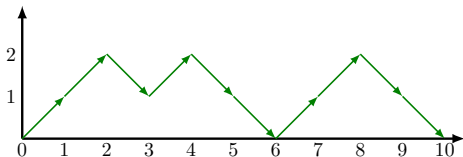


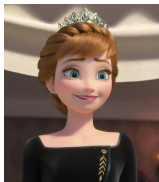
Sorting probability for Young diagrams

Swee Hong Chan (Rutgers)

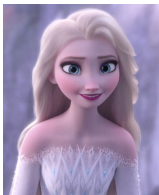
joint with Igor Pak and Greta Panova

1	2	4	7	8
3	5	6	9	10





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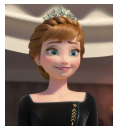


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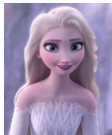




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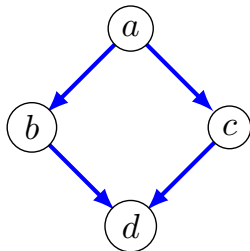


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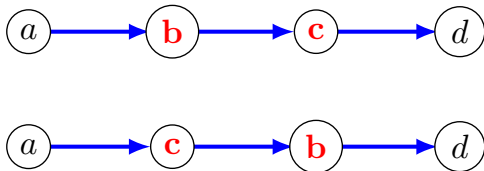
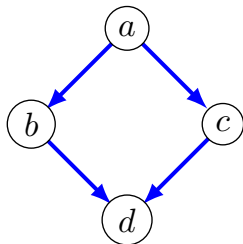
Partially ordered set

A poset P is a set X with a partial order \preccurlyeq on X .



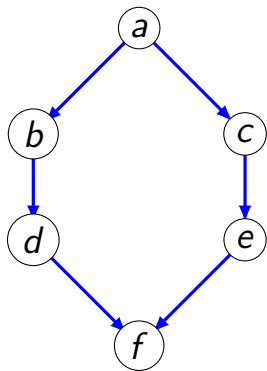
Linear extension

A linear extension L is a complete order of \preccurlyeq .



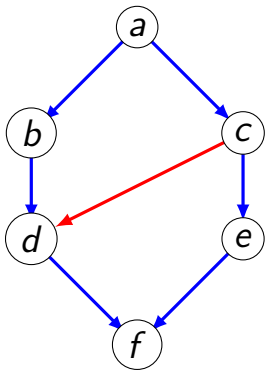
We write $e(P)$ for number of linear extensions of P .

How many steps needed to complete a partial order?



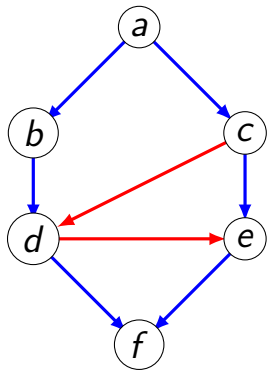
How many steps needed to complete a partial order?

We first compare c and d , and get $c \preceq d$.



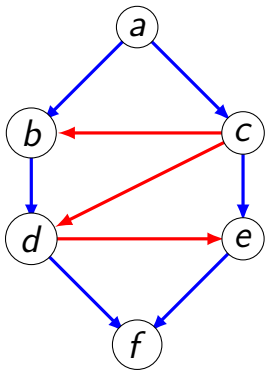
How many steps needed to complete a partial order?

We then compare d and e , and get $d \preceq e$.



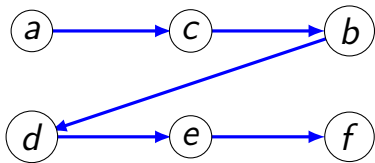
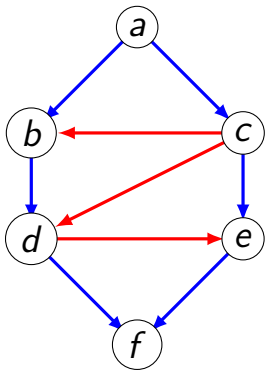
How many steps needed to complete a partial order?

We continue with b and c , and get $c \preceq b$.



How many steps needed to complete a partial order?

