Log-concavity and Cross Product Inequalities in Order Theory

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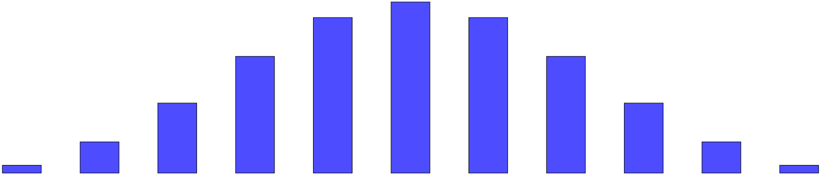
joint with Igor Pak and Greta Panova



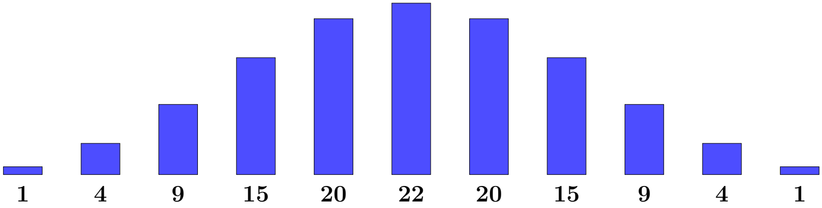
What is log-concavity?

A sequence $a_1, \ldots, a_n \in \mathbb{R}_{\geq 0}$ is log-concave if

$$a_k^2 \;\geq\; a_{k+1}\, a_{k-1} \qquad \text{for all } \; 1 < k < n.$$

Log-concavity (and positivity) implies unimodality:

$$a_1 \;\leq\; \cdots \;\leq\; a_m \;\geq\; \cdots \;\geq\; a_n \quad \text{for some } \; 1 \leq m \leq n.$$



Example: binomial coefficients

$$a_k = \binom{n}{k} \qquad k = 0, 1, \ldots, n.$$

This sequence is log-concave because

$$\frac{a_k^2}{a_{k+1}\, a_{k-1}} = \frac{\binom{n}{k}^2}{\binom{n}{k+1}\binom{n}{k-1}} = \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right),$$

which is greater than 1.

Example: permutations with k inversions

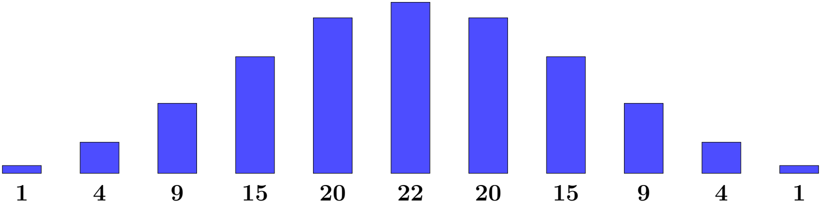
$$a_k = \text{number of } \pi \in S_n \text{ with } k \text{ inversions},$$

where inversion of π is pair $i < j$ s.t. $\pi_i > \pi_j$.

This sequence is log-concave because

$$\sum_{0 \le k \le \binom{n}{2}} a_k\, q^k = [n]_q! = (1+q) \ldots (1+q \ldots + q^{n-1})$$

is a product of log-concave polynomials.

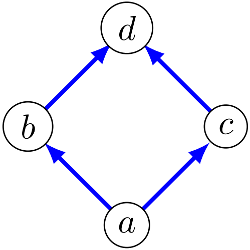


Log-concavity appears in different objects
for different reasons.

Today we focus on reasons for **posets**.

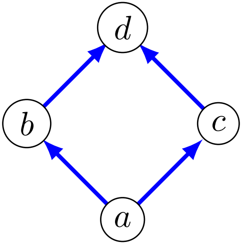
Partially ordered sets

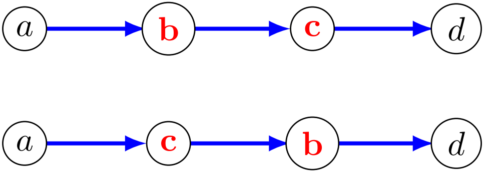
A poset P is a set X with a partial order $\prec$ on X.



