

Combinatorial Atlas for Log-concave Inequalities

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joint with Igor Pak

What is log-concavity?

A sequence $a_1, \dots, a_n \in \mathbb{R}_{\geq 0}$ is **log-concave** if

$$a_k^2 \geq a_{k+1} a_{k-1} \quad (1 \leq k < n).$$

Equivalently,

$$\log a_k \geq \frac{\log a_{k+1} + \log a_{k-1}}{2} \quad (1 \leq k < n).$$

1 4 9 15 20 22 20 15 9 4 1

Example: binomial coefficients

$$a_k = \binom{n}{k} \quad k = 0, 1, \dots, n.$$

This sequence is **log-concave** because

$$\frac{a_k^2}{a_{k+1} a_{k-1}} = \frac{\binom{n}{k}^2}{\binom{n}{k+1} \binom{n}{k-1}} = \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right),$$

which is greater than 1.

Example: permutations with k inversions

a_k = number of $\pi \in S_n$ with k inversions,

where **inversion** of π is pair $i < j$ s.t. $\pi_i > \pi_j$.

This sequence is **log-concave** because

$$\sum_{0 \leq k \leq \binom{n}{2}} a_k x^k = (1+x) \dots (1+x+\dots+x^{n-1})$$

is a product of log-concave polynomials.

Log-concavity appears in many objects:

**algebras, matroids, mixed volumes,
measures, posets, random walks.**

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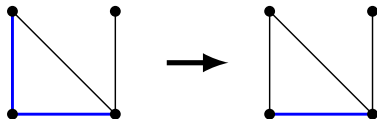
algebras, **matroids**, mixed volumes,
measures, **posets**, random walks

Today we focus on **matroids** and **posets**.

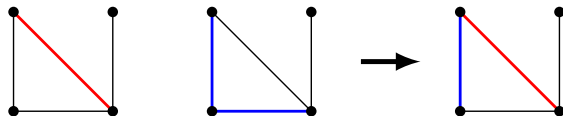
Matroids

Matroid \mathcal{M} is ground set X with
collection of independent sets $\mathcal{I} \subseteq 2^X$,

- $S \subseteq T$ and $T \in \mathcal{I}$ implies $S \in \mathcal{I}$.



- If $S, T \in \mathcal{I}$ and $|S| < |T|$, then there is $x \in T \setminus S$ such that $S \cup \{x\} \in \mathcal{I}$.



Examples: Matroids

Graphical matroids

- X = edges of a graph G ,
- \mathcal{I} = forests in G .

Realizable matroids

- X = finite set of vectors over field \mathbb{F} ,
- \mathcal{I} = sets of linearly independent vectors.

Mason's Conjecture (1972)

For every matroid and $k \geq 1$,

$$(1) \quad I_k^2 \geq I_{k+1} I_{k-1};$$

$$(2) \quad I_k^2 \geq \left(1 + \frac{1}{k}\right) I_{k+1} I_{k-1};$$

$$(3) \quad I_k^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right) I_{k+1} I_{k-1}.$$

I_k is number of ind. sets of size k , and $n = |X|$.

Why $(1 + \frac{1}{k}) (1 + \frac{1}{n-k})$?

Mason (3) is equivalent to **ultra log-concavity**,

$$\frac{I_k^2}{\binom{n}{k}^2} \geq \frac{I_{k+1}}{\binom{n}{k+1}} \frac{I_{k-1}}{\binom{n}{k-1}}.$$

Equality occurs **if** every $(k + 1)$ -subset is independent.

Solution to Mason (1)

Theorem (Adiprasito-Huh-Katz '18)

For every matroid and $k \geq 1$,

$$I_k^2 \geq I_{k+1} I_{k-1}.$$

Proof used [combinatorial Hodge theory](#) for
matroids.

Solution to Mason (2)

Theorem (Huh-Schröter-Wang '18)

For every matroid and $k \geq 1$,

$$I_k^2 \geq \left(1 + \frac{1}{k}\right) I_{k+1} I_{k-1}.$$

Proof used **combinatorial Hodge theory** for
correlation bound on matroids.

Solution to Mason (3)

Theorem (Anari-Liu-Gharan-Vinzant, Brändén-Huh '20)

For every matroid and $k \geq 1$,

$$I_k^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right) I_{k+1} I_{k-1}.$$

Proof used theory of **strong log-concave polynomials** /
Lorentzian polynomials.