Completing the partial order took 3 steps.



Strategy to complete the partial order

At each step, compare x and y that satisfies

$$\frac{1}{2} - c \leq P[x \preceq y] \leq \frac{1}{2} + c,$$

where P is uniform on linear extensions of P .

Runtime is $\Theta(\log e(P))$ steps.

$\frac{1}{3} - \frac{2}{3}$ Conjecture

Conjecture (Kislitsyn '68, Fredman '75, Linial '84)

For every finite poset that is not completely ordered, there exists x, y :

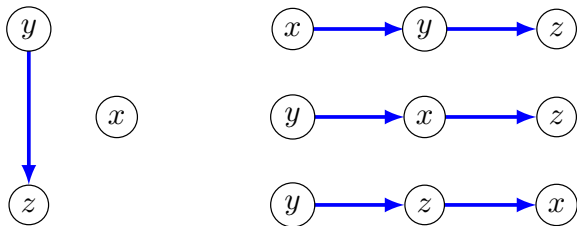
$$\frac{1}{3} \leq P[x \preceq y] \leq \frac{2}{3}.$$

(Brightwell-Felsner-Trotter '95)

“This problem remains one of the most intriguing problems in the combinatorial theory of posets.”

Why $\frac{1}{3}$ and $\frac{2}{3}$?

The upper, lower bound are achieved by this poset:



$$P[x \preceq y] = \frac{1}{3}; \quad P[y \preceq x] = \frac{2}{3}.$$

What is known so far

Theorem (Kahn-Saks '84)

For every finite poset, there always exists x, y :

$$\frac{3}{11} \leq P[x \preceq y] \leq \frac{8}{11},$$

roughly between 0.273 and 0.727.

Proof is by applying **mixed-volume inequalities** to **order polytopes**.

What is known so far

Theorem (Brightwell-Felsner-Trotter '95)

For every finite poset, there always exists x, y :

$$\frac{5 - \sqrt{5}}{10} \leq P[x \preceq y] \leq \frac{5 + \sqrt{5}}{10},$$

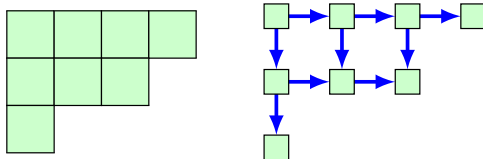
roughly between 0.276 and 0.724.

This bound cannot be improved for **infinite posets**.

Young diagrams

Elements of P_λ are **cells** of Young diagram of shape λ .

$x \preccurlyeq y$ if y lies to the Southeast of x .



Young diagram of shape $\lambda = (4, 3, 1)$

We write n for **number of cells** of Young diagram.

Young diagrams

Linear extensions of P_λ correspond to **standard Young tableau** of the Young diagram.

1	2	5	6
3	4	7	
8			

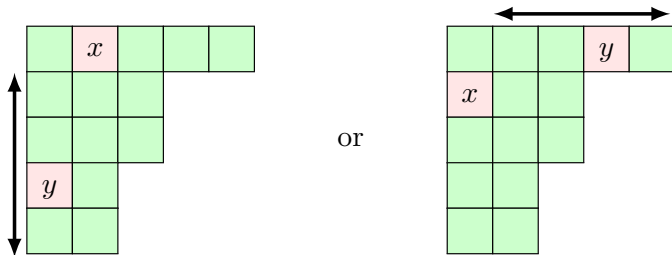
Linear extensions are counted by **hook-length formulas**.

What is known for Young diagrams

Theorem 1 (Olson–Sagan '18)

For *Young diagrams*, there always exists x, y :

$$\frac{1}{3} \leq P[x \preceq y] \leq \frac{2}{3}.$$



What is known for Young diagrams

Theorem 1 (Olson–Sagan '18)

For Young diagrams, there always exists x, y :

$$\frac{1}{3} \leq P[x \preceq y] \leq \frac{2}{3}.$$

We sketch an alternative proof for Young diagrams using Naruse hook-length formulas.