Linear extension

A linear extension L is a complete order of $\prec$.





We write $L(x) = k$ if x is k-th smallest in L.

Stanley's inequality

Fix $z \in P$.

N_k *is* number of linear extensions with $L(z) = k$.

Theorem (Stanley '81)

For every poset and $k \geq 1$,

$$N_k{}^2 \;\geq\; N_{k+1}\, N_{k-1}.$$

The inequality was initially conjectured by Chung-Fishburn-Graham, and was proved using Aleksandrov-Fenchel inequality for mixed volumes.

When is equality achieved?

Theorem (Shenfeld-van Handel '22)

Suppose $N_k > 0$. *Then*

$$N_k{}^2 = N_{k+1}\, N_{k-1}$$

if and only if

$$N_k \;=\; N_{k+1} \;=\; N_{k-1}.$$

Proof used classifications of extremals of Aleksandrov-Fenchel inequality for convex polytopes.

Our contribution

Problem

(Folklore, Graham '83, Biró-Trotter '11, Stanley '14)

Give a combinatorial proof of Stanley's inequality.

Answer (C.–Pak)

New combinatorial proof for Stanley's inequality, with generalizations to weighted version.

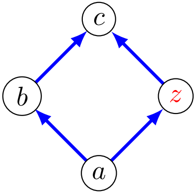
Order-reversing weight

A weight $\omega : X \to \mathbb{R}_{>0}$ is order-reversing if

$$\omega(x) \;\geq\; \omega(y) \qquad \text{whenever} \qquad x \prec y.$$

Weight of linear extension L is

$$\omega(L) \;:=\; \prod_{L(x)<L(z)} \omega(x).$$



$$a \to z \to b \to c$$

$$\omega(L) = \omega(a)$$

$$a \to b \to z \to c$$

$$\omega(L) = \omega(a)\omega(b)$$

Weighted Stanley's inequality

Fix $z \in P$.

$N_{\omega,k}$ is sum of ω-weight of lin. exts. with $L(z) = k$.

Theorem (C. Pak)

For every poset and $k \geq 1$,

$$N_{\omega,k}{}^2 \;\geq\; N_{\omega,k+1}\, N_{\omega,k-1}.$$

Proof used combinatorial atlas method,

a new tool to establish log-concave inequalities.

When is equality achieved?

Theorem (C.-Pak)

Suppose $N_{w,k} > 0$. *Then*

$$N_{w,k}{}^2 = N_{w,k+1}\, N_{w,k-1}$$

if and only if

for every linear extension L *with* $L(z) = k$,

- $w(x_{k+1}) \;=\; w(x_{k-1}) = s$, *and*
- z *is incomparable to both* x_{k-1} *and* x_{k+1},

where s *is fixed constant, and*

$$x_{k-1} \;:=\; L^{-1}(k-1), \quad x_{k+1} \;:=\; L^{-1}(k+1).$$

What the equality condition means

For every linear extension L with $L(z) = k$,

$$\circ \to \cdots \to \boxed{x_{k-1}} \to z \to \boxed{x_{k+1}} \to \cdots \to \circ$$

we have linear extension K with weight $s^{-1}\omega(L)$,

$$\circ \to \cdots \to z \to \boxed{x_{k-1}} \to \boxed{x_{k+1}} \to \cdots \to \circ$$

and linear extension M with weight $s\,\omega(L)$,

$$\circ \to \cdots \to \boxed{x_{k-1}} \to \boxed{x_{k+1}} \to z \to \cdots \to \circ$$

so $\quad N_{\omega,k+1} \;=\; s\,N_{\omega,k} \;=\; s^2\,N_{\omega,k-1}.$

Log-concavity of order polyomials

Order polynomials

Fix poset $P = (X, \prec)$.