Solution to Mason (3)

Theorem (Anari-Liu-Gharan-Vinzant, Brändén-Huh '20)

For every matroid and $k \geq 1$,

$$I_k^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right) I_{k+1} I_{k-1}.$$

Theorem (Murai-Nagaoka-Yazawa '21)

Equality occurs if and only if every $(k+1)$ -subset is independent.

Our contribution

Method: Combinatorial atlas

Results: Log-concave inequalities, and
if and only if conditions for equality

- Matroids (refined);
- Morphism of matroids (refined);
- Discrete polymatroids;
- Stanley's poset inequality (refined);
- Poset antimatroids;
- Branching greedoid (log-convex).

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Matroids

Warmup: graphical matroids refinement

Corollary (C.-Pak)

For graphical matroid of simple connected graph $G = (V, E)$ that is not tree, and $k = |V| - 2$,

$$(I_k)^2 \geq \frac{3}{2} \left(1 + \frac{1}{k}\right) I_{k+1} I_{k-1},$$

with equality *if and only if* G is cycle graph.

Numerically *better* than Mason (3), because

$$\frac{3}{2} \geq 1 + \frac{1}{n-k} = 1 + \frac{1}{|E| - |V| + 2}.$$

Comparison with Mason (3)

Our bound gives

$$\frac{(I_k)^2}{I_{k+1} I_{k-1}} \geq \frac{3}{2} \quad \text{when } |E| - |V| \rightarrow \infty,$$

Meanwhile, Mason (3) bound only gives

$$\frac{(I_k)^2}{I_{k+1} I_{k-1}} \geq 1 \quad \text{when } |E| - |V| \rightarrow \infty.$$

Our bound is **better** numerically and asymptotically.

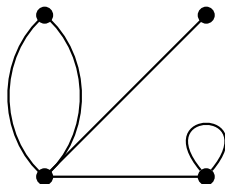
Parallel classes of matroid \mathcal{M}

Loop is $x \in X$ such that $\{x\} \notin \mathcal{I}$.

Non-loops x, y are **parallel** if $\{x, y\} \notin \mathcal{I}$.

Parallelship equiv. relation: $x \sim y$ if $\{x, y\} \notin \mathcal{I}$.

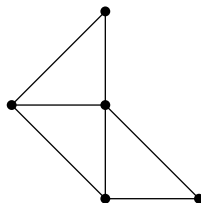
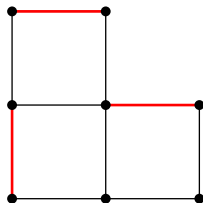
Parallel class = equivalence class of \sim .



Matroid contraction

Contraction of $S \in \mathcal{I}$ is matroid \mathcal{M}_S with

$$X_S = X \setminus S, \quad \mathcal{I}_S = \{T \setminus S : S \subseteq T\}.$$

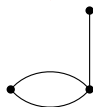


$\text{prl}(S) :=$ number of parallel classes of \mathcal{M}_S

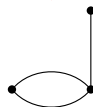
Parallel number

The k -parallel number is

$$\text{prl}(k) := \max\{\text{prl}(S) \mid S \in \mathcal{I} \text{ with } |S| = k\}.$$



$$\text{prl}(S) = 2$$



$$\text{prl}(S) = 2$$



$$\text{prl}(S) = 2$$



$$\text{prl}(S) = 3$$

$$\text{prl}(1) = 3$$

Refinement for Mason (3)

Theorem 1 (C.-Pak)

For every matroid and $k \geq 1$,

$$I_k^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{\text{prl}(k-1) + 1}\right) I_{k+1} I_{k-1}.$$

This refines Mason (3),

$$I_k^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right) I_{k+1} I_{k-1},$$

since

$$\text{prl}(k-1) \leq n - k + 1.$$

When is equality achieved?

- When every $(k + 1)$ -subset is independent,
 $\text{prl}(k - 1) = n - k + 1$.
- Graphical matroid when G is a cycle,
 $\text{prl}(k - 1) = 3$.
- Full realizable matroids over finite field \mathbb{F}_q ,
 $\text{prl}(k - 1) = \frac{n}{q^{k-1}} - 1$.
- (k, m, n) -Steiner system matroid,
 $\text{prl}(k - 1) = \frac{n - k + 1}{m - k + 1}$.