Hook-length formulas

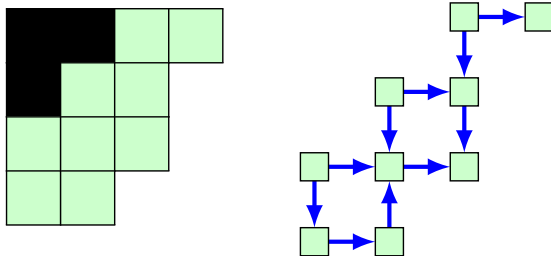
Number of standard Young tableau of shape λ is

$$f^\lambda := \frac{n!}{\prod_{x \in \lambda} h_\lambda(x)}.$$

7	6	4	1
5	4	2	
4	3	1	
2	1		

$$f^\lambda = \frac{12!}{7 \cdot 6 \cdot 4 \cdot 1 \cdot 5 \cdot 4 \cdot 2 \cdot 4 \cdot 3 \cdot 1 \cdot 2 \cdot 1} = 2970$$

Skew Young diagrams



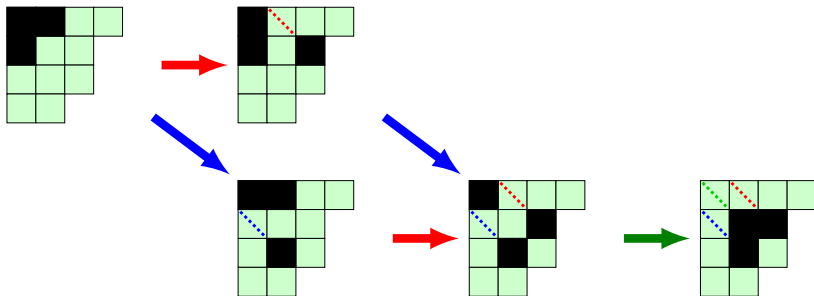
Skew Young diagram of shape λ/μ ,
 $\lambda = (5, 3, 3, 1)$ and $\mu = (2, 1)$.

We write n for number of cells in λ ,
and m for number of cells in μ .

Excited diagrams

At each step, move a black box on SouthEast direction

- Boxes cannot leave the green diagram,
- Boxes cannot move if blocked by other boxes.



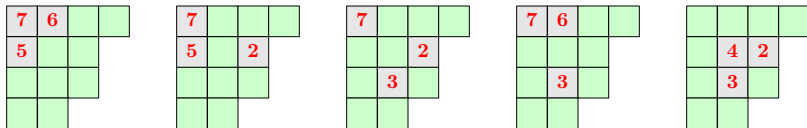
Naruse hook-length formulas

Theorem (Naruse '14, Morales-Pak-Panova '17)

Number of skew Young tableau of shape λ/μ is

$$f^{\lambda/\mu} := f^{\lambda} \frac{(n-m)!}{n!} \sum_{\substack{\text{excited} \\ \text{diagrams } B}} \prod_{\substack{\text{black cells} \\ x \in B}} h_{\lambda}(x).$$

Naruse hook-length formulas



The number of SYT of shape λ/μ is equal to

$$2970 \frac{9!}{12!} (7 \cdot 6 \cdot 5 + 7 \cdot 5 \cdot 2 + 7 \cdot 2 \cdot 3 + 7 \cdot 6 \cdot 3 + 4 \cdot 2 \cdot 3) \\ = 1062.$$

Note: Every term in NHLF is **nonnegative**.

Proof of Theorem Olson–Sagan

y_1	x		

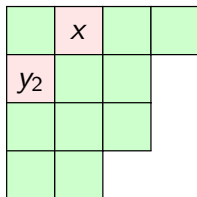
$$P[x \preceq y_1] = \underbrace{\quad\quad\quad}_0 \quad 1$$

The i -th jump probability p_i is

$$p_i := P[y_i \preceq x \preceq y_{i+1}],$$

where y_i is the i -th element in 1st column.

