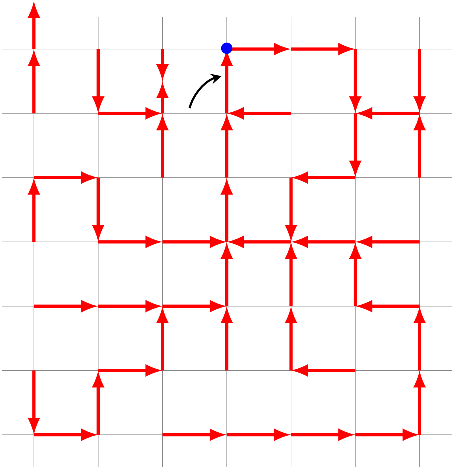

Do the opposite.

p-rotor walk on $\mathbb{Z}^2$

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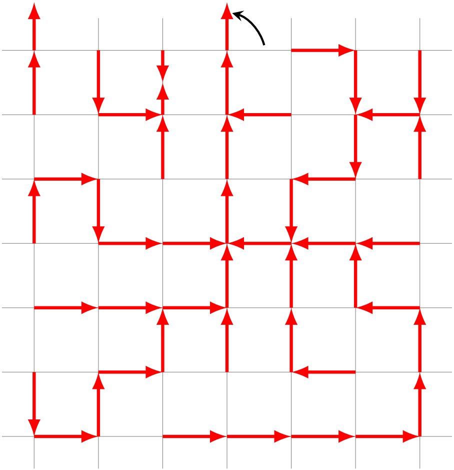

Ops...

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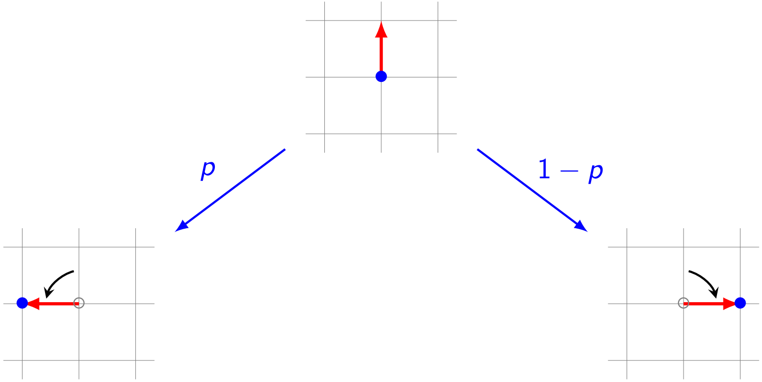
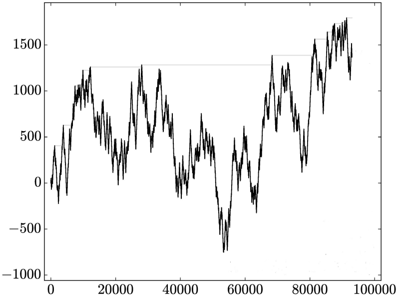


Recover the rotor walk if $p = 1$.

Scaling limit for p-rotor walk on $\mathbb{Z}$

(Huss, Levine, Sava-Huss 18) The scaling limit for p-rotor walk on $\mathbb{Z}$ is a perturbed Brownian motion $(Y(t))_{t\geq 0}$,

$$Y(t) = \underbrace{B(t)}_{\substack{\text{standard}\\ \text{Brownian}\\ \text{motion}}} + \underbrace{a \sup_{0\leq s\leq t} Y(s)}_{\substack{\text{perturbation at}\\ \text{maximum}}} + \underbrace{b \inf_{0\leq s\leq t} Y(s)}_{\substack{\text{perturbation at}\\ \text{minimum}}}, \qquad t \geq 0.$$



$Y(t)$ for $a = -0.998$, and $b = 0$ (by Wilfried Huss).

Scaling limit for p-rotor walk on $\mathbb{Z}^2$

Question: Is the scaling limit for p-rotor walk on $\mathbb{Z}^2$ is a "2-D perturbed Brownian motion"?

Problem: How to define "2-D perturbed Brownian motion"?.

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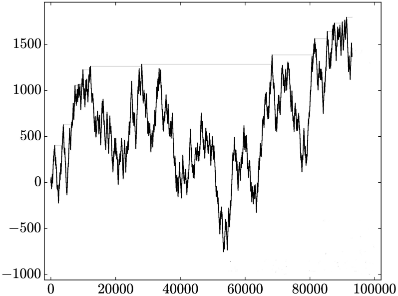

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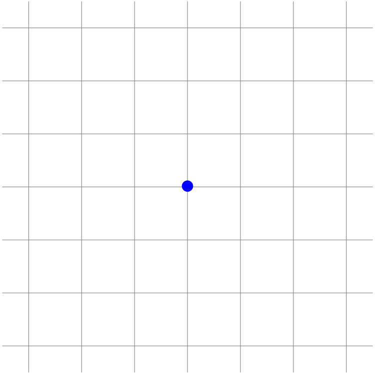
Problem: How to define "2-D perturbed Brownian motion"?.