Equality conditions

Theorem 2 (C.-Pak)

For every matroid and $k \geq 1$,

$$I_k^2 = \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{\text{prl}(k-1) + 1}\right) I_{k+1} I_{k-1}$$

if and only if

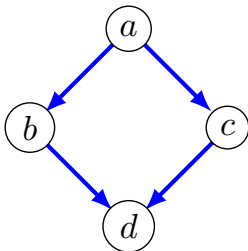
for every $S \in \mathcal{I}$ with $|S| = k - 1$,

- S has $\text{prl}(k - 1)$ parallel classes; and
- Every parallel class of S has same size.

Stanley's poset inequality

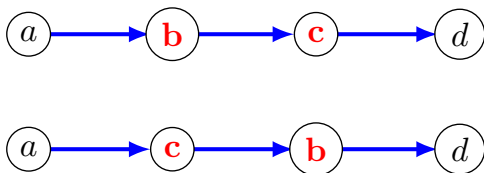
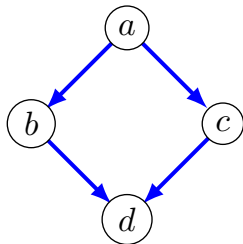
Partially ordered sets

A poset P is a set X with a partial order \prec on X .



Linear extension

A linear extension L is a complete order of \prec .



We write $L(x) = k$ if x is k -th smallest in L .

Stanley's inequality

Fix $z \in P$.

N_k is number of linear extensions with $L(z) = k$.

Theorem (Stanley '81)

For every poset and $k \geq 1$,

$$N_k^2 \geq N_{k+1} N_{k-1}.$$

Proof used Aleksandrov-Fenchel inequality for mixed volumes.

When is equality achieved?

Theorem (Shenfeld-van Handel)

Suppose $N_k > 0$. Then

$$N_k^2 = N_{k+1} N_{k-1}$$

if and only if

$$N_k = N_{k+1} = N_{k-1}.$$

Proof used classifications of extremals of
Aleksandrov-Fenchel inequality for convex polytopes.

Kahn–Saks inequality

Fix $x, y \in P$.

E_k is number of lin. exts. with $L(y) - L(x) = k$.

Theorem (Kahn–Saks '84)

For every poset and $k \geq 1$,

$$E_k^2 \geq E_{k+1} E_{k-1}.$$

Crucial component in the proof of first known
bound for $\frac{1}{3} - \frac{2}{3}$ Conjecture.

Kahn–Saks equality

Conjecture

Suppose $E_k > 0$. Then

$$E_k^2 = E_{k+1} E_{k-1}$$

if and only if

$$E_k = E_{k+1} = E_{k-1}.$$

Verified by C.–Pak–Panova for **width two** posets.

Our contribution

We give new **combinatorial proof** for Stanley's ineq.
and extend to **weighted version**.

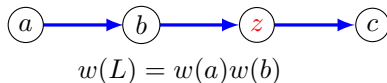
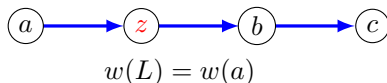
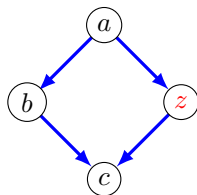
Order-reversing weight

A weight $w : X \rightarrow \mathbb{R}_{>0}$ is **order-reversing** if

$$w(x) \geq w(y) \quad \text{whenever} \quad x \prec y.$$

Weight of linear extension L is

$$w(L) := \prod_{L(x) < L(z)} w(x).$$



Weighted Stanley's inequality

Fix $z \in P$.

$N_{w,k}$ is w -weight of linear extensions with $L(z) = k$.

Theorem 3 (C. Pak)

For every poset and $k \geq 1$,

$$N_{w,k}^2 \geq N_{w,k+1} N_{w,k-1}.$$

When is equality achieved?

Theorem 4 (C.-Pak)

Suppose $N_{w,k} > 0$. Then

$$N_{w,k}^2 = N_{w,k+1} N_{w,k-1}$$

if and only if

for every linear extension L with $L(z) = k$,

$$w(L^{-1}(k+1)) = w(L^{-1}(k-1)) =: s,$$

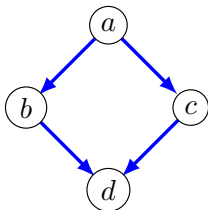
and

$$\frac{N_{w,k}}{s^k} = \frac{N_{w,k+1}}{s^{k+1}} = \frac{N_{w,k-1}}{s^{k-1}}.$$

Poset antimatroids

Feasible words of a poset

A word $\alpha \in X^*$ is **feasible** if no repeating elements, and y occurs in α and $x \prec y \Rightarrow x$ occurs in α before y .