Proof of Theorem Olson–Sagan



$$P[x \preceq y_2] =$$

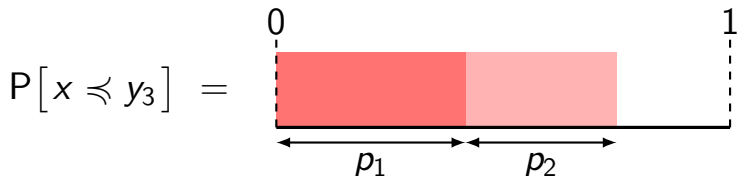
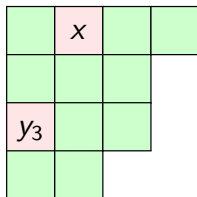
A unit interval $[0, 1]$ on a horizontal axis. A red rectangle is drawn from 0 to p_1 . Dashed vertical lines are at 0 and 1. A double-headed arrow below the axis indicates the length p_1 .

The i -th jump probability p_i is

$$p_i := P[y_i \preceq x \preceq y_{i+1}],$$

where y_i is the i -th element in 1st column.

Proof of Theorem Olson–Sagan

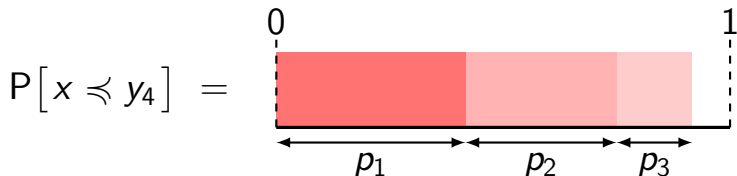
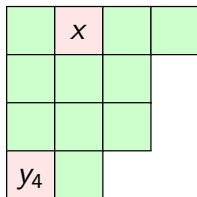


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Proof of Theorem Olson–Sagan

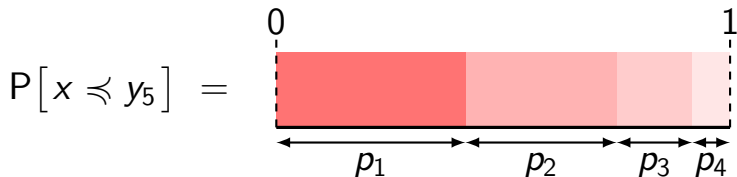
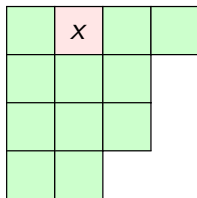


The i -th jump probability p_i is

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Proof of Theorem Olson–Sagan



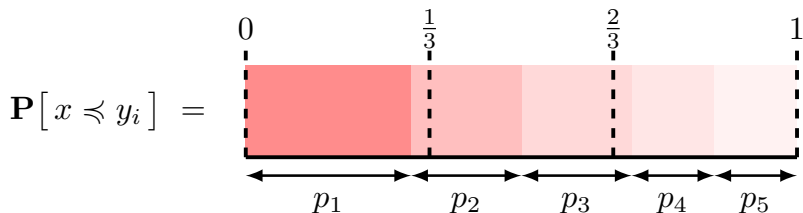
The i -th jump probability p_i is

$$p_i := P[y_i \preceq x \preceq y_{i+1}],$$

where y_i is the i -th element in 1st column.

Linial-type argument

Suppose that p_1, p_2, p_3, \dots are all $< \frac{1}{3}$.



Look at when the probability exceeds $\frac{1}{3}$. Then

$$\frac{1}{3} \leq \mathbf{P}[x \preceq y_{i+1}] \leq \frac{2}{3}.$$

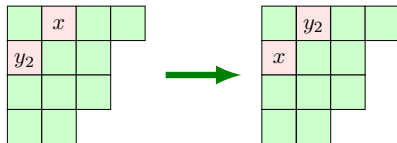
Proof of $p_1 < \frac{1}{3}$

Suppose to the contrary that $p_1 \geq \frac{1}{3}$. Then

- If $\frac{1}{3} \leq p_1 \leq \frac{2}{3}$, then

$$\frac{1}{3} \leq p_1 = P[x \preccurlyeq y_2] \leq \frac{2}{3}.$$

- If $p_1 > \frac{2}{3}$, then substitute $x \leftrightarrow y_2$ so $p_1 < \frac{1}{3}$.



- So we assume $p_1 < \frac{1}{3}$.