A map $\phi : X \to \{1, \ldots, t\}$ is order-preserving if

$$x \prec y \qquad \text{implies} \qquad \phi(x) \leq \phi(y).$$

Fix $z \in P$ and positive integer k.

The k-order polynomial $\Omega_k(t)$ is

$$\Omega_k(t) \;:=\; \text{number of order-preserving maps from } X \text{ to } \{1, \ldots, t\} \text{ such that } \phi(z) = k.$$

Graham's conjecture

Conjecture (Graham '83)

$$\Omega_k(t)^2 \;\geq\; \Omega_{k+1}(t)\,\Omega_{k-1}(t).$$

Conjecture can be viewed as order polynomial version of Stanley's inequality .

Quote (Graham '83)

"So far, no one has been able to establish [Stanley's inequality] for order-preserving maps, although it must certainly be true".

Asymptotic Graham's inequality

Theorem (C.–Pak–Panova)

For sufficiently large $t > t(k)$,

$$\Omega_k(t)^2 \;\geq\; \Omega_{k+1}(t)\,\Omega_{k-1}(t).$$

Proof used FKG inequality applied to Shepp's lattice and a series of asymptotic analysis.

Project

Prove the full version of Graham's conjecture.

Applications of log-concavity

$\frac{1}{3} - \frac{2}{3}$ Conjecture

Conjecture (Kislitsyn '68, Fredman '75, Linial '84)

For finite poset that is not completely ordered, there exist elements x, y:

$$\frac{1}{3} \quad \leq \quad \mathbb{P}\big[\mathcal{L}(x) < \mathcal{L}(y)\big] \quad \leq \quad \frac{2}{3},$$

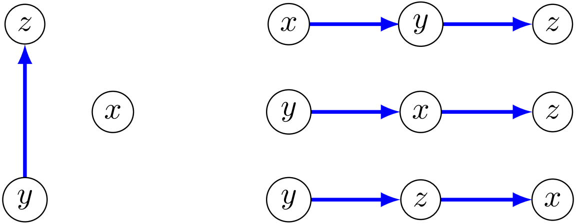
where $\mathcal{L}$ is uniform random linear extension of P.

Quote (Brightwell-Felsner-Trotter '95)

"This problem remains one of the most intriguing problems in the combinatorial theory of posets."

Why $\frac{1}{3}$ and $\frac{2}{3}$?

The upper,lower bound are achieved by this poset:



$$\mathbb{P}\left[\mathcal{L}(x) < \mathcal{L}(y)\right] = \frac{1}{3}; \qquad \mathbb{P}\left[\mathcal{L}(y) < \mathcal{L}(x)\right] = \frac{2}{3}.$$

Wikipedia: unsolved problems in Combinatorics

List of unsolved problems in mat

en.wikipedia.org/wiki/List_of_unsolved_problems_in_mathematics#Combinatorics

- Regularity of solutions of Vlasov–Maxwell equations

Combinatorics

Main article: Combinatorics

- The 1/3–2/3 conjecture – does every finite partially ordered set that is not totally ordered contain two elements *x* and *y* such that the probability that *x* appears before *y* in a random linear extension is between 1/3 and 2/3?[24]
- Problems in Latin squares – open questions concerning Latin squares
- The lonely runner conjecture – if k runners with pairwise distinct speeds run round a track of unit length, will every runner be "lonely" (that is, be at least a distance $1/k$ from each other runner) at some time?[25]
- The sunflower conjecture: can the number of k size sets required for the existence of a sunflower of r sets be bounded by an exponential function in k for every fixed $r > 2$?
- No-three-in-line problem – how many points can be placed in the $n \times n$ grid so that no three of them lie on a line?
- Frankl's union-closed sets conjecture – for any family of sets closed under sums there exists an element (of the underlying space) belonging to half or more of the sets[26]
- Give a combinatorial interpretation of the Kronecker coefficients[27]
- The values of the Dedekind numbers $M(n)$ for $n \geq 9$[28]
- The values of the Ramsey numbers, particularly $R(5, 5)$
- The values of the Van der Waerden numbers
- Finding a function to model n-step self-avoiding walks[29]