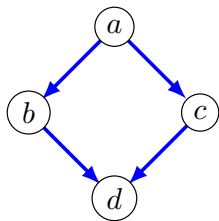
Feasible: \emptyset , a , ab , ac , abc , acb , $abcd$.

Not feasible: aa , bc , ba .

Chain weight

For $x \in P$, chain weight is

$\omega(x)$ = number of maximal chains that starts with x .

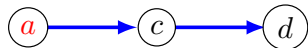
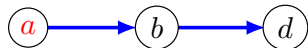


$$\omega(a) = 2$$

$$\omega(b) = 1$$

$$\omega(c) = 1$$

$$\omega(d) = 1$$



Weight of word α is $\omega(\alpha) := \omega(\alpha_1) \dots \omega(\alpha_\ell)$.

Log-concave inequality for poset antimatroids

$F_{\omega,k}$ is sum of ω -weight of feasible words of length k .

Theorem 5 (C.-Pak)

For every poset and $k \geq 1$,

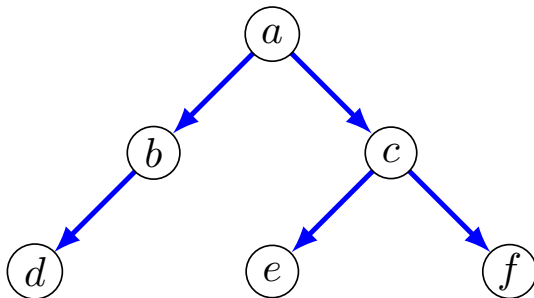
$$F_{\omega,k}^2 \geq F_{\omega,k+1} F_{\omega,k-1}.$$

When is equality achieved?

Theorem 6 (C.-Pak)

*Equality occurs for $k = 1, \dots, \text{height}(P) - 1$
if and only if*

*Hasse diagram of P is a forest where every leaf is of
the same level.*



Method: Combinatorial atlas

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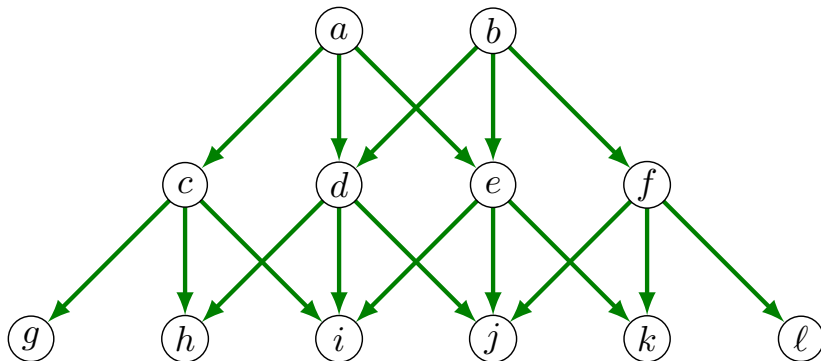
Combinatorial atlas

The strategy

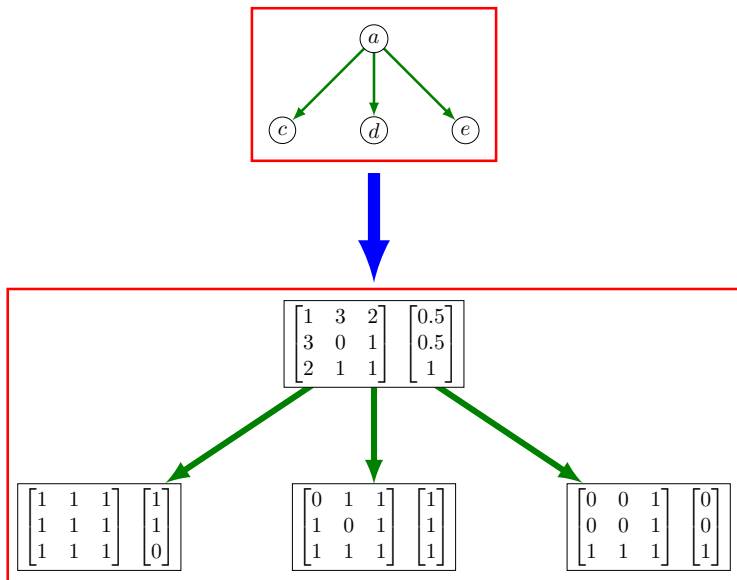
Input: Acyclic digraph \mathcal{A} , where each vertex v is associated with

- $r \times r$ nonnegative symmetric matrix M ;
- nonnegative r -vector h .