Skew diagrams enter the scene

It suffices to show $p_1 \geq p_2 \geq p_3 \geq \dots$

$$p_1 = P[y_1 \preccurlyeq x \preccurlyeq y_2] = \frac{\# \text{ of SYTs of } f^\lambda}{f^\lambda}$$

1	2		

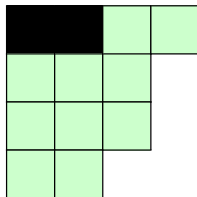
$$p_2 = P[y_2 \preccurlyeq x \preccurlyeq y_3] = \frac{\# \text{ of SYTs of } f^\lambda}{f^\lambda}$$

1	3		
2			

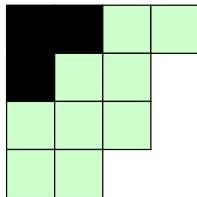
Skew diagrams enter the scene

It suffices to show $p_1 \geq p_2 \geq p_3 \geq \dots$

$$p_1 = P[y_1 \preceq x \preceq y_2] = \frac{\# \text{ of SYTs of } f^\lambda}{f^\lambda}$$



$$p_2 = P[y_2 \preceq x \preceq y_3] = \frac{\# \text{ of SYTs of } f^\lambda}{f^\lambda}$$



We can now use **NHLF**.

Proof of $p_1 \geq p_2$



$(10!)(7)(6)$



$(10!)(7)(2)$



$(10!)(4)(2)$

$$p_1 = \frac{(10! \cdot 7 \cdot 6 + 10! \cdot 7 \cdot 2 + 10! \cdot 4 \cdot 2)}{12!} = \frac{9!}{12!} 640.$$



$(9!)(7)(6)(5)$



$(9!)(7)(6)(3)$



$(9!)(7)(2)(5)$



$(9!)(7)(2)(3)$



$(9!)(4)(2)(3)$

$$p_2 = \frac{(9! \cdot 7 \cdot 6 \cdot 8 + 9! \cdot 7 \cdot 2 \cdot 8 + 9! \cdot 4 \cdot 2 \cdot 3)}{12!} = \frac{9!}{12!} 472.$$

Thus we complete the proof of this theorem.

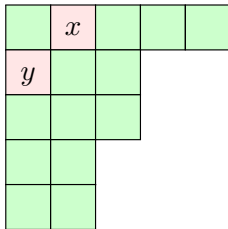
Theorem (Olson-Sagan '18)

There always exists x, y :

$$\frac{1}{3} \leq P[x \preceq y] \leq \frac{2}{3},$$

for poset P_λ of Young diagram of shape λ .

Back to previous example



Comparison probability for this Young diagram is

$$P[x \preceq y] = \frac{16}{33} \approx 0.4848,$$

which is closer to $\frac{1}{2}$ than $\frac{1}{3}$, $\frac{2}{3}$.

What we will do next

Previously, we want to find x, y :

$$\frac{1}{3} \leq \mathbb{P}[x \preceq y] \leq \frac{2}{3},$$

Now, we want to find x, y :

$$\frac{1}{2} - \delta \leq \mathbb{P}[x \preceq y] \leq \frac{1}{2} + \delta,$$

Sorting probability

Sorting probability of a poset P is

$$\delta(P) := \min_{\text{distinct } x, y} |P[x \prec y] - P[y \prec x]|.$$

In particular, there exists x, y :

$$\frac{1}{2} - \frac{\delta(P)}{2} \leq P[x \preceq y] \leq \frac{1}{2} + \frac{\delta(P)}{2}.$$

Kahn–Saks Conjecture

Conjecture (Kahn-Saks '84)

For every finite poset,

$$\delta(P) \rightarrow 0 \quad \text{as} \quad \text{width}(P) \rightarrow \infty.$$

Here $\text{width}(P)$ is the largest size of anti-chains in P .

Komlós '90 proved such a result for posets with $\Omega\left(\frac{n}{\log \log \log n}\right)$ minimal elements.