Compromise: Work with the unperturbed p-rotor walk.

Conjecture: The scaling limit for p-rotor walk on $\mathbb{Z}^2$ when $p = \frac{1}{2}$ is the standard 2-D Brownian motion.

Uniform spanning tree plus one edge (UST$^+$)



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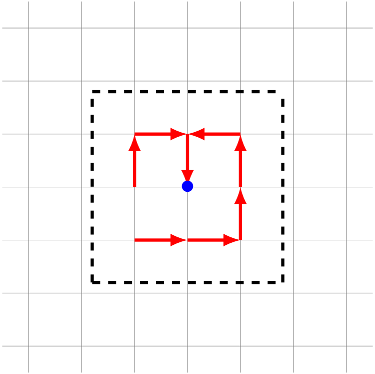
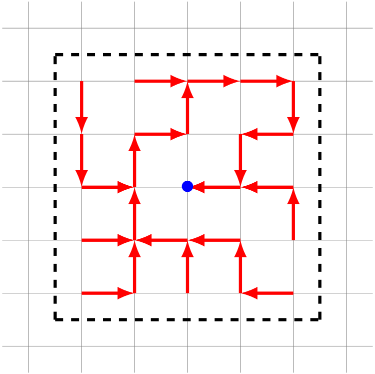


Pick a spanning tree of the black box directed to the origin (uniformly at random).

Uniform spanning tree plus one edge (UST$^+$)



Take the limit as the black box grows until it covers $\mathbb{Z}^2$.

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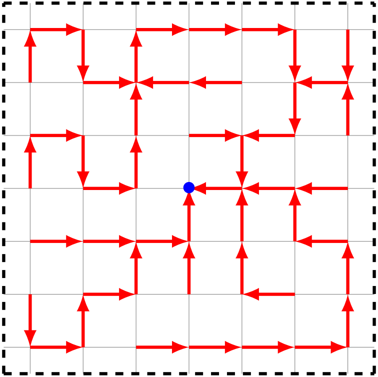

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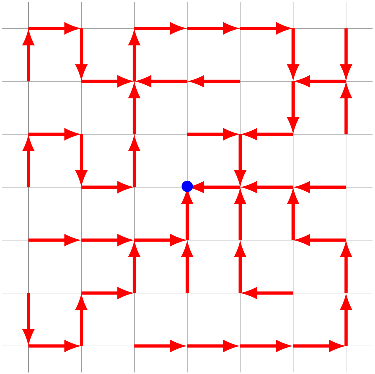

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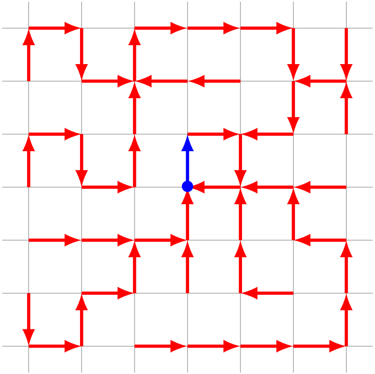


Add a signpost from the origin, uniform among the four directions.

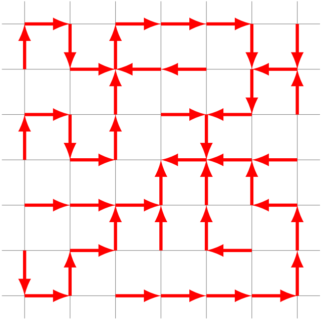
Our scaling limit result

Theorem (C., Greco, Levine, Li '18+)

Let $p = \frac{1}{2}$ and let the uniform spanning tree plus one edge be the initial signposts configuration. Then, with probability 1, the p-rotor walk on $\mathbb{Z}^2$ scales to the standard 2-D Brownian motion:

$$\frac{1}{\sqrt{n}}(\underbrace{X_{[nt]}}_{\substack{\text{location of the}\\ \text{walker at time } [nt]}})_{t\geq 0} \overset{n\to\infty}{\Longrightarrow} \frac{1}{\sqrt{2}}(\underbrace{B_1(t), B_2(t)}_{\substack{\text{independent}\\ \text{Brownian motions}}})_{t\geq 0}.$$

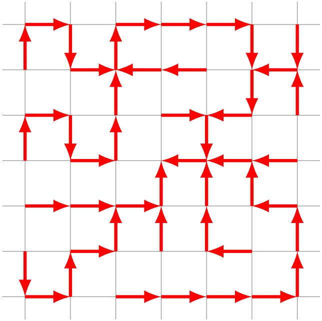
- Can be extended to $\frac{1}{2}$-rotor walk on $\mathbb{Z}^d$ for $d > 2$.
- Implies transience when $d > 2$; the case $d = 2$ remains open.



Preprint coming soon to an arXiv server near you!

Can be contacted at: sweehong@math.cornell.edu

"**Randomness makes life easier**"



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