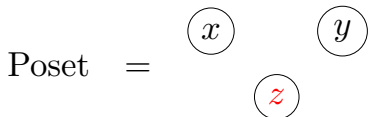
Combinatorial atlas: example



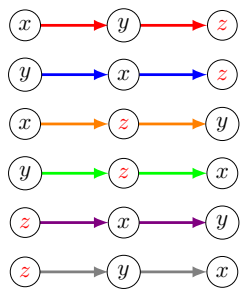
Combinatorial atlas: example (zoomed in)



Combinatorial atlas: Example Stanley's Inequality



x_{min}	y_{min}	x_{max}	y_{max}	
0	1	0	1	x_{min}
1	0	1	0	y_{min}
0	1	0	1	x_{max}
1	0	1	0	y_{max}



The strategy

Input: Acyclic digraph \mathcal{A} , where each vertex v is associated with

- $r \times r$ nonnegative symmetric matrix M ;
- A nonnegative r -vector h .

Goal: Show every M has hyperbolic inequality.

Hyperbolic inequality

M has hyperbolic inequality property if

$$\langle \mathbf{x}, M\mathbf{y} \rangle^2 \geq \langle \mathbf{x}, M\mathbf{x} \rangle \langle \mathbf{y}, M\mathbf{y} \rangle,$$

for every $\mathbf{x} \in \mathbb{R}^r$, $\mathbf{y} \in \mathbb{R}_{\geq 0}^r$.

Note: This property already known to be important in Lorentzian polynomials and Bochner's method proof of Aleksandrov-Fenchel inequality.

How to get log-concave inequalities?

Assume a_{k-1}, a_k, a_{k+1} can be computed by

$$a_k = \langle \mathbf{g}, \mathbf{M}\mathbf{h} \rangle, \quad a_{k+1} = \langle \mathbf{g}, \mathbf{M}\mathbf{g} \rangle, \quad a_{k-1} = \langle \mathbf{h}, \mathbf{M}\mathbf{h} \rangle,$$

for specific $\mathbf{M}, \mathbf{g}, \mathbf{h}$ in the atlas.

$$\langle \mathbf{g}, \mathbf{M}\mathbf{h} \rangle^2 \geq \langle \mathbf{g}, \mathbf{M}\mathbf{g} \rangle \langle \mathbf{h}, \mathbf{M}\mathbf{h} \rangle \quad (\text{hyperbolic ineq.})$$

then implies

$$a_k^2 \geq a_{k+1} a_{k-1} \quad (\text{log-concave ineq.})$$

The strategy

Input: Acyclic digraph \mathcal{A} , where each vertex v is associated with

- $r \times r$ nonnegative symmetric matrix M ,
- A nonnegative r -vector h .

Goal: Show every M has hyperbolic inequality.

Method: Verify three conditions:

- **Irreducibility condition;**
- **Inheritance condition;**
- **Subdivergence condition.**

Irreducibility condition

- Matrix M associated to v is irreducible when restricted to its support;
- Vector h is associated to v is a positive vector.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Inheritance condition

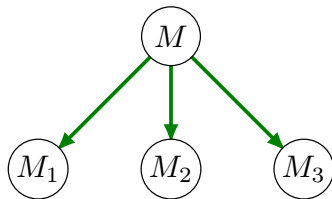
The i -th edge $e = (v, v_i)$ of v is associated with

$$\text{linear map } T_i : \mathbb{R}^r \rightarrow \mathbb{R}^r$$

such that, for every $\mathbf{x} \in \mathbb{R}^r$,

$$i\text{-th coordinate of } \mathbf{M}\mathbf{x} = \langle T_i \mathbf{x}, \mathbf{M}_i T_i \mathbf{h} \rangle,$$

where \mathbf{M} and \mathbf{h} are associated to v , while \mathbf{M}_i is associated to v_i .