Our results

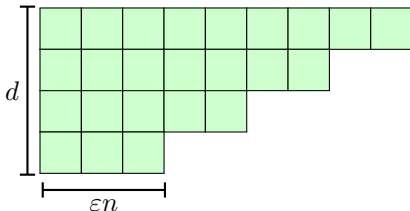
First result

Theorem (C.-Pak-Panova '21)

Let $\lambda_1 \geq \dots \geq \lambda_d \geq \varepsilon n$. For poset P_λ of *Young diagram* of λ ,

$$\delta(P_\lambda) \leq \frac{C}{\sqrt{n}},$$

for some $C = C(d, \varepsilon) > 0$.



First result

Theorem (C.-Pak-Panova '21)

Let $\lambda_1 \geq \dots \geq \lambda_d \geq \varepsilon n$. For poset P_λ of *Young diagram* of λ ,

$$\delta(P_\lambda) \leq \frac{C}{\sqrt{n}},$$

for some $C = C(d, \varepsilon) > 0$.

Proof ingredient:

NHLF + Random walk intuition

Where is the improvement?

Before: x is 2nd element in 1st row, y is in 1st column.

Intuition: Probability of SRW on \mathbb{Z} to visit 0
at 2nd step is of constant order.

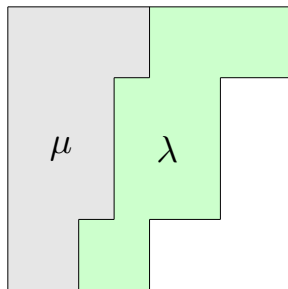
Now: x is midpoint of 1st row, y is in 2nd row.

Intuition: Probability of SRW on \mathbb{Z} to visit 0
at $\frac{n}{2}$ -th step is of the order of $\frac{1}{\sqrt{n}}$.

Sketch of proof

After reductions using [Hoeffding's inequality](#),

$$\delta(P_\lambda) \leq \sum_{\mu} \frac{\text{SYTs of } \mu}{f^\lambda}$$



$$\text{with } \mu \approx \left(\frac{\lambda_1}{2} \pm \sqrt{n}, \dots, \frac{\lambda_d}{2} \pm \sqrt{n} \right).$$

Right side is then upper-bounded via [NHLF](#).

Back to first result

Theorem (C.-Pak-Panova '21)

Let $\lambda_1 \geq \dots \geq \lambda_d \geq \varepsilon n$. For poset P_λ of *Young diagram* of λ ,

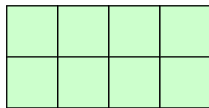
$$\delta(P_\lambda) \leq \frac{C}{\sqrt{n}},$$

for some $C = C(d, \varepsilon) > 0$.

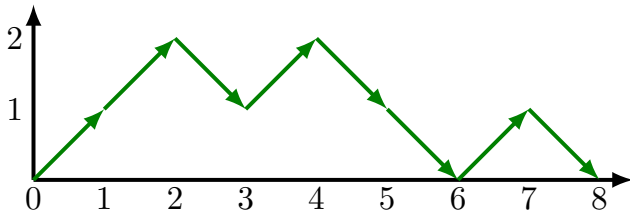
Next: better bound for Catalan posets.

Catalan posets, $\lambda = (\frac{n}{2}, \frac{n}{2})$

Young diagram is rectangle with 2 rows and n cells.



1	2	4	7
3	5	6	8



Second result

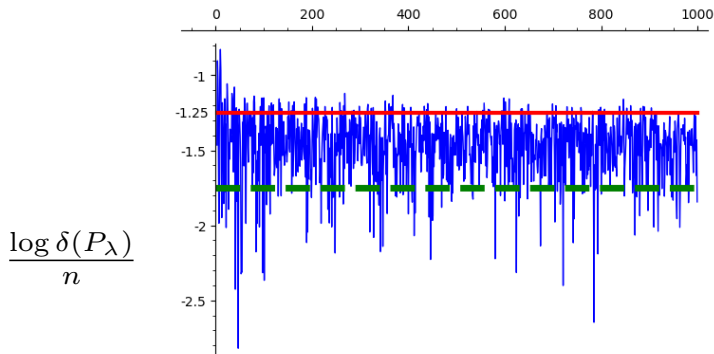
Theorem (C.-Pak-Panova '21)

For Catalan posets with n cells,

$$\delta(P_\lambda) \leq Cn^{-\frac{5}{4}},$$

for some $C > 0$.