The big breakthrough

Theorem (Kahn-Saks '84)

For poset that is not completely ordered, there exist elements x, y:

$$\frac{3}{11} \quad \leq \quad \mathbb{P}\big[\, L(x) < L(y) \,\big] \quad \leq \quad \frac{8}{11}\,,$$

roughly between 0.273 *and* 0.727.

Proof sketch of Kahn-Saks Theorem

Find $x, y \in P$ such that

$$|h(y) - h(x)| \;\leq\; 1,$$

where $h(x) := \mathbb{E}[\mathcal{L}(x)]$ and $h(y) := \mathbb{E}[\mathcal{L}(y)]$.

Let $F_k =$ number of lin. ext. with $L(y) - L(x) = k$.

$$\mathbb{P}\big[\,\mathcal{L}(x) < \mathcal{L}(y)\,\big] \;=\; (F_1 + F_2 + \cdots + F_n)/e(P),$$

$$\mathbb{P}\big[\,\mathcal{L}(y) < \mathcal{L}(x)\,\big] \;=\; (F_{-1} + F_{-2} + \cdots + F_{-n})/e(P).$$

Proof sketch of Kahn-Saks Theorem

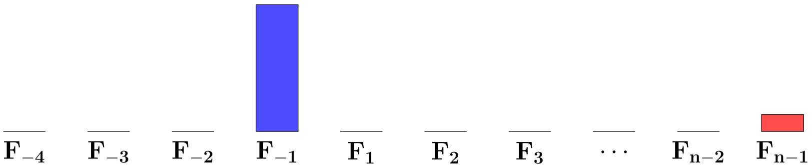
Since $|h(y) - h(x)|$ is small,

$$F_1 + 2F_2 + \cdots + nF_n \;\approx\; F_{-1} + 2F_{-2} + \cdots + nF_{-n}.$$

One can hope this implies

$$F_1 + F_2 + \cdots + F_n \;\approx\; F_{-1} + F_{-2} + \cdots + F_{-n}.$$

But things can go really wrong:



Log-concavity comes to rescue

Theorem (Kahn–Saks '84)

For $k \neq 0$,

$$F_k{}^2 \;\geq\; F_{k+1}\, F_{k-1},$$

$$F_{-k}{}^2 \;\geq\; F_{-(k+1)}\, F_{-(k-1)}.$$

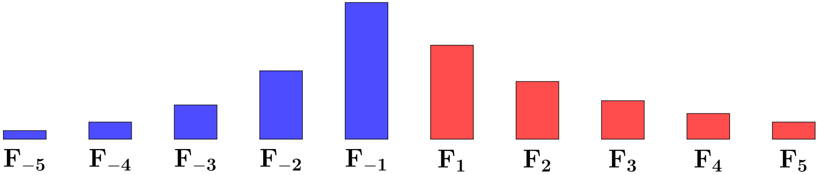
This generalizes Stanley's inequality, and was proved by Aleksandrov-Fenchel inequality.

Proof sketch of Kahn-Saks Theorem

Log-concavity (and other ineqs.) imply:

- $\mathbb{P}[\mathcal{L}(x) < \mathcal{L}(y)]$ is maximized (resp. minimized) when $F_1, F_2, \ldots, F_n$ is geometric sequence,

- $\mathbb{P}[\mathcal{L}(y) < \mathcal{L}(x)]$ is minimized (resp. maximized) when $F_{-1}, F_{-2}, \ldots, F_{-n}$ is geometric sequence.



Combined with $|h(y) - h(x)| \leq 1$, the result follows.