Subdivergence condition

For every $\mathbf{x} \in \mathbb{R}^r$,

$$\sum_{i=1}^r h_i \langle T_i \mathbf{x}, \mathbf{M}_i T_i \mathbf{x} \rangle \geq \langle \mathbf{x}, \mathbf{M} \mathbf{x} \rangle,$$

where $h_i = i$ -th coordinate of \mathbf{h} .

Note: Often **hardest** condition to check,
usually done through **injective** arguments.

Note: Equality occurs for **matroids**.

Bottom-to-top principle for inequalities

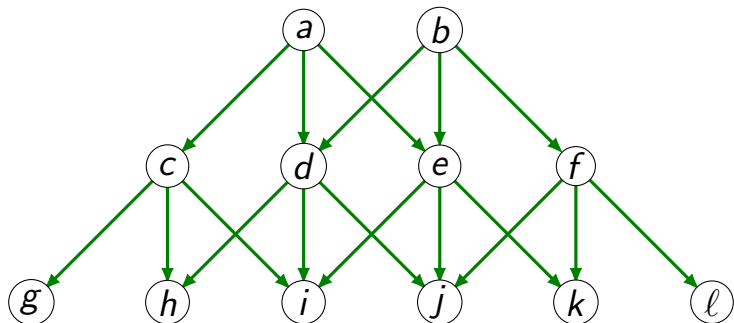
Proposition

Assume *irreducibility, inheritance, subdivergence*.

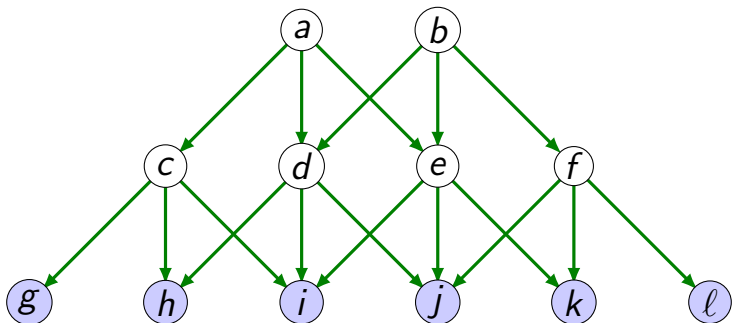
If M_1, \dots, M_r has hyperbolic inequality property, then so does M .

Bottom-to-top principle reduces **Goal** to checking hyperbolic inequality only for *sink vertices*, which are usually *easy* to check.

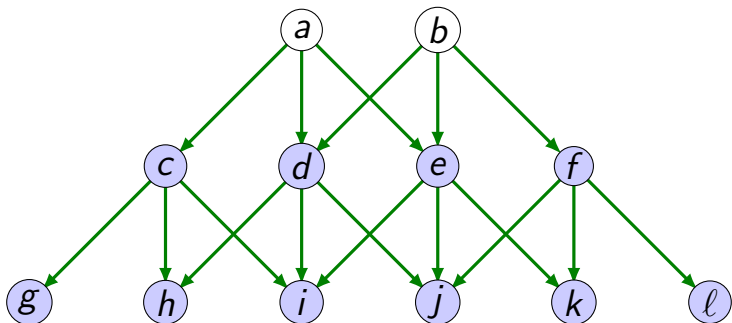
Bottom-to-top principle



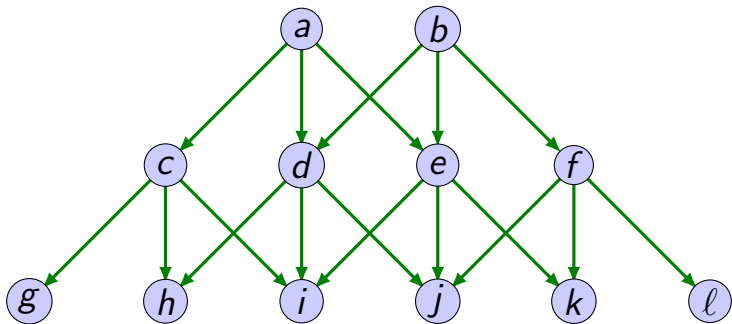
Bottom-to-top principle



Bottom-to-top principle



Bottom-to-top principle



How about equalities?