How good is this bound?



Open Problem

Show that

$$\limsup_{n \rightarrow \infty} \frac{\log \delta(P_\lambda)}{n} = -\frac{5}{4}; \quad \liminf_{n \rightarrow \infty} \frac{\log \delta(P_\lambda)}{n} < -\frac{5}{4}.$$

Where is the improvement?

For each x in 1st row, find $y(x)$ in 2nd row minimizing

$$\delta(x, y(x)) := \left| P[x \prec y(x)] - P[y(x) \prec x] \right|.$$

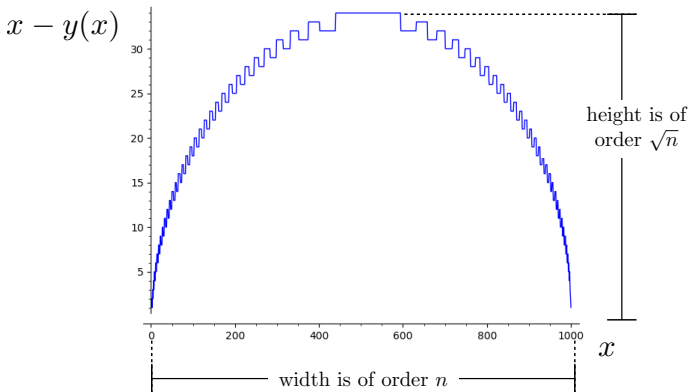
Before: x is fixed at midpoint of 1st row,

$$\delta(P_\lambda) \leq \delta(x, y(x)).$$

Now: Optimize over all x 's in 1st row,

$$\delta(P_\lambda) \leq \min_{x \text{ in 1st row}} \delta(x, y(x)).$$

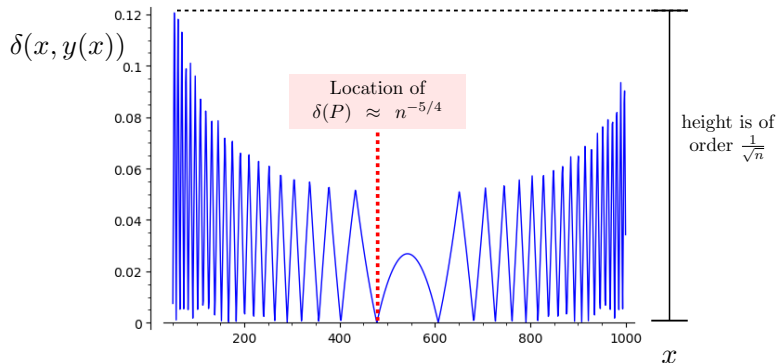
Location of the minimizer $y(x)$ for $n = 2000$



Semicircle shape is because of Brownian excursion.

Discrete pattern does not vanish in the limit.

Sorting probability $\delta(P)$ for $n = 2000$



Choosing x to be slightly left of midpoint gives smaller sorting probability because of zigzag pattern.

Back to second result

Theorem (C.-Pak-Panova '21)

For *Catalan posets with n cells*,

$$\delta(P_\lambda) \leq C n^{-\frac{5}{4}},$$

for some $C > 0$.

Important: Estimates are not done by **NHLF**,
but by direct computation.

Better upper bound for **general Young diagrams**
remain open.

What is next?

Theorem (C.-Pak-Panova '21)

Let $\lambda_1 \geq \dots \geq \lambda_d \geq \varepsilon n$. For poset P_λ of *Young diagram* of λ , there exists x, y :

$$\delta(P_\lambda) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Open Problem

Prove same result for other *families of posets*, e.g., *k-dimensional Young diagrams* and *periodic posets*.

arXiv link: 2005.08390 and 2005.13686.

Webpage: <http://math.rutgers.edu/~sc2518/>

THANK YOU!

arXiv link: [2005.08390](#) and [2005.13686](#).

Webpage: <http://math.rutgers.edu/~sc2518/>