Cross Product Conjecture

Best known bound for $\frac{1}{3} - \frac{2}{3}$ Conjecture

Theorem (Brightwell-Felsner-Trotter '95)

For poset that is not completely ordered, there exist elements x, y:

$$\frac{5-\sqrt{5}}{10} \quad \leq \quad \mathbb{P}\big[\mathcal{L}(x) < \mathcal{L}(y)\big] \quad \leq \quad \frac{5+\sqrt{5}}{10},$$

roughly between 0.276 *and* 0.724*.*

This bound cannot be improved for infinite posets.

New ingredient: Cross Product Conjecture

Fix $x, y, z \in P$. Let $F(k, \ell)$ be number of lin. ext. with $L(y) - L(x) = k$ and $L(z) - L(y) = \ell$.

Conjecture (Brightwell-Felsner-Trotter '95)

For $k, \ell \geq 1$,

$$F(k, \ell)\, F(k+1, \ell+1) \;\leq\; F(k+1, \ell)\, F(k, \ell+1).$$

Equivalently,

$$\det \begin{bmatrix} F(k, \ell) & F(k, \ell+1) \\ F(k+1, \ell) & F(k+1, \ell+1) \end{bmatrix} \;\leq\; 0.$$

What was known

Conjecture (Brightwell-Felsner-Trotter '95)

For $k, \ell \geq 1$,

$$F(k,\ell)\,F(k+1,\ell+1) \;\leq\; F(k+1,\ell)\,F(k,\ell+1).$$

Brightwell-Felsner-Trotter proved the case $k = \ell = 1$ by Ahlswede–Daykin inequality.

Combined with Kahn–Saks proof, this gives the $\frac{5\pm\sqrt{5}}{10}$ bound for $\frac{1}{3}-\frac{2}{3}$ Conjecture.

What was known

Conjecture (Brightwell-Felsner-Trotter '95)

For $k, \ell \geq 1$,

$$F(k,\ell)\,F(k+1,\ell+1) \;\leq\; F(k+1,\ell)\,F(k,\ell+1).$$

Quote (Brightwell-Felsner-Trotter '95)

"Something more powerful seems to be needed to prove general form of Cross Product Conjecture."

Our results

Theorem 1 (C.-Pak-Panova '22)
Cross Product Conjecture is true for posets of width two.

Proved algebraically using matrix algebra argument and combinatorially through Lindström–Gessel–Viennot type argument.

Our results

Theorem 2 (C.-Pak)

For every poset and $k, \ell \geq 1$,

$$F(k,\ell)\,F(k+1,\ell+1) \;<\; 2F(k+1,\ell)\,F(k,\ell+1).$$

Proved using Bézout's inequality for mixed volumes.

The constant 2 is tight for general geometric objects.

Project

Prove the full version of Cross Product Conjecture.

Important open problem

Kahn-Saks Conjecture

$\delta(P)$ is largest number such that there exist $x, y \in P$:

$$\delta(P) \ \leq \ \mathbb{P}\big[\mathcal{L}(x) < \mathcal{L}(y)\big] \ \leq \ 1 - \delta(P).$$

Note that $\frac{1}{3} - \frac{2}{3}$ Conjecture is equivalent to
$\delta(P) \geq \frac{1}{3}$ for P not completely ordered.

Conjecture (Kahn-Saks '84)

$$\delta(P) \to \frac{1}{2} \quad \textit{as} \quad \text{width}(P) \to \infty.$$

Kahn-Saks Conjecture

Conjecture (Kahn-Saks '84)

$$\delta(P) \to \frac{1}{2} \quad \textit{as} \quad \mathrm{width}(P) \to \infty.$$

Komlós '90 proved Conjecture for posets with $\Omega(\frac{n}{\log\log\log n})$ minimal elements.

C.-Pak-Panova '21 proved Conjecture for Young diagram posets with fixed width.

THANK YOU!

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