The strategy

Input:

- An acyclic digraph $\mathcal{A} := (\mathcal{V}, \mathcal{E})$ satisfying **previous** conditions;
- Vectors $\mathbf{g}, \mathbf{h} \in \mathbb{R}_{\geq 0}$;

Goal: Show “every” M has **hyperbolic equality**,

$$\langle \mathbf{g}, M\mathbf{h} \rangle^2 = \langle \mathbf{g}, M\mathbf{g} \rangle \langle \mathbf{h}, M\mathbf{h} \rangle.$$

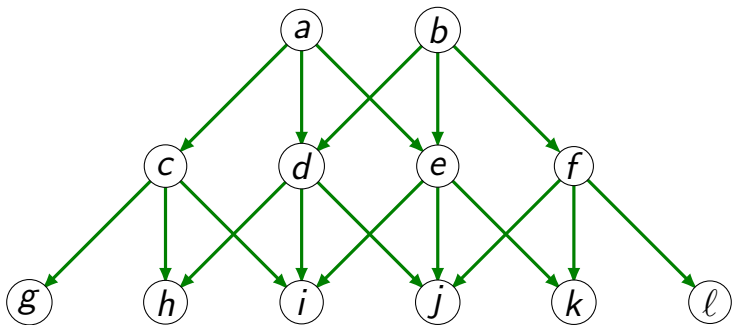
Top-to-bottom principle for equalities

Proposition

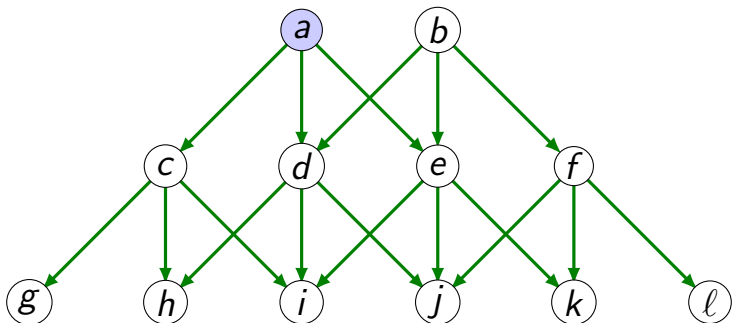
Assume *regularity* condition. If M has hyperbolic equality property, then so do M_1, \dots, M_r .

Top-to-bottom principle expands hyperbolic equality to sink vertices, which usually gives combinatorial characterizations.

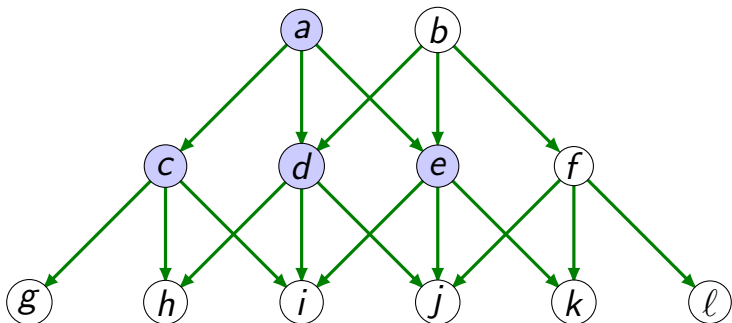
Top-to-bottom principle



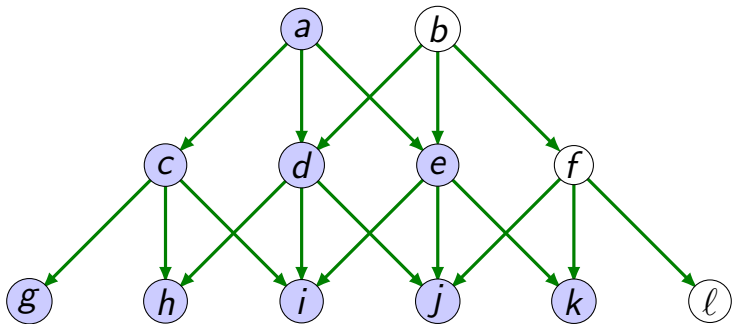
Top-to-bottom principle



Top-to-bottom principle



Top-to-bottom principle



Conclusion

Problem: Log-concave **inequalities** and **equalities**.

Strategy:

- Build a **combinatorial atlas**;
- Verify the required conditions;
- Use **hyperbolic inequality property** to derive **log-concave inequalities**;
- Use **hyperbolic equality** to derive **log-concave equalities**.

THANK YOU!

Preprint to appear soon in your nearest arXiv server